Optimal Control Theory and the Efficiency of the Swimming Mechanism of the Copepod Zooplankton

Jérémy Rouot, Piernicola Bettiol, Bernard Bonnard

Optimal Control Theory and the Efficiency of the Swimming Mechanism of the Copepod Zooplankton

Jérémy Rouot, LAAS-CNRS, Toulouse

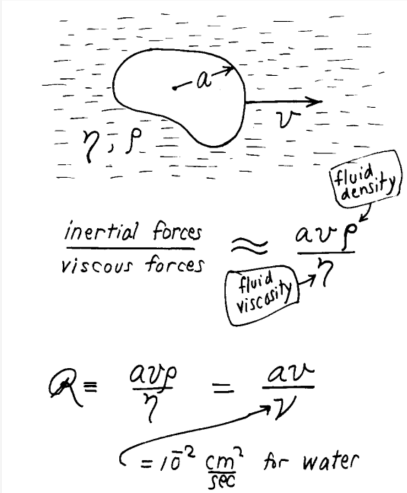
with Piernicola Bettiol (LBMA, Brest) and Bernard Bonnard (INRIA, Sophia Antipolis).

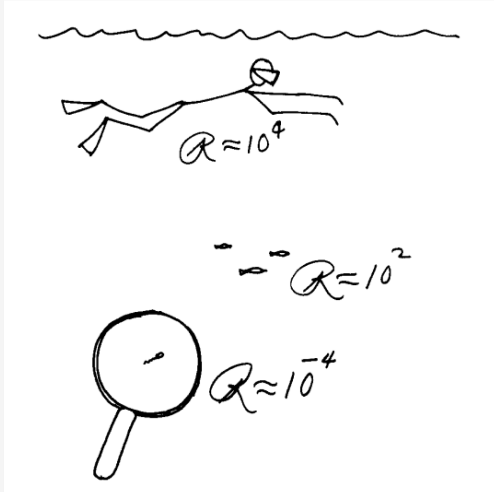
2017, 10th July

IFAC 2017 World Congress, Toulouse, France

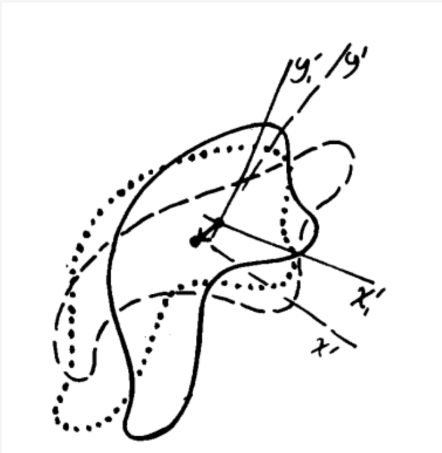
jrouot@laas.fr

Life at low Reynolds number - Purcell, 1977

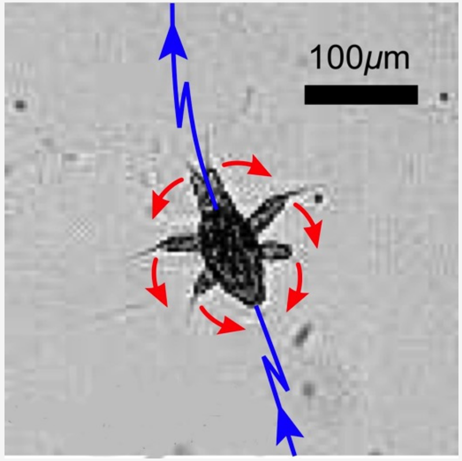




Reynolds number

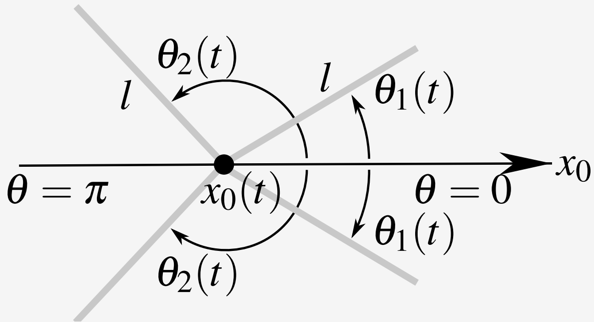


Shape deformations of a microswimmer



Zooplankton

2-link symmetric swimmer.



Dynamic.

$$\dot{x}_0 = \frac{\sum_{i=1}^{2} l^2\, \dot{\theta}_i \sin(\theta_i)}{2 + \sum_{i=1}^{2} \sin^2(\theta_i)}, \quad \dot{\theta}_i = u_i,\ i = 1,2 \quad \text{(state constraint : } 0 \leq \theta_1 \leq \theta_2 \leq \pi).$$

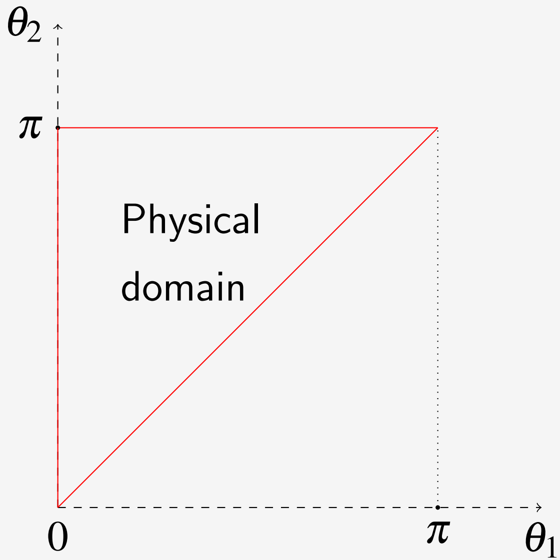
Criterion. minimize drag forces : $\dot{q} S(q) \dot{q}^{\intercal}$, $\boldsymbol{q = (\theta_1, \theta_2, x_0)}$ and S is positive definite $\Longrightarrow$ quadratic form in (u_1, u_2).

Aim

- Compute **optimal strokes** minimizing an efficiency.

- Stroke : T-**periodic motion of the shape variables** θ s.t. $x_0(T) - x_0(0) > 0$.

- Admissible stroke : find closed curves in the θ-plane **contained in the triangle** :



$$\mathcal{E}(q(\cdot)) = \textbf{Geometric Efficiency} = x_0(T)/l_{SR}(q)$$

SR Length l_{SR}.

$$l_{SR}(q) = \int_0^T \sqrt{L(q,u)}\,\mathrm{d}t, \qquad L(q,u) = a(q)\,u_1^2 + 2b(q)\,u_1u_2 + c(q)\,u_2^2$$

$$\min_{u(\cdot)} l_{SR}(q) \Leftrightarrow \min_{u(\cdot)} \int_0^T L(q,u)\mathrm{d}t$$

Two steps.

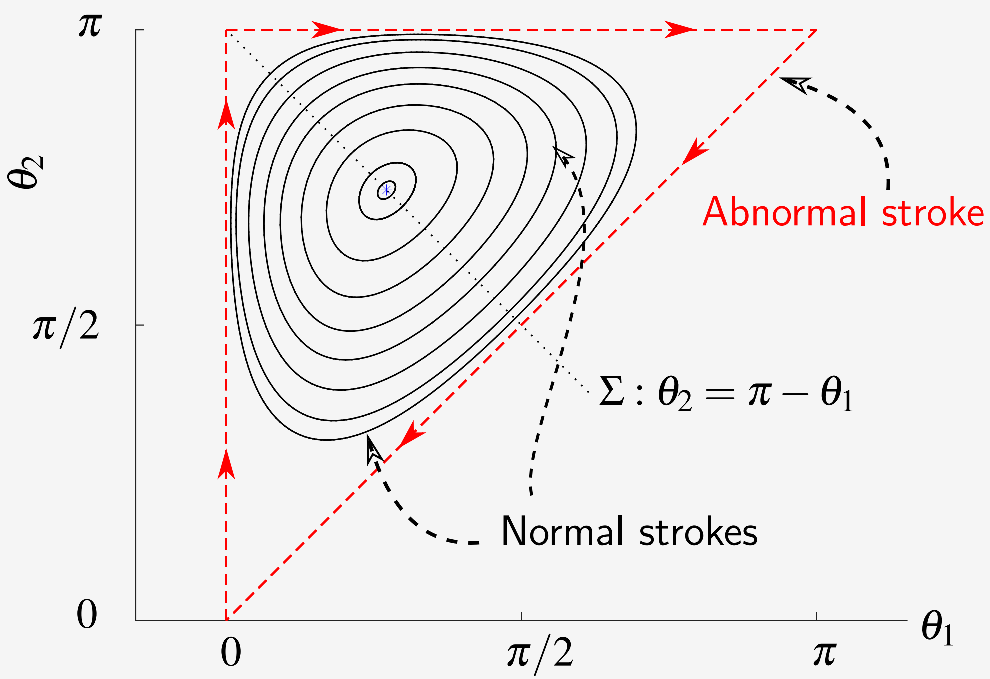
$$\dot{q} = \sum_{i=1}^{2} u_i F_i \leftarrow \min_{u(\cdot)} \int_0^T L(q,u)\mathrm{d}t,$$

$$\max_{u(\cdot),\,x_0(T)} \mathcal{E} \Leftrightarrow \left(\min_{u(\cdot)} \int_0^T L(q,u)\mathrm{d}t \;\; \text{with } \boldsymbol{x_0(T)} \textbf{ fixed}, \text{ then } \underbrace{\text{select} \max_{x_0(T)} \mathcal{E}}_{\text{Transversality cond.}} \right)$$

Compute points on the Sub-Riemannian sphere $\rightarrow$
provide candidate solutions for the maximum of efficiency problem.

$$\int_0^T L(q,u) \leftarrow \min_{u(\cdot)}$$

Proposition 1. *There is a one parameter family of normal simple loop strokes parameterized by the displacement* $x_0(T)$.

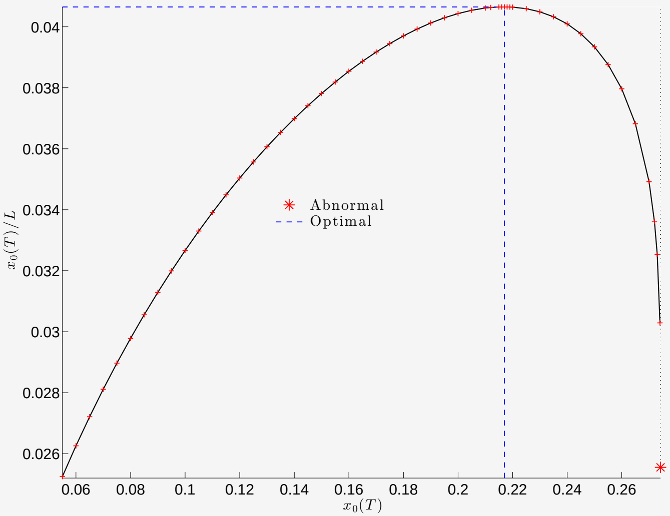


Select on the point on the SR sphere with maximum of efficiency

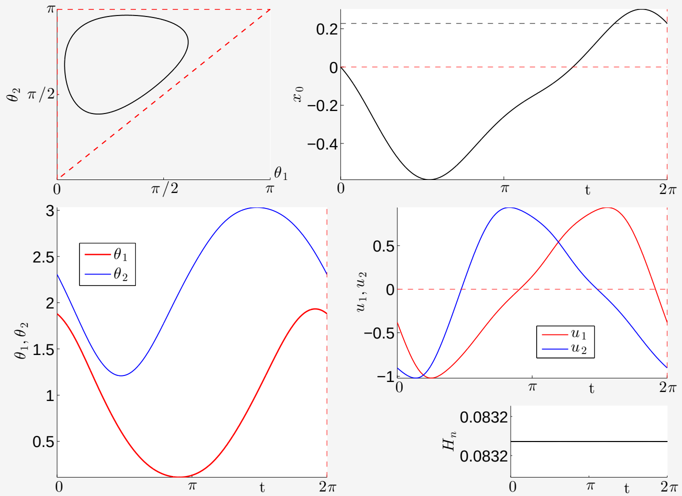
Geometric efficiency : $\mathscr{E} = x_0(T)/l_{SR}(q)$, $\quad$ **($x_0(T)$ free)**

Transversality condition of the maximum Principle

$$p_{x_0}(T) = q^0(T)/x_0(T), \qquad p^0(T) = -1/2$$



Efficiency curve with continuation on $x_0(T)$.



Optimal normal stroke

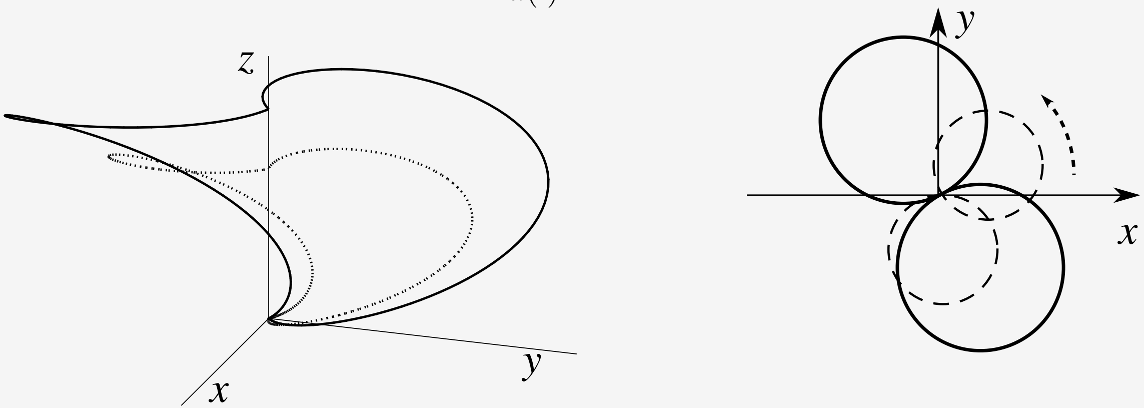
Theorem 2. *The abnormal stroke is not minimizing.*

Normal form of order -1 Nilpotent model of order -1, the Brockett-Heisenberg model :

$$\hat{F} = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \quad \hat{G} = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$$

where (weight of x, y)=1 and (weight of z)=2 (privileged coordinates).

$$u_1^2 + u_2^2 \leftarrow \min_{u(\cdot)}, \quad \dot{q} = u_1 \hat{F}(q) + u_2 \hat{G}(q)$$



Geodesics $q(\cdot)$ for the model of order -1.

These families of circles are **not generic**

Theorem 3 (Brockett, Chakir et al)**.** *The model of order* -1 *is equivalent to the model of order* 0*.*

Application to the Copepod z cannot be identified to the displacement x_0.
$x = \theta_1 - \theta_{10},\ y = \theta_2 - \theta_{20},\ \theta_{20} = \pi - \theta_{10}$. (contact point)

Perturbation of the model of order -1 by terms of order 0 :

$$F_1(x,y,z) = \frac{\partial}{\partial x} + (a_{11}(\theta_{10})xy + a_{02}(\theta_{10})y^2 + y)\frac{\partial}{\partial z}$$
$$F_2(x,y,z) = \frac{\partial}{\partial y} + (a_{11}(\theta_{10})xy + a_{02}(\theta_{10})x^2 - x)\frac{\partial}{\partial z}$$

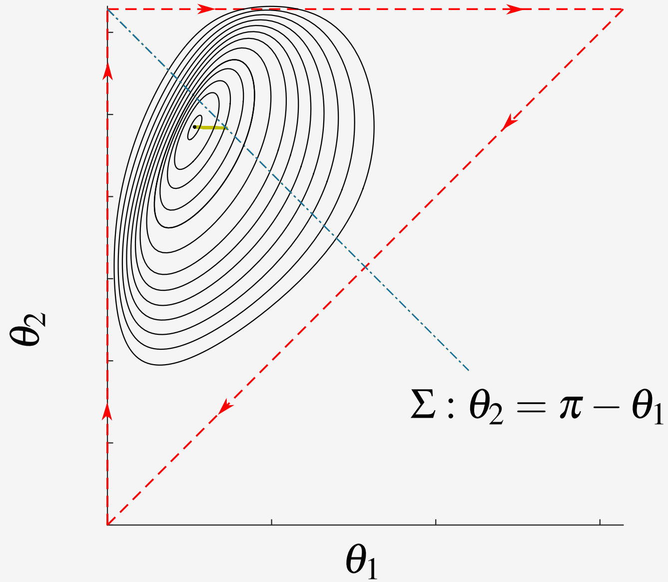
$$(F_1, F_2) \underset{\varphi}{\Longleftrightarrow} (\hat{F}, \hat{G})$$

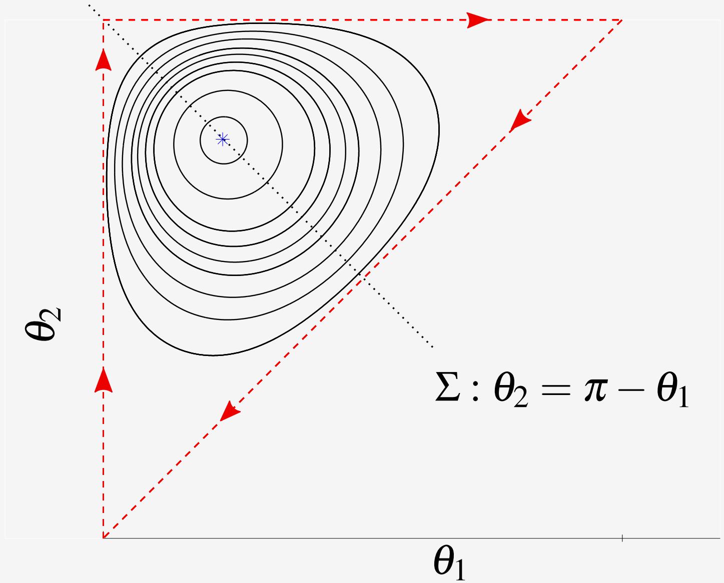
Lemma 4. *The only value of* θ_{10} *such that the transformation* φ *doesn't mix up the* **shape variables (x,y) and the displacement z** *corresponds to the center of swimming of the family.*

$$\varphi(x,y,z) = (x + c_{011}(\theta_{10})\,\boldsymbol{yz} + c_{001}(\theta_{10})\,\boldsymbol{z},\; y - c_{011}(\theta_{10})\,\boldsymbol{xz} + c_{001}(\theta_{10})\,\boldsymbol{z},\; z + P(x,y,z,\theta_{10}))$$

$$\underset{\text{Lem. 5}}{\Longrightarrow} c_{011}(\theta_{10}) = c_{001}(\theta_{10}) = 0 \quad \Leftrightarrow \quad \theta_{10} = \text{center of the family.}$$

Theorem 5. *The center of swimming is a SR-invariant.*





One parameter family of simple loops with different metrics.

SR-invariant and inverse problem : given two swimmers with their centers of swimming, we can recover the metric.

Second order sufficient optimality conditions

for a Copepod stroke

$$\begin{cases} \min & J(q(.),u(.)) := c(q(0),q(T)) \\ \text{s.t.} & \dot{q}(t) \ = \ F(q(t),u(t)) \\ & m(q(0),q(T)) = 0 \,, \end{cases}$$

$H(q,u,p) := p \cdot F(q,u)$ and $h(q_0,q_T) := c(q_0,q_T) + \nu \cdot m(q_0,q_T)$.
Consider a **normal extremal** $(\overline{q},\overline{p})$ associated with $\overline{u}$.

Second variation

$$\delta^2 J(\delta q(.),\delta u(.)) := 1/2\,[\delta q(0)^\intercal \quad \delta q(T)^\intercal]\,\boldsymbol{C}\,[\delta q(0) \quad \delta q(T)]^\intercal$$
$$+\,1/2\int_0^T (\delta q(t)^\intercal \boldsymbol{\partial}_{qq}\boldsymbol{H}(t)\delta q(t) + 2\,\delta q(t)^\intercal \boldsymbol{\partial}_{qu}\boldsymbol{H}(t)\delta u(t) + \delta u(t)^\intercal \boldsymbol{\partial}_{uu}\boldsymbol{H}(t)\delta u(t))\,\mathrm{d}t$$

where $\quad \boldsymbol{C} := \mathrm{Hessian}_{q_0,q_T}(h)$.

$$\begin{cases} \min & \delta^2 J(\delta q(.), \delta u(.)) \\ \text{s.t.} & \dot{\delta q} := \boldsymbol{\partial}_q \boldsymbol{F}(t)\,\delta q(t) + \boldsymbol{\partial}_u \boldsymbol{F}(t)\,\delta u(t) \\ & \nabla_{q_0} m(\overline{q}(0), \overline{q}(T))\,\delta q(0) + \nabla_{q_T} m(\overline{q}(0), \overline{q}(T))\,\delta q(T) = 0 \end{cases}$$

Classical optimality conditions

- *2nd order necessary conditions :* $\delta^2 J(\delta q(.), \delta u(.)) \geq 0$,
- *2nd order sufficient conditions :* $\delta^2 J(\delta q(.), \delta u(.))$ coercive.

Monodromy matrix. $\Phi(.,.)$ associated with the linearized Hamiltonian system :

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,s) = \boldsymbol{Z}\,\Phi(t,s) \\ \Phi(s,s) = \mathrm{Id}, \end{cases}$$

where

$$Z := \begin{bmatrix} \partial_q F - \partial_u F[\partial_{uu} H]^{-1} \partial_{qu} H^T & -\partial_u F[\partial_{uu} H]^{-1} \partial_u F^T \\ -\partial_{qq} H + \partial_{qu} H[\partial_{uu} H]^{-1} \partial_{qu} H^T & -\partial_q F[\partial_{uu} H]^{-1} \partial_u F^T \end{bmatrix}.$$

Define

$$\mathscr{W} := \begin{bmatrix} \phi_{22}\phi_{12}^{-1} & \phi_{21} - \phi_{22}\phi_{12}^{-1}\phi_{11} \\ -\phi_{12} & \phi_{12}^{-1}\phi_{11} \end{bmatrix}, \quad \Phi(0,T) =: \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

Theorem 6 (**Standard conditions**). *Assume*

(i) : $\boldsymbol{\partial}_{uu}\boldsymbol{H}(t) \leq -\varepsilon$ *Id on* $[0,T]$, $\overline{u}(.)$ *bounded and* $(\boldsymbol{\partial_q F}(.), \boldsymbol{\partial_u F}(.))$ *is controllable on* $[0,T]$,

(ii) : the extremal $(\overline{q}(.), \overline{u}(.), \overline{p}(.))$ **doesn't have conjugate points** *on* $[0,T]$,

(iii) : there exists $\gamma > 0$ *t.q.*

$$[\xi_0^T \ \xi_1^T]\, \mathscr{W} \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \geq \gamma \left\| \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \right\|^2 ,$$

for all vectors $\xi_0, \xi_1 \in \mathbb{R}^n \setminus \{0\}$ *s.t.*

$$\nabla_{q_0} m((\overline{q}(0), \overline{q}(T)))\xi_0 + \nabla_{q_T} m((\overline{q}(0), \overline{q}(T)))\xi_1 = 0.$$

Then $(\overline{q}(.), \overline{u}(.))$ *is a* $W^{1,\infty}$*-minimizer and* **locally unique***.*

Boundary values

$$\theta_j(0) = \theta_j(T) \quad j = 1,2,$$
$$x_0(0) = 0,\ x(T) = x_T, \quad x_T \text{ is fixed}$$

Proposition 7. *Take $I = (-\varepsilon, \varepsilon)$, $\varepsilon > 0$ and let $(\overline{q}(\cdot), \overline{u}(\cdot), \overline{p}(\cdot))$ be a normal extremal. For all $a \in I$ and $t \in [0,T]$, we define $q^a(\cdot) = (\theta_1^a(\cdot), \theta_2^a(\cdot), x^a(\cdot))$, $u_1^a(\cdot), u_2^a(\cdot)$ and $p^a(\cdot)$ by*

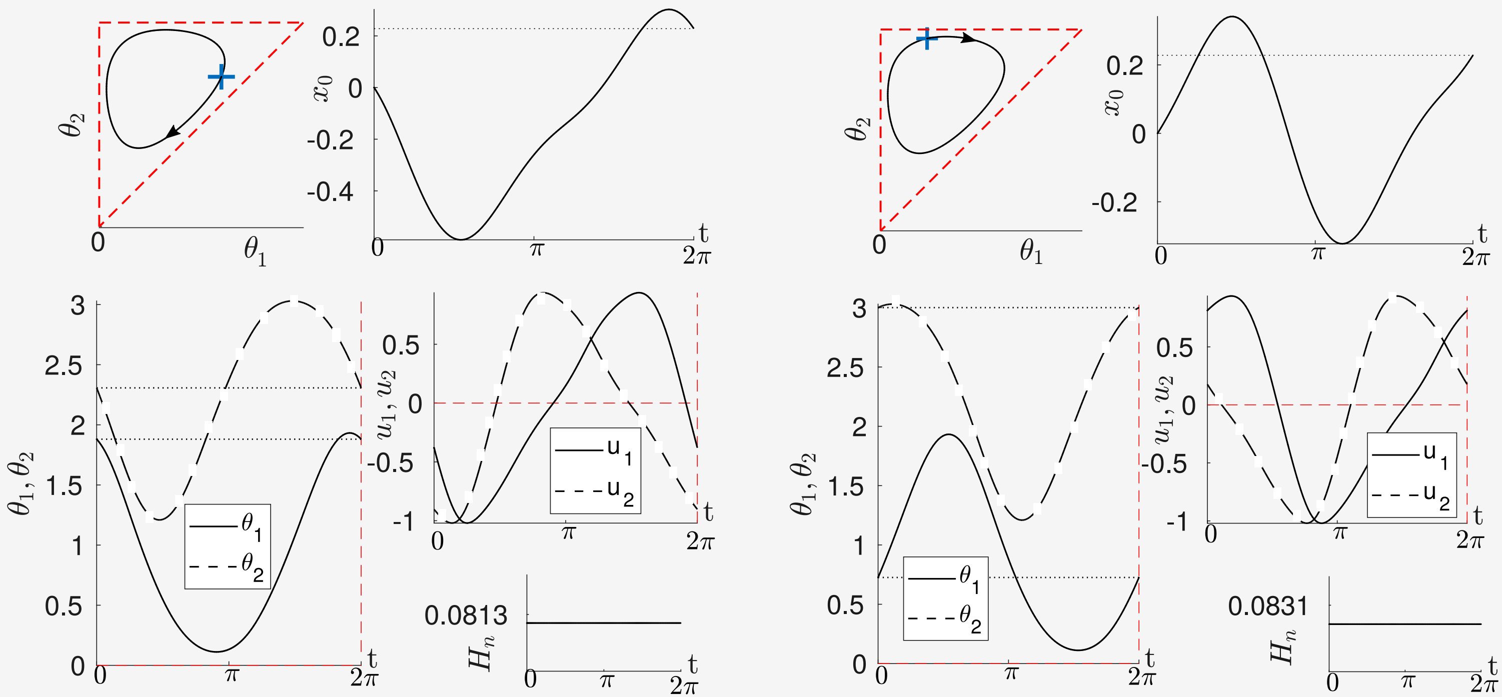
$$\theta_j^a(t) = \overline{\theta}_j(t+a), \quad u_j^a(t) = \overline{u}_j(t+a) \quad \text{for } j = 1,2,$$
$$x^a(t) = \overline{x}(t+a) - \overline{x}(a),\ p^a(t) = (\overline{p}_1(t), \overline{p}_2(t+a), \overline{p}_3(t+a)).$$

Then, for $\varepsilon > 0$ small enough, the normal extremal $(\overline{q}(.), \overline{p}(.), \overline{u}(.))$ is continuously embedded in the family of extremals $(q^a(.), p^a(.), u^a(.))_{a \in I}$.

These strokes have the SAME COST and satisfy the SAME BOUNDARY CONDITIONS

$\Longrightarrow$ **Standard conditions fail** because of non-unique minimizers.

Families of extremals with same cost and same boundary conditions



Theorem 8 (**Refined Conditions**, Gavriel, Vinter (2014)). *Assume the reference normal extremal* $(\overline{q}(.),\overline{u}(.),\overline{p}(.))$ *is continuously embedded in a family of extremals and*

(i) : $\boldsymbol{\partial}_{uu}\boldsymbol{H} \leq -\varepsilon$ *Id on* $[0,T]$, $(\boldsymbol{\partial}_{\boldsymbol{q}}\boldsymbol{F}(.),\boldsymbol{\partial}_{\boldsymbol{u}}\boldsymbol{F}(.))$ *is controllable on* $[0,T]$,

(ii) : the extremal $(\overline{q}(.),\overline{u}(.),\overline{p}(.))$ **doesn't have conjugate points** *on* $[0,T]$,

(iii) : there exists $\gamma > 0$ *s.t.*

$$[\xi_0^T \; \xi_1^T]\, \mathscr{W} \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \geq \gamma \left\| \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \right\|^2 ,$$

for all vectors $\xi_0, \xi_1 \neq 0$ *s.t.*

$$\nabla_{q_0} m((\overline{q}(0),\overline{q}(T)))\xi_0 + \nabla_{q_T} m((\overline{q}(0),\overline{q}(T)))\xi_1 = 0 \qquad \text{and} \qquad \boldsymbol{\Gamma}^T \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} = 0 \, .$$

where $\Gamma := \begin{bmatrix} \nabla_a q^a(0) \\ \nabla_a q^a(T) \end{bmatrix} \Big|_{a=0}$.

Then $(\overline{q}(.),\overline{u}(.))$ *is a* **local** $W^{1,\infty}$**-minimizer**.

Computation. Define the matrix N_s from the subspace $\mathscr{L}_s$ s.t.

$$\mathscr{L}_s = \{\, (\xi_0, \xi_T) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \nabla_{q_0,q_T} m(q_0,q_T)\, (\xi_0 \quad \xi_T)^{\intercal} = 0 \,\} =: \mathrm{Im}(N_s)$$

Standard conditions. Does the matrix $\mathscr{W}_s := N_s^{\intercal}\,(\mathscr{W}^{\intercal} + \mathscr{W})\,N_s \in \mathscr{M}_2$ is positive-definite ?

Consider

$$\Gamma_r = \big(\nabla_a q^a(0) \;\; \nabla_a q^a(T)\big)_{a=0} = \big(\dot{\bar{q}}(0) \;\; \dot{\bar{q}}(T)\big)$$

and the linear subspace $\mathscr{L}_r$ s.t.

$$\mathscr{L}_r := \mathscr{L}_s \cap \{(\xi_0, \xi_T) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \Gamma_r^{\intercal}\,(\xi_0 \quad \xi_T)^{\intercal} = 0\} =: \mathrm{Im}(N_r)$$

Refined conditions. Does $\mathscr{W}_r := N_r^{\intercal}\,(\mathscr{W}^{\intercal} + \mathscr{W})\,N_r > 0$?

Relative tolerance	(Standard condition) $\mathrm{Spec}(W_s)$	(Refined condition) $\mathrm{Spec}(W_r)$
10^{-5}	**6.89e-4** 3.42	22.5
10^{-8}	**-9.12e-7** 3.42	22.5

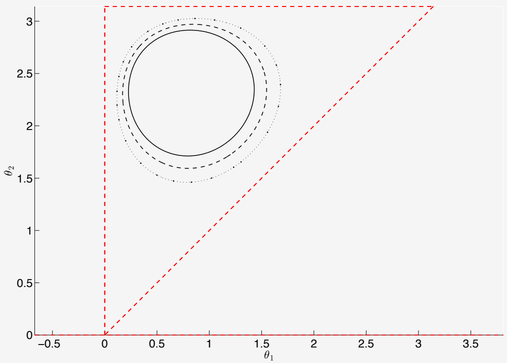
— Standard conditions fail : $\mathscr{W}_s$ has a zero eigenvalue.
— **BUT** Refined conditions are satisfied : $\mathscr{W}_r$ is positive-definite.

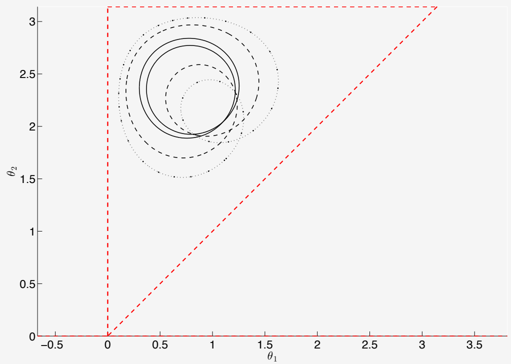
Theorem 9 (Numerical)**.** *The simple loop normal stroke* $(\overline{q}, \overline{u})$ *is* $W^{1,\infty}-optimal.$

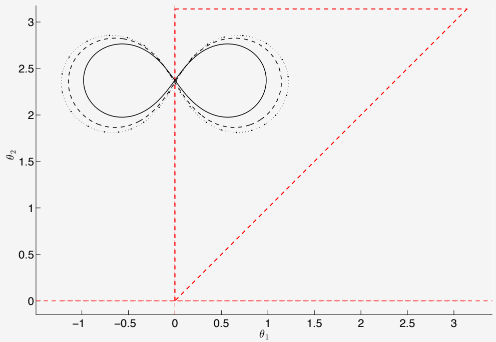
Perspectives

- *Contact point* : expression of the generic normal form at any point inside the triangle $\rightarrow$ unique family of simple loops.
- *Martinet point* : compute the normal form for a point on the edges $\rightarrow$ locate the eight loops.
- estimation of the first conjugate time using normal forms.
- swimmer model with more than 2 pairs of links.

Generate normal strokes



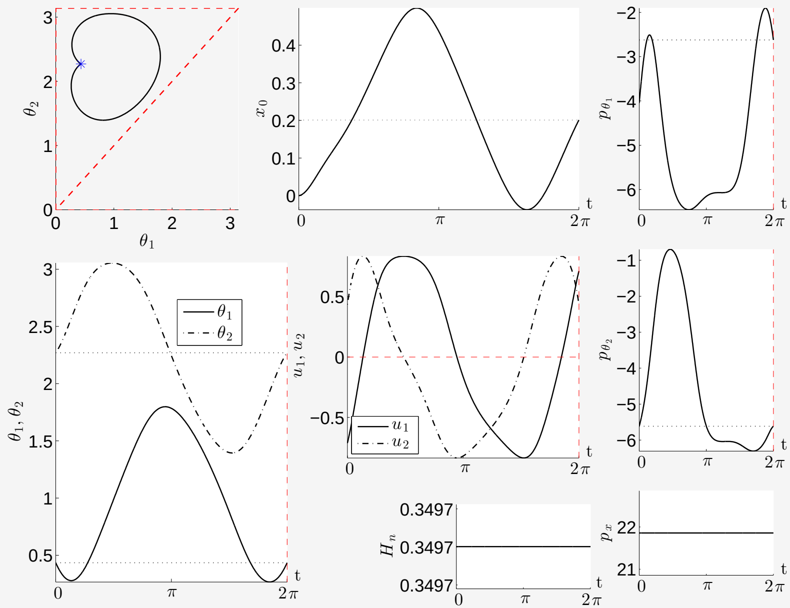




$$\mathscr{E}' = \frac{x_0(T)\,\boldsymbol{m(\theta(0))}}{l(q)}, \quad (m \text{ smooth})$$

Transversality condition of the maximum Principle

$$p_\theta(0) - p_\theta(2\pi) = \lambda \frac{\partial \mathscr{E}'}{\partial \theta(0)}$$



Theorem 10 (Chakir, Gauthier, Kupka, 1996)**.** *The* **generic model** *is given by the normal form of order* 1

$$F = \hat{F} + yQ(x,y)\frac{\partial}{\partial z},\ G = \hat{G} - xQ(x,y)\frac{\partial}{\partial z},$$

Q *quadratic in* (x,y)*.*

Remark 11. *This normal form can be used to approximate the one parameter family of simple strokes.*