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Abstract

Considering a retarded nonlinear system, this note proposes several modifications of the Lyapunov-Razumikhin approach guaranteeing the existence of an upper estimate on convergence rate of the system solutions. The cases of exponential, finite-time and fixed-time (with respect to a ball) convergences are studied. The proposed approach is illustrated by simulation of academic examples.

1 Introduction

There exist two generic frameworks assessing asymptotic stability of time-delay systems, which are based on analysis of a Lyapunov-Razumikhin function or a Lyapunov-Krasovskii functional [8, 10]. The latter method has been proven to be equivalent to the asymptotic stability property for some particular classes of the time-delay systems [4, 15, 16], and it can also be used to establish finite-time stability [13]. The former approach is only sufficient for the asymptotic stability [8, 10], and it is less intuitive while obtaining the rate of solution convergence [3, 6, 14]. An advantage of Lyapunov-Razumikhin approach with respect to Lyapunov-Krasovskii one is that in many nonlinear cases it is more simple to find a Lyapunov-Razumikhin function than a Lyapunov-Krasovskii functional [5, 7] (e.g., a Lyapunov function for the delay-free case can be tested).

The objective of this work is to overcome one of the main drawbacks of the Lyapunov-Razumikhin function approach, and to propose its several extensions, which allow the rate of solution convergence to be estimated using the method. Three cases will be considered: the systems with asymptotic rate of convergence of solutions to the origin (the expansion of [14] is given), with a finite time of convergence and a fixed-time one with respect to a ball (the definitions of these kinds of comportment are given below). The obtained results are used to formulate examples of differential inequalities for Lyapunov-Razumikhin function providing the studied convergence rates.

The paper is organized as follows. Preliminaries are given in Section 2. The problem statement is presented in Section 3, and the main results are formulated in Section 4. The performed simulations are described in Section 5. The final remarks and discussion are presented in Section 6.

2 Preliminaries

The real numbers are denoted by $\mathbb{R}$, $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$. Euclidean norm for a vector $x \in \mathbb{R}^n$ is denoted as $|x|$. We denote by $C_{[a,b]}$, $-\infty < a < b < +\infty$ the Banach space of continuous functions $\phi : [a,b] \to \mathbb{R}^n$ with the uniform norm $\|\phi\| = \sup_{a\leq\varsigma\leq b} |\phi(\varsigma)|$.

Consider an autonomous functional differential equation of retarded type [11]:

$$dx(t)/dt = f(x_t),\ t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $x_t \in C_{[-\tau,0]}$ is the state function, $x_t(s) = x(t+s)$, $-\tau \leq s \leq 0$ and $\tau > 0$ is a finite delay; $f : C_{[-\tau,0]} \to \mathbb{R}^n$ is a continuous function, $f(0) = 0$, and is such that solutions in forward time for the system (1) exist and are unique [11]. Denote such a unique solution $x(t, x_0)$ satisfying the initial condition $x_0 \in C_{[-\tau,0]}$ and $x_t(s, x_0) = x(t + s, x_0)$ for $-\tau \leq s \leq 0$, which is defined on some finite time interval $[-\tau, T)$ with $0 < T \leq +\infty$ (we will use the notation $x(t)$ to reference $x(t, x_0)$ if the origin of x_0 is clear from the context). The representation (1) includes pointwise or distributed time-delay systems.

For a locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ the upper directional Dini derivative is defined as follows:

$$D^+V(x)v = \limsup_{h\to 0^+} \frac{V(x+hv) - V(x)}{h}$$

for any $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$.

A continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$; it belongs to class

$\mathcal{K}_\infty$ if it is also radially unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{KL}$ if $\beta(\cdot, r) \in \mathcal{K}$ and $\beta(r, \cdot)$ is decreasing to zero for any fixed $r \in \mathbb{R}_+$.

2.1 Stability definitions

Let Ω be a neighborhood of zero in $C_{[-\tau,0]}$.

Definition 1 *[8, 10, 13] At the origin the system* (1) *is said to be*

(a) *stable if there is $\sigma \in \mathcal{K}$ such that for any $x_0 \in \Omega$, the solutions are defined and $|x(t, x_0)| \le \sigma(\|x_0\|)$ for all $t \ge 0$;*

(b) *asymptotically stable if it is stable and*

$$\lim_{t\to+\infty} |x(t, x_0)| = 0$$

for any $x_0 \in \Omega$;

(c) *finite-time stable if it is stable and for any $x_0 \in \Omega$ there exists $0 \le T^{x_0} < +\infty$ such that $x(t, x_0) = 0$ for all $t \ge T^{x_0}$. The functional $T_0(x_0) = \inf\{T^{x_0} \ge 0 : x(t, x_0) = 0 \ \forall t \ge T^{x_0}\}$ is called the settling time of the system* (1)*;*

(d) *fixed-time to a ball stable if it is stable and for any $\varrho > 0$ there exists $0 < T_\varrho < +\infty$ such that $|x(t, x_0)| \le \varrho$ for all $t \ge T_\varrho$ and all $x_0 \in \Omega$.*

If $\Omega = C_{[-\tau,0]}$, then the corresponding properties are called global stability/asymptotic stability/finite-time stability/fixed-time to a ball stability.

The definition of global asymptotic stability can be given in terms of existence of a function $\beta \in \mathcal{KL}$ such that $|x(t, x_0)| \le \beta(\|x_0\|, t)$ for all $x_0 \in C_{[-\tau,0]}$ and $t \ge 0$.

Remark 1 *The property of fixed-time to a ball stability is conceptually different from the fixed-time stability notion studied in [17, 18], since in the former case $\sup_{\varrho>0} T_\varrho \le +\infty$, i.e., at the origin a fixed-time to a ball stable system* (1) *may be just asymptotically stable. The difference between fixed-time to a ball stable and asymptotically stable systems becomes important for an unbounded set Ω only.*

For finite-time stability of (1) there exists the following sufficient condition:

Proposition 1 *[13] Let the system* (1) *admit uniqueness of solutions in the forward time. If there exist a continuous functional $V : \Omega \to \mathbb{R}_+$, $\eta_1, \eta_2 \in \mathcal{K}_\infty$, $\rho \in \mathcal{K}$ and $\epsilon > 0$ such that $\dot z(t) = -\rho(z(t))$ has a flow for all $z(0), t \in \mathbb{R}_+$ with*

$$\int_0^\epsilon \frac{dz}{\rho(z)} < +\infty,$$

and for all $\phi \in \Omega$:

$$\eta_1(|\phi(0)|) \le V(\phi) \le \eta_2(\|\phi\|),\ \dot V(\phi) \le -\rho(V(\phi)),$$

then the system (1) *is finite-time stable at the origin with the settling time satisfying an upper estimate:*

$$T_0(\phi) \le \int_0^{V(\phi)} \frac{dz}{\rho(z)}.$$

A usual example of such a function ρ includes

$$\rho(z) = az^\alpha$$

for $a > 0$ and $\alpha \in [0, 1)$. Then it is straightforward to check that under conditions of Proposition 1, the solutions of system (1) admit an upper estimate for $x_0 \in \Omega$ and all $t \ge 0$:

$$|x(t, x_0)| \le \max\{0, \eta_1^{-1} \circ \left(\eta_2^{1-\alpha}(\|x_0\|) - a(1-\alpha)t\right)^{\frac{1}{1-\alpha}}\}.$$

If $\alpha = 1$, then the system is asymptotically stable and it admits an exponential convergence rate:

$$|x(t, x_0)| \le \eta_1^{-1}\left(\eta_2(\|x_0\|)\exp(-at)\right)$$

for all $t \ge 0$ and $x_0 \in \Omega$. Note that for any $\beta \in \mathcal{KL}$ there exist $\theta_1, \theta_2 \in \mathcal{K}_\infty$ such that [19]:

$$\beta(s, t) \le \theta_1\left(\theta_2(s)\exp(-t)\right) \quad \forall s \ge 0,\ t \ge 0,$$

then under a suitable bound substitution, an estimate with exponential convergence rate can be proposed for any asymptotically stable system. Finally, if $\alpha > 1$, then (1) is fixed-time to a ball stable:

$$|x(t, x_0)| \le \eta_1^{-1}\left(\frac{1}{\left(\eta_2^{1-\alpha}(\|x_0\|) + a(\alpha-1)t\right)^{\frac{1}{\alpha-1}}}\right)$$

for all $t \ge 0$ and $x_0 \in \Omega$. Such a type of convergence is also called polynomial (in time). However, since for big deviations of $\|x_0\|$ the speed of convergence is faster than exponential, and to any ball the time of convergence is uniform in the initial conditions $x_0 \in C_{[-\tau,0]}$, in order to highlight these features, this property will be refereed here as the fixed-time to a ball stability. Nevertheless, close to the origin the system converges slower than in any precedent case.

Moreover, the following counterpart of Proposition 1 for fixed-time to a ball stable systems can be proposed:

Proposition 2 *Let the system* (1) *admit uniqueness of solutions in the forward time. If there exist a continuous functional $V : \Omega \to \mathbb{R}_+$, $\eta_1, \eta_2 \in \mathcal{K}_\infty$ and $\rho \in \mathcal{K}$ such that $\dot z(t) = -\rho(z(t))$ has a flow for all $z(0), t \in \mathbb{R}_+$ with*

$$\int_\vartheta^{+\infty} \frac{dz}{\rho(z)} < +\infty$$

for any $\vartheta > 0$, and for all $\phi \in \Omega$:

$$\eta_1(|\phi(0)|) \le V(\phi) \le \eta_2(\|\phi\|),\ \dot V(\phi) \le -\rho(V(\phi)),$$

then the system (1) *is fixed-time to a ball stable.*

PROOF. The stability follows from standard arguments. From any initial conditions in Ω the time of reaching any ball of radius $\varrho > 0$ can be evaluated as follows:

$$T_\varrho = \int_0^{T_\varrho} dt \le -\int_{V(0)}^{\eta_1(\varrho)} \frac{dV}{\rho(V)} \le \int_{\eta_1(\varrho)}^{+\infty} \frac{dV}{\rho(V)} < +\infty.$$

3 Problem statement

As we mentioned above, there exist two methods evaluating asymptotic stability of the system (1) based on a Lyapunov-Razumikhin function or a Lyapunov-Krasovskii functional. The Lyapunov-Krasovskii approach is used in propositions 1 and 2 to establish finite-time/fixed-time to a ball stability of (1), and there are also converse results for asymptotic stability in [4, 15, 16], while the Lyapunov-Razumikhin method can be formulated as follows:

Theorem 1 *[11] Let there exist a locally Lipschitz continuous Lyapunov-Razumikhin function* $V : \mathbb{R}^n \to \mathbb{R}_+$ *such that*

(i) for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *and all* $x \in \mathbb{R}^n$*:*

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|);$$

(ii) for some $\alpha, \gamma \in \mathcal{K}$*, with* $\gamma(s) > s$ *for all* $s > 0$*, and all* $\varphi \in C_{[-\tau,0]}$*:*

$$\max_{\theta\in[-\tau,0]} V(\varphi(\theta)) \le \gamma \circ V(\varphi(0)) \Rightarrow$$
$$D^+V(\varphi(0)) f(\varphi) \le -\alpha(|\varphi(0)|).$$

Then the system (1) *is globally asymptotically stable at the origin.*

The shortage of the Lyapunov-Razumikhin approach is that there is no result relating the behavior of $V(x)$ and the rate of convergence in nonlinear systems (as we have demonstrated after Proposition 1, the Lyapunov-Krasovskii method can be applied for this purpose).

The goal of the present note is to overcome the last drawback, and to propose mild modifications of the Lyapunov-Razumikhin approach allowing the exponential, finite-time and fixed-time convergence rates to be estimated using the method.

4 Main result

The previous evaluations by the Lyapunov-Razumikhin approach of the convergence rate for an asymptotically stable system have been presented in [3, 12, 14].

Theorem 2 *Let there exist a locally Lipschitz continuous Lyapunov-Razumikhin function* $V : \mathbb{R}^n \to \mathbb{R}_+$ *such that*

(i) for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *and all* $x \in \mathbb{R}^n$*:*

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|);$$

(ii) for some $\gamma' > 1$*,* $\alpha' > 0$ *and all* $\varphi \in C_{[-\tau,0]}$*:*

$$\max_{\theta\in[-\tau,0]} V(\varphi(\theta)) \le \gamma' V(\varphi(0)) \Rightarrow$$
$$D^+V(\varphi(0)) f(\varphi) \le -\alpha' V(\varphi(0)).$$

Then the origin is globally asymptotically stable for the system (1) *with exponential rate of convergence, and for all* $x_0 \in C_{[-\tau,0]}$ *and* $t \ge 0$*:*

$$|x(t, x_0)| \le \alpha_1^{-1}\left(\exp\left(-\min\{\alpha', \frac{\ln\gamma'}{\tau}\}t\right)\alpha_2(\|x_0\|)\right).$$

PROOF. Since all conditions of Theorem 1 are satisfied with $\alpha(s) = \alpha'\alpha_1(s)$ and $\gamma(s) = \gamma' s$, then the system (1) is globally asymptotically stable at the origin. Take any $x_0 \in C_{[-\tau,0]}$, the solution $x(t, x_0)$ is well defined for all $t \ge 0$, and in particular

$$V(x(t)) \le \max_{\theta\in[-\tau,0]} V(x_0(\theta))$$

for all $t \ge 0$ [10].

First, assume that the implication given in the formulation of the theorem is not satisfied and

$$\max_{\theta\in[-\tau,0]} V(x_t(\theta)) > \gamma' V(x(t))$$

for an interval of time $t \in [0, t_1]$, where $t_1 \ge 0$ is (possibly infinite) instant of time that the relation stated in Theorem 1 holds. This inequality implies that for all $t \in [0, t_1]$ there exists

$$\theta_t = \min\{\vartheta \in [-\tau, 0] : V(x_t(\vartheta)) = \max_{\theta\in[-\tau,0]} V(x_t(\theta))\},$$

where we introduce the minimum over $\vartheta \in [-\tau, 0]$ to resolve the non-uniqueness issue. Note that the inequality $\theta_t \le -\varepsilon_{x_0}$ is satisfied for some $\varepsilon_{x_0} \in (0, \tau]$ dependent on initial conditions x_0, since the maximum is calculated under the restriction that $\max_{\theta\in[-\tau,0]} V(x_t(\theta)) > \gamma' V(x(t))$ with $\gamma' > 1$ and the solution $x(t, x_0)$ is bounded for $t \ge 0$. Thus,

$$V(x(t)) < \frac{1}{\gamma'} V(x_t(\theta_t)) = \exp(-\ln\gamma') V(x_t(\theta_t))$$
$$\le \exp\left(\ln\gamma' \frac{\theta_t}{\tau}\right) V(x_t(\theta_t)).$$

Recursively applying this estimate, i.e.,

$$V(x_t(\theta_t)) = V(x(t + \theta_t))$$
$$< \exp\left(\ln\gamma' \frac{\theta_{t+\theta_t}}{\tau}\right) V(x_{t+\theta_t}(\theta_{t+\theta_t})),$$

we obtain

$$V(x(t)) < \exp\left(\ln\gamma' \frac{\theta_t + \theta_{t+\theta_t}}{\tau}\right) V(x_{t+\theta_t}(\theta_{t+\theta_t})),$$

and by induction,

$$V(x(t)) \le \exp\left(-\frac{\ln\gamma'}{\tau}t\right) \max_{\theta\in[-\tau,0]} V(x_0(\theta))$$
$$\le \exp\left(-\min\{\alpha', \frac{\ln\gamma'}{\tau}t\}\right) \max_{\theta\in[-\tau,0]} V(x_0(\theta)) \quad (2)$$

for all $t \in [0, t_1]$ (i.e., for $t \ge 0$ sufficiently small it could be $t + \theta_t < 0$ and the sum $\theta_t + \theta_{t+\theta_t} + \ldots$ above belongs to the interval $[-t - \tau, -t]$).

Now, suppose that for $t \in [t_1, t_2)$ the implication $\max_{\theta\in[-\tau,0]} V(x_t(\theta)) \leq \gamma' V(x(t))$ holds, where $t_2 \geq t_1$ is (possibly again infinite) time instant that the relation stated in Theorem 1 fails for the first time higher than t_1 (by their definitions, $t_1 + t_2 > 0$). Obviously,

$$D^+ V(x(t)) f(x_t) \leq -\alpha' V(x(t))$$

for all $t \in [t_1, t_2)$ and, consequently,

$$V(x(t)) \leq \exp(-\alpha'(t - t_1)) V(x(t_1)).$$

Hence, we obtain that the estimate (2) is satisfied for all $t \in [0, t_2)$. Next, the analysis above can be iterated for all $t \geq 0$.

Therefore, let us check by contradiction that this estimate (2) is actually valid for all $t \geq 0$. Recall that $V(x(t)) \leq \max_{\theta\in[-\tau,0]} V(x_0(\theta))$ for all $t \geq 0$ and let $t_3 \geq 0$ be a time instant such that

$$V(x(t_3)) = \exp\left(-\min\{\alpha', \frac{\ln\gamma'}{\tau} t\} t_3\right) \max_{\theta\in[-\tau,0]} V(x_0(\theta))$$

and

$$V(x(t)) \leq \exp\left(-\min\{\alpha', \frac{\ln\gamma'}{\tau} t\} t\right) \max_{\theta\in[-\tau,0]} V(x_0(\theta))$$

for all $t \in [0, t_3)$, i.e., the estimate (2) is reached exactly at the instant t_3. These properties imply that $\max_{\theta\in[-\tau,0]} V(x_{t_3}(\theta)) \leq \gamma' V(x(t_3))$ and, consequently, $D^+ V(x(t_3)) f(x_{t_3}) \leq -\alpha' V(x(t_3))$, which means that (2) cannot be violated, and the inequality has to be also preserved at the instant t_3. The same arguments can be applied further for all $t \geq t_3$, and the required exponential convergence rate in (1) follows.

Remark 2 *The result is formulated for the case of global asymptotic stability, and its modification for a local analysis is straightforward (the same for other results of the paper).*

The next result is a counterpart of Proposition 1:

Theorem 3 *Let there exist a locally Lipschitz continuous Lyapunov-Razumikhin function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that*

(i) for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and all $x \in \mathbb{R}^n$:

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|);$$

(ii) for some $\mu \in (0, 1)$, $c > 0$, $\alpha' > 0$ and all $\varphi \in C_{[-\tau,0]}$:

$$\max_{\theta\in[-\tau,0]} V^{1-\mu}(\varphi(\theta)) \leq V^{1-\mu}(\varphi(0)) + c\tau \Rightarrow$$
$$D^+ V(\varphi(0)) f(\varphi) \leq -\alpha' V^{\mu}(\varphi(0)).$$

Then the origin is globally finite-time stable for the system (1), and for all $x_0 \in C_{[-\tau,0]}$ and $t \geq 0$:

$$|x(t, x_0)| \leq \max\{0, \alpha_1^{-1} \circ (\alpha_2^{1-\mu}(\|x_0\|) - \min\{\alpha'(1-\mu), c\} t)^{\frac{1}{1-\mu}}\}.$$

PROOF. The conditions of Theorem 1 for asymptotic stability are not satisfied. However, the system (1) is globally stable at the origin since we have an implication:

$$\max_{\theta\in[-\tau,0]} V(\varphi(\theta)) \leq V(\varphi(0)) \Rightarrow$$
$$\max_{\theta\in[-\tau,0]} V^{1-\mu}(\varphi(\theta)) \leq V^{1-\mu}(\varphi(0)) + c\tau \Rightarrow$$
$$D^+ V(\varphi(0)) f(\varphi) \leq 0,$$

then for any $x_0 \in C_{[-\tau,0]}$ the solution $x(t, x_0)$ is well defined for all $t \geq 0$, and

$$V(x(t)) \leq \max_{\theta\in[-\tau,0]} V(x_0(\theta))$$

for all $t \geq 0$ [10].

First, assume that the implication given in the formulation of the theorem is not satisfied and

$$\max_{\theta\in[-\tau,0]} V^{1-\mu}(x_t(\theta)) > V^{1-\mu}(x(t)) + c\tau$$

for an interval of time $t \in [0, t_1]$, where $t_1 \geq 0$ is (possibly infinite) instant of time that the relation is failed. This inequality implies that for all $t \in [0, t_1]$ there exists $\theta_t \in [-\tau, 0]$ as in Theorem 2 such that

$$V^{1-\mu}(x(t)) \leq \max\{0, V^{1-\mu}(x_t(\theta_t)) - c\tau\}$$
$$\leq \max\{0, V^{1-\mu}(x_t(\theta_t)) + c\theta_t\}.$$

Recursively applying this estimate, i.e., $V^{1-\mu}(x_t(\theta_t)) = V^{1-\mu}(x(t+\theta_t)) \leq \max\{0, V^{1-\mu}(x_{t+\theta_t}(\theta_{t+\theta_t})) + c\theta_{t+\theta_t}\}$, we obtain

$$V^{1-\mu}(x(t)) \leq \max\{0, V^{1-\mu}(x_{t+\theta_t}(\theta_{t+\theta_t})) + c(\theta_t + \theta_{t+\theta_t})\},$$

and by induction,

$$\begin{aligned} V^{1-\mu}(x(t)) &\leq \max\{0, \max_{\theta\in[-\tau,0]} V^{1-\mu}(x_0(\theta)) - ct\} \\ &\leq \max\{0, \max_{\theta\in[-\tau,0]} V^{1-\mu}(x_0(\theta)) \\ &\quad - \min\{\alpha'(1-\mu), c\} t\} \end{aligned} \tag{3}$$

for all $t \in [0, t_1]$. Note that this estimate implies that once $x(t') = 0$ for some $t' \in [0, t_1]$, then $x(t) = 0$ for all $t \in [t', t_1]$, which corresponds to the observations given in [6].

Now, suppose that for $t \in [t_1, t_2)$ the implication $\max_{\theta\in[-\tau,0]} V^{1-\mu}(x_t(\theta)) \leq V^{1-\mu}(x(t)) + c\tau$ holds, where $t_2 \geq t_1$ is (possibly again infinite) time instant that this relation is broken for the first time after t_1. By the conditions of the theorem:

$$D^+ V(x(t)) f(x_t) \leq -\alpha' V^{\mu}(x(t))$$

for all $t \in [t_1, t_2)$ and, consequently,

$$V^{1-\mu}(x(t)) \leq \max\{0, V^{1-\mu}(x(t_1)) - \alpha'(1-\mu)(t - t_1)\}.$$

Therefore, we obtain that the estimate (3) is satisfied for all $t \in [0, t_2)$. As in Theorem 2, this consideration can be iterated for all $t \geq 0$.

Let us demonstrate by contradiction that the estimate (3) is actually valid for all $t \geq 0$. Recall that $V(x(t)) \leq \max_{\theta\in[-\tau,0]} V(x_0(\theta))$ for all $t \geq 0$ and let $t_3 \geq 0$ be a time instant such that

$$V^{1-\mu}(x(t_3)) = \max\{0, \max_{\theta\in[-\tau,0]} V^{1-\mu}(x_0(\theta)) - \min\{\alpha'(1-\mu), c\}t_3\}$$

and

$$V^{1-\mu}(x(t)) \leq \max\{0, \max_{\theta\in[-\tau,0]} V^{1-\mu}(x_0(\theta)) - \min\{\alpha'(1-\mu), c\}t\}$$

for all $t \in [0, t_3)$. These properties imply that

$$\max_{\theta\in[-\tau,0]} V^{1-\mu}(x_{t_3}(\theta)) \leq V^{1-\mu}(x(t_3)) + c\tau$$

and, consequently, $D^+V(x(t_3)) f(x_{t_3}) \leq -\alpha' V^{\mu}(x(t_3))$, which means that (3) cannot be violated. The same arguments can be applied further for all $t \geq t_3$, and the required estimate on the solutions of (1) follows.

This theorem contains more important modifications of Theorem 1 than Theorem 2 since a finite-time convergence rate is established. However, it preserves the main idea of the Lyapunov-Razumikhin approach, when the derivative decrease of V is asked only under a special relations between $\max_{\theta\in[-\tau,0]} V(x_t(\theta))$ and $V(x(t))$.

The last result formulates the Lyapunov-Razumikhin conditions for the fixed-time to a ball stability completing Proposition 2:

Theorem 4 *Let there exist a locally Lipschitz continuous Lyapunov-Razumikhin function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that*

(i) for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and all $x \in \mathbb{R}^n$:

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|);$$

(ii) for some $\mu > 1$, $c > 0$, $\alpha' > 0$ and all $\varphi \in C_{[-\tau,0]}$:

$$\frac{1}{\left(\max_{\theta\in[-\tau,0]} V(\varphi(\theta))\right)^{1-\mu} + c\tau} \leq V^{\mu-1}(\varphi(0)) \Rightarrow D^+V(\varphi(0)) f(\varphi) \leq -\alpha' V^{\mu}(\varphi(0)).$$

Then the system (1) is globally fixed-time to a ball stable at the origin, and for all $x_0 \in C_{[-\tau,0]}$ and $t \geq 0$:

$$|x(t, x_0)| \leq \alpha_1^{-1}\left(\frac{1}{\left(\alpha_2^{1-\mu}(\|x_0\|) + \min\{\alpha'(\mu-1), c\}t\right)^{\frac{1}{\mu-1}}}\right).$$

PROOF. The conditions of Theorem 1 for global stability are satisfied in this case since $\mu > 1$ and all $\varphi \in C_{[-\tau,0]}$, $\varphi \neq 0$:

$$\frac{1}{\left(\max_{\theta\in[-\tau,0]} V(\varphi(\theta))\right)^{1-\mu} + c\tau} = \frac{\max_{\theta\in[-\tau,0]} V^{\mu-1}(\varphi(\theta))}{1 + c\tau \max_{\theta\in[-\tau,0]} V^{\mu-1}(\varphi(\theta))},$$

then

$$\max_{\theta\in[-\tau,0]} V(\varphi(\theta)) \leq V(\varphi(0)) \Rightarrow$$
$$\max_{\theta\in[-\tau,0]} V^{\mu-1}(\varphi(\theta)) \leq \left(1 + c\tau \max_{\theta\in[-\tau,0]} V^{\mu-1}(\varphi(\theta))\right) V^{\mu-1}(\varphi(0))$$
$$\Rightarrow D^+V(\varphi(0)) f(\varphi) \leq -\alpha' \alpha_1^{\mu}(|\varphi(0)|) \leq 0,$$

and even a version of conditions for asymptotic stability holds due to $1 + c\tau \max_{\theta\in[-\tau,0]} V^{\mu-1}(\varphi(\theta)) > 1$ for $\varphi \neq 0$. Anyway, for any $x_0 \in C_{[-\tau,0]}$ the solution $x(t, x_0)$ is well defined for all $t \geq 0$, and

$$V(x(t)) \leq \max_{\theta\in[-\tau,0]} V(x_0(\theta))$$

for all $t \geq 0$ [10].

First, assume that the implication given in the formulation of the theorem is not satisfied and

$$\frac{1}{\left(\max_{\theta\in[-\tau,0]} V(x_t(\theta))\right)^{1-\mu} + c\tau} > V^{\mu-1}(x(t))$$

for an interval of time $t \in [0, t_1]$, where $t_1 \geq 0$ is (possibly infinite) instant of time that the relation is failed. This inequality implies that for all $t \in [0, t_1]$ there exists $\theta_t = \min\{\vartheta \in [-\tau, 0] : V(x_t(\vartheta)) = \max_{\theta\in[-\tau,0]} V(x_t(\theta))\}$ such that

$$V^{\mu-1}(x(t)) < \frac{1}{V^{1-\mu}(x_t(\theta_t)) + c\tau} \leq \frac{1}{V^{1-\mu}(x_t(\theta_t)) - c\theta_t}.$$

Recursively applying this estimate, i.e., $V^{1-\mu}(x_t(\theta_t)) = V^{1-\mu}(x(t+\theta_t)) > V^{1-\mu}(x_{t+\theta_t}(\theta_{t+\theta_t})) - c\theta_{t+\theta_t}$, we obtain

$$V^{\mu-1}(x(t)) < \frac{1}{V^{1-\mu}(x_{t+\theta_t}(\theta_{t+\theta_t})) - c(\theta_{t+\theta_t} + \theta_t)},$$

and by induction,

$$V^{\mu-1}(x(t)) \leq \frac{1}{\left(\max_{\theta\in[-\tau,0]} V(x_0(\theta))\right)^{1-\mu} + ct} \leq \frac{1}{\left(\max_{\theta\in[-\tau,0]} V(x_0(\theta))\right)^{1-\mu} + \min\{\alpha'(\mu-1), c\}t} \quad (4)$$

for all $t \in [0, t_1]$.

Now, suppose that for $t \in [t_1, t_2)$ the implication $\frac{1}{\left(\max_{\theta\in[-\tau,0]} V(x_t(\theta))\right)^{1-\mu} + c\tau} \leq V^{\mu-1}(x(t))$ holds, where $t_2 \geq t_1$ is (possibly again infinite) time instant that this relation is broken for the first time after t_1. By the conditions of the theorem:

$$D^+V(x(t)) f(x_t) \leq -\alpha' V^{\mu}(x(t))$$

for all $t \in [t_1, t_2)$ and, consequently,

$$V^{\mu-1}(x(t)) \leq \frac{1}{V^{1-\mu}(x(t_1)) + \alpha'(\mu-1)(t-t_1)}.$$

Consequently, we obtain that the estimate (4) is satisfied for all $t \in [0, t_2)$. As in theorems 2 and 3, this conclusion can be extended to all $t \geq 0$.

Similarly to theorems 2 and 3, we can also verify by contradiction that (4) is valid for all $t \geq 0$, and the required estimate on the solutions of (1) follows.

In [1, 2] the local type of this convergence has been investigated using the condition:

$$\max_{\theta\in[-\tau,0]} V(\varphi(\theta)) \leq \gamma' V(\varphi(0)) \Rightarrow$$
$$D^+V(\varphi(0)) f(\varphi) \leq -\alpha' V^\mu(\varphi(0))$$

that has to be satisfied for some $\mu > 1$, $\gamma' > 1$, $\alpha' > 0$ and all sufficiently small $\varphi \in C_{[-\tau,0]}$, which is a valid replacement with respect to Theorem 4 since close to the origin the exponential convergence followed the negation of the inequality $\max_{\theta\in[-\tau,0]} V(\varphi(\theta)) \leq \gamma' V(\varphi(0))$ is faster than the polynomial in time one resulted from $D^+V(\varphi(0)) f(\varphi) \leq -\alpha' V^\mu(\varphi(0))$.

As we can conclude from the results of theorems 2, 3 and 4, the Lyapunov-Razumikhin approach can be used for estimation of the rate of solution convergence, but the conditions have to be formulated differently in accordance with the rate, and the functions α and γ (from Theorem 1) both influence the decay rate.

5 Simulations

Let us demonstrate by simulations that under the conditions introduced in theorems 2, 3 and 4, the solutions of the system (1) have the corresponding convergence rates. For simplicity of illustration we will consider several examples for a scalar variable $V(t) \in \mathbb{R}_+$, which represents a possible behavior of a Lyapunov-Razumikhin function, and we will restrict ourselves to the scenarios with pointwise constant delay $\tau > 0$.

First, assume that for all $t \geq 0$ and $V_0 \in C_{[-\tau,0]}$:

$$\dot{V}(t) \leq -aV(t) + bV(t-\tau),$$

where $a > 0$ and $b > 0$ are parameters, $V(t) = V(t, V_0)$, then for $\gamma' > 1$ the Lyapunov-Razumikhin relation is satisfied:

$$V(t-\tau) < \gamma' V(t) \Rightarrow \dot{V}(t) \leq -(a - b\gamma')V(t),$$

and the conditions of Theorem 2 are verified for $a > b\gamma'$. Moreover, let

$$a \geq b\gamma' + \frac{\ln \gamma'}{\tau}, \tag{5}$$

which imposes the convergence exponent $\frac{\ln \gamma'}{\tau}$. It is worth to stress that if $\tau \to 0$, then simultaneously we may ask for $\gamma' \to 1$, in order to recover in the limit delay-free case the standard condition $a > b$. In addition, using Halanay's inequality [9] the exponential rate of convergence $\delta > 0$ in this case can be evaluated as the solution of the equation

$$a - \delta = be^{\delta\tau},$$

in our case $\delta = \frac{\ln \gamma'}{\tau}$ and after its substitution in the Halanay's inequality we obtain the restriction on the parameters (5).

Second, let for all $t \geq 0$ and $V_0 \in C_{[-\tau,0]}$ with $\|V_0\| \leq 1$:

$$\dot{V}(t) \leq -aV^\mu(t) + bV^\eta(t)V^\rho(t-\tau),$$

where $a > 0$, $b > 0$, $\mu \in (0,1)$, $\rho > 0$ and $\eta \geq \mu$ are parameters, then for $c > 0$ the Lyapunov-Razumikhin relation is satisfied:

$$V^{1-\mu}(t-\tau) < V^{1-\mu}(t) + c\tau \Rightarrow$$
$$\dot{V}(t) \leq -aV^\mu(t) + bV^\eta(t)\left(V^{1-\mu}(t) + c\tau\right)^{\frac{\rho}{1-\mu}}.$$

Since (for the second line Jensen's inequality has been applied)

$$\left(V^{1-\mu}(t) + c\tau\right)^{\frac{\rho}{1-\mu}} \leq \left(V^\rho(t) + (c\tau)^{\frac{\rho}{1-\mu}}\right)$$
$$\times \begin{cases} 1 & \rho \in (0, 1-\mu] \\ 2^{\frac{\rho}{1-\mu}-1} & \rho > 1-\mu \end{cases},$$

we obtain

$$V^{1-\mu}(t-\tau) < V^{1-\mu}(t) + c\tau \Rightarrow$$
$$\dot{V}(t) \leq -aV^\mu(t) + \max\{1, 2^{\frac{\rho}{1-\mu}-1}\}b$$
$$\times[V^{\eta+\rho}(t) + V^\eta(t)(c\tau)^{\frac{\rho}{1-\mu}}]$$
$$\leq -[a - \max\{1, 2^{\frac{\rho}{1-\mu}-1}\}b(1 + (c\tau)^{\frac{\rho}{1-\mu}})]V^\mu(t),$$

and the conditions of Theorem 3 are verified locally for

$$a > \max\{1, 2^{\frac{\rho}{1-\mu}-1}\}b(1 + (c\tau)^{\frac{\rho}{1-\mu}}).$$

Third, let for all $t \geq 0$ and $V_0 \in C_{[-\tau,0]}$:

$$\dot{V}(t) \leq -aV^\mu(t) + \frac{bV(t)V^{1-\mu}(t-\tau)}{1 + c\tau V^{1-\mu}(t-\tau)},$$

where $a > 0$, $b > 0$, $\mu > 1$, $c > 0$ are parameters, then the Lyapunov-Razumikhin relation is satisfied:

$$\frac{1}{V^{1-\mu}(t-\tau) + c\tau} < V^{\mu-1}(t) \Rightarrow$$
$$\dot{V}(t) \leq -(a-b)V^\mu(t),$$

and the conditions of Theorem 4 are verified for

$$a > b.$$

The results of simulations for all these three cases are shown in figures 1, 2 and 3, respectively, for $\tau = 0.5$, with the explicit Euler method and a fixed discretization step corresponding 10^{-4} of the time of simulation. The values of $V(t)$ are plotted on vertical axis in the logarithmic scale, the time t is given in the horizontal axis. Two system trajectories are presented (for the equality mode) for different initial constant conditions, solid and dash red lines, with

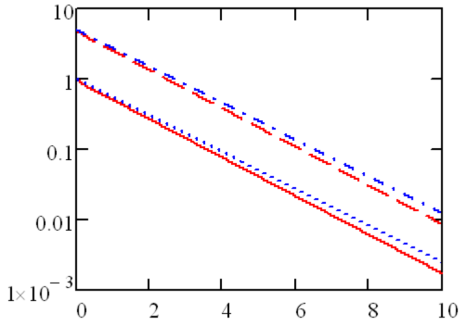

Fig. 1. The results of simulation for exponential convergence ($a = 2$, $b = 1$, $\gamma' = 1.35$)

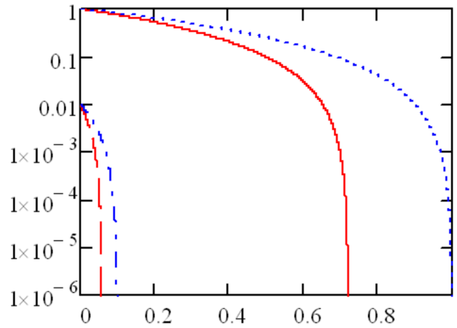

Fig. 2. The results of simulation for finite-time convergence ($a = 3.75$, $c = b = 1$, $\mu = \eta = 0.5$, $\rho = \frac{1-\mu}{2}$)

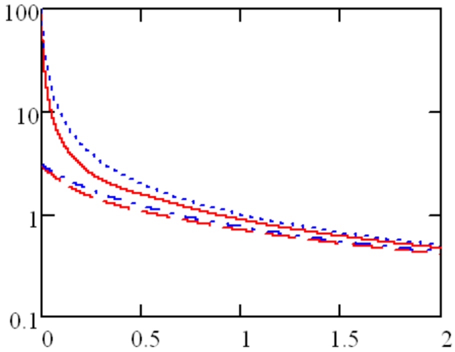

Fig. 3. The results of simulation for fixed-time convergence to a ball ($a = 2$, $b = c = 1$, $\mu = 2$)

two corresponding upper estimates obtained in the paper, blue dot and dash-dot lines, respectively.

The presented results of simulation confirm the theoretical findings of our work.

6 Conclusions

Three extensions of the Lyapunov-Razumikhin function approach are proposed, which allow an upper estimate on the rate of decreasing of solutions to be obtained for a time-delay system. The case of the exponential convergence just adds an additional parametric restriction to the conventional method. While the cases of finite-time and fixed-time to a ball stability need more severe modifications of the approach. The latter property is also introduced in this work together with a Lyapunov-Krasovskii sufficient condition. The obtained results are illustrated in simulations. Application of the proposed method to the analysis and design of control and estimation algorithms is a direction of future research.

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