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Cooperative colorings of trees and of bipartite graphs

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Abstract

Given a system (G_1, \dots, G_m) of graphs on the same vertex set V , a cooperative coloring is a choice of vertex sets I_1, \dots, I_m , such that I_j is independent in G_j and $\bigcup_{j=1}^m I_j = V$. For a class \mathcal{G} of graphs, let $m_{\mathcal{G}}(d)$ be the minimal m such that every m graphs from \mathcal{G} with maximum degree d have a cooperative coloring. We prove that $\Omega(\log \log d) \leq m_{\mathcal{T}}(d) \leq O(\log d)$ and $\Omega(\log d) \leq m_{\mathcal{B}}(d) \leq O(d/\log d)$, where \mathcal{T} is the class of trees and \mathcal{B} is the class of bipartite graphs.

Mathematics Subject Classifications: 05C15, 05C69

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1 Introduction

A set of vertices in a graph is called *independent* if no two vertices in it form an edge. A *coloring* of a graph G is a covering of $V(G)$ by independent sets. Given a system (G_1, \dots, G_m) of graphs on the same vertex set V , a *cooperative coloring* is a choice of vertex sets $\{I_j \subseteq V : j \in [m]\}$ such that I_j is independent in G_j and $\bigcup_{j=1}^m I_j = V$. If all G_j 's are the same graph G , then a cooperative coloring is just a proper vertex coloring of G by m independent sets.

A basic fact about vertex coloring is that every graph G of maximum degree d is $(d+1)$ -colorable. It is therefore natural to ask whether $d+1$ graphs, each of maximum degree d , always have a cooperative coloring. This was shown to be false:

Theorem 1 (Theorem 5.1 of Aharoni, Holzman, Howard and Sprüssel [AHHS15]). *For every $d \geq 2$, there exist $d+1$ graphs of maximum degree d that do not have a cooperative coloring.*

Using the fundamental result on independent transversals of Haxell [Hax01, Theorem 2], it can be shown that $2d$ graphs of maximum degree d always have a cooperative coloring. Let $m(d)$ be the minimal m such that every m graphs of maximum degree d have a cooperative coloring. By the above, $m(1) = 2$ and

$$d+2 \leq m(d) \leq 2d, \text{ for every } d \geq 2. \quad (1)$$

The theorem of Loh and Sudakov [LS07, Theorem 4.1] on independent transversals in locally sparse graphs implies that $m(d) = d + o(d)$. Neither the lower bound nor the upper bound in (1) has been improved for general d ; even $m(3)$ is not known. However, restricting the graphs to specific classes, better upper bounds can be obtained.

Definition 2. For a class \mathcal{G} of graphs, denote by $m_{\mathcal{G}}(d)$ the minimal m such that every m graphs belonging to \mathcal{G} , each of maximum degree at most d , have a cooperative coloring.

For example, the following was proved:

Theorem 3 (Corollary 3.3 of Aharoni et al. [ABZ07] and Theorem 6.6 of Aharoni et al. [AHHS15]). *Let \mathcal{C} be the class of chordal graphs and let \mathcal{P} be the class of paths. Then $m_{\mathcal{C}}(d) = d+1$ for all d , and $m_{\mathcal{P}}(2) = 3$.*

In this paper, we prove some bounds on $m_{\mathcal{G}}(d)$ for two more classes:

Theorem 4. *Let \mathcal{T} be the class of trees, and let \mathcal{B} be the class of bipartite graphs. Then for $d \geq 2$,*

$$\begin{aligned} \log_2 \log_2 d &\leq m_{\mathcal{T}}(d) \leq (1 + o(1)) \log_{4/3} d, \\ \log_2 d &\leq m_{\mathcal{B}}(d) \leq (1 + o(1)) \frac{2d}{\ln d}. \end{aligned}$$

Remark 5. Let \mathcal{F} be the class of forests. It is evident that $m_{\mathcal{F}}(d) \geq m_{\mathcal{T}}(d)$ as $\mathcal{F} \supset \mathcal{T}$. Conversely, when $d \geq 2$, given $m = m_{\mathcal{T}}(d)$ forests F_1, \dots, F_m of maximum degree d , we can add edges to F_i to obtain a tree F'_i of maximum degree d , and the cooperative coloring for F'_1, \dots, F'_m is also a cooperative coloring for F_1, \dots, F_m . Therefore $m_{\mathcal{F}}(d) = m_{\mathcal{T}}(d)$ for $d \geq 2$.

The notions of cooperative coloring and of list coloring have a common generalization: given a system (G_1, \dots, G_m) of graphs with vertex sets V_1, \dots, V_m (which are not necessarily the same vertex set), a *cooperative list coloring* is then a choice of independent sets in G_i whose union equals $V := V_1 \cup \dots \cup V_m$. The notion of cooperative coloring is obtained by taking $V_i = V$, and list colorings are formed when G_i is an induced subgraph of the same graph G for all i . The upper bounds in Theorem 4 generalize to cooperative list colorings. For example, our proof of Theorem 4 for bipartite graphs readily gives the following result.

Theorem 6. *For every system (G_1, \dots, G_m) of bipartite graphs with maximum degree d with vertex sets V_1, \dots, V_m , there is a cooperative list coloring if for every $v \in V_1 \cup \dots \cup V_m$, the number of its occurrences in V_1, \dots, V_m , that is $|\{i \in [m] : v \in V_i\}|$, is at least $(1 + o(1)) \frac{2d}{\ln d}$.*

A conjecture of Alon and Krivelevich [AK98, Conjecture 5.1] states that the choice number of any bipartite graph with maximum degree d is at most $O(\log d)$ (see [AR08] for a result in this direction). This conjecture would follow if the term $(1 + o(1)) \frac{2d}{\ln d}$ in Theorem 6 was strengthened to $\Omega(\log d)$.

The rest of the paper is organized as follows. In Section 2 and Section 3, we prove Theorem 4 for trees and bipartite graphs respectively. In Section 4 we discuss a further generalization of cooperative colorings.

2 Trees

Proof of the lower bound on $m_{\mathcal{T}}(d)$. Note that the system \mathcal{T}_2 , consisting of two paths in Figure 1 (one in thin red, the other in bold blue), does not have a cooperative coloring.

Suppose now that $\mathcal{S} = (F_1, F_2, \dots, F_m)$ is a system of forests on a vertex set V , not having a cooperative coloring. We shall construct a system $\mathcal{Q}(\mathcal{S})$ of $m + 1$ new forests $F'_1, F'_2, \dots, F'_m, F'_{m+1}$, again not having a cooperative coloring.

The vertex set common to the new forests is $V' = (V \cup \{z\}) \times V$, namely the vertex set consists of $|V| + 1$ copies of V . For every $u \in V \cup \{z\}$ and every $i \in [m]$, take a copy



Figure 1: Construction of two paths without a cooperative coloring.

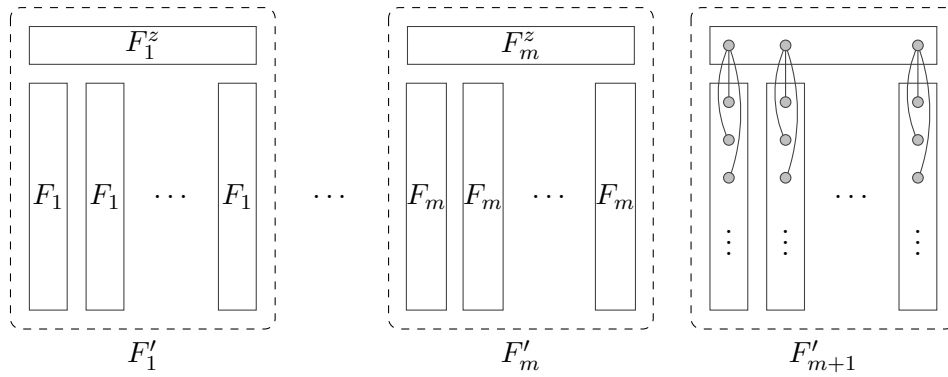


Figure 2: Construction of $Q(\mathcal{S}) = (F'_1, \dots, F'_m, F'_{m+1})$ from $\mathcal{S} = (F_1, \dots, F_m)$.

F_i^u of F_i on the vertex set $\{(u, v) : v \in V\}$. Let F'_i consist of $|V| + 1$ disjoint copies of F_i :

$$F'_i := \bigcup_{u \in V \cup \{z\}} F_i^u, \quad \text{for all } i \in [m].$$

To these we add the $(m + 1)$ st forest F'_{m+1} obtained by joining (z, u) to (u, v) for all $u, v \in V$. So F'_{m+1} is a disjoint union of stars, each with $|V|$ leaves.

Assume that there is a cooperative coloring $(I_1, I_2, \dots, I_m, I_{m+1})$ for the system $Q(\mathcal{S})$. Since the forests $F_1^u, F_2^u, \dots, F_m^u$ do not have a cooperative coloring, I_{m+1} must contain a vertex from $\{u\} \times V$ for all $u \in V \cup \{z\}$. In particular, I_{m+1} contains a vertex $(z, u) \in I_{m+1}$ for some $u \in V$ and a vertex (u, v) for some $v \in V$. Since (z, u) is connected in F'_{m+1} to (u, v) , this is contrary to our assumption that I_{m+1} is independent.

Note that $|V'| = |V|^2 + |V| \leq 2|V|^2$. Note also that the maximum degree of $Q(\mathcal{S})$ is attained in F'_{m+1} , and it is equal to $|V|$. Recursively define the system $\mathcal{T}_m := Q(\mathcal{T}_{m-1})$ consisting of m forests for $m \geq 3$. Because the base \mathcal{T}_2 has 4 vertices, one can check inductively that $|V(\mathcal{T}_m)|$ is at most $2^{3 \cdot 2^{m-2} - 1}$ using $|V(\mathcal{T}_m)| \leq 2|V(\mathcal{T}_{m-1})|^2$. Thus the maximum degree of \mathcal{T}_m is at most $2^{3 \cdot 2^{m-3} - 1} \leq 2^{2^{m-1}}$.

Given the maximum degree $d \geq 2$, choose $m := \lceil \log_2 \log_2 d \rceil$. By the choice of m , the maximum degree of \mathcal{T}_m is at most $2^{2^{m-1}} \leq d$. By adding a few edges between the leaves in each forest of \mathcal{T}_m , we can obtain a system of m trees of maximum degree d that does not have a cooperative coloring. This means $m_{\mathcal{T}}(d) > m > \log_2 \log_2 d$. \square

Proof of the upper bound on $m_{\mathcal{T}}(d)$. Let (T_1, T_2, \dots, T_m) be a system of trees of maximum degree d . We shall find a cooperative coloring by a random construction if $m \geq (1 + o(1)) \log_{4/3} d$.

Choose arbitrarily for each tree T_i a root so that we can specify the parent or a sibling of a vertex that is not the root of T_i . For each T_i , choose independently a random vertex set S_i , in which each vertex is included in S_i independently with probability $1/2$. Set

$$R_i := \{v \in S_i : \text{the parent of } v \text{ is not in } S_i, \text{ or } v \text{ is a root}\}.$$

Since among any two adjacent vertices in T_i one is the parent of the other, R_i is independent in T_i .

We shall show that with positive probability the sets R_i form a cooperative coloring. For each vertex v , let B_v be the event that $v \notin \bigcup_{i=1}^m R_i$. If v is the root of T_i , then $\Pr(v \in R_i) = 1/2$; otherwise $\Pr(v \in R_i) = 1/4$. In any case, $\Pr(v \notin R_i) \leq 3/4$, and so $\Pr(B_v) \leq (3/4)^m$. Notice that B_v is only dependent on the events B_u for u that is the parent, a sibling or a child of v in some T_i . Since the degree of v is at most d , it follows that B_v is dependent on less than $2md$ other events. By the symmetric version of the Lovász Local Lemma (see for example [AS16, Chapter 5]), if

$$e \times \left(\frac{3}{4}\right)^m \times 2md \leq 1, \tag{2}$$

then with positive probability no B_v occurs, meaning that the sets R_i form a cooperative coloring. The inequality (2) indeed holds under the assumption that $m \geq (1+o(1)) \log_{4/3} d$. \square

3 Bipartite graphs

Proof of the lower bound on $m_{\mathcal{B}}(d)$. Given d , take $m = \lceil \log_2 d \rceil$. Let the vertex set be $\{0, 1\}^m$, and for $j \in [m]$ let G_j be the complete bipartite graph between V_j^0 and V_j^1 where

$$V_j^k = \{v \in \{0, 1\}^m : v_j = k\}, \quad \text{for } k \in \{0, 1\}.$$

Note that the degree of G_j is $2^{m-1} \leq d$.

Suppose that I_1, \dots, I_m are independent sets in G_1, \dots, G_m respectively. As each G_j is a complete bipartite graph, $I_j \subseteq V_j^{k_j}$ for some $k_j \in \{0, 1\}$. Thus $(1 - k_1, \dots, 1 - k_m)$ is not in any I_j , and so I_1, \dots, I_m do not form a cooperative coloring. This means $m_{\mathcal{B}}(d) > m \geq \log_2 d$. \square

Proof of the upper bound on $m_{\mathcal{B}}(d)$. Let $\mathcal{G} = (G_1, \dots, G_m)$ be a system of bipartite graphs on the same vertex set V with maximum degree d . By a semi-random construction, we shall find a cooperative coloring if $m \geq (1 + \varepsilon) \frac{2d}{\ln d}$ for fixed $\varepsilon > 0$ and d sufficiently large. We may assume that $m = O(d)$ because of (1).

For each $j \in [m]$, let (L_j, R_j) be a bipartition of G_j . Define $J_L(v) := \{j \in [m] : v \in L_j\}$ and $J_R(v) := \{j \in [m] : v \in R_j\}$ for each vertex $v \in V$, and let $A := \{v \in V : |J_L(v)| \geq m/2\}$. Set $B := V \setminus A$. Clearly, we have

$$|J_L(a)| \geq m/2, \quad \text{for all } a \in A; \tag{3a}$$

$$|J_R(b)| \geq m/2, \quad \text{for all } b \in B. \tag{3b}$$

Consider the following random process.

1. For each $a \in A$, choose $j = j(a) \in J_L(a)$ uniformly at random, and put a in the set I_j .
2. For each $b \in B$, choose arbitrarily $j \in J_R(b) \setminus \{j(a) : a \in A, (a, b) \in E(G_j)\} =: J'_R(b)$ as long as it is possible, and put b in the set I_j .

For any $a, a' \in A \cap I_j$, $a, a' \in L_j$ and so $(a, a') \notin G_j$. This means $A \cap I_j$ is independent, and similarly $B \cap I_j$ is independent. For any $b \in B \cap I_j$ and $(a, b) \in E(G_j)$, by the definition of $J'_R(b)$, $j(a) \neq j$ and so $a \notin I_j$. Therefore I_j is independent for all $j \in [m]$.

To prove the existence of a cooperative coloring it suffices to show that $J'_R(b)$ is nonempty for all $b \in B$ with positive probability. For a vertex $b \in B$, let E_b be the contrary event, that is, the event that $J'_R(b)$ is empty.

For a fixed $b \in B$, let us estimate from above the probability of E_b . For every $j \in J_R(b)$, let E^j be the event that $j \notin J'_R(b)$, that is the event that $j(a) = j$ for some $a \in A$ that is a neighbor of b in G_j . For each $a \in A$ that is a neighbor of b in G_j , we have

$$\Pr(j(a) = j) = \frac{1}{|J_L(a)|} \stackrel{(3a)}{\leq} \frac{2}{m} \leq \frac{\ln d}{(1 + \varepsilon)d}.$$

As there are at most d neighbors of b in G_j , we have for sufficiently large d that

$$1 - \Pr(E^j) \geq \left(1 - \frac{\ln d}{(1 + \varepsilon)d}\right)^d \geq \exp(-(1 - \varepsilon) \ln d) = d^{\varepsilon - 1} \geq \frac{8 \ln d}{m}. \quad (4)$$

We claim that the events E^j , $j \in J_R(b)$, are negatively correlated. This is easier to see with the complementary events \bar{E}^j , $j \in J_R(b)$. We have to show that for any choice of indices $j_1, \dots, j_t \in J_R(b)$ there holds

$$\Pr(E^j \mid \bar{E}^{j_1} \cap \bar{E}^{j_2} \cap \dots \cap \bar{E}^{j_t}) \geq \Pr(E^j).$$

The event $\bar{E}^{j_1} \cap \bar{E}^{j_2} \cap \dots \cap \bar{E}^{j_t}$ means that for all $a \in A$ if a is a neighbor of b in G_{j_i} then $j(a) \neq j_i$. Then, for any $j \notin \{j_1, \dots, j_t\}$, for those vertices $a \in A$ that are neighbors of b in G_j , knowing that $j(a) \neq j_i$ for certain $i \in [t]$ increases the probability that $j(a) = j$, and therefore increases the probability of E^j .

By the claim, the inequality (4) and the fact that $E_b = \bigcap_{j \in J_R(b)} E^j$, we have

$$\Pr(E_b) \leq \prod_{j \in J_R(b)} \Pr(E^j) \stackrel{(3b)}{\leq} \left(1 - \frac{8 \ln d}{m}\right)^{\frac{m}{2}} \leq \exp\left(-\frac{8 \ln d}{m} \cdot \frac{m}{2}\right) = \frac{1}{d^4}.$$

The event E_b is dependent on less than md^2 other events $E_{b'}$, since for such dependence to exist it is necessary that $b' \in B$ is at distance at most 2 from b in some graph G_j . Thus, by the Lovász Local Lemma, for the positive probability that none of E_b occurs it suffices that

$$e \times \frac{1}{d^4} \times md^2 \leq 1,$$

which indeed holds for d sufficiently large as $m = O(d)$. □

4 Cooperative covers

Cooperative coloring of graphs is a special case of a more general concept.

Definition 7. Given a system (C_1, \dots, C_n) of (abstract) simplicial complexes, all sharing the same vertex set V , a *cooperative cover* is a choice of faces $f_i \in C_i$ such that $\bigcup_{i=1}^n f_i = V$.

A cooperative coloring for (G_1, \dots, G_n) is the special case in which C_i is the independence complex $I(G_i)$ of G_i , that is, the collection of all independent sets in G_i .

Definition 8. Given a hypergraph C with vertex set V , the *edge covering number* $\rho(C)$ is the minimal number of hyperedges from C whose union is V . For a class \mathcal{C} of simplicial complexes, let $n_{\mathcal{C}}(b)$ denote the minimal number n , such that every system (C_1, \dots, C_n) of simplicial complexes belonging to \mathcal{C} on the same vertex set V satisfying $\rho(C_i) \leq b$ for all $i \leq n$, has a cooperative cover. Let $n_{\mathcal{C}}(b) = \infty$ if no such n exists.

For example, consider the class \mathcal{I} of all the independence complexes of graphs. If G is bipartite, then $\rho(I(G)) \leq 2$. Hence the fact that $m_{\mathcal{B}}(d) \geq \log_2(d)$ for all $d \geq 2$ (see Theorem 4) implies $n_{\mathcal{I}}(2) = \infty$.

There are natural classes \mathcal{C} of hypergraphs for which $n_{\mathcal{C}}$ is finite. One of these is the class of simplicial complexes associated to polymatroids, as introduced in [Edm70]. A *polymatroid* (V, r) is defined via a rank function $r: 2^V \rightarrow \mathbb{N}$, that is submodular, monotone increasing and is 0 on the empty set. A *k-polymatroid* is a polymatroid in which every singleton set has rank at most k . For example, a k -uniform hypergraph H endowed with the function $r(E) = |\cup E|$, for every subset of hyperedges E in H , is a k -polymatroid.

Following the notation in [LP86, Section 11], given a k -polymatroid (V, r) , a set $M \subseteq V$ is called a *matching* if $r(M) = k|M|$. By the submodularity of the rank function r , the matchings in a k -polymatroid form a simplicial complex on V , which we call the *matching complex* of a k -polymatroid.

Theorem 9. Let \mathcal{M}_k be the class of all the matching complexes of k -polymatroids. Then $n_{\mathcal{M}_k}(b) \leq kb$ for every b .

The proof uses the (homotopic) connectivity $\eta(C)$ of a complex C . We refer to [AB06, Section 2] for background. We shall use the following two topological tools. Given a complex C on V and $U \subseteq V$, we denote by $C[U]$ the simplicial subcomplex induced on U .

Theorem 10 (Topological Hall's theorem). Let C be a simplicial complex on the vertex set V and let $\bigcup_{i=1}^m W_i$ be a partition of V . If for all $I \subseteq [m]$

$$\eta \left(C \left[\bigcup_{i \in I} W_i \right] \right) \geq |I|,$$

then C contains a face σ such that $|\sigma \cap W_i| = 1$ for all $i \in [m]$.

Theorem 11. If C is a matching complex on V of a k -polymatroid, then the connectivity $\eta(C)$ of C is at least $\nu(C)/k$, where $\nu(C)$ is the maximal size of faces in C .

The above formulation of Theorem 10 first appeared in [Mes01], attributed to the first author of the present paper (see the remark after Theorem 1.3 in [Mes01]). Theorem 11 is an unpublished result of the first two authors. The special case, where the k -polymatroid is the sum of k matroids on the same vertex set, is proved in [AB06, Theorem 6.5].

Proof of Theorem 9. Let $n = kb$, and let C_1, \dots, C_n be simplicial complexes associated to k -polymatroids $(V, r_1), \dots, (V, r_n)$ on the same vertex set V such that the edge covering number of each C_i is at most b . Let C be the join of C_1, \dots, C_n on $V \times [n]$, that is,

$$C := \left\{ \bigcup_{i=1}^n \sigma_i \times \{i\} : \sigma_i \in C_i \text{ for all } i \in [n] \right\}.$$

A cooperative cover can be viewed as a face $\sigma \in C$ such that $|\sigma \cap (\{v\} \times [n])| = 1$ for all $v \in V$. By the topological Hall's theorem, it suffices to prove that

$$\eta(C[U \times [n]]) \geq |U| \text{ for all } U \subseteq V.$$

Let U be a subset of V . Note that $C_i[U]$ is the matching complex of the k -polymatroid $(U, r_i|_U)$. By Theorem 11, $\eta(C_i[U]) \geq \nu(C_i[U])/k$. Since $\nu(C_i[U])$ is the maximal size of faces in $C_i[U]$ and the edge covering number of $C_i[U]$ is at most b , we obtain $\nu(C_i[U])b \geq |U|$, and so $\eta(C_i[U]) \geq |U|/(kb)$. Notice that $C[U \times [n]]$ is the join of $C_1[U], \dots, C_n[U]$. Using the superadditivity of η with respect to the join operator and Theorem 11, we obtain the required condition for the topological Hall's theorem

$$\eta(C[U \times [n]]) \geq \sum_{i=1}^n \eta(C_i[U]) \geq \sum_{i=1}^n |U|/(kb) = |U|. \quad \square$$

Remark 12. It is of interest to explore the sharpness of this result.

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