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Léa Boittin, François Bouchut, Marie-Odile Bristeau, Anne Mangeney, Jacques Sainte-Marie, Fabien Souillé

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L. Boittin[1], F. Bouchut[2], M.-O. Bristeau[1], A. Mangeney[1,3], J. Sainte-Marie[1], and F. Souillé[1]

[1]Inria Paris, 2 rue Simone Iff, CS 42112, 75589 Paris Cedex 12 and Sorbonne Université, univ. Paris-Diderot SPC, CNRS, Laboratoire Jacques-Louis Lions, LJLL, F-75005 Paris
[2]Laboratoire d'Analyse et de Mathématiques Appliquées (UMR 8050), CNRS, Univ. Gustave Eiffel, UPEC, F-77454, Marne-la-Vallée, France
[3]Univ. Paris Diderot, Sorbonne Paris Cité, Institut de Physique du Globe de Paris, Seismology Group, 1 rue Jussieu, Paris F-75005, France

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Abstract

We are interested in free surface flows where density variations coming e.g. from temperature or salinity differences play a significant role in the hydrodynamic regime. In water, acoustic waves travel much faster than gravity and internal waves, hence the study of models arising from compressible fluid mechanics often requires a decoupling between these waves. Starting from the compressible Navier-Stokes system, we derive the so-called Navier-Stokes-Fourier system in an "incompressible" regime using the low-Mach scaling, hence filtering the acoustic waves, neglecting the density dependency on the fluid pressure but keeping its variations in terms of temperature and salinity. A slightly modified low-Mach asymptotics is proposed to obtain a model with thermo-mechanical compatibility. The case when the density depends only on the temperature is studied first. Then the variations of the fluid density with respect to temperature and salinity are considered, and it seems to be the first time that salinity dependency is considered in this low Mach limit. We give a layer-averaged formulation of the obtained models in an hydrostatic context, allowing to derive numerical schemes endowed with strong stability properties that are presented in a companion paper. Several stability properties of the layer-averaged Navier-Stokes-Fourier system are proved.

Keywords: Navier-Stokes equations, compressible and incompressible fluids, free surface flows, variable density flows, low-Mach approximation, layer-averaged formulation

1 Introduction

In oceans and lakes, one of the predominant driving forces is the difference in density, caused by salinity and temperature variations: increasing the salinity and lowering the temperature of a fluid both increase its density (Rahmstorf, 2003). Oceans and lakes are stratified: the water density varies along the vertical direction. In the present work, we aim at describing and simulating variable density flows with free surface.

The Navier-Stokes equations – or possibly simplified versions of these equations – are the cornerstone of the modeling of hydrodynamics in lakes and oceans. Compared to situations where the density is constant, when considering variable density flows with free surface, two characteristics appear:

- the variations of the fluid density can lead to acoustic waves,
- the fluid density adds nonlinearities in the Navier-Stokes equations.

These two aspects are detailed hereafter.

The density variations are usually small and a common assumption in the case of geophysical flows is the Boussinesq approximation (Boussinesq, 1903). It is widely used to simplify the Navier-Stokes equations with variable density and consists in ignoring density variations in momentum conservation equations except in the buoyancy force term. The underlying assumptions are 1) the density variations are very small with respect to the mean density and 2) the vertical scale of the motion is small compared to the vertical scale of variations in density, and the density and pressure variations due to motion are not larger than the total static variations of pressure and density (Spiegel and Veronis, 1960; Lu, 2000) The consequences of the Boussinesq approximation are listed in (Auclair et al., 2018). Notably, under the Boussinesq approximation, the volume of the water is conserved, while the mass is not conserved. With this approximation, the density can be defined as a function of any given tracer, the temperature or a pollutant for instance. The Boussinesq approximation is the basis of many ocean models such as ICON (ICON-ESM software), NEMO (NEMO software) and POM (POM software).

Several authors have shown the benefits of taking into account non-Boussinesq effects in lake and ocean models, either for the propagation of internal waves (Soontiens et al., 2013) or for sea level variations induced by expansion/contraction processes (Mellor and Ezer, 1995; Lu, 2000; Greatbatch et al., 2001). In the case of ocean water, density is in general a function of pressure, temperature and salinity. Equations of state for water based on experiments are proposed in (Safarov et al., 2009; Unesco, 2010).

Consequently, approaches to take into account the non-Boussinesq effects have been developed. A possibility is to adopt pressure coordinates (Huang et al., 2001; Song and Hou, 2006). The non-Boussinesq equations written in pressure coordinates are isomorphic to the Boussinesq equations in z-coordinates, which allows to use the same algorithm for the non-Boussinesq model as for the Boussinesq model. So far, however, this approach is not widely used in ocean modeling, though it is available in the code MITgcm (MIT). A review of the different types of vertical coordinates used for ocean modeling is given in (Griffies et al., 2000). In (Auclair et al., 2018), a non-hydrostatic non-Boussinesq model is presented. A non-hydrostatic pressure anomaly is related to a compressible (non-Boussinesq) density anomaly. Yet the authors of (Auclair et al., 2018) are primarily concerned with the simulation of acoustic waves and thermal dilation is not investigated numerically. This model has recently been included in CROCO (CROCO software). In the present work, we focus on the inclusion of dilation effects in the model.

The approach chosen here to propose a non-Boussinesq model is different from the previous ones. Starting from the compressible Navier-Stokes equations, we propose a formulation of the Navier-Stokes-Fourier system in the asymptotics of an "incompressible" fluid. The term "Fourier" refers here to Fourier's law of thermal conduction. For most geophysical flows, the water can be considered as incompressible in the sense that the variation of its density with respect to the fluid pressure is small (Safarov et al., 2009; Unesco, 2010) and in this case, the acoustic waves can be filtered out, this is the so called low Mach number limit. This is advantageous from the computational point of view because a restrictive condition must be imposed on the time step when the acoustic waves are included (Paolucci, 1982). Even if in some very particular situations (Kadri and Stiassnie, 2013), a coupling between gravity and acoustic waves can occur in the context of fluid density variations, we consider here situations (in terms of water depth and wave frequency) when the phenomena are decoupled.

The incompressible limit of the Navier-Stokes-Fourier system has been extensively studied mathematically, see (Alazard, 2006; Feireisl and Novotný, 2007) and the references therein. The incompressible limit is a low-Mach approximation of the Navier-Stokes-Fourier system. It consists in neglecting the pressure dependency in the fluid state law (but not the temperature and salinity dependencies). This limit is singular and requires not necessarily intuitive scalings, in particular for numerical approximations (Bouchut et al., 2017). Since the studied phenomena strongly couple mechanics (fluid motion and rheology) with thermodynamics (temperature and salinity variations), we pay close attention that the obtained models do not violate the second principle of thermodynamics. The most simple way of performing the incompressible limit yields a model for which the energy balance exhibits discrepancies with respect to the energy balance of the original compressible system. To recover an energy balance that is close to that of the original system, which is important for the robustness of numerical methods, some corrections terms are incorporated in the incompressible model. These do not alter the accuracy of the approximation. Instead of conserving the volume of fluid and not the mass (which is what the Boussinesq approximation implies), the model derived here strictly conserves the mass and is enriched by the thermohaline expansion effects i.e. the volume is no longer conserved. In particular the velocity is not divergence free, but the divergence of the velocity field is equal to a right-hand side involving temperature and salinity gradients, as well as viscous dissipation. Our incompressible models include the dependency of density on salinity additionally to the dependency on temperature, and it seems to be the first time that this is considered in detail, including Onsager's principles. The proposed model is close to the one in (Audusse et al., 2011a), yet it is rigorously derived here. While solving the model in Audusse et al. (2011a) required inverting a non-linear system, the numerical resolution of the present model is more simple, see (Boittin et al., 2018). Moreover, the model in the present work is 3D while it was only 2D in (Audusse et al., 2011a).

In the second section of this paper, a layer-averaged model is derived from the Navier-Stokes-Fourier system, in the hydrostatic context. Vertically averaged multilayer models (Allgeyer et al., 2019; Bristeau et al., 2017; Audusse et al., 2018, 2011a) are a way to describe stratified flows and to overcome the limitations inherent to isopycnal models. Layer-averaged models can deal with situations when the density stratification is broken due to external forcing terms (e.g. the wind during upwelling phenomena) and when mixing of fresh/cold/salted water occurs. We also prove that the multilayer models obtained (one for the Euler-Fourier system and one for the Navier-Stokes-Fourier system) satisfy an energy balance and we show that the stable equilibria of the multilayer model for the Euler system with variable density are those of the classical Euler system. Moreover, the multilayer approach does not require moving meshes. As the equations obtained on each layer are similar to the classical one-layer Shallow Water equations, we can use the existing robust and accurate techniques developed for the Shallow Water equations. A numerical scheme and numerical test cases are presented in a companion paper (Boittin et al., 2018).

The paper is organized as follows. The incompressible Navier-Stokes-Fourier and Euler-Fourier models are derived in Section 2. In Section 3, the multilayer formulations of the incompressible Euler-Fourier and Navier-Stokes-Fourier systems are given and the properties of the multilayer models are analyzed.

2 The 3d Navier-Stokes-Fourier system

⟨sec:NSF⟩ 2.1 The compressible Navier-Stokes-Fourier system

We consider the classical compressible Navier-Stokes system describing a free surface gravitational flow over a bottom topography $z_b(x,y)$,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0, \tag{2.1}$$ eq:mass_cons

$$\frac{\partial (\rho \mathbf{U})}{\partial t} + \nabla \cdot (\rho \mathbf{U} \otimes \mathbf{U}) + \nabla p - \nabla \cdot \sigma = \rho \mathbf{g}, \tag{2.2}$$ eq:NSF_2d3

$$\frac{\partial}{\partial t}\left(\rho \frac{|\mathbf{U}|^2}{2} + \rho e\right) + \nabla \cdot \left(\left(\rho \frac{|\mathbf{U}|^2}{2} + \rho e + p - \sigma\right)\mathbf{U}\right) = -\nabla \cdot Q_T + \rho \mathbf{g} \cdot \mathbf{U}, \tag{2.3}$$ eq:NSF_energy

where $\mathbf{U}(t,x,y,z) = (u,v,w)^T$ is the velocity, ρ is the mass density, p is the fluid pressure, σ is the viscosity stress and $\mathbf{g} = (0,0,-g)^T$ represents the gravity forces. The internal specific (i.e. per mass unit) energy is denoted by e, the temperature by T. The heat flux Q_T obeys the Fourier law $Q_T = -\lambda \nabla T$, hence the name "Navier-Stokes-Fourier", λ being the heat conductivity. The quantity ∇ denotes $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)^T$. In the following, we will also use the notations $\mathbf{u}$ and $\nabla_{x,y}$, $\mathbf{u}(t,x,y,z) = (u,v)^T$ is the horizontal velocity and $\nabla_{x,y}$ corresponds to the projection of ∇ on the horizontal plane i.e. $\nabla_{x,y} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)^T$. The square norm of the velocity vector is $|\mathbf{U}|^2 = u^2 + v^2 + w^2$.

We consider a free surface flow (see Fig. 1), therefore we assume

$$z_b(x,y) \leq z \leq \eta(t,x,y) := h(t,x,y) + z_b(x,y)$$

with $z_b(x,y)$ the bottom elevation and $h(t,x,y)$ the water depth.

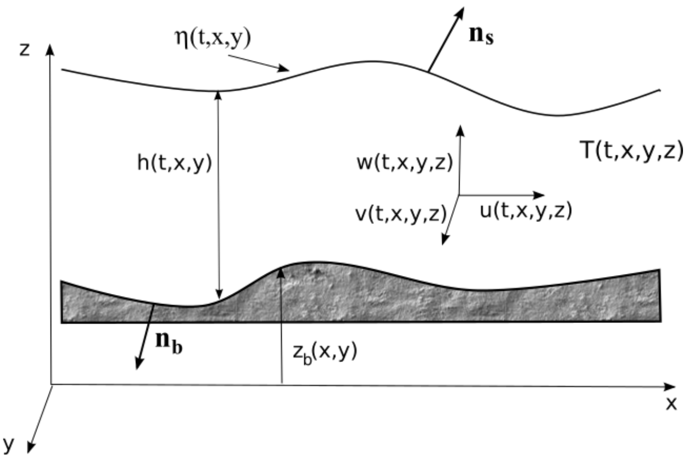

Figure 1: Flow domain with water height $h(t,x,y)$, free surface $\eta(t,x,y)$ and bottom $z_b(x,y)$.

tations3d_rho⟩

Notice that, because of the free surface and of the gravitational forces, the term $\rho\mathbf{g}.\mathbf{U} = -\rho g w$ in the right hand side of (2.3) prevents this equation from being directly a local energy conservation law. Nevertheless one can write (see Remark 2.1)

$$\rho g w = \frac{\partial(\rho g z)}{\partial t} + \nabla.(\rho g z \mathbf{U}),$$

which leads to a conservative version of (2.3). In the sequel, we will mainly work with the energy balance written under the form (2.3).

Regarding constitutive equations, we assume that the fluid is Newtonian i.e. the viscous part of the Cauchy stress depends linearly on the velocity gradient. Hence the stress tensor $\mathbf{\Sigma}$ is given by

$$\mathbf{\Sigma} \equiv -p\mathbb{1} + \sigma = -p\mathbb{1} + \zeta\nabla\cdot\mathbf{U}\mathbb{1} + 2\mu D(\mathbf{U}). \tag{2.4}$$ eq:stress_tens

where μ is the viscosity coefficient, ζ is the second viscosity and $D(\mathbf{U}) = (\nabla\mathbf{U} + (\nabla\mathbf{U})^T)/2$. Further on we shall use the notation $A : B$ for the scalar product between two square matrices A, B, and $|A|^2 = A : A$.

Among the thermodynamic variables ρ, p, T, e, only two of them are independent. This implies in particular that we have a state law under the form

$$f(\rho, T, p) = 0. \tag{2.5}$$ eq:NSF_state_e

The thermodynamic variables are linked by the identity

$$de = \frac{p}{\rho^2} d\rho + T ds, \tag{2.6}$$

eq:thermo_id

where s is the specific entropy of the fluid. Classically, in order to have a good entropy structure one has to assume that $-s$ is a convex function of $1/\rho, e$. In Section 2.4 the case for which there is an additional thermodynamic variable S, the specific salinity, is described.

Energy equations can be deduced from the above equations. Multiplying (2.2) by $\mathbf{U}$ yields the kinetic energy equation

$$\frac{\partial}{\partial t}\left(\rho\frac{|\mathbf{U}|^2}{2}\right) + \nabla\cdot\Big(\big(\rho\frac{|\mathbf{U}|^2}{2} + p - \sigma\big)\mathbf{U}\Big) = p\nabla\cdot\mathbf{U} - \sigma : D(\mathbf{U}) + \rho\mathbf{g}\cdot\mathbf{U}. \tag{2.7}$$

eq:NSF_kin_ene

Subtracting (2.7) to (2.3) gives the equation for the internal energy

$$\frac{\partial \rho e}{\partial t} + \nabla\cdot(\rho e\mathbf{U}) = -p\nabla\cdot\mathbf{U} + \sigma : D(\mathbf{U}) - \nabla\cdot Q_T,$$

or equivalently

$$\rho\frac{De}{Dt} = -p\nabla\cdot\mathbf{U} + \sigma : D(\mathbf{U}) - \nabla\cdot Q_T, \tag{2.8}$$

eq:NSF_interna

with the classical notation $D/Dt \equiv \partial/\partial t + \mathbf{U}\cdot\nabla$. We can write the continuity equation (2.1) as

$$\rho\frac{D\rho}{Dt} + \rho^2\nabla\cdot\mathbf{U} = 0. \tag{2.9}$$

eq:mass_cons_c

With the thermodynamic relation (2.6) one can write $ds = de/T - (p/T\rho^2)d\rho$, thus multiplying (2.8) by $1/T$ and (2.9) by $-p/T\rho^2$ we obtain

$$\rho\frac{Ds}{Dt} = \frac{1}{T}\sigma : D(\mathbf{U}) - \frac{1}{T}\nabla\cdot Q_T.$$

This can be written also

$$\frac{\partial \rho s}{\partial t} + \nabla\cdot(\rho s\mathbf{U}) = \frac{1}{T}\sigma : D(\mathbf{U}) - \nabla\cdot\frac{Q_T}{T} - Q_T\cdot\frac{\nabla T}{T^2}, \tag{2.10}$$

eq:NSF_specif

which gives the increase with time of $\int \rho s$, the second principle of thermodynamics.

2.2 Boundary conditions

2.2.1 Bottom and free surface

tom_freesurf⟩? Let $\mathbf{n}_b$ and $\mathbf{n}_s$ be the unit outward normals at the bottom and at the free surface respectively, defined by (see Fig 1)

$$\mathbf{n}_b = \frac{1}{\sqrt{1+|\nabla_{x,y}z_b|^2}}\begin{pmatrix}\nabla_{x,y}z_b\\ -1\end{pmatrix}, \qquad \mathbf{n}_s = \frac{1}{\sqrt{1+|\nabla_{x,y}\eta|^2}}\begin{pmatrix}-\nabla_{x,y}\eta\\ 1\end{pmatrix}.$$

On the bottom we prescribe an impermeability condition

$$\mathbf{U} \cdot \mathbf{n}_b = 0, \tag{2.11}$$

eq:bottom

and a friction condition given e.g. by a Navier law

$$(\mathbf{\Sigma} \cdot \mathbf{n}_b) \cdot \mathbf{t}_i = -\kappa \mathbf{U} \cdot \mathbf{t}_i, \quad i = 1, 2, \tag{2.12}$$

eq:fric

with κ a Navier coefficient and $(\mathbf{t}_i, i = 1, 2)$ two tangential vectors. For some applications, we rather use more specific friction laws and the equation (2.12) is then replaced by

$$(\Sigma \cdot \mathbf{n}_b) \cdot \mathbf{t}_i = -\kappa(h, \mathbf{U}) \cdot \mathbf{t}_i, \quad i = 1, 2,$$

with $\kappa(h, \mathbf{U}).\mathbf{U} \geq 0$. On the free surface, we use the kinematic boundary condition

$$\frac{\partial \eta}{\partial t} + \mathbf{u}(t, x, y, \eta) \cdot \nabla_{x,y} \eta - w(t, x, y, \eta) = 0, \tag{2.13}$$

eq:free_surf

and the no stress condition

$$\mathbf{\Sigma} \cdot \mathbf{n}_s = -p^a(t, x, y)\mathbf{n}_s + W(t, x, y)\mathbf{t}_s, \tag{2.14}$$

eq:bound3ns

where $p^a(t, x, y)$, $W(t, x, y)$ are two given quantities, p^a (resp. W) mimics the effects of the atmospheric pressure (resp. the wind blowing at the free surface) and $\mathbf{t}_s$ is a given unit horizontal vector. Throughout the paper $p^a = cst$, $W = 0$. For the temperature, Neumann or Dirichlet boundary conditions can be taken, see Subsection 2.5.

k conspotgrav⟩ **Remark 2.1** *Computing the quantity* $\int_{z_b}^{\eta} z \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U})\right) dz$ *and using the boundary conditions* (2.13), (2.11) *one finds*

$$\frac{\partial}{\partial t} \int_{z_b}^{\eta} g\rho z dz + \nabla_{x,y} \cdot \int_{z_b}^{\eta} g\rho z \mathbf{u} dz = \int_{z_b}^{\eta} g\rho w dz,$$

which is the integrated local conservation of gravitational potential energy.

2.2.2 Fluid boundaries and solid walls

d_boundaries⟩? On solid walls we prescribe a non-penetration condition (neglecting viscosity)

$$\mathbf{U} \cdot \mathbf{n} = 0,$$

$\mathbf{n}$ being the outward normal to the considered wall, together with appropriate conditions for the temperature.

In this paper we consider fluid boundaries on which we prescribe zero, one or two of the following conditions depending on the type of the flow (fluvial or torrential): water level $h + z_b(x, y)$ given, discharge $h\mathbf{U} \cdot \mathbf{n}$ given.

2.2.3 Boundary conditions for the temperature

The heat flux in Eq. (2.3) requires to define boundary conditions for the temperature. Moreover when the state law (2.5) will be precised, the definition of the temperature at each boundary will be mandatory. We can choose either Neumann or Dirichlet conditions namely at the bottom

$$\lambda \frac{\partial T}{\partial \mathbf{n}_b} = FT_b^0, \tag{2.15}$$

BC:neumann_bot

or

$$T_b = T_b^0, \tag{2.16}$$

BC:dirichlet_b

and at the free surface

$$\lambda \frac{\partial T}{\partial \mathbf{n}_s} = FT_s^0, \tag{2.17}$$

BC:neumann_su

or

$$T_s = T_s^0, \tag{2.18}$$

BC:dirichlet_s

where FT_b^0, FT_s^0 are two given temperature fluxes and T_b^0, T_s^0 are two given temperatures.

The system is completed with some initial conditions

$$h(0,x,y) = h^0(x,y), \quad \rho(0,x,y) = \rho^0(x,y), \quad \mathbf{U}(0,x,y,z) = \mathbf{U}^0(x,y,z).$$

2.3 The low Mach limit

essible_limit⟩ In this section, the low Mach limit of the compressible Navier-Stokes equations is performed in order to get an incompressible system. As already mentioned in the introduction, one of the motivations for this limit is that the density of the water varies very little with pressure variations, and removing acoustic waves from the model is advantageous from the computational point of view. Therefore, we now consider the state equation of the fluid (2.5) under the form

$$\tilde{f}\big(\rho, T, \varepsilon(p - p_{ref})\big) = 0, \tag{2.19}$$

eq:state_eq

where $\varepsilon \ll 1$ is a small parameter and with p_{ref} a reference pressure constant in space and time. In other words, taking into account the implicit function theorem, this means to assume the particular form for the pressure

$$p = p_{ref} + \frac{p_0}{\varepsilon}, \tag{2.20}$$

eq:scaling_p

where the law $p_0(\rho, T)$ has no stiff scale.

Remark 2.2 *When writing equation* (2.19)*, we assume that the density of the water depends very weakly on the pressure, and this is true in practice. Water and seawater compressibility values at various temperatures and pressures are given in (Fine and*

Millero, 1973; Safarov et al., 2009). A possible state law for seawater (involving the salinity S, which we will include later on in Section 2.4) is to write (2.19) *as*

$$\rho(S,T,p) = \frac{\rho(S,T,p_{ref})}{1 - \frac{p-p_{ref}}{K(S,T,p-p_{ref})}},$$

where in the fraction $(p - p_{ref})/K(S,T,p - p_{ref})$, the denominator is very large with respect to the numerator, so that this law could actually be written

$$\rho(S,T,p) = \frac{\rho(S,T,p_{ref})}{1 - \varepsilon(p - p_{ref})}.$$

This law was published in (UNESCO, 1981), where values of the density ρ at different pressures and constant S,T are also given. One can see that the density varies slowly with respect to the pressure. As reported in Appendix A of (Massel, 2015), for $S = 8PSU$ and $T = 10°\mathrm{C}$, the water density for $p = 0$ (atmospheric pressure) is $\rho(8,10,0) = 1005.945659 kg.m^{-3}$ while for $p = 10bar$, the density is $\rho(8,10,10) = 1006.41797 kg.m^{-3}$. This justifies the assumption (2.19).

Remark 2.3 *One could consider that the reference pressure p_{ref} varies in time, for instance because of changes in the boundary conditions of the system - p_{ref} adapts to temperature fluxes and mass fluxes at the boundaries. Here, for the sake of simplicity, we consider that p_{ref} is constant in space and in time. Yet the model derivation should not be significantly different with $p_{ref} = p_{ref}(t)$.*

⟨rem:pref⟩

For the thermodynamic identity (2.6) to be compatible with (2.20), it is necessary to consider a rescaling for e and s, leading to the following rescaled thermodynamic identities

$$e + \frac{p_{ref}}{\rho} = \frac{e_0}{\varepsilon}, \qquad s = \frac{s_0}{\varepsilon}, \quad \text{with} \quad de_0 = \frac{p_0}{\rho^2} d\rho + T ds_0, \tag{2.21}$$

eq:scaling_the

where e_0 and s_0 do not involve stiff scales. The low Mach limit is performed by letting ε go to 0 in the Navier-Stokes-Fourier system with the relations (2.20), (2.21). As p is the physical pressure, it has to remain finite. Therefore according to (2.20) at the limit we get $p_0(\rho, T) = 0$. This can be written

$$T = T^{eq}(\rho),$$

or equivalently $\rho = \rho(T^{eq})$. In all the remainder of the paper, the superscript ^{eq} (as in e_0^{eq}) is used for the quantities "at equilibrium", i.e. quantities in which ρ and T are constrained by the relation $p_0(\rho, T) = 0$.

One can define the specific enthalpy $H = e + p/\rho$, and the specific heat capacity at constant pressure $c_p = \left(\frac{\partial H}{\partial T}\right)_p$. Then with (2.20), (2.21) one has $H = H_0/\varepsilon$ with $H_0 = e_0 + p_0/\rho$, and $c_p = c_{p_0}/\varepsilon$ with $c_{p_0} = \left(\frac{\partial H_0}{\partial T}\right)_{p_0}$.

We have the following result.

⟨prop:lowmach⟩ **Proposition 2.4** *The system*

$$\nabla \cdot \mathbf{U} = -\frac{\rho'(T^{eq})}{\rho^2 c_p} \nabla \cdot (\lambda \nabla T^{eq}), \tag{2.22}$$ eq:compatibili

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0, \tag{2.23}$$?eq:masscons_l

$$\frac{\partial (\rho \mathbf{U})}{\partial t} + \nabla \cdot (\rho \mathbf{U} \otimes \mathbf{U}) + \nabla p - \nabla \cdot \sigma = \rho \mathbf{g}, \tag{2.24}$$ eq:momentum_lo

with the relation $T = T^{eq}(\rho)$ and where p is a Lagrange multiplier, is the formal limit of the system (2.1)-(2.3)*, with* (2.20)*,* (2.21) *as ε goes to 0. The energy balance verified by* (2.22)-(2.24) *is*

$$\frac{\partial}{\partial t}\left(\rho \frac{|\mathbf{U}|^2}{2} - p_{ref} + \rho \frac{e_0^{eq}}{\varepsilon}\right) + \nabla \cdot \left(\left(\rho \frac{|\mathbf{U}|^2}{2} - p_{ref} + \rho \frac{e_0^{eq}}{\varepsilon} + p - \sigma\right)\mathbf{U}\right) \\ = \nabla \cdot \left(\frac{\lambda_0}{\varepsilon} \nabla T^{eq}\right) + \rho \mathbf{g} \cdot \mathbf{U} + (p - p_{ref}) \nabla \cdot \mathbf{U} - \sigma : D(\mathbf{U}). \tag{2.25}$$ eq:energy_lowm

The energy balance (2.25) *is expressed in the rescaled variables defined by* (2.20)*,* (2.21) *in order to show clearly the order of magnitude of each term. The quantity λ_0 is defined by* (2.28)*.*

Proof *We first rewrite* (2.1) *under the form*

$$\rho \frac{D\rho}{Dt} = -\rho^2 \nabla \cdot \mathbf{U}. \tag{2.26}$$ eq:mass_cons1

When $\varepsilon \to 0$ we get $\rho = \rho(T^{eq})$, and we can multiply (2.26) *by $dT^{eq}/d\rho$ to get an equation for the temperature*

$$\rho \frac{DT^{eq}}{Dt} = -\rho^2 \frac{dT^{eq}}{d\rho} \nabla \cdot \mathbf{U}. \tag{2.27}$$ eq:temperature

In the sequel, we consider that for the heat conduction λ and the fluid viscosity μ, we are in the following asymptotic regime,

$$\lambda = \frac{\lambda_0}{\varepsilon}, \quad \textit{and} \quad \mu \sim 1. \tag{2.28}$$ eq:scaling_lam

Notice that if λ or μ is smaller than these scales, then the corresponding terms in the incompressible system (2.22)-(2.25) *will simply vanish. Hence, taking* (2.8)$-p_{ref}/\rho^2 \times$(2.9)*, multiplying by ε and taking the limit, we get according to* (2.21)

$$\rho \frac{De_0^{eq}}{Dt} = \nabla \cdot (\lambda_0 \nabla T^{eq}). \tag{2.29}$$ eq:NSF_interna

Writing then as above the rescaled enthalpy $H_0 = e_0 + p_0/\rho$, one gets that at equilibrium

$$H_0^{eq} = e_0^{eq},$$

and $H_0^{eq} = H_0^{eq}(T^{eq})$. *One has then according to the definition of* c_{p_0}

$$dH_0^{eq} = c_{p_0} dT^{eq}.$$

We thus get a second equation for the temperature

$$\rho c_{p_0} \frac{DT^{eq}}{Dt} = \nabla \cdot (\lambda_0 \nabla T^{eq}). \qquad (2.30)$$ eq:temperature

Comparing (2.27) *and* (2.30) *we obtain* (2.22), *where we have replaced the scaled quantities* c_{p_0} *and* λ_0 *by their physical values* εc_p *and* $\varepsilon\lambda$ *respectively. The pressure* p *in* (2.24) *can finally be interpreted as a Lagrange multiplier for the equation* (2.22). *The momentum equation* (2.24) *together with the mass equation* (2.26) *gives again the kinetic energy equation* (2.7). *Adding it to* (2.29) *divided by* ε *and to trivial terms in* p_{ref} *finally gives the energy balance* (2.25). ■

Remark 2.5 *At the limit, the thermodynamic identity* (2.21) *becomes*

$$de_0^{eq} = T^{eq} ds_0^{eq}.$$

From (2.29) *we obtain the equation for the evolution of the entropy*

$$\frac{\partial}{\partial t}(\rho s_0^{eq}) + \nabla \cdot (\rho s_0^{eq} \mathbf{U}) - \frac{1}{T^{eq}} \nabla \cdot (\lambda_0 \nabla T^{eq}) = 0.$$

Written in the conservative/dissipative form, this gives

$$\frac{\partial}{\partial t}(\rho s_0^{eq}) + \nabla \cdot (\rho s_0^{eq} \mathbf{U}) - \nabla \cdot \left(\lambda_0 \frac{\nabla T^{eq}}{T^{eq}}\right) = \lambda_0 \frac{|\nabla T^{eq}|^2}{(T^{eq})^2},$$

which shows that in accordance with the second law of thermodynamics, the total entropy $\int \rho s_0^{eq}$ *can only increase.*

The energy balance (2.25) exhibits discrepancies of the order ε with respect to the original energy balance (2.3). Namely, the terms $(p - p_{ref})\nabla \cdot \mathbf{U} - \sigma : D(\mathbf{U})$ are of size ε with respect to the leading order terms and they are not present in (2.3). In order to get an energy balance closer to the original compressible equations, we go one step further and some corrections of size ε are incorporated into the system. These do not modify the order of accuracy of the approximation however, since other error terms of order ε are not corrected.

ected_system⟩? **Proposition 2.6** *The system*

$$\nabla \cdot \mathbf{U} = -\frac{\rho'(T^{eq})}{\rho^2 c_p} (\nabla \cdot (\lambda \nabla T^{eq}) + \sigma : D(\mathbf{U})), \qquad (2.31)$$ eq:NSF_1

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0, \qquad (2.32)$$?eq:NSF_2?

$$\frac{\partial(\rho\mathbf{U})}{\partial t} + \nabla\cdot(\rho\mathbf{U}\otimes\mathbf{U}) + \nabla p - \nabla\cdot\sigma = \rho\mathbf{g}, \tag{2.33}$$ eq:NSF_3

with T a given function of ρ

$$T = T^{eq}(\rho), \tag{2.34}$$ eq:rho_T

and where p is a Lagrange multiplier, is an approximation of order ε of the formal limit (2.22)-(2.24) *of the system* (2.1)-(2.3) *with* (2.20), (2.21). *The system* (2.31)-(2.33) *satisfies the energy balance equation*

$$\begin{aligned}\frac{\partial}{\partial t}\left(\rho\frac{|\mathbf{U}|^2}{2} - p_{ref} + \rho\frac{e_0^c}{\varepsilon}\right) + \nabla\cdot\left((\rho\frac{|\mathbf{U}|^2}{2} - p_{ref} + \rho\frac{e_0^c}{\varepsilon} + p - \sigma)\mathbf{U}\right) \\ = \nabla\cdot\left(\frac{\lambda_0}{\varepsilon}\nabla T^{eq}\right) + \rho\mathbf{g}\cdot\mathbf{U},\end{aligned} \tag{2.35}$$ eq:NSF_energy_

where e_0^c is an independent (rescaled) corrected energy variable such that $e_0^c - e_0^{eq} = O(\varepsilon)$.

Proof *The proof is similar to that of Proposition 2.4. At the limit, the variables are constrained by the relation* (2.34). *To obtain an energy balance close to* (2.3), *we define an independent corrected energy variable e_0^c ($e_0^c \neq e_0^{eq}$) satisfying*

$$\rho\frac{De_0^c}{Dt} = \nabla\cdot(\lambda_0\nabla T^{eq}) - \varepsilon(p - p_{ref})\nabla\cdot\mathbf{U} + \varepsilon\sigma : D(\mathbf{U}). \tag{2.36}$$ eq:NSF_interna

With this correction, the energy balance is now (2.35). *The internal energy at equilibrium e_0^{eq} satisfies* (2.36) *but without the term $\varepsilon(p - p_{ref})\nabla\cdot\mathbf{U}$, which implies that $e_0^c - e_0^{eq}$ is of order ε. Recalling that $H_0^{eq} = e_0^{eq}$, we get the equation for the enthalpy at equilibrium H_0^{eq}*

$$\rho\frac{DH_0^{eq}}{Dt} = \nabla\cdot(\lambda_0\nabla T^{eq}) + \varepsilon\sigma : D(\mathbf{U}). \tag{2.37}$$ eq:NSF_enthal

Eq. (2.37) *appears as a correction of* (2.29). *Notice that if the correction $-\varepsilon(p-p_{ref})\nabla\cdot\mathbf{U}$ were incorporated, the model obtained would be even more accurate, but it would contain derivatives of p, which makes the equations much more difficult to handle since p is a Lagrange multiplier. Here we restrict ourselves to the correction $\varepsilon\sigma : D(\mathbf{U})$, which is viscous dissipation, to keep the model simple.*

From equation (2.37), *we get a corrected equation on the temperature*

$$\rho c_{p_0}\frac{DT^{eq}}{Dt} = \nabla\cdot(\lambda_0\nabla T^{eq}) + \varepsilon\sigma : D(\mathbf{U}).$$

Consequently, comparing with (2.27), *we get the equation on $\nabla\cdot\mathbf{U}$ (in the rescaled variables)*

$$\nabla\cdot\mathbf{U} = -\frac{\rho'(T^{eq})}{\rho^2 c_{p_0}}(\nabla\cdot(\lambda_0\nabla T^{eq}) + \varepsilon\sigma : D(\mathbf{U})),$$

which is (2.31). *In physical variables, the equation for the temperature can also be written*

$$\rho c_p\frac{DT^{eq}}{Dt} = -\nabla\cdot Q_T + \sigma : D(\mathbf{U}). \tag{2.38}$$ eq:temperature

■

Remark 2.7 *The entropy equation for the model* (2.31)-(2.33) *is*

$$\frac{\partial}{\partial t}(\rho s_0^{eq}) + \nabla\cdot(\rho s_0^{eq}\mathbf{U}) - \nabla\cdot\left(\lambda_0 \frac{\nabla T^{eq}}{T^{eq}}\right) = \lambda_0 \frac{|\nabla T^{eq}|^2}{(T^{eq})^2} + \frac{\varepsilon}{T^{eq}}\sigma : D(\mathbf{U}).$$

As $\sigma : D(\mathbf{U}) \geq 0$, *we obtain again that the total entropy* $\int \rho s_0^{eq}$ *can only increase. Here there is no discrepancy with the original entropy equation* (2.10).

Remark 2.8 *In models* (2.22)-(2.24) *and* (2.31)-(2.33), *the temperature* T^{eq} *is no longer an independent variable of the system. It is recovered by inverting the state law* $\rho = \rho(T^{eq})$.

Remark 2.9 *The model* (2.31)-(2.33) *is very similar to what is classically obtained in the literature when the low-Mach limit of the Navier-Stokes equations with thermal conduction is taken, see for instance (Paolucci, 1982). The limit is usually performed by expanding the variables of the system in power series of the Mach number. The method we have used here is different. As we have given ourselves only a generic state law, we cannot express the Mach number, let alone make it appear in the equations. Instead, the result is obtained via a rescaling of the variable part of the pressure, and this enables to consider general state laws.*

Remark 2.10 *Even if a gas and water behave very differently, an example of equation of state to which the previous asymptotics can be applied is the stiffened gas law (Harlow and Amsden, 1971)*

$$p = (\gamma - 1)\rho e - \gamma p_\infty, \qquad \frac{c_p}{\gamma}T = e - \frac{p_\infty}{\rho},$$

with the constraint $e - p_\infty/\rho > 0$, *where* $\gamma > 1$, $p_\infty > 0$, $c_p > 0$ *are constants. Here the entropy is given by* $s = \frac{c_p}{\gamma}\log(T/\rho^{\gamma-1})$. *Then the scaling assumptions* (2.20), (2.21) *are satisfied when*

$$p_\infty = -p_{ref} + \frac{p_{\infty 0}}{\varepsilon}, \qquad c_p = \frac{c_{p_0}}{\varepsilon},$$

with $p_{\infty 0}$ *and* c_{p_0} *constants independent of* ε. *At equilibrium we get the relation* $\rho T^{eq} = \frac{\gamma}{\gamma-1}\frac{p_{\infty 0}}{c_{p_0}}$, *thus* T^{eq} *is inversely proportional to* ρ.

Remark 2.11 *In (Audusse et al., 2011a) the authors focused on the following 2D* (x, z) *model*

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho w}{\partial z} = 0, \tag{2.39}$$

eq:jcp_cont

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho u w}{\partial z} + \frac{\partial p}{\partial x} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z}, \tag{2.40}$$

?eq:jcp_moment

$$\frac{\partial p}{\partial z} = -\rho g + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z}, \tag{2.41}$$

?eq:jcp_pressu

$$\rho = \rho(T), \tag{2.42}$$

?eq:jcp_state?

$$\frac{\partial \rho T}{\partial t} + \frac{\partial \rho u T}{\partial x} + \frac{\partial \rho w T}{\partial z} = \frac{\lambda}{c_p}\frac{\partial^2 T}{\partial x^2} + \frac{\lambda}{c_p}\frac{\partial^2 T}{\partial z^2}. \tag{2.43}$$
eq:jcp_T

Rewriting equation (2.43) *in the non-conservative form, we get*

$$\rho\frac{\partial T}{\partial t} + \rho u\frac{\partial T}{\partial x} + \rho w\frac{\partial T}{\partial z} = \frac{\lambda}{c_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right).$$

Multiplying this equation by $\rho'(T)/\rho$ *gives an equation for* ρ

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z} = \frac{\rho'(T)}{\rho}\frac{\lambda}{c_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right). \tag{2.44}$$
eq:jcp_cont_2

Finally, subtracting (2.39) *to* (2.44) *and rearranging the terms gives a compatibility condition similar to* (2.22) *for a constant* λ

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = -\frac{\rho'(T)}{\rho^2 c_p}\lambda\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right).$$

Indeed (2.43) *is similar to* (2.30). *This shows that the model* (2.39)-(2.43) *corresponds to the model* (2.22)-(2.24) *with a hydrostatic assumption, see paragraph 2.5.*

Because of the stability inherited from the energy balance (2.35), that is consistent with (2.3), in the sequel we consider the system (2.31)-(2.33) instead of (2.22)-(2.24).

2.4 The Navier-Stokes-Fourier system with salinity

imit_salinity⟩ We now consider the situation where the fluid density depends on the temperature T and on another internal variable, the specific salinity S. This is the case of sea water. The compressible Navier-Stokes-Fourier system with temperature and salinity can be written

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{U}) = 0, \tag{2.45}$$
eq:mass_cons_s

$$\frac{\partial(\rho\mathbf{U})}{\partial t} + \nabla\cdot(\rho\mathbf{U}\otimes\mathbf{U}) + \nabla p - \nabla\cdot\sigma = \rho\mathbf{g}, \tag{2.46}$$
eq:NSF_2d3_sa

$$\frac{\partial}{\partial t}\left(\rho\frac{|\mathbf{U}|^2}{2} + \rho e\right) + \nabla\cdot\Big(\big(\rho\frac{|\mathbf{U}|^2}{2} + \rho e + p - \sigma\big)\mathbf{U}\Big) = -\nabla\cdot\mathbf{F}^T + \rho\mathbf{g}\cdot\mathbf{U}, \tag{2.47}$$
eq:NSF_energy_

$$\frac{\partial(\rho S)}{\partial t} + \nabla\cdot(\rho S\mathbf{U}) = -\nabla\cdot\mathbf{F}^S. \tag{2.48}$$
eq:transpS

The local mass and momentum conservation equations are identical to (2.1) and (2.2), whereas the energy equation is slightly modified: the heat flux is now $\mathbf{F}^T$. The conservation equation (2.48) on the mass fraction of chlorides S can also be written as

$$\rho\frac{DS}{Dt} = -\nabla\cdot\mathbf{F}^S, \tag{2.49}$$
eq:transpS_2

with $\mathbf{F}^S$ the salt flux. According to Unesco (2010), the molecular fluxes of heat and salt $\mathbf{F}^T$ and $\mathbf{F}^S$ are expressed in terms of the thermodynamic Onsager forces related to the entropy equation (2.55) below,

$$\mathbf{F}^S = A\nabla\left(\frac{-\mu_S}{T}\right) + B\nabla\left(\frac{1}{T}\right), \tag{2.50}$$
def:fluxS_TEOS

$$\mathbf{F}^T = B\nabla\left(\frac{-\mu_S}{T}\right) + C\nabla\left(\frac{1}{T}\right), \tag{2.51}$$
def:fluxT_TEOS

where A, B and C are three independent coefficients to be discussed later, and μ_S is the chemical potential of seawater. The particular symmetric form of (2.50), (2.51) is related to Onsager's reciprocity principle and the dissipation of entropy, see (2.56) below. The state equation of the fluid is

$$f(\rho, T, S, p) = 0,$$

and the thermodynamic identity now reads

$$de = \frac{p}{\rho^2}d\rho + Tds + \mu_S dS. \tag{2.52}$$
eq:thermo_id_s

As in classical gas dynamics, a simple assumption that ensures the hyperbolic structure of the model is that $-s$ is a convex function of $1/\rho, e, S$. From (2.47) and using (2.46), (2.45) we get the equation on the internal energy

$$\rho\frac{De}{Dt} = -p\nabla\cdot\mathbf{U} + \sigma : D(\mathbf{U}) - \nabla\cdot\mathbf{F}^T. \tag{2.53}$$
eq:NSF_interna

Let us explain how the formulas (2.50), (2.51) lead to the second law of thermodynamics. The equation for the entropy is obtained using the thermodynamic identity (2.52) combined with the mass, internal energy and salinity equations (2.9), (2.53), (2.49),

$$\rho\left(\frac{\partial s}{\partial t} + \mathbf{U}\cdot\nabla s\right) = \frac{1}{T}\left(\sigma : D(\mathbf{U}) - \nabla\cdot\mathbf{F}^T + \mu_S\nabla\cdot\mathbf{F}^S\right), \tag{2.54}$$
?eq:entropy_sa

that can be written under conservative/dissipative form

$$\frac{\partial\rho s}{\partial t}+\nabla\cdot(\rho s\mathbf{U}) = \frac{1}{T}\sigma : D(\mathbf{U})-\nabla\cdot\left(\frac{1}{T}\mathbf{F}^T - \frac{\mu_S}{T}\mathbf{F}^S\right)+\mathbf{F}^T\cdot\nabla\left(\frac{1}{T}\right)-\mathbf{F}^S\cdot\nabla\left(\frac{\mu_S}{T}\right). \tag{2.55}$$
eq:conservati

Substituting the expressions (2.50), (2.51) in the right-hand side of (2.55), we obtain the following quadratic form for the nonconservative terms in the right-hand side of (2.55),

$$\mathbf{F}^T\cdot\nabla\left(\frac{1}{T}\right)-\mathbf{F}^S\cdot\nabla\left(\frac{\mu_S}{T}\right) = C\left|\nabla\left(\frac{1}{T}\right)\right|^2-2B\nabla\left(\frac{1}{T}\right)\cdot\nabla\left(\frac{\mu_S}{T}\right)+A\left|\nabla\left(\frac{\mu_S}{T}\right)\right|^2. \tag{2.56}$$
eq:quadsalini

For this quadratic form to be nonnegative (which is required for entropy dissipation), the three constraints are $A > 0$, $C > 0$ and $AC > B^2$. With these constraints the expressions (2.50), (2.51) of $\mathbf{F}^S$ and $\mathbf{F}^T$ can be written in terms of the gradients of the salinity S, temperature T and pressure p (as in (Unesco, 2010), equations (B.26) and (B.27)) by writing $\mu_S = \mu_S(T, S, p)$ and assuming $\partial_S \mu_S > 0$,

$$\mathbf{F}^S = -\rho k^S \left(\nabla S + \frac{\partial_p \mu_S}{\partial_S \mu_S} \nabla p \right) - \left(\frac{\rho k^S T}{\partial_S \mu_S} \partial_T \left(\frac{\mu_S}{T} \right) + \frac{B}{T^2} \right) \nabla T, \tag{2.57}$$ eq:FSk

$$\mathbf{F}^T = -\rho c_p k^T \nabla T + \frac{B \partial_S \mu_S}{\rho k^S T} \mathbf{F}^S, \tag{2.58}$$ eq:FTk

where $k^T > 0$ and $k^S > 0$ are the thermal and molecular diffusivities of salt, related to A, B, C by

$$A = \frac{\rho k^S T}{\partial_S \mu_S}, \qquad C = \rho c_p k^T T^2 + \frac{B^2}{A}. \tag{2.59}$$ eq:relACkSkT

The free model coefficients are thus now k^S, k^T and B. Note that $\mathbf{F}^T$ in (2.58) is written as a gradient of T (as in the case where the temperature is the only tracer), plus another term, due to the presence of salt. Using (2.59) and (2.50), the quadratic form (2.56) can be rewritten

$$C \left| \nabla \left(\frac{1}{T} \right) \right|^2 - 2B \nabla \left(\frac{1}{T} \right) \cdot \nabla \left(\frac{\mu_S}{T} \right) + A \left| \nabla \left(\frac{\mu_S}{T} \right) \right|^2 = \rho c_p k^T T^2 \left| \nabla \frac{1}{T} \right|^2 + \frac{1}{A} \left| \mathbf{F}^S \right|^2, \tag{2.60}$$ eq:finalquadS

which shows that it is indeed nonnegative. Thus with (2.55) the total entropy $\int \rho s$ can only increase, in accordance with the second principle of thermodynamics.

We now perform the low Mach limit as in Section 2.3. We introduce the state equation of the fluid under the form

$$\tilde{f}\big(\rho, T, S, \varepsilon(p - p_{ref})\big) = 0, \tag{2.61}$$ eq:state_eq_sa

with $\varepsilon \ll 1$. Taking into account the thermodynamic identity (2.52), p, e, s are thus rescaled as in (2.20), (2.21) and μ_S scales as $1/\varepsilon$, which yields

$$p = p_{ref} + \frac{p_0}{\varepsilon}, \quad e + \frac{p_{ref}}{\rho} = \frac{e_0}{\varepsilon}, \quad s = \frac{s_0}{\varepsilon}, \quad \mu_S = \frac{\mu_{S0}}{\varepsilon}, \tag{2.62}$$ eq:scaling_the

$$\text{with} \quad de_0 = \frac{p_0}{\rho^2} d\rho + T ds_0 + \mu_{S0} dS. \tag{2.63}$$ eq:scaling_the

We still define the specific enthalpy $H = e + p/\rho$, and the specific heat capacity at constant pressure $c_p = \left(\frac{\partial H}{\partial T}\right)_{p,S}$. Then one has $H = H_0/\varepsilon$ with $H_0 = e_0 + p_0/\rho$, and $c_p = c_{p_0}/\varepsilon$ with $c_{p_0} = \left(\frac{\partial H_0}{\partial T}\right)_{p_0,S}$. As $\varepsilon \to 0$, the finiteness of p yields the equilibrium relation $p_0(\rho, T, S) = 0$ or equivalently $T = T^{eq}(\rho, S)$.

Proposition 2.12 *The system*

$$\nabla \cdot \mathbf{U} = \frac{1}{\rho^2 c_p} \left(\frac{\partial \rho}{\partial T^{eq}} \right)_S \left((T^{eq})^2 \left(\frac{\partial \left(\mu_S / T^{eq} \right)}{\partial T^{eq}} \right)_S \nabla \cdot \mathbf{F}^S + \nabla \cdot \mathbf{F}^T - \sigma : D(\mathbf{U}) \right)$$

$$+\frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial S}\right)_{T^{eq}}\nabla\cdot\mathbf{F}^S, \tag{2.64}$$
eq:NSF_1_sali

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{U})=0, \tag{2.65}$$
?eq:NSF_2_sali

$$\frac{\partial(\rho\mathbf{U})}{\partial t}+\nabla\cdot(\rho\mathbf{U}\otimes\mathbf{U})+\nabla p-\nabla\cdot\sigma=\rho\mathbf{g}, \tag{2.66}$$
?eq:NSF_3_sali

$$\frac{\partial(\rho S)}{\partial t}+\nabla\cdot(\rho S\mathbf{U})=-\nabla\cdot\mathbf{F}^S, \tag{2.67}$$
eq:NSF_4_sali

$$\mathbf{F}^S=-\rho k^S\nabla S-\left(\frac{\rho k^S T}{\partial_S\mu_S}\partial_T\left(\frac{\mu_S}{T}\right)+\frac{B}{T^2}\right)\nabla T, \tag{2.68}$$
eq:FSk-incomp

$$\mathbf{F}^T=-\rho c_p k^T\nabla T+\frac{B\partial_S\mu_S}{\rho k^S T}\mathbf{F}^S, \tag{2.69}$$
eq:FTk-incomp

with $T=T^{eq}(\rho,S)$ and where p is a Lagrange multiplier, is an approximation of order ε of the formal limit of the system (2.45)-(2.48), (2.57), (2.58), (2.62).

Proof *At the limit there remain only two independent thermodynamic variables, and one can take T and S. We consider that $\mathbf{F}^S$ is bounded but $\mathbf{F}^T\sim 1/\varepsilon$. With* (2.50), (2.51) *this means that $A\sim\varepsilon$, $B\sim 1$, $C\sim 1/\varepsilon$, and using* (2.59) *that*

$$k^S\sim 1,\quad c_p k^T\sim 1/\varepsilon. \tag{2.70}$$
?eq:scaling_di

At equilibrium the term in ∇p in (2.57) *disappears in the expression of $\mathbf{F}^S$ since μ_S depends only weakly on p (this is a consequence of the scaling assumption* (2.61)*) and ∇p remains bounded, giving* (2.68), (2.69)*. The mass conservation equation* (2.45)*, the momentum equation* (2.46) *and the salinity equation* (2.48) *are unchanged. Consider then the enthalpy $H=e+p/\rho=H_0/\varepsilon$. Taking* (2.53)$-p_{ref}/\rho^2\times$(2.9)*, multiplying by ε and taking the limit, we get according to* (2.62) *and using the correction as in Section* 2.3 *the equation for the rescaled enthalpy at equilibrium $H_0^{eq}=e_0^{eq}$,*

$$\rho\frac{DH_0^{eq}}{Dt}=-\varepsilon\nabla\cdot\mathbf{F}^T+\varepsilon\sigma:D(\mathbf{U}). \tag{2.71}$$
eq:NSF_enthal

One can write at equilibrium

$$dH^{eq}=\left(\frac{\partial H^{eq}}{\partial T^{eq}}\right)_S dT^{eq}+\left(\frac{\partial H^{eq}}{\partial S}\right)_{T^{eq}}dS. \tag{2.72}$$
eq:dH_TS

Combining (2.71) *(written in the physical variables) with* (2.72) *gives*

$$\rho\left(\frac{\partial H^{eq}}{\partial T^{eq}}\right)_S\frac{DT^{eq}}{Dt}-\left(\frac{\partial H^{eq}}{\partial S}\right)_{T^{eq}}\nabla\cdot\mathbf{F}^S+\nabla\cdot\mathbf{F}^T-\sigma:D(\mathbf{U})=0. \tag{2.73}$$
eq:incomprsal_

We have similarly for the density

$$d\rho=\left(\frac{\partial\rho}{\partial T^{eq}}\right)_S dT^{eq}+\left(\frac{\partial\rho}{\partial S}\right)_{T^{eq}}dS. \tag{2.74}$$
?eq:massTS?

The quantities $\left(\frac{\partial\rho}{\partial T^{eq}}\right)_S$ *and* $\left(\frac{\partial\rho}{\partial S}\right)_{T^{eq}}$ *are known from the state law of salted water, and using* (2.26), (2.49) *we deduce another equation on the temperature*

$$-\rho^2\nabla\cdot\mathbf{U} = \left(\frac{\partial\rho}{\partial T^{eq}}\right)_S \rho\frac{DT^{eq}}{Dt} - \left(\frac{\partial\rho}{\partial S}\right)_{T^{eq}} \nabla\cdot\mathbf{F}^S. \tag{2.75}$$

eq:rhoTSeq_TS

Combining (2.73) *with* (2.75) *gives an expression for* $\rho^2\nabla\cdot\mathbf{U}$

$$\rho^2\nabla\cdot\mathbf{U} = \frac{\left(\frac{\partial\rho}{\partial T^{eq}}\right)_S}{\left(\frac{\partial H^{eq}}{\partial T^{eq}}\right)_S}\left(-\left(\frac{\partial H^{eq}}{\partial S}\right)_{T^{eq}}\nabla\cdot\mathbf{F}^S + \nabla\cdot\mathbf{F}^T - \sigma : D(\mathbf{U})\right) + \left(\frac{\partial\rho}{\partial S}\right)_{T^{eq}}\nabla\cdot\mathbf{F}^S, \tag{2.76}$$

eq:divu_TS

that generalizes (2.31). *We recall that by definition* $c_p = \left(\frac{\partial H}{\partial T}\right)_{p,S}$, *thus* $c_p = c_{p0}/\varepsilon$ *and at equilibrium* $c_{p0} = \left(\frac{\partial H_0}{\partial T^{eq}}\right)_S$, *or in physical variables*

$$c_p = \left(\frac{\partial H^{eq}}{\partial T^{eq}}\right)_S, \tag{2.77}$$

eq:defcp

which enables to express the denominator in (2.76). *Note that we obtain* (2.76) *without using the thermodynamic identity* (2.52), *and without involving* μ_S. *Next in* (2.76) *it remains to express* $\partial H^{eq}/\partial S$. *One has at equilibrium (see Unesco (2010), equations (A.11.1) and (A.11.2))*

$$\left(\frac{\partial H^{eq}}{\partial S}\right)_{T^{eq}} = \mu_S - T^{eq}\frac{\partial\mu_S}{\partial T^{eq}} = -(T^{eq})^2\frac{\partial}{\partial T^{eq}}(\mu_S/T^{eq}), \tag{2.78}$$

eq:lawHmu

thus finally (2.76) *gives* (2.64). *The relation* (2.78) *can be deduced from the limit of the thermodynamic identity* (2.63). *At equilibrium we have*

$$ds_0^{eq} = \frac{dH_0^{eq}}{T^{eq}} - \frac{\mu_{S0}}{T^{eq}}dS, \tag{2.79}$$

eq:ds0eq-S

that can be interpreted as $dH^{eq} = T^{eq}ds^{eq} + \mu_S dS$, *analogous to* (2.52). *We reformulate* (2.79) *as*

$$d\left(s_0^{eq} - \frac{H_0^{eq}}{T^{eq}}\right) = \frac{H_0^{eq}}{(T^{eq})^2}dT^{eq} - \frac{\mu_{S0}}{T^{eq}}dS.$$

The left-hand side is an exact differential form, therefore we can write that the two cross derivatives with respect to T, S *and* S, T *are equal. It yields*

$$\frac{\partial}{\partial T^{eq}}\left(-\frac{\mu_{S0}}{T^{eq}}\right) = \frac{\partial}{\partial S}\left(\frac{H_0^{eq}}{(T^{eq})^2}\right),$$

which gives (2.78). ■

Remark 2.13 *Because of* (2.79), (2.71), *the entropy equation* (2.55) *is still valid for our model* (2.64)-(2.69), *and the quadratic form on the right-hand side takes the form* (2.60).

Remark 2.14 *A criterion of well-posedness of our incompressible system* (2.64)-(2.69) *can be derived as follows. We write that the second-order terms in the coupled S and T^{eq} equations* (2.67), (2.73) *give a diffusion matrix with positive eigenvalues. With* (2.68), (2.69) *we have at equilibrium (using* (2.78)*)*

$$\mathbf{F}^S = -\rho k^S \nabla S - E\nabla T, \qquad \textit{with } E = \frac{\rho k^S T}{\partial_S \mu_S}\partial_T\left(\frac{\mu_S}{T}\right) + \frac{B}{T^2}, \tag{2.80}$$ eq:FSeq

$$\begin{aligned}\mathbf{F}^T - \left(\frac{\partial H^{eq}}{\partial S}\right)_T \mathbf{F}^S &= -\rho c_p k^T \nabla T + \left(\frac{B\partial_S\mu_S}{\rho k^S T} + T^2\partial_T(\mu_S/T)\right)\mathbf{F}^S \\ &= -\rho c_p k^T \nabla T + E\frac{T\partial_S\mu_S}{\rho k^S}\mathbf{F}^S.\end{aligned} \tag{2.81}$$ eq:FTcombSeq

The diffusion matrix of the system is thus, taking into account (2.77),

$$\begin{pmatrix} \rho k^S & E \\ E\frac{T\partial_S\mu_S}{c_p} & E^2\frac{T\partial_S\mu_S}{\rho k^S c_p} + \rho k^T \end{pmatrix}.$$

We obtain positive eigenvalues under the natural conditions also mentioned in (Unesco, 2010)

$$k^S > 0, \quad k^T > 0, \quad \partial_S\mu_S > 0, \quad c_p > 0. \tag{2.82}$$ eq:parabcondS

Remark 2.15 *For the particular choice of B such that $E = 0$ (the relation between E and B is* (2.80)*) we have a diagonal diffusion matrix in S, T. In this case we have particular formulas*

$$\mathbf{F}^S = -\rho k^S \nabla S, \tag{2.83}$$ eq:FSdiag

and from (2.81), (2.78)

$$\mathbf{F}^T + T^2\partial_T\left(\frac{\mu_S}{T}\right)\mathbf{F}^S = -\rho c_p k^T \nabla T. \tag{2.84}$$?eq:FTdiag?

We deduce that

$$\nabla\cdot\mathbf{F}^T + T^2\partial_T\left(\frac{\mu_S}{T}\right)\nabla\cdot\mathbf{F}^S = -\nabla\cdot\left(\rho c_p k^T\nabla T\right) - \mathbf{F}^S\cdot\nabla\Big(T^2\partial_T\left(\frac{\mu_S}{T}\right)\Big), \tag{2.85}$$ eq:FTdiagdiv

that can be used in (2.64)*. Next we write* (2.73) *using* (2.77), (2.78) *and* (2.85) *to get*

$$\rho c_p\frac{DT}{Dt} = \nabla\cdot\left(\rho c_p k^T\nabla T\right) + \mathbf{F}^S\cdot\nabla\Big(T^2\partial_T\left(\frac{\mu_S}{T}\right)\Big) + \sigma : D(\mathbf{U}). \tag{2.86}$$ {?}

We can write

$$\begin{aligned}\nabla\Big(T^2\partial_T\left(\frac{\mu_S}{T}\right)\Big) &= \nabla\Big(-\mu_S + T\partial_T\mu_S\Big) \\ &= -\partial_T\mu_S\nabla T - \partial_S\mu_S\nabla S + \nabla\Big(T\partial_T\mu_S\Big) \\ &= -\partial_S\mu_S\nabla S + T\nabla\big(\partial_T\mu_S\big),\end{aligned}$$

thus we obtain

$$\rho c_p \frac{DT}{Dt} = \nabla \cdot \left(\rho c_p k^T \nabla T\right) + T\, \mathbf{F}^S \cdot \nabla\left(\partial_T \mu_S\right) - \partial_S \mu_S\, \mathbf{F}^S \cdot \nabla S + \sigma : D(\mathbf{U}), \tag{2.87}$$

eq:DTDtdiag

that generalizes (2.38). *We therefore obtain the equations* (2.67) *and* (2.87) *on* S *and* T*, with the value of* $\mathbf{F}^S$ *given by* (2.83). *With the conditions* (2.82) *we can see that* T *remains nonnegative and* S *verifies the maximum principle (i.e. it remains within its initial lower and upper bounds). Notice that with* (2.77), (2.78), *the model finally relies on the knowledge of* $\rho^{eq}(T,S)$, $H^{eq}(T,S)$, $k^T(T,S)$, $k^S(T,S)$.

2.5 The incompressible and hydrostatic Navier-Stokes-Fourier system

er_continuous⟩ We consider the incompressible Navier-Stokes-Fourier system (2.31)-(2.33) derived in paragraph 2.3, for the sake of simplicity we consider that the density only depends on the temperature $\rho = \rho(T)$, and we neglect the salinity S. For the sake of lightness, the exponent ^{eq} is dropped in this part and in the rest of the present document. The system is completed with the boundary conditions (2.11)-(2.14).

The hydrostatic assumption consists in neglecting the vertical acceleration of the fluid

$$\rho\left(\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial vw}{\partial y} + \frac{\partial w^2}{\partial z}\right) \approx 0, \tag{2.88}$$

eq:verticacc

where we recall that $\mathbf{U} = (\mathbf{u}, w)$ with $\mathbf{u} = (u, v)$. We refer to (Brenier, 1999; Grenier, 1999; Masmoudi and Wong, 2012) for the analysis of hydrostatic models and for their asymptotic derivation (Audusse, 2005; Ferrari and Saleri, 2004; Marche, 2007).

Using the assumption (2.88) and the defintion (2.4), the hydrostatic approximation of the incompressible Navier-Stokes-Fourier system (2.31)-(2.33) therefore reads

$$\nabla \cdot \mathbf{U} = -\frac{\rho'(T)}{\rho^2 c_p}\Big(\nabla \cdot (\lambda \nabla T) + \zeta\left(\nabla_{x,y} \cdot \mathbf{u} + \partial_z w\right)^2 + 2\mu\Big(|D_{x,y}\mathbf{u}|^2 + \frac{1}{2}|\nabla_{x,y} w + \partial_z \mathbf{u}|^2 + (\partial_z w)^2\Big)\Big), \tag{2.89}$$

eq:comp

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0, \tag{2.90}$$

eq:mass_cons_r

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla_{x,y} \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \frac{\partial (\rho \mathbf{u} w)}{\partial z} + \nabla_{x,y}\, p = \nabla_{x,y}\Big(\zeta\big(\nabla_{x,y} \cdot \mathbf{u} + \partial_z w\big)\Big) + \nabla_{x,y} \cdot \Big(2\mu D_{x,y}\mathbf{u}\Big) + \partial_z\Big(\mu\left(\nabla_{x,y} w + \partial_z \mathbf{u}\right)\Big), \tag{2.91}$$

?eq:ns_2d3?

$$\partial_z p = -\rho g + \partial_z\Big(\zeta\big(\nabla_{x,y} \cdot \mathbf{u} + \partial_z w\big)\Big) + \nabla_{x,y} \cdot \Big(\mu(\nabla_{x,y} w + \partial_z \mathbf{u})\Big) + \partial_z(2\mu \partial_z w). \tag{2.92}$$

eq:p_hyd

with $T = T(\rho)$ a given function. According to Remark 2.3, p_{ref} is taken to be the atmospheric pressure p^a that is supposed constant.

The energy balance for the hydrostatic system (2.89)-(2.92) is

$$\frac{\partial}{\partial t}\left(\rho\frac{|\mathbf{u}|^2}{2}+\rho e^c\right)+\nabla\cdot\left(\left(\rho\frac{|\mathbf{u}|^2}{2}+\rho e^c+p-p^a\right)\mathbf{U}\right)$$
$$+\nabla_{x,y}\cdot\Bigg(-\zeta(\nabla_{x,y}\cdot\mathbf{u}+\partial_z w)\mathbf{u}-2\mu(D_{x,y}\mathbf{u})\mathbf{u}-\mu\left(\nabla_{x,y}w+\partial_z\mathbf{u}\right)w\Bigg)$$
$$+\partial_z\Bigg(-\zeta(\nabla_{x,y}\cdot\mathbf{u}+\partial_z w)w-\mu\left(\nabla_{x,y}w+\partial_z\mathbf{u}\right)\cdot\mathbf{u}-2\mu w\partial_z w\Bigg)=\nabla.(\lambda\nabla T)-\rho g w,$$

with the corrected internal energy e^c governed by

$$\begin{aligned}\frac{\partial}{\partial t}(\rho e^c)+\nabla\cdot(\rho e^c\mathbf{U})&=-(p-p^a)\left(\nabla_{x,y}\cdot\mathbf{u}+\partial_z w\right)+\nabla\cdot(\lambda\nabla T)\\&+\zeta\left(\nabla_{x,y}\cdot\mathbf{u}+\partial_z w\right)^2+2\mu\Big(|D_{x,y}\mathbf{u}|^2+\frac{1}{2}|\nabla_{x,y}w+\partial_z\mathbf{u}|^2+(\partial_z w)^2\Big).\end{aligned}\tag{2.93}$$

eq:e_hydro

Notice that (2.93) is similar to (2.36) divided by ε. The energy $e(T)$ defined by $de/dT=c_p$ (e corresponds to e^{eq} in paragraph 2.3) satisfies the same equation (2.93) except the term $-(p-p^a)\nabla\cdot\mathbf{U}$ in the right-hand side. It is obtained by multiplying (2.90) by $c_p\,dT/d\rho$ and by using (2.89).

In order to make the numerical approximation easier and without loss of applicability we propose a few usual simplifications for the viscous terms. First, let us notice that for water the second viscosity cannot be neglected compared to the dynamic viscosity, see Dukhin and Goetz (2009). But in practice, the chosen value for the coefficient μ (typically $\mu\approx 0.1$ kg.m^{-1}.s^{-1}) is much bigger than the value of the dynamic viscosity ($\approx 10^{-3}$ kg.m^{-1}.s^{-1}) in order to model the turbulence effects. Hence hereafter, we can neglect the second viscosity i.e. we consider $\zeta=0$.

Instead of (2.89)-(2.92) we propose to consider the simplified system with constant μ

$$\nabla\cdot\mathbf{U}=-\frac{\rho'(T)}{\rho^2 c_p}\left(\nabla\cdot(\lambda\nabla T)+\mu\left|\nabla_{x,y}\mathbf{u}\right|^2+\mu\left|\frac{\partial\mathbf{u}}{\partial z}\right|^2\right),\tag{2.94}$$

eq:comp_mod

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{U})=0,\tag{2.95}$$

?eq:mass_cons_

$$\frac{\partial(\rho\mathbf{u})}{\partial t}+\nabla_{x,y}\cdot(\rho\mathbf{u}\otimes\mathbf{u})+\frac{\partial(\rho\mathbf{u}w)}{\partial z}+\nabla_{x,y}\int_z^\eta\rho g dz=\mu\nabla_{x,y}.\nabla_{x,y}\mathbf{u}+\mu\frac{\partial^2\mathbf{u}}{\partial z^2}\tag{2.96}$$

eq:ns_2d3_mod

Notice that we have used that from Eq. (2.94) we have $\nabla\cdot\mathbf{U}=\mathcal{O}\big((\lambda+\mu)\rho'/(\rho^2c_p)\big)$ and the terms in $\mathcal{O}\big(\mu(\lambda+\mu)\rho'/(\rho^2c_p)\big)$ appearing in Eq. (2.96) have been neglected. Therefore the formulation of the rheology terms appearing in (2.96) is similar to what has been studied by some of the authors in (Allgeyer et al., 2019). Moreover in order to obtain (2.96), the viscous terms in Eq. (2.92) have been neglected.

Concerning the dynamical boundary conditions for Eq. (2.96), at the bottom we impose a friction condition given e.g. by a Navier law

$$\frac{\mu}{\rho}\sqrt{1+|\nabla_{x,y}z_b|^2}\frac{\partial\mathbf{u}}{\partial\mathbf{n}_b}=-\kappa\mathbf{u},\tag{2.97}$$

?eq:uboundihmf

with κ a Navier coefficient. For some applications, one can choose $\kappa = \kappa(h, \mathbf{u}|_b)$. At the free surface, we impose the no stress condition

$$\frac{\mu}{\rho}\frac{\partial \tilde{\mathbf{u}}}{\partial \mathbf{n}_s} - p\mathbf{n}_s = -p^a(t,x,y)\mathbf{n}_s + W(t,x,y)\mathbf{t}_s. \tag{2.98}$$

?eq:uboundihm?

where $\tilde{\mathbf{u}} = (\mathbf{u}, 0)^T$, $p^a(t,x,y)$ and $W(t,x,y)$ are two given quantities, p^a (resp. W) mimics the effects of the atmospheric pressure (resp. the wind blowing at the free surface) and $\mathbf{t}_s$ is a given unit horizontal vector.

2.6 The hydrostatic Euler-Fourier system

Assuming that the viscous terms are small, the incompressible Navier-Stokes-Fourier system (2.31)-(2.33) becomes the incompressible Euler-Fourier system

$$\nabla \cdot \mathbf{U} = -\frac{\rho'(T)}{\rho^2 c_p}\nabla \cdot (\lambda \nabla T), \tag{2.99}$$

eq:divu_incomp

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0, \tag{2.100}$$

?eq:mass_cons_

$$\frac{\partial (\rho \mathbf{U})}{\partial t} + \nabla \cdot (\rho \mathbf{U} \otimes \mathbf{U}) + \nabla p = \rho \mathbf{g}. \tag{2.101}$$

eq:NSF_2d3_inc

Therefore, the hydrostatic approximation of the system (2.99)-(2.101) consists in the hydrostatic Euler-Fourier model

$$\nabla \cdot \mathbf{U} = -\frac{\rho'(T)}{\rho^2 c_p}\nabla \cdot (\lambda \nabla T), \tag{2.102}$$

eq:incomp0

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0, \tag{2.103}$$

?eq:mass_cons_

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla_{x,y} \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \frac{\partial (\rho \mathbf{u} w)}{\partial z} + \nabla_{x,y} \int_z^\eta \rho g dz = 0, \tag{2.104}$$

eq:euler_2d30

completed with the relation $\rho = \rho(T)$ and the boundary conditions (2.11), (2.13).

2.7 Remark on the Boussinesq assumption

ubsec:boussi⟩? In geophysical water flows, density variations are often considered as small and this allows justifying the Boussinesq assumption, which consists in considering the density variations only in the gravitational forces. More precisely, assuming

$$\rho = \rho(T) = \rho_0 + f(T),$$

with $f(T) \ll \rho_0$, this leads to writing the incompressible hydrostatic Euler system (2.102)-(2.104) under the form

$$\nabla \cdot \mathbf{U} = 0, \tag{2.105}$$

eq:div_boussi

$$\rho_0 c_p \left(\frac{\partial T}{\partial t} + \mathbf{U} \cdot \nabla T \right) = \nabla \cdot (\lambda \nabla T), \qquad (2.106)$$?eq:temp_bouss

$$\rho_0 \left(\frac{\partial \mathbf{u}}{\partial t} + \nabla_{x,y} \cdot (\mathbf{u} \otimes \mathbf{u}) + \frac{\partial (\mathbf{u} w)}{\partial z} \right) + \nabla_{x,y} \int_z^\eta \rho g dz = 0. \qquad (2.107)$$?eq:mom_boussi

Notice that whereas in (2.102), the divergence of the velocity field equals the expansion due to the temperature effects, the Boussinesq assumption implies the divergence free condition (2.105).

The Boussinesq assumption is valid in various regimes (McDougall et al., 2002; Greatbatch et al., 2001) but

- it does not ensure a conservation of the kinetic energy since $\rho_0 \frac{|\mathbf{u}|^2}{2}$ is conserved instead of $\rho \frac{|\mathbf{u}|^2}{2}$,
- for long time phenomena (sloshing, wave propagation,...) significant differences appear when the Boussinesq assumption is made, see (Audusse et al., 2011a, paragraph 6.2).

In this work, the Boussinesq assumption is not done and some remarks about its validity are given in the companion paper, see also (McDougall et al., 2002; Greatbatch et al., 2001).

3 Layer-averaging for hydrostatic models

⟨sec:av_euler⟩ Following the same strategy as in Fernandez-Nieto et al. (2018), the layer-averaging could be performed for the non-hydrostatic system (2.31)-(2.33), but due to the complexity of the problem and in order to be consistent with the proposed numerical scheme in the companion paper, we focus on hydrostatic models. In this section we propose a layer-averaged formulation of the hydrostatic Navier-Stokes-Fourier system (2.94)-(2.96). To simplify the presentation, in a first step we neglect the viscous effects within the fluid and the diffusion terms for the temperature, therefore we consider the incompressible and hydrostatic Euler system with variable density and free surface given in paragraph 3.1. Then in paragraph 3.2, the dissipative and diffusion terms will be considered.

In order to describe and simulate complex flows where the velocity field cannot be approximated by its vertical mean, multilayer models have been developed (Audusse, 2005; Audusse and Bristeau, 2007; Audusse et al., 2008; Bouchut and Zeitlin, 2010; Castro et al., 2004, 2001). Unfortunately these models are physically relevant for non miscible fluids. In (Fernández-Nieto et al., 2014; Audusse et al., 2011b,a; Sainte-Marie, 2011), some authors have proposed a simpler and more general formulation for multilayer model with mass exchanges between the layers. The obtained model has the form of a conservation law with source terms. The layer-averaged approximation of the 3d Navier-Stokes system with constant density is studied in (Allgeyer et al., 2019). Compared to the constant density case, when considering the density variations, additional source terms appear, see remark 3.2. Notice that in (Audusse et al., 2011a) the hydrostatic Navier-Stokes equations with variable density is tackled but only in the 2d context.

With respect to commonly used Euler or Navier-Stokes approximations, the appealing features of the proposed multilayer approach are the easy handling of the free surface, which does not require moving meshes (e.g. (Decoene and Gerbeau, 2009)), and the possibility to take advantage of robust and accurate numerical techniques developed in extensive amount for classical one-layer Saint-Venant equations.

We consider a discretization of the fluid domain by layers (see Fig. 2) where the layer α contains the points of coordinates (x, y, z) with $z \in L_\alpha(t, x, y) = (z_{\alpha-1/2}, z_{\alpha+1/2})$ and $\{z_{\alpha+1/2}\}_{\alpha=1,\dots,N}$ is defined by

$$\left\{ \begin{array}{l} z_{\alpha+1/2}(t,x,y) = z_b(x,y) + \sum_{j=1}^{\alpha} h_j(t,x,y), \quad \alpha \in [0,\dots,N], \\ h_\alpha(t,x,y) = z_{\alpha+1/2}(t,x,y) - z_{\alpha-1/2}(t,x,y) = l_\alpha h(t,x,y), \end{array} \right. \tag{3.1}$$

eq:layer

and $\sum_{\alpha=1}^{N} l_\alpha = 1$.

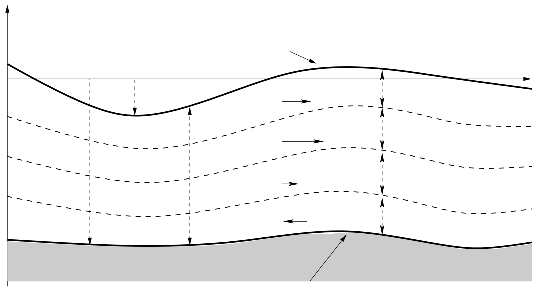

Figure 2: Notations for the layerwise discretization.

⟨fig:free⟩

Using the notations (3.1), let us consider the space $\mathbb{P}_{0,h}^{N,t}$ of piecewise constant functions defined by

$$\mathbb{P}_{0,h}^{N,t} = \left\{ \mathbb{1}_{z\in L_\alpha(t,x,y)}(z), \quad \alpha \in \{1,\dots,N\} \right\}, \tag{3.2}$$

eq:P0_space

where $\mathbb{1}_{z\in L_\alpha(t,x,y)}(z)$ is the characteristic function of the layer $L_\alpha(t,x,y)$. Using this formalism, the projection of ρ, u, v and w on $\mathbb{P}_{0,h}^{N,t}$ is a piecewise constant function defined by

$$X^N(t,x,y,z,\{z_\alpha\}) = \sum_{\alpha=1}^{N} \mathbb{1}_{z\in L_\alpha(t,x,y)}(z) X_\alpha(t,x,y), \tag{3.3}$$

eq:ulayer

for $X \in (\rho, u, v, w)$. When the quantities $\{\rho_\alpha(t,x,y)\}_{\alpha=1,\dots,N}$ are known, if the function $T = T(\rho)$ is known, it is possible to recover the temperature using the formula

$$T^N(t,x,z,\{z_\alpha\}) = \sum_{\alpha=1}^{N} \mathbb{1}_{z\in L_\alpha(t,x,y)}(z) T(\rho_\alpha(t,x,y)).$$

The layer-averaging process for the 2d hydrostatic Euler and Navier-Stokes systems is precisely described in the paper Bristeau et al. (2017) with a general rheology and in Allgeyer et al. (2019) for the 3d Navier-Stokes system with constant density, the reader can refer to it.

3.1 The layer-averaged hydrostatic Euler system with variable density

ayer_av_euler⟩ In the following, we present a Galerkin type approximation of the incompressible hydrostatic Euler system with variable density

$$\nabla \cdot \mathbf{U} = 0, \tag{3.4}$$

eq:incomp

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0, \tag{3.5}$$

eq:mass_cons_e

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla_{x,y} \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \frac{\partial (\rho \mathbf{u} w)}{\partial z} + \nabla_{x,y} \int_z^\eta \rho g dz = 0, \tag{3.6}$$

eq:euler_2d3

completed with the boundary conditions (2.11), (2.13).

The three following propositions hold.

Proposition 3.1 *Using the space $\mathbb{P}_{0,h}^{N,t}$ defined by* (3.2) *and the decomposition* (3.3)*, the Galerkin approximation of the incompressible and hydrostatic Euler equations* (3.4)-(3.6)*,* (2.11)*,* (2.13) *leads to the system*

$$\frac{\partial h}{\partial t} + \sum_{\alpha=1}^{N} \nabla_{x,y} \cdot (h_\alpha \mathbf{u}_\alpha) = 0, \tag{3.7}$$

eq:massesvm

$$\frac{\partial \rho_\alpha h_\alpha}{\partial t} + \nabla_{x,y} \cdot (\rho_\alpha h_\alpha \mathbf{u}_\alpha) = \rho_{\alpha+1/2} G_{\alpha+1/2} - \rho_{\alpha-1/2} G_{\alpha-1/2}, \qquad \alpha = 1, ..., N, \tag{3.8}$$

eq:massesvm

$$\frac{\partial \rho_\alpha h_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla_{x,y} \cdot (\rho_\alpha h_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha) + \nabla_{x,y} \big(h_\alpha p_\alpha \big) = p_{\alpha+1/2} \nabla_{x,y} z_{\alpha+1/2} - p_{\alpha-1/2} \nabla_{x,y} z_{\alpha-1/2}$$
$$+ \mathbf{u}_{\alpha+1/2} \rho_{\alpha+1/2} G_{\alpha+1/2} - \mathbf{u}_{\alpha-1/2} \rho_{\alpha-1/2} G_{\alpha-1/2}, \qquad \alpha = 1, ..., N, \tag{3.9}$$

eq:mvtsvml

where the pressure terms p_α, $p_{\alpha+1/2}$ are given by

$$p_\alpha = g \left(\frac{\rho_\alpha h_\alpha}{2} + \sum_{j=\alpha+1}^{N} \rho_j h_j \right) \qquad \text{and} \qquad p_{\alpha+1/2} = g \sum_{j=\alpha+1}^{N} \rho_j h_j. \tag{3.10}$$

eq:palpha1

The quantity $G_{\alpha+1/2}$ (resp. $G_{\alpha-1/2}$) corresponds to mass exchange across the interface $z_{\alpha+1/2}$ (resp. $z_{\alpha-1/2}$) and $G_{\alpha+1/2}$ is defined by

$$G_{\alpha+1/2} = \sum_{j=1}^{\alpha} \left(\frac{\partial h_j}{\partial t} + \nabla_{x,y} \cdot (h_j \mathbf{u}_j) \right) = - \sum_{j=1}^{N} \Big(\sum_{p=1}^{\alpha} l_p - \mathbb{1}_{j \leq \alpha} \Big) \nabla_{x,y} \cdot (h_j \mathbf{u}_j), \tag{3.11}$$

eq:Qalphabis

for $\alpha = 1, \ldots, N$. The velocities $\mathbf{u}_{\alpha+1/2}$ and the densities $\rho_{\alpha+1/2}$ at the interfaces are defined by

$$v_{\alpha+1/2} = \left\{ \begin{array}{ll} v_\alpha & \text{if } G_{\alpha+1/2} \leq 0 \\ v_{\alpha+1} & \text{if } G_{\alpha+1/2} > 0 \end{array} \right. \tag{3.12}$$

eq:upwind_uT

for $v = \mathbf{u}, \rho$.

⟨thm:model_ml⟩

_source_terms⟩ **Remark 3.2** *In the constant density case, the integration of the pressure term gives*

$$\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \nabla_{x,y} p dz = \nabla_{x,y} \left(\rho_0 g \frac{h h_\alpha}{2} \right) + \rho_0 g h_\alpha \nabla_{x,y} z_b,$$

which is the sum of a conservative term and a source term depending on the given topography z_b*. In the variable density case, the integration of the pressure term yields the terms*

$$\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \nabla_{x,y} p dz = \nabla_{x,y}(h_\alpha p_\alpha) - p_{\alpha+1/2} \nabla_{x,y} z_{\alpha+1/2} + p_{\alpha-1/2} \nabla_{x,y} z_{\alpha-1/2}.$$

Note that $z_{\alpha+1/2}$*,* $z_{\alpha-1/2}$ *are not given data, they depend on the unknown* h*. Therefore, the pressure source terms are more difficult to handle in the variable density case.*

The smooth solutions to (3.7), (3.9) satisfy an energy balance and we have the following proposition.

nergy_balance⟩ **Proposition 3.3** *The system* (3.7), (3.9) *admits, for smooth solutions, the energy balance*

$$\begin{aligned}
&\frac{\partial}{\partial t} E_\alpha + \nabla_{x,y} \cdot \left(\mathbf{u}_\alpha \left(E_\alpha + h_\alpha p_\alpha \right) \right) \\
&\quad = \left(\rho_{\alpha+1/2} \frac{|\mathbf{u}_{\alpha+1/2}|^2}{2} + g \rho_{\alpha+1/2} z_{\alpha+1/2} \right) G_{\alpha+1/2} + p_{\alpha+1/2} \big(G_{\alpha+1/2} - \frac{\partial z_{\alpha+1/2}}{\partial t} \big) \\
&\quad - \left(\rho_{\alpha-1/2} \frac{|\mathbf{u}_{\alpha-1/2}|^2}{2} + g \rho_{\alpha-1/2} z_{\alpha-1/2} \right) G_{\alpha-1/2} - p_{\alpha-1/2} \big(G_{\alpha-1/2} - \frac{\partial z_{\alpha-1/2}}{\partial t} \big) \\
&\quad - \frac{1}{2} \left(\rho_{\alpha+1/2} |\mathbf{u}_{\alpha+1/2} - \mathbf{u}_\alpha|^2 + g h_\alpha (\rho_{\alpha+1/2} - \rho_\alpha) \right) G_{\alpha+1/2} \\
&\quad + \frac{1}{2} \left(\rho_{\alpha-1/2} |\mathbf{u}_{\alpha-1/2} - \mathbf{u}_\alpha|^2 - g h_\alpha (\rho_{\alpha-1/2} - \rho_\alpha) \right) G_{\alpha-1/2},
\end{aligned} \tag{3.13}$$

eq:energy_mcl

with

$$E_\alpha = \rho_\alpha \frac{h_\alpha |\mathbf{u}_\alpha|^2}{2} + \frac{\rho_\alpha g}{2} \big(z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2 \big). \tag{3.14}$$

?eq:energ_al?

The sum of eqs. (3.13) *for* $\alpha = 1, \ldots, N$ *gives the energy balance*

$$\begin{aligned}
&\frac{\partial}{\partial t} \sum_{\alpha=1}^{N} E_\alpha + \sum_{\alpha=1}^{N} \nabla_{x,y} \cdot \mathbf{u}_\alpha \left(E_\alpha + h_\alpha p_\alpha \right) \\
&\qquad = \; - \sum_{\alpha=1}^{N} \rho_{\alpha+1/2} \frac{|\mathbf{u}_{\alpha+1} - \mathbf{u}_\alpha|^2}{2} |G_{\alpha+1/2}| \\
&\qquad\quad - \frac{g}{2} \sum_{\alpha=1}^{N} \left(h_\alpha (\rho_{\alpha+1/2} - \rho_\alpha) + h_{\alpha+1} (\rho_{\alpha+1/2} - \rho_{\alpha+1}) \right) G_{\alpha+1/2}.
\end{aligned} \tag{3.15}$$

eq:energy_glo

op:energy_bal⟩

In the energy balance (3.15), the first line of the right hand side is non positive due to the upwinding (3.12). Concerning the second, it is a third order term since we have

$$h_\alpha(\rho_{\alpha+1/2} - \rho_\alpha) + h_{\alpha+1}(\rho_{\alpha+1/2} - \rho_{\alpha+1}) \approx h_\alpha^3 \left.\frac{\partial^2 \rho}{\partial z^2}\right|_\alpha = \mathcal{O}(l_\alpha^3).$$

Due to the hydrostatic assumption, the vertical velocity is no more a variable of the momentum equations (3.9). This is an advantage of this formulation over the hydrostatic model where the vertical velocity is needed in the momentum equation (2.104) and is deduced from the incompressibility condition (2.102). Even if the vertical velocity w no more appears in the model (3.7)-(3.9), it can be obtained as follows.

Proposition 3.4 *The piecewise constant approximation of the vertical velocity w satisfying* (3.3) *is given by*

$$w_\alpha = k_\alpha - z_\alpha \nabla_{x,y} \cdot \mathbf{u}_\alpha \tag{3.16}$$ eq:def_w

with

$$k_1 = \nabla_{x,y} \cdot (z_b \mathbf{u}_1), \qquad k_{\alpha+1} = k_\alpha + \nabla_{x,y} \cdot \big(z_{\alpha+1/2}(\mathbf{u}_{\alpha+1} - \mathbf{u}_\alpha)\big).$$

The quantities $\{w_\alpha\}_{\alpha=1}^N$ are obtained only using a post-processing of the variables governing the system (3.7)-(3.9).

Notice that relation (3.16) *is equivalent to*

$$\frac{\partial z_\alpha}{\partial t} + \mathbf{u}_\alpha . \nabla_{x,y} z_\alpha = w_\alpha + \frac{G_{\alpha+1/2} + G_{\alpha-1/2}}{2}, \tag{3.17}$$ eq:w_alpha0

and using (3.19) *is also equivalent to*

$$\frac{\partial}{\partial t}\left(\rho_\alpha \frac{z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2}{2}\right) + \nabla_{x,y} \cdot \left(\rho_\alpha \frac{z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2}{2} \mathbf{u}_\alpha\right) = \rho_\alpha h_\alpha w_\alpha$$
$$+ (z_\alpha \rho_{\alpha+1/2} + \rho_\alpha \frac{h_\alpha}{2}) G_{\alpha+1/2} - (z_\alpha \rho_{\alpha-1/2} - \rho_\alpha \frac{h_\alpha}{2}) G_{\alpha-1/2}, \tag{3.18}$$ eq:w_alpha

see (Bristeau et al., 2017, paragraph 4.2).

⟨prop:def_w⟩

Proof of prop. 3.1 *Considering the divergence free condition* (3.4) *we get*

$$0 = \int_{\mathbb{R}} \mathbf{1}_{z \in L_\alpha(t,x,y)} \nabla \cdot \mathbf{U} dz = \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \nabla \cdot \mathbf{U} dz,$$

and from the definition (3.1)*, simple computations give*

$$0 = \int_{\mathbb{R}} \mathbb{1}_{z \in L_\alpha(t,x,y)} \nabla \cdot \mathbf{U} dz = \frac{\partial h_\alpha}{\partial t} + \frac{\partial}{\partial x} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} u \, dz + \frac{\partial}{\partial y} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} v \, dz - G_{\alpha+1/2} + G_{\alpha-1/2},$$

with $G_{\alpha\pm1/2}$ corresponding to

$$G_{\alpha+1/2} = \frac{\partial z_{\alpha+1/2}}{\partial t} + \mathbf{u}_{\alpha+1/2} \cdot \nabla_{x,y} z_{\alpha+1/2} - w_{\alpha+1/2},$$

and it will be precised below. Now from the definition (3.3)*, we obtain the mass balance for the layer α under the form*

$$\frac{\partial h_\alpha}{\partial t} + \nabla_{x,y} \cdot (h_\alpha \mathbf{u}_\alpha) = G_{\alpha+1/2} - G_{\alpha-1/2}. \qquad (3.19)$$ eq:c0_mc

The sum for $\alpha = 1, \ldots, N$ of the above relations gives (3.7) *where the kinematic boundary conditions* (2.11)*,* (2.13) *corresponding to*

$$G_{1/2} = G_{N+1/2} = 0, \qquad (3.20)$$ eq:G_boundary

have been used. Similarly, the sum for $j = 1, \ldots, \alpha$ of the relations (3.19) *with* (3.20) *gives the expression* (3.11) *for $G_{\alpha+1/2}$.*

The same strategy is used to obtain a layer-averaged version of (3.5)*,* (3.6)*,*

$$\int_{\mathbb{R}} \mathbf{1}_{z\in L_\alpha(t,x,y)} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) dz \right) dz = 0,$$

and

$$\int_{\mathbb{R}} \mathbb{1}_{z\in L_\alpha(t,x,y)} \left(\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla_{x,y} \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \frac{\partial(\rho \mathbf{u} w)}{\partial z} + \nabla_{x,y} \int_z^\eta \rho g dz \right) dz = 0,$$

leading, after simple computations, to (3.8)*,* (3.9)*.* ■

Proof of prop. 3.3 *In order to obtain* (3.13) *we multiply* (3.8) *by $gz_\alpha - |\mathbf{u}_\alpha|^2/2$ and* (3.9) *by $\mathbf{u}_\alpha$, we sum the two obtained equations and we perform simple manipulations. More precisely, the momentum equation along the x axis multiplied by u_α writes*

$$\left(\frac{\partial}{\partial t}(\rho_\alpha h_\alpha u_\alpha) + \frac{\partial}{\partial x}\left(\rho_\alpha h_\alpha u_\alpha^2 + h_\alpha p_\alpha\right) + \frac{\partial}{\partial y}\left(\rho_\alpha h_\alpha u_\alpha v_\alpha\right) \right) u_\alpha = \left(p_{\alpha+1/2} \frac{\partial z_{\alpha+1/2}}{\partial x} - p_{\alpha-1/2} \frac{\partial z_{\alpha-1/2}}{\partial x} + u_{\alpha+1/2} G_{\alpha+1/2} - u_{\alpha-1/2} G_{\alpha-1/2} \right) u_\alpha.$$

The pressure terms are treated separately from the other terms. The previous equation is rewritten as

$$I_{u,\alpha} + \frac{\partial}{\partial x}(h_\alpha u_\alpha p_\alpha) = I_{p,u,\alpha},$$

with

$$I_{u,\alpha} = \left(\frac{\partial}{\partial t}(\rho_\alpha h_\alpha u_\alpha) + \frac{\partial}{\partial x}\left(\rho_\alpha h_\alpha u_\alpha^2\right) + \frac{\partial}{\partial y}\left(\rho_\alpha h_\alpha u_\alpha v_\alpha\right) - u_{\alpha+1/2} G_{\alpha+1/2} + u_{\alpha-1/2} G_{\alpha-1/2} \right) u_\alpha,$$

and

$$I_{p,u,\alpha} = h_\alpha p_\alpha \frac{\partial u_\alpha}{\partial x} + p_{\alpha+1/2} u_\alpha \frac{\partial z_{\alpha+1/2}}{\partial x} - p_{\alpha-1/2} u_\alpha \frac{\partial z_{\alpha-1/2}}{\partial x}.$$

Using (3.8) *multiplied by* $-u_\alpha^2/2$, *the term* $I_{u,\alpha}$ *becomes*

$$\begin{aligned} I_{u,\alpha} = {} & \frac{\partial}{\partial t}\left(\frac{\rho_\alpha h_\alpha u_\alpha^2}{2}\right) + \frac{\partial}{\partial x}\left(u_\alpha \frac{\rho_\alpha h_\alpha u_\alpha^2}{2}\right) + \frac{\partial}{\partial y}\left(v_\alpha \frac{\rho_\alpha h_\alpha u_\alpha^2}{2}\right) \\ & - \rho_{\alpha+1/2} \frac{u_{\alpha+1/2}^2}{2} G_{\alpha+1/2} + \rho_{\alpha-1/2} \frac{u_{\alpha-1/2}^2}{2} G_{\alpha-1/2} \\ & + \rho_{\alpha+1/2} \frac{(u_{\alpha+1/2} - u_\alpha)^2}{2} G_{\alpha+1/2} - \rho_{\alpha-1/2} \frac{(u_{\alpha-1/2} - u_\alpha)^2}{2} G_{\alpha-1/2}. \end{aligned} \tag{3.21}$$

eq:energ_ec_0

When the second component of (3.9) *is multiplied by* v_α, *we write in a similar manner*

$$I_{v,\alpha} + \frac{\partial}{\partial y}(h_\alpha v_\alpha p_\alpha) = I_{p,v,\alpha},$$

and a similar expression is obtained for $I_{v,\alpha}$.

The pressure terms $I_{p,u,\alpha}, I_{p,v,\alpha}$ *are handled together. First we notice that*

$$p_{\alpha+1/2} = p_\alpha - \frac{\rho_\alpha g h_\alpha}{2}, \quad \text{and} \quad p_{\alpha-1/2} = p_\alpha + \frac{\rho_\alpha g h_\alpha}{2},$$

so that

$$\begin{aligned} I_{p,u,\alpha} + I_{p,v,\alpha} &= h_\alpha p_\alpha \nabla_{x,y} \cdot \mathbf{u}_\alpha + p_{\alpha+1/2} \mathbf{u}_\alpha \cdot \nabla_{x,y} z_{\alpha+1/2} - p_{\alpha-1/2} \mathbf{u}_\alpha \cdot \nabla_{x,y} z_{\alpha-1/2} \\ &= p_\alpha \nabla_{x,y} \cdot (h_\alpha \mathbf{u}_\alpha) - g\rho_\alpha h_\alpha \mathbf{u}_\alpha \cdot \nabla_{x,y} z_\alpha. \end{aligned}$$

Using (3.19), *the sum of the pressure terms becomes*

$$I_{p,u,\alpha} + I_{p,v,\alpha} = p_\alpha \left(G_{\alpha+1/2} - G_{\alpha-1/2} - \frac{\partial h_\alpha}{\partial t}\right) - g\rho_\alpha h_\alpha \mathbf{u}_\alpha \cdot \nabla_{x,y} z_\alpha, \tag{3.22}$$

eq:sum_ipu_ip

or equivalently

$$\begin{aligned} I_{p,u,\alpha} + I_{p,v,\alpha} = {} & p_{\alpha+1/2} G_{\alpha+1/2} - p_{\alpha-1/2} G_{\alpha-1/2} - p_\alpha \frac{\partial h_\alpha}{\partial t} \\ & + g\rho_\alpha \frac{h_\alpha}{2} \big(G_{\alpha+1/2} - G_{\alpha-1/2}\big) - g\rho_\alpha h_\alpha \mathbf{u}_\alpha \cdot \nabla_{x,y} z_\alpha. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} I_{p,u,\alpha} + I_{p,v,\alpha} = {} & p_{\alpha+1/2} \left(G_{\alpha+1/2} - \frac{\partial z_{\alpha+1/2}}{\partial t}\right) - p_{\alpha-1/2} \left(G_{\alpha-1/2} - \frac{\partial z_{\alpha-1/2}}{\partial t}\right) \\ & + g\rho_\alpha \frac{h_\alpha}{2} \big(G_{\alpha+1/2} - G_{\alpha-1/2}\big) - g\rho_\alpha h_\alpha \mathbf{u}_\alpha \cdot \nabla_{x,y} z_\alpha - g\rho_\alpha h_\alpha \frac{\partial z_\alpha}{\partial t}, \end{aligned} \tag{3.23}$$

eq:sum_ipu_ip

where
$$z_{\alpha\pm 1/2} = z_\alpha \pm \frac{h_\alpha}{2}$$
has been used. Next, we multiply (3.8) *by* gz_α *and we arrange the right-hand-side to get*

$$gz_\alpha \frac{\partial(\rho_\alpha h_\alpha)}{\partial t} + gz_\alpha \nabla_{x,y}\cdot(\rho_\alpha h_\alpha) = gz_{\alpha+1/2}\rho_{\alpha+1/2}G_{\alpha+1/2} - gz_{\alpha-1/2}\rho_{\alpha-1/2}G_{\alpha-1/2} - g\frac{h_\alpha}{2}\big(\rho_{\alpha+1/2}G_{\alpha+1/2} + \rho_{\alpha-1/2}G_{\alpha-1/2}\big) \quad (3.24)$$ eq:potential_t

Summing
$$I_{u,\alpha} + I_{v,\alpha} + \nabla_{x,y}\cdot(h_\alpha \mathbf{u}_\alpha p_\alpha) = I_{p,u,\alpha} + I_{p,v,\alpha} \quad (3.25)$$ eq:energy_dec

with (3.24) *gives the result.*

Finally summing the relations (3.13) *for* $\alpha = 1,\dots,N$ *gives* (3.15) *which completes the proof.* ■

Proof of prop. 3.4 *Using the boundary condition* (2.11)*, an integration from* z_b *to* z *of the divergence free condition* (3.4) *easily gives*
$$w = -\nabla_{x,y}\cdot\int_{z_b}^{z} \mathbf{u}\ dz.$$
Replacing formally in the above equation $\mathbf{u}$ *(resp.* w*) by* $\mathbf{u}^N$ *(resp.* w^N*) defined by* (3.3) *and performing an integration over the layer* L_1 *of the obtained relation yields*
$$h_1 w_1 = -\int_{z_b}^{z_{3/2}} \nabla_{x,y}\cdot\int_{z_b}^{z} \mathbf{u}_1\ dzdz_1 = h_1\nabla_{x,y}\cdot(z_b\mathbf{u}_1) - \frac{z_{3/2}^2 - z_b^2}{2}\nabla_{x,y}\cdot\mathbf{u}_1,$$
i.e.
$$w_1 = \nabla_{x,y}\cdot(z_b\mathbf{u}_1) - z_1\nabla_{x,y}\cdot\mathbf{u}_1,$$
corresponding to (3.16) *for* $\alpha = 1$*. A similar computation for the layers* $L_2,\dots,L_N$ *proves the result* (3.16) *for* $\alpha = 2,\dots,N$*.*

A more detailed version of this proof is given in Bristeau et al. (2017).

Notice that from
$$\int_{z_\alpha}^{z_{\alpha+1/2}} \nabla\cdot\mathbf{U}dz = 0, \quad \textit{and} \quad \int_{z_{\alpha-1/2}}^{z_\alpha} \nabla\cdot\mathbf{U}dz = 0,$$
performing computations similar to those depicted in prop. 3.1 and using the assumption (3.3)*, it comes*
$$w_{\alpha+1/2} - w_\alpha + \frac{h_\alpha}{2}\nabla_{x,y}\cdot\mathbf{u}_\alpha + (\mathbf{u}_\alpha - \mathbf{u}_{\alpha+1/2})\cdot\nabla_{x,y}z_{\alpha+1/2} = 0, \quad (3.26)$$ eq:w_alpha_p

and
$$w_\alpha - w_{\alpha-1/2} + \frac{h_\alpha}{2}\nabla_{x,y}\cdot\mathbf{u}_\alpha + (\mathbf{u}_{\alpha-1/2} - \mathbf{u}_\alpha)\cdot\nabla_{x,y}z_{\alpha-1/2} = 0. \quad (3.27)$$ eq:w_alpha_m

The two relations (3.26), (3.27) *are consistent with the definition* (3.16) *in the sense that the sum of* (3.26), (3.27) *gives* (3.19) *whereas the subtraction of* (3.26) *and* (3.27) *gives* (3.17)*. Finally, equation* (3.18) *is rewritten as*

$$\rho_\alpha h_\alpha \frac{\partial z_\alpha}{\partial t} + z_\alpha \frac{\partial \rho_\alpha h_\alpha}{\partial t} + \rho_\alpha h_\alpha \mathbf{u}_\alpha \cdot \nabla_{x,y} z_\alpha + z_\alpha \nabla_{x,y} \cdot (\rho_\alpha h_\alpha \mathbf{u}_\alpha) = \rho_\alpha h_\alpha w_\alpha \\ + \left(z_\alpha \rho_{\alpha+1/2} + \rho_\alpha \frac{h_\alpha}{2} \right) G_{\alpha+1/2} - \left(z_\alpha \rho_{\alpha-1/2} - \rho_\alpha \frac{h_\alpha}{2} \right) G_{\alpha-1/2}$$

and simplified into

$$\rho_\alpha h_\alpha \frac{\partial z_\alpha}{\partial t} + \rho_\alpha h_\alpha \mathbf{u}_\alpha \cdot \nabla_{x,y} z_\alpha = \rho_\alpha h_\alpha w_\alpha + \rho_\alpha h_\alpha \frac{G_{\alpha+1/2} + G_{\alpha-1/2}}{2}$$

using (3.19)*. Dividing by* $\rho_\alpha h_\alpha$ *gives equation* (3.17). ■

ic_equilibria⟩ **Proposition 3.5** *For equally distributed layers, the static equilibria of system* (3.7)-(3.9) *verify*

$$\nabla_{x,y} \tilde{\rho}_{\alpha+1/2} - \left. \frac{\partial \rho}{\partial z} \right|_{z_{\alpha+1/2}} \nabla_{x,y} z_{\alpha+1/2} = 0, \\ \nabla_{x,y} \eta = O\left(\frac{1}{N}\right), \tag{3.28}$$

eq:static_equi

with $\tilde{\rho}_{\alpha+1/2} = \dfrac{\rho_{\alpha+1} + \rho_\alpha}{2}$*. The first relation in* (3.28) *can be re-interpreted as*

$$\nabla_{x,y} \tilde{\rho}_{z_{\alpha+1/2}} = 0.$$

Remark 3.6 *For* N *large enough, the free surface is almost flat and the conditions* (3.28) *correspond to the static equilibria of the Euler system.*
Moreover, for N *large enough and* $\|\nabla_{x,y} z_b\|$ *small, all the interfaces between the layers are flat and we get* $\nabla_{x,y} \tilde{\rho}_{\alpha+1/2} = 0$*. Such a condition allows "checkerboard modes" for the density. However, as explained in the proof of Proposition 3.7, these checkerboard modes are not stable equilibria.*

Proof of prop. 3.5 *Inserting* $u_\alpha = 0$ *and replacing all the time derivatives by 0 in system* (3.7)-(3.9) *gives*

$$\nabla_{x,y}(h_\alpha p_\alpha) = p_{\alpha+1/2} \nabla_{x,y} z_{\alpha+1/2} - p_{\alpha-1/2} \nabla_{x,y} z_{\alpha-1/2}, \quad \alpha = 1, \ldots, N,$$

which we simplify to get

$$\nabla_{x,y} p_\alpha = -g \rho_\alpha \nabla_{x,y} z_\alpha, \quad \alpha = 1, \ldots, N. \tag{3.29}$$

pressure_equi

For $\alpha = N$*, equation* (3.29) *becomes*

$$g \frac{h_N}{2} \nabla_{x,y} \rho_N = -g \rho_N \nabla_{x,y} \eta.$$

Note that the left-hand side of the previous relation depends on the number of layers N. Assuming that the layers have the same size, we get $g\frac{h}{2N}\nabla_{x,y}\rho_N = -g\rho_N\nabla_{x,y}\eta$, which means that $\nabla_{x,y}\eta = O(1/N)$. The difference of (3.29) written for $\alpha+1$ and for α gives

$$\nabla_{x,y}(p_{\alpha+1} - p_\alpha) = -g\rho_{\alpha+1}\nabla_{x,y}z_{\alpha+1} + g\rho_\alpha\nabla_{x,y}z_\alpha.$$

We use relation (3.10) to express p_α and $p_{\alpha+1}$ and we assume that the layers are equally distributed, so that we get

$$gh_{\alpha+1/2}\nabla_{x,y}\left(\frac{\rho_{\alpha+1}+\rho_{\alpha+1/2}}{2}\right) = g\rho_{\alpha+1}\nabla_{x,y}z_{\alpha+1/2} - g\rho_\alpha\nabla_{x,y}z_{\alpha+1/2}, \qquad (3.30)$$ eq:pressure_st

where $h_{\alpha+1/2} = h_\alpha = h_{\alpha+1}$. Finally, we divide equation (3.30) by $h_{\alpha+1/2}$ and we define $\frac{\partial\rho}{\partial z}\Big|_{z_{\alpha+1/2}} = \frac{\rho_{\alpha+1}-\rho_\alpha}{h_{\alpha+1/2}}$ to get the result. ■

tratification⟩ **Proposition 3.7** *For $\nabla_{x,y}z_b$ small enough and for equally distributed layers, the stable equilibria of system (3.7)-(3.9) verify*

$$\partial_z\rho|_{z_\alpha} < 0, \quad \alpha = 1,\dots,N.$$

Proof of prop. 3.7 *Let us define a perturbation around a static equilibrium*

$$u_\alpha = u'_\alpha, \qquad w_\alpha = w'_\alpha$$
$$\rho_\alpha = R_\alpha + \rho'_\alpha, \qquad \nabla_{x,y}\eta = 0$$

The superscript $'$ denotes a first-order term. R_α is constant in space and time. The perturbation of the free surface is neglected. As a consequence, the space derivatives of h_α and z_α are zero for all α. The equations (3.7)-(3.9) are linearized around the static equilibrium. The Boussinesq approximation is performed for the sake of simplicity.

$$G'_{\alpha+1/2} = \sum_{j=1}^{\alpha}(\nabla_{x,y}\cdot(h_ju'_j)), \quad \alpha = 1,\dots,N, \qquad (3.31)$$?eq:lin_freedi

$$\partial_t\rho'_\alpha = \frac{\rho_{\alpha+1}-\rho_\alpha}{h_\alpha}G'_{\alpha+1/2} - \frac{\rho_{\alpha-1}-\rho_\alpha}{h_\alpha}G'_{\alpha-1/2}, \quad \alpha = 1,\dots,N, \qquad (3.32)$$ eq:lin_masscv

$$\rho_0\partial_t u'_\alpha + \nabla_{x,y}p_\alpha = 0, \quad \alpha = 1,\dots,N. \qquad (3.33)$$ eq:lin_moment

In (3.32), the terms $\frac{\rho_{\alpha+1}-\rho_\alpha}{h_\alpha}$ and $\frac{\rho_{\alpha-1}-\rho_\alpha}{h_\alpha}$ are interpreted as $\frac{1}{2}\partial_z\rho|_{z_\alpha}$ and $-\frac{1}{2}\partial_z\rho|_{z_\alpha}$ respectively, so that we get

$$\partial_t\rho'_\alpha = K_\alpha\frac{G'_{\alpha+1/2}+G'_{\alpha-1/2}}{2}, \quad \alpha = 1,\dots,N, \qquad (3.34)$$ eq:lin_masscv2

where we have used the notation $K_\alpha = \partial_z\rho|_{z_\alpha}$. Next, $G_{\alpha+1/2}, G_{\alpha-1/2}$ are replaced by their expressions given by (3.34). In (3.33), the expression given by (3.10) for the pressure is substituted. We get

$$\partial_t\rho'_\alpha = K_\alpha\left(\sum_{j=1}^{\alpha}h_j\nabla_{x,y}\cdot u'_j - \frac{h_\alpha}{2}\nabla_{x,y}\cdot u'_\alpha\right), \quad \alpha = 1,\dots,N,$$

$$\rho_0 \partial_t u'_\alpha + g \left(\sum_{j=\alpha+1}^{N} h_j \nabla_{x,y} \rho'_j + \frac{h_\alpha}{2} \nabla_{x,y} \rho_\alpha \right) = 0, \quad \alpha = 1, \dots, N.$$

We describe the perturbations u'_α, ρ'_α *as plane waves*

$$u'_\alpha = \begin{pmatrix} u_{0,\alpha,x} \\ u_{0,\alpha,y} \end{pmatrix} e^{i(\Omega t - k_\alpha x - l_\alpha y)} \quad \rho'_\alpha = \rho_{0,\alpha} e^{i(\Omega t - k_\alpha x - l_\alpha y)}, \quad \alpha = 1, \dots, N,$$

with $\Omega, k_\alpha, l_\alpha$ *positive real numbers. The linearized equations become*

$$\omega \rho_{0,\alpha} + K_\alpha \frac{H_0}{N} \left(\sum_{j=1}^{\alpha} u_{0,j} (k_\alpha + l_\alpha) - \frac{u_{0,\alpha}}{2} (k_\alpha + l_\alpha) \right) = 0, \quad \alpha = 1, \dots, N,$$
$$\rho_0 \Omega u_{0,\alpha} - g \frac{H_0}{N} \left(\sum_{j=\alpha+1}^{N} \rho_{0,j} (k_\alpha + l_\alpha) + \frac{\rho_{0,\alpha}}{2} (k_\alpha + l_\alpha) \right) = 0, \quad \alpha = 1, \dots, N.$$

Let the two vectors $u_0 = (u_{0,1}, \dots, u_{0,N})^T$ *and* $\rho_0 = (\rho_{0,1}, \dots, \rho_{0,N})^T$*. The previous system is rewritten as*

$$\Omega I_N \rho_0 + K T_+^- u_0 = 0,$$
$$T_-^+ \rho_0 + \Omega I_N u_0 = 0,$$

where I_N *is the identity matrix of size* N*,* T_-^+ *is an upper triangular matrix with negative coefficients and* KT_+^- *is a lower triangular matrix, where the coefficients of line* α *have the sign of* K_α*. Then, necessarily, for the system to admit a solution, the coefficients* $K_\alpha, \quad \alpha = 1, \dots, N$ *must be negative.* ■

3.2 The layer-averaged hydrostatic Navier-Stokes-Fourier system

sec:layer_NSF⟩ Following the same strategy as in paragraph 3.1, we derive a layer-averaged version for the hydrostatic Navier-Stokes-Fourier system (2.94)-(2.96).

?⟨layerav_nsf⟩? **Proposition 3.8** *Using the space* $\mathbb{P}_{0,h}^{N,t}$ *defined by* (3.2) *and the decomposition* (3.3)*, the layer-averaged approximation of the hydrostatic Navier-Stokes-Fourier system* (2.94)-(2.96) *completed with* (2.11), (2.12), (2.13), (2.14) *leads to the system*

$$\frac{\partial h}{\partial t} + \sum_{\alpha=1}^{N} \nabla_{x,y} \cdot (h_\alpha \mathbf{u}_\alpha) = - \sum_{\alpha=1}^{N} \frac{\rho'(T_\alpha)}{\rho_\alpha^2 c_p} (\mathcal{S}_{T,\alpha} - \mathcal{S}_{\mu,\alpha}), \tag{3.35}$$ eq:massesvm

$$\frac{\partial \rho_\alpha h_\alpha}{\partial t} + \nabla_{x,y} \cdot (\rho_\alpha h_\alpha \mathbf{u}_\alpha) = \rho_{\alpha+1/2} G_{\alpha+1/2} - \rho_{\alpha-1/2} G_{\alpha-1/2}, \qquad \alpha = 1, ..., N, \tag{3.36}$$?eq:massesvm

$$\frac{\partial \rho_\alpha h_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla_{x,y} \cdot (\rho_\alpha h_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha) + \nabla_{x,y} \big(h_\alpha p_\alpha \big) = p_{\alpha+1/2} \nabla_{x,y} z_{\alpha+1/2} - p_{\alpha-1/2} \nabla_{x,y} z_{\alpha-1/2}$$
$$+ \mathbf{u}_{\alpha+1/2} \rho_{\alpha+1/2} G_{\alpha+1/2} - \mathbf{u}_{\alpha-1/2} \rho_{\alpha-1/2} G_{\alpha-1/2} + \nabla_{x,y} \cdot (\mu h_\alpha \nabla_{x,y} \mathbf{u}_\alpha)$$

$$+\Gamma_{\alpha+1/2}(\mathbf{u}_{\alpha+1}-\mathbf{u}_\alpha)-\Gamma_{\alpha-1/2}(\mathbf{u}_\alpha-\mathbf{u}_{\alpha-1})-\kappa_\alpha\mathbf{u}_\alpha,\qquad \alpha=1,...,N, \tag{3.37}$$ eq:mvtsvml_r

where p_α and $p_{\alpha+1/2}$ are defined by (3.10) *and with*

$$G_{\alpha+1/2}=-\sum_{j=1}^{N}\Big(\sum_{p=1}^{\alpha}l_p-\mathbb{1}_{j\le\alpha}\Big)\nabla_{x,y}\cdot(h_j\mathbf{u}_j)+\sum_{j=1}^{\alpha}\frac{\rho'(T_j)}{\rho_j^2c_p}(S_{T,j}-S_{\mu,j}), \tag{3.38}$$ eq:Qalphabis_T

$$\kappa_\alpha=\begin{cases}\kappa & if\quad \alpha=1\\ 0 & if\quad \alpha\neq 1\end{cases}, \tag{3.39}$$?eq:Kalpha?

$$\mathcal{S}_{T,\alpha}=\left(\lambda\nabla_{x,y}\cdot(h_\alpha\nabla_{x,y}T_\alpha)+2\lambda_{\alpha+1/2}\frac{T_{\alpha+1}-T_\alpha}{h_{\alpha+1}+h_\alpha}-2\lambda_{\alpha-1/2}\frac{T_\alpha-T_{\alpha-1}}{h_\alpha+h_{\alpha-1}}\right), \tag{3.40}$$?eq:S_T,alpha?

$$\lambda_{\alpha+1/2}=\lambda\quad for\quad \alpha=1,\dots,N-1.$$

For $\alpha=0$, we impose $2\lambda_{\alpha+1/2}\frac{T_{\alpha+1}-T_\alpha}{h_{\alpha+1}+h_\alpha}=FT_b^0$ if the Neumann boundary condition (2.15) *is chosen, or $h_0=h_1$, $T_0=T_b^0$ if the Dirichlet boundary condition* (2.16) *is chosen. Likewise, for $\alpha=N$, we impose $2\lambda_{\alpha+1/2}\frac{T_{\alpha+1}-T_\alpha}{h_{\alpha+1}+h_\alpha}=FT_s^0$ with the boundary condition* (2.17)*, or $h_{N+1}=h_N$, $T_{N+1}=T_s^0$ with the boundary condition* (2.18)*. The terms $\mathcal{S}_{\mu,\alpha}$, $\Gamma_{\alpha+1/2}$, $\mu_{\alpha+1/2}$ are respectively defined as*

$$\mathcal{S}_{\mu,\alpha}=-h_\alpha\mu|\nabla_{x,y}\mathbf{u}_\alpha|^2-\Gamma_{\alpha+1/2}\frac{|\mathbf{u}_{\alpha+1}-\mathbf{u}_\alpha|^2}{2}-\Gamma_{\alpha-1/2}\frac{|\mathbf{u}_\alpha-\mathbf{u}_{\alpha-1}|^2}{2}-\kappa_\alpha|\mathbf{u}_\alpha|^2$$

$$\Gamma_{\alpha+1/2}=\frac{2\mu_{\alpha+1/2}}{h_{\alpha+1}+h_\alpha},$$

$$\mu_{\alpha+1/2}=\begin{cases}0 & if\quad \alpha=0\\ \mu & if\quad \alpha=1,\dots,N-1\\ 0 & if\quad \alpha=N.\end{cases}$$

The term $|\nabla_{x,y}\mathbf{u}_\alpha|^2$ actually denotes

$$\begin{aligned}|\nabla_{x,y}\mathbf{u}_\alpha|^2 &= (\nabla_{x,y}\mathbf{u}_\alpha):(\nabla_{x,y}\mathbf{u}_\alpha)^T\\ &= \left(\frac{\partial u_\alpha}{\partial x}\right)^2+\left(\frac{\partial u_\alpha}{\partial y}\right)^2+\left(\frac{\partial v_\alpha}{\partial x}\right)^2+\left(\frac{\partial v_\alpha}{\partial y}\right)^2.\end{aligned}$$

_averaged_NSF⟩

The temperature and viscosity terms have been simplified. In particular, the terms $\mu(\nabla_{x,y}u)|_{\alpha+1/2}\nabla_{x,y}z_{\alpha+1/2}, \mu(\nabla_{x,y}u)|_{\alpha-1/2}\nabla_{x,y}z_{\alpha-1/2}$ have been simplified, which is reasonable because the problems of interest are much vaster in the horizontal direction than in the vertical direction. Providing a detailed treatment of these terms is out of the scope of the present work. Notably, it would lead to very complicated terms in the fully discretized equations. For an exact integration of the viscosity terms, see (Bristeau et al., 2017), and for a simplified rheology, see (Allgeyer et al., 2019). The term $\mathcal{S}_{\mu,\alpha}$ is exactly the dissipative term that is obtained when the quantity $\mathbf{u}_\alpha.(\nabla_{x,y}\cdot(\mu h_\alpha\nabla_{x,y}\mathbf{u}_\alpha)+\Gamma_{\alpha+1/2}(\mathbf{u}_{\alpha+1}-\mathbf{u}_\alpha)-\Gamma_{\alpha-1/2}(\mathbf{u}_\alpha-\mathbf{u}_{\alpha-1})-\kappa_\alpha\mathbf{u}_\alpha)$ is reformulated as a conservative term plus a dissipative term.

Proof of prop 3.8 *The "Euler part" of the hydrostatic Navier-Stokes system is integrated as in the proof of Proposition 3.1. Here, we deal only with the viscosity terms and the fact that* $\nabla \cdot \mathbf{U}$ *is no longer equal to 0. For the integration of the viscosity term in the momentum equation, we refer to (Allgeyer et al., 2019). The temperature diffusion term is integrated in the same manner. Integrating the equation on the divergence gives*

$$\frac{\partial h_\alpha}{\partial t} + \nabla_{x,y} \cdot (h_\alpha \mathbf{u}_\alpha) = G_{\alpha+1/2} - G_{\alpha-1/2} - \frac{\rho'(T_\alpha)}{\rho_\alpha^2 c_p}(\mathcal{S}_{T,\alpha} - \mathcal{S}_{\mu,\alpha}). \tag{3.41}$$ eq:c0_mc_T

The sum for $j = 1, \ldots, \alpha$ *of the relations* (3.41) *with the boundary conditions* (3.20) *gives the expression* (3.38) *for* $G_{\alpha+1/2}$.

■

gy_balance_ns⟩ **Proposition 3.9** *The system* (3.35)-(3.37) *completed with the equation*

$$\frac{\partial}{\partial t}(\rho_\alpha h_\alpha e_\alpha) + \nabla_{x,y} \cdot (\rho_\alpha h_\alpha \mathbf{u}_\alpha e_\alpha) = \rho_{\alpha+1/2} e_{\alpha+1/2} G_{\alpha+1/2} - \rho_{\alpha-1/2} e_{\alpha-1/2} G_{\alpha-1/2} + p_\alpha \frac{\rho'(T_\alpha)}{\rho_\alpha^2 c_p}(\mathcal{S}_{T,\alpha} - \mathcal{S}_{\mu,\alpha}) + \mathcal{S}_{T,\alpha} - \mathcal{S}_{\mu,\alpha} \tag{3.42}$$ eq:intenergyml

admits, for smooth solutions, the energy balance

$$\begin{aligned}
&\frac{\partial}{\partial t} E_\alpha + \nabla_{x,y} \cdot \left(\mathbf{u}_\alpha \left(E_\alpha + h_\alpha p_\alpha - \mu h_\alpha \nabla_{x,y} \mathbf{u}_\alpha\right)\right) \\
&+ \Gamma_{\alpha+1/2} \frac{|\mathbf{u}_{\alpha+1}|^2 - |\mathbf{u}_\alpha|^2}{2} - \Gamma_{\alpha-1/2} \frac{|\mathbf{u}_\alpha|^2 - |\mathbf{u}_{\alpha-1}|^2}{2} \\
&\quad = \left(\rho_{\alpha+1/2} \frac{|\mathbf{u}_{\alpha+1/2}|^2}{2} + g \rho_{\alpha+1/2} z_{\alpha+1/2}\right) G_{\alpha+1/2} + p_{\alpha+1/2}\big(G_{\alpha+1/2} - \frac{\partial z_{\alpha+1/2}}{\partial t}\big) \\
&\quad - \left(\rho_{\alpha-1/2} \frac{|\mathbf{u}_{\alpha-1/2}|^2}{2} + g \rho_{\alpha-1/2} z_{\alpha-1/2}\right) G_{\alpha-1/2} - p_{\alpha-1/2}\big(G_{\alpha-1/2} - \frac{\partial z_{\alpha-1/2}}{\partial t}\big) \\
&\quad - \frac{1}{2} \left(\rho_{\alpha+1/2} (\mathbf{u}_{\alpha+1/2} - \mathbf{u}_\alpha)^2 + g h_\alpha (\rho_{\alpha+1/2} - \rho_\alpha)\right) G_{\alpha+1/2} \\
&\quad + \frac{1}{2} \left(\rho_{\alpha-1/2} (\mathbf{u}_{\alpha-1/2} - \mathbf{u}_\alpha)^2 - g h_\alpha (\rho_{\alpha-1/2} - \rho_\alpha)\right) G_{\alpha-1/2} + \mathcal{S}_{T,\alpha},
\end{aligned} \tag{3.43}$$ eq:energy_mcl_

with

$$E_\alpha = \rho_\alpha \frac{h_\alpha |\mathbf{u}_\alpha|^2}{2} + \frac{\rho_\alpha g}{2} \big(z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2\big) + e_\alpha. \tag{3.44}$$?eq:energ_al_N

Note that in (3.43), we use the notation

$$\mathbf{u}_\alpha \nabla_{x,y} \mathbf{u}_\alpha = \begin{pmatrix} u_\alpha \frac{\partial u_\alpha}{\partial x} + v_\alpha \frac{\partial v_\alpha}{\partial x} \\ u_\alpha \frac{\partial u_\alpha}{\partial y} + v_\alpha \frac{\partial v_\alpha}{\partial y} \end{pmatrix}.$$

Remark 3.10 *The energy* e_α *plays a role analogous to that of* e *in the continuous model in Section 2. While in the Euler model it was enough to work with the kinetic and* *potential energies because $de/dt = 0$, here, an internal energy is needed in order to obtain an energy balance. The term $\mathcal{S}_{T,\alpha}$ is a heat flux, so its sign is unknown. The sum of $\mathcal{S}_{T,\alpha}$ over the layers gives*

$$\sum_{\alpha=1}^{N} \mathcal{S}_{T,\alpha} = \lambda \sum_{\alpha=1}^{N} \nabla_{x,y} \cdot (h_\alpha \nabla_{x,y} T_\alpha) - \nabla T|_s \cdot \mathbf{n}_s + \nabla T|_b.\mathbf{n}_b.$$

Proof of prop. 3.9 *The proof is very similar to the proof of Proposition 3.3. The kinetic energy contribution is the same as before, plus a contribution from the viscosity term. The quantity $I_{u,\alpha}$ described in (3.21) becomes*

$$I_{u,\alpha} = \frac{\partial}{\partial t}\left(\frac{\rho_\alpha h_\alpha u_\alpha^2}{2}\right) + \frac{\partial}{\partial x}\left(u_\alpha\left(\frac{\rho_\alpha h_\alpha u_\alpha^2}{2} - \mu h_\alpha \frac{\partial u_\alpha}{\partial x}\right)\right) + \frac{\partial}{\partial y}\left(v_\alpha \frac{\rho_\alpha h_\alpha u_\alpha^2}{2} - \mu h_\alpha u_\alpha \frac{\partial u_\alpha}{\partial y}\right)$$
$$- \rho_{\alpha+1/2}\frac{u_{\alpha+1/2}^2}{2} G_{\alpha+1/2} + \rho_{\alpha-1/2}\frac{u_{\alpha-1/2}^2}{2} G_{\alpha-1/2}$$
$$+ \rho_{\alpha+1/2}\frac{(u_{\alpha+1/2} - u_\alpha)^2}{2} G_{\alpha+1/2} - \rho_{\alpha-1/2}\frac{(u_{\alpha-1/2} - u_\alpha)^2}{2} G_{\alpha-1/2} + \mathcal{S}_{\mu,x,\alpha}$$

For the contribution of the pressure terms, the beginning of the proof is the same, but there is a difference when substituting $\nabla_{x,y} \cdot (h_\alpha \mathbf{u}_\alpha)$ in the sum of the pressure terms. Instead of (3.22), we now obtain

$$I_{p,u,\alpha} + I_{p,v,\alpha} = p_\alpha \left(G_{\alpha+1/2} - G_{\alpha-1/2} - \frac{\partial h_\alpha}{\partial t} - \frac{\rho'(T_\alpha)}{\rho_\alpha^2 c_p}(\mathcal{S}_{T,\alpha} - \mathcal{S}_{\mu,\alpha})\right) - g\rho_\alpha h_\alpha \mathbf{u}_\alpha.\nabla_{x,y} z_\alpha,$$

so that instead of (3.23), we get

$$I_{p,u,\alpha} + I_{p,v,\alpha} = p_{\alpha+1/2}\left(G_{\alpha+1/2} - \frac{\partial z_{\alpha+1/2}}{\partial t}\right) - p_{\alpha+1/2}\left(G_{\alpha-1/2} - \frac{\partial z_{\alpha-1/2}}{\partial t}\right)$$
$$+ g\rho_\alpha \frac{h_\alpha}{2}(G_{\alpha+1/2} - G_{\alpha-1/2}) - g\rho_\alpha h_\alpha \mathbf{u}_\alpha.\nabla_{x,y} z_\alpha - g\rho_\alpha h_\alpha \frac{\partial z_\alpha}{\partial t} - p_\alpha \frac{\rho'(T_\alpha)}{\rho_\alpha^2 c_p}(\mathcal{S}_{T,\alpha} - \mathcal{S}_{\mu,\alpha}),$$

The sum of (3.25) with (3.24) and (3.42) gives the final result. ■

Remark 3.11 *The layer-averaged Navier-Stokes system obtained in Prop. 3.8 has the form*

$$\frac{\partial U}{\partial t} + \nabla_{x,y} \cdot F(U) = \mathcal{S}_p(U, z_b) + S_e(U, \partial_t U, \partial_x U) + S_{v,f}(U), \qquad (3.45)$$ eq:glo

where the vector of unknowns is

$$U = (h, \rho_1 h_1, \ldots, \rho_N h_N, q_{x,1}, \ldots, q_{x,N}, q_{y,1}, \ldots, q_{y,N})^T,$$

with $q_{x,\alpha} = \rho_\alpha h_\alpha u_\alpha$, $q_{y,\alpha} = \rho_\alpha h_\alpha v_\alpha$. We denote by $F(U) = (F_x(U), F_y(U))^T$ the fluxes of the conservative part and by

$$\mathcal{S}_p(U, z_b) = \left(0, \ldots, p_{3/2}\frac{\partial z_{3/2}}{\partial x} - p_{1/2}\frac{\partial z_{1/2}}{\partial x}, \ldots, p_{3/2}\frac{\partial z_{3/2}}{\partial y} - p_{1/2}\frac{\partial z_{1/2}}{\partial y}, \ldots\right)^T,$$

the non-conservative part of the pressure terms. The source terms are $S_e(U, \partial_t U, \partial_x U)$ *and* $S_{v,f}(U)$*, representing respectively the mass and momentum exchanges and the viscous and friction effects. A numerical scheme for the simulation of the layer-averaged Navier-Stokes system is proposed in the companion paper (Boittin et al., 2018); it relies on the form* (3.45)*.*

4 Conclusion

In this paper we have derived, using a low-Mach approximation, an incompressible model for variable density flows. This model is obtained from compressible Navier-Stokes equations by assuming that the density depends weakly on the pressure, thus involving a small parameter ε, that can eventually be identified as the Mach number. As $\varepsilon \to 0$, the pressure dependency is lost, while the dependency of the density on temperature and eventually salinity remains. It leads to an "incompressible" model where the divergence of the velocity is equal to source terms involving temperature and salinity fluxes. In order to obtain a more accurate model in terms of energy balance, a correction of order ε is incorporated, which is indeed the viscous dissipation. The resulting model does not rely on the Boussinesq approximation. It is mass-conservative; expansion or contraction can be observed as a result of the variation of a tracer concentration. Our models generalize classical incompressible models to quite general state laws that can depend on salinity. They include entropy consistency via Onsager's principles. A layer-averaged model for the hydrostatic Navier-Stokes-Fourier system is then proposed. The layer boundaries do not correspond to isopycnal surfaces and mass exchanges between the layers are allowed. The equilibria of the layer-averaged Euler-Fourier model (in which the diffusion and viscosity effects are neglected) are found to be those of the classical Euler system. For smooth solutions, the layer-averaged model verifies an energy balance.

In (Boittin et al., 2018), a numerical scheme is proposed and analyzed and the behaviour of the model is illustrated by means of several test cases.

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References

MITgcm [1] Software MITgcm. `http://mitgcm.org`.

azard:2006 [2] T. Alazard. Low Mach number limit of the full Navier-Stokes equations. *Archive for Rational Mechanics and Analysis*, 180(1):1–73, 2006.

art_3d [3] S. Allgeyer, M.-O. Bristeau, D. Froger, R. Hamouda, V. Jauzein, A. Mangeney, J. Sainte-Marie, F. Souillé, and M. Vallée. Numerical approximation of the 3d hydrostatic Navier-Stokes system with free surface. *ESAIM: M2AN*, 53(6):1981–2024, 2019. doi: https://doi.org/10.1051/m2an/2019044.

clair:2018 [4] F. Auclair, L. Bordois, Y. Dossmann, T. Duhaut, A. Paci, C. Ulses, and C. Nguyen. A non-hydrostatic non-Boussinesq algorithm for free-surface ocean modelling. *Ocean Modelling*, 2018.

audusse [5] E. Audusse. A multilayer Saint-Venant model : Derivation and numerical validation. *Discrete Contin. Dyn. Syst. Ser. B*, 5(2):189–214, 2005.

bristeau3 [6] E. Audusse and M.-O. Bristeau. Finite-volume solvers for a multilayer Saint-Venant system. *Int. J. Appl. Math. Comput. Sci.*, 17(3):311–319, 2007.

bristeau2 [7] E. Audusse, M.-O. Bristeau, and A. Decoene. Numerical simulations of 3d free surface flows by a multilayer Saint-Venant model. *Internat. J. Numer. Methods Fluids*, 56(3): 331–350, 2008.

JSM_JCP [8] E. Audusse, M.-O. Bristeau, M. Pelanti, and J. Sainte-Marie. Approximation of the hydrostatic Navier-Stokes system for density stratified flows by a multilayer model. Kinetic interpretation and numerical validation. *J. Comp. Phys.*, 230:3453–3478, 2011a. doi: 10.1016/j.jcp.2011.01.042.

JSM_M2AN [9] E. Audusse, M.-O. Bristeau, B. Perthame, and J. Sainte-Marie. A multilayer Saint-Venant system with mass exchanges for Shallow Water flows. Derivation and numerical validation. *ESAIM: M2AN*, 45:169–200, 2011b. doi: 10.1051/m2an/2010036. URL `http://dx.doi.org/10.1051/m2an/2010036`.

kin_entro [10] E. Audusse, M.-O. Bristeau, and J. Sainte-Marie. Kinetic entropy for layer-averaged hydrostatic Navier-Stokes equations. *submitted*, 2018.

sf_partII [11] L. Boittin, F. Bouchut, M.-O. Bristeau, A. Mangeney, J. Sainte-Marie, and F. Souille. The incompressible Navier-Stokes-Fourier system with free surface, Part II: Numercial scheme & validation. 2018.

multi-bz [12] F. Bouchut and V. Zeitlin. A robust well-balanced scheme for multi-layer shallow water equations. *Discrete Contin. Dyn. Syst. Ser. B*, 13:739–758, 2010.

-01661275 [13] F. Bouchut, C. Chalons, and S. Guisset. An entropy satisfying two-speed relaxation system for the barotropic Euler equations. Application to the numerical approximation of low Mach number flows. working paper or preprint, Dec. 2017. URL `https://hal.archives-ouvertes.fr/hal-01661275`.

nesq:1903 [14] J. V. Boussinesq. *Théorie analytique de la chaleur mise en harmonie avec la thermodynamique et avec la théorie mécanique de la lumière*, volume 2. Paris: Gathier-Villars, 1903.

brenier [15] Y. Brenier. Homogeneous hydrostatic flows with convex velocity profiles. *Nonlinearity*, 12(3):495–512, 1999.

BDGSM [16] M.-O. Bristeau, B. Di-Martino, C. Guichard, and J. Sainte-Marie. Layer-averaged Euler and Navier-Stokes equations. *Commun. Math. Sci.*, 15(5):1221–1246, June 2017. URL `https://hal.inria.fr/hal-01202042`.

pares [17] M. Castro, J. García-Rodríguez, J. González-Vida, J. Macías, C. Parés, and M. Vázquez-Cendón. Numerical simulation of two-layer shallow water flows through channels with irregular geometry. *J. Comput. Phys.*, 195(1):202–235, 2004.

pares1 [18] M.-J. Castro, J. Macías, and C. Parés. A Q-scheme for a class of systems of coupled conservation laws with source term. Application to a two-layer 1-D shallow water system. *M2AN Math. Model. Numer. Anal.*, 35(1):107–127, 2001.

croco [19] CROCO software. Home page. `https://www.croco-ocean.org`.

e_telemac [20] A. Decoene and J.-F. Gerbeau. Sigma transformation and ALE formulation for three-dimensional free surface flows. *Internat. J. Numer. Methods Fluids*, 59(4):357–386, 2009.

dukhin [21] A. Dukhin and P. Goetz. Bulk viscosity and compressibility measurement using acoustic spectroscopy. *The Journal of chemical physics*, 130:124519, 04 2009. doi: 10.1063/1.3095471.

eisl:2007 [22] E. Feireisl and A. Novotný. The Low Mach Number Limit for the Full Navier–Stokes–Fourier System. *Archive for Rational Mechanics and Analysis*, 186:77–107, 2007.

to_chacon [23] E. Fernández-Nieto, E. Koné, and T. Chacón Rebollo. A multilayer method for the hydrostatic Navier-Stokes equations: a particular weak solution. *Journal of Scientific Computing*, 60(2):408–437, 2014. ISSN 0885-7474. doi: 10.1007/s10915-013-9802-0. URL `http://dx.doi.org/10.1007/s10915-013-9802-0`.

-01324012 [24] E. D. Fernandez-Nieto, M. Parisot, Y. Penel, and J. Sainte-Marie. A hierarchy of dispersive layer-averaged approximations of Euler equations for free surface flows. *Communications in Mathematical Sciences*, 16(5):1169–1202, Dec. 2018. doi: 10.4310/CMS.2018.v16.n5.a1. URL `https://hal.archives-ouvertes.fr/hal-01324012`.

saleri [25] S. Ferrari and F. Saleri. A new two-dimensional Shallow Water model including pressure effects and slow varying bottom topography. *M2AN Math. Model. Numer. Anal.*, 38 (2):211–234, 2004.

fine:1973 [26] R. Fine and F. Millero. Compressibility of Water as a Function of Temperature and Pressure. *The Journal of Chemical Physics*, 59(10):5529–5536, 1973.

N1975353? [27] C. H. Gibson, L. A. Vega, and R. B. Williams. Turbulent diffusion of heat and momentum in the ocean. In F. Frenkiel and R. Munn, editors, *Turbulent Diffusion in Environmental Pollution*, volume 18 of *Advances in Geophysics*, pages

353–370. Elsevier, 1975. doi: https://doi.org/10.1016/S0065-2687(08)60471-9. URL http://www.sciencedirect.com/science/article/pii/S0065268708604719.

reatbatch [28] R.-J. Greatbatch, Y. Lu, and Y. Cai. Relaxing the boussinesq approximation in ocean circulation models. *Journal of Atmospheric and Oceanic Technology*, 18(11):1911–1923, 2001. doi: 10.1175/1520-0426(2001)018<1911:RTBAIO>2.0.CO;2.

grenier [29] E. Grenier. On the derivation of homogeneous hydrostatic equations. *ESAIM: M2AN*, 33(5):965–970, 1999.

griffies [30] S. M. Griffies, C. BÃűning, F. Bryan, E. Chassignet, R. Gerdes, H. Hasumi, A. Hirst, A.-M. Treguier, and D. Webb. Developments in ocean climate modelling. *Ocean Modelling*, 2(3):123 – 192, 2000. ISSN 1463-5003. doi: https://doi.org/10.1016/S1463-5003(00)00014-7. URL http://www.sciencedirect.com/science/article/pii/S1463500300000147.

rlow:1971 [31] F. Harlow and A. Amsden. Fluid dynamics. Technical Report LA-4700, Los Alamos National Laboratory, 1971.

uang:2001 [32] R. X. Huang, X. Jin, and X. Zhang. An Oceanic General Circulation Model in Pressure Coordinates. *Advances in Atmospheric Sciences*, 18(1), 2001.

icon [33] ICON-ESM software. Home page. https://www.mpimet.mpg.de/en/science/models/icon-esm/.

snie_2013 [34] U. Kadri and M. Stiassnie. Generation of an acoustic-gravity wave by two gravity waves, and their subsequent mutual interaction. *Journal of Fluid Mechanics*, 735:R6, 2013. doi: 10.1017/jfm.2013.539.

youyu [35] Y. Lu. Including Non-Boussinesq Effects in Boussinesq Ocean Circulation Models. *Journal Of Physical Oceanography*, 31, 2000.

marche [36] F. Marche. Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects. *European Journal of Mechanic /B*, 26:49–63, 2007.

masmoudi [37] N. Masmoudi and T. Wong. On the Hs theory of hydrostatic Euler equations. *Archive for Rational Mechanics and Analysis*, 204(1):231–271, 2012. ISSN 0003-9527. doi: 10.1007/s00205-011-0485-0. URL http://dx.doi.org/10.1007/s00205-011-0485-0.

ssel:2015 [38] S. R. Massel. *Internal Gravity Waves in the Shallow Seas*. Springer, 2015.

mcdougall [39] T. J. McDougall, R. J. Greatbatch, and Y. Lu. On conservation equations in oceanography: How accurate are boussinesq ocean models? *Journal of Physical Oceanography*, 32(5):1574–1584, 2002. doi: 10.1175/1520-0485(2002)032<1574:OCEIOH>2.0.CO;2.

mellor [40] G. L. Mellor and T. Ezer. Sea level variations induced by heating and cooling: An evaluation of the Boussinesq approximation in ocean models . *J. Geophys. Res.*, 100 (C10):20565–20577, 1995.

nemo [41] NEMO software. Home page. https://www.nemo-ocean.eu.

ucci:1982 [42] S. Paolucci. On the filtering of sound from the Navier-Stokes equations. Technical Report 82-8257, Sandia National Laboratories, 1982.

pom [43] POM software. Home page. http://www.ccpo.odu.edu/POMWEB/.

rahmstorf [44] S. Rahmstorf. Thermohaline circulation: The current climate. *Nature*, 421(699), 2003. doi: 10.1038/421699a. URL http://dx.doi.org/10.1038/421699a.

arov:2009 [45] J. Safarov, F. Millero, R. Feistel, A. Heintz, and E. Hassel. Thermodynamic properties of standard seawater: extensions to high temperatures and pressures. *Ocean Science*, 5:235–246, 2009.

JSM_M3AS [46] J. Sainte-Marie. Vertically averaged models for the free surface Euler system. Derivation and kinetic interpretation. *Math. Models Methods Appl. Sci. (M3AS)*, 21(3):459–490, 2011. doi: 10.1142/S0218202511005118.

song:2006 [47] T. T. Song and T. Y. Hou. Parametric vertical coordinate formulation for multiscale, Boussinesq, and non-Boussinesq ocean modeling. *Ocean Modelling*, 11:298–332, 2006.

soontiens [48] N. Soontiens, M. Stastna, and M. L. Waite. Trapped internal waves over topography: Non-Boussinesq effects, symmetry breaking and downstream recovery jumps. *Physics of Fluids*, 25(Âğ):066602, 2013.

l_veronis [49] E. A. Spiegel and G. Veronis. On the boussinesq approximation for a compressible fluid. *Astrophysical Journal*, 131:442–447, 1960.

esco:1981 [50] UNESCO. Tenth report of the joint panel on oceanographic tables and standards. Technical report, UNESCO Technical Papers in Marine Science, 1981.

unesco [51] Unesco. Thermodynamic equation of seawater: Calculation and use of thermodynamic properties. http://unesdoc.unesco.org/images/0018/001881/188170e.pdf, 2010.