Boundedness of the Kitanidis Filter for Optimal Robust State Estimation

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Abstract: The Kitanidis filter is a natural extension of the Kalman filter to systems subject to arbitrary disturbances or unknown inputs. Though the optimality of the Kitanidis filter was founded for general time varying systems more than 30 years ago, its stability analysis is still limited to time invariant systems, to the author's knowledge. In the framework of general time varying systems, this paper establishes upper bounds for the error covariance of the Kitanidis filter and for all the auxiliary variables involved in the filter.

Keywords: State estimation, disturbance rejection, stability analysis, unknown input observer, Kalman filter, time varying system.

1. INTRODUCTION

This paper considers linear time varying (LTV) stochastic systems in the form of

$$x_{k+1} = A_k x_k + B_k u_k + E_k d_k + w_k \tag{1a}$$

$$y_k = C_k x_k + v_k, \tag{1b}$$

where $x_k \in \mathbb{R}^n$ is the state, $y_k \in \mathbb{R}^m$ the output, $u_k \in \mathbb{R}^p$ the (known) input, $d_k \in \mathbb{R}^q$ some *disturbance* (or unknown input), $w_k \in \mathbb{R}^n$ the state noise of covariance Q_k, $v_k \in \mathbb{R}^m$ the output noise of covariance R_k, and A_k, B_k, C_k, E_k are known matrices of appropriate sizes at each discrete time instant $k = 0, 1, 2, \dots$.

The disturbance d_k, also known as *unknown input*, is a *totally arbitrary and unknown* vector sequence.

In this framework, state estimation is said *robust* in the sense of being *insensitive to the disturbance* d_k. Such results are useful for robust prediction (Kitanidis, 1987), for robust control (Ioannou and Sun, 1996), and for fault diagnosis (Chen and Patton, 1999).

The Kitanidis filter (Kitanidis, 1987) provides an *optimal* solution to this robust state estimation problem, by minimizing an error covariance criterion while being insensitive to the unknown disturbance. It is a natural extension of the classical Kalman filter to systems subject to unknown disturbances. Nevertheless, more than 30 years after the publication of this algorithm, an important piece of the theory is still missing or incomplete: the stability analysis of the Kitanidis filter. The importance of stability analysis is obvious: as a recursive algorithm, the boundedness property guarantees absence of data overflow, and the error dynamics stability ensures the well-behavedness of numerical computations.

The Kitanidis filter (Kitanidis, 1987) has been designed for general LTV (time varying) systems as formulated in (1), and it has been later studied in the same framework (Darouach and Zasadzinski, 1997). However, when its stability is analyzed, the result reported in the last cited reference is restricted to linear time *invariant* (LTI) systems. As this stability analysis is related to transfer functions, it has no obvious generalization to *time varying* systems. Stability was also considered in early studies on *unknown input observers* (Yang and Wilde, 1988; Darouach et al., 1994; Chen and Patton, 1999), but these stability results are all restricted to the LTI (time invariant) case. Indeed, there is a true difficulty to study the stability of *time varying* systems.

The main purpose of this paper is to establish an upper bound of the error covariance matrix of the Kitanidis filter, in the general framework of LTV systems as formulated in (1). It will also be shown that the Kitanidis filter gain matrix is bounded, as well as all the auxiliary variables involved in the filter. The main idea is to build a non-optimal filter insensitive to the disturbance d_k, for which an upper bound of the error covariance matrix can be first established. Then due to the optimality of the Kitanidis filter, its error covariance matrix cannot be larger, hence its boundedness is established.

Note on notations
In this paper, lower case letters denote scalars and vectors, whereas upper case letters are reserved to matrices. The $n \times n$ identity matrix is denoted by I_n. For a vector v, its Euclidean norm is denoted by $\|v\|$. For a matrix M, its matrix-norm induced by the Euclidean vector norm is denoted by $\|M\|$, which is equal to its largest singular value. For a symmetric positive (semi)-definite matrix M, $\|M\|$ is also equal to its largest eigenvalue. For a matrix M, the sum of its main diagonal entries is denoted by $\mathsf{Trace}(M)$. For a random variable vector x, its mathematical expectation is denoted by $\mathsf{E}(x)$, and its covariance matrix by $\mathsf{Cov}(x)$.

2. PROBLEM FORMULATION

Consider LTV stochastic systems as formulated in (1). Among all recursive linear filters of the form

$$\hat{x}_{k+1} = A_k\hat{x}_k + B_k u_k + L_{k+1}(y_{k+1} - C_{k+1}A_k\hat{x}_k - C_{k+1}B_k u_k), \quad (2)$$

with the state estimate $\hat{x}_k \in \mathbb{R}^n$ and the filter gain matrix $L_k \in \mathbb{R}^{n\times m}$, the Kitanidis filter, characterized by an optimal gain matrix L^*_{k+1}, is the unbiased minimum variance filter, in the sense that

$$\text{(Optimal gain)} \quad L^*_k = \arg\min_{L_k} \mathsf{Trace}\,\mathsf{Cov}(\tilde{x}_k|L_k) \quad (3)$$

subject to

$$\text{(Unbiasedness)} \quad \mathsf{E}(\tilde{x}_k|L^*_k) = 0, \quad (4)$$

where the dependence of the filter error

$$\tilde{x}_k \triangleq x_k - \hat{x}_k \quad (5)$$

on the filter gain matrix is indicated in the notations of *error mean* (mathematical expectation) $\mathsf{E}(\,\cdot\,|L^*_k)$ and *error covariance* $\mathsf{Cov}(\,\cdot\,|L_k)$.

Note that the unbiasedness (4) holds despite the disturbance $d_k \in \mathbb{R}^q$, which is totally unknown and arbitrary, random or not.

The Kitanidis filter, as presented in (Kitanidis, 1987), is given by

$$\hat{x}_{k+1} = A_k\hat{x}_k + B_k u_k + L^*_{k+1}(y_{k+1} - C_{k+1}A_k\hat{x}_k - C_{k+1}B_k u_k) \quad (6)$$

with the optimal gain L^*_{k+1} recursively computed as

$$P_{k+1|k} = A_k P_{k|k} A_k^T + Q_k \quad (7a)$$

$$\Sigma_{k+1} = C_{k+1}P_{k+1|k}C^T_{k+1} + R_{k+1} \quad (7b)$$

$$\Gamma_{k+1} = E_k - P_{k+1|k}C^T_{k+1}\Sigma^{-1}_{k+1}C_{k+1}E_k \quad (7c)$$

$$\Xi_{k+1} = E_k^T C^T_{k+1}\Sigma^{-1}_{k+1}C_{k+1}E_k \quad (7d)$$

$$\Lambda_{k+1} = \Gamma_{k+1}\Xi^{-1}_{k+1} \quad (7e)$$

$$P_{k+1|k+1} = P_{k+1|k} - P_{k+1|k}C^T_{k+1}\Sigma^{-1}_{k+1}C_{k+1}P_{k+1|k} + \Lambda_{k+1}\Xi_{k+1}\Lambda^T_{k+1} \quad (7f)$$

$$L^*_{k+1} = P_{k+1|k}C^T_{k+1}\Sigma^{-1}_{k+1} + \Gamma_{k+1}\Xi^{-1}_{k+1}E_k^T C^T_{k+1}\Sigma^{-1}_{k+1}. \quad (7g)$$

The purpose of this paper is to study the boundedness of the Kitanidis filter, mainly the existence of upper bounds for the error covariance matrix $P_{k|k} = \mathsf{Cov}(\tilde{x}_k|L^*_k)$ and for the Kitanidis gain matrix L^*_k. The reported results will also ensure that all the auxiliary variables involved in the filter are bounded, under easy to check conditions.

3. BOUNDEDNESS OF A NON-OPTIMAL FILTER

The main idea for establishing an upper bound of the error covariance matrix $P_{k|k} = \mathsf{Cov}(\tilde{x}_k|L^*_k)$ is to build a non-optimal filter, corresponding to a gain matrix $\bar{L}_k$, such that it is easier to establish an upper bound of the error covariance $\mathsf{Cov}(\tilde{x}_k|\bar{L}_k)$. Then an upper bound of $P_{k|k}$ will be obtained through $\mathsf{Trace}\,\mathsf{Cov}(\tilde{x}_k|L^*_k) \leq \mathsf{Trace}\,\mathsf{Cov}(\tilde{x}_k|\bar{L}_k)$.

3.1 Assumptions

The results of this paper will be based on some assumptions.

Basic assumptions.

(i) A_k, B_k, C_k, E_k are bounded matrix sequences for all $k \geq 0$.

(ii) The initial state $x_0 \in \mathbb{R}^n$ is a random vector following the Gaussian distribution $\mathcal{N}(\bar{x}_0, P_0)$, with a mean vector $\bar{x}_0$ and a positive definite covariance matrix P_0.

(iii) w_k and v_k are zero mean white Gaussian noises independent of each other and of x_0, with bounded covariance matrices $\mathrm{E}(w_k w_k^T) = Q_k$ and $\mathrm{E}(v_k v_k^T) = R_k$ for all $k \geq 0$. The inverse matrix R_k^{-1} is also bounded for all $k \geq 0$.

These assumptions are usually made in the classical LTV system Kalman filter theory, apart from the involved matrix E_k that did not exist in the classical case.

Disturbance subspace assumption

(iv) For all $k \geq 0$, the matrix product $C_{k+1}E_k$ has a full column rank and a bounded Moore-Penrose inverse.

This assumption is for the purpose of reliable rejection of disturbances.

3.2 Building a non-optimal filter

For the purpose of bounding the error covariance matrix of the (optimal) Kitanidis filter, let us build a non-optimal filter corresponding to a gain matrix $\bar{L}_k$, satisfying the following requirements:

- $\bar{L}_k$ is bounded for all $k \geq 0$.
- $\bar{L}_k$ leads to an unbiased filter, *i.e.*, $\mathsf{E}(\tilde{x}|\bar{L}_k) = 0$.
- $\bar{L}_k$ is simple enough so that an upper bound for the resulting filter error covariance matrix $\mathsf{Cov}(\tilde{x}|\bar{L}_k)$ can be established.

The construction of $\bar{L}_k$ presented below consists of two components, the first one ensuring the unbiasedness of the filter, and the second one stabilizing the filter error dynamics. This construction may appear complex, but the result will indeed satisfy the above requirements, as analyzed in the following subsection.

First define a matrix sequence, for $k \geq 0$,

$$G_{k+1} \triangleq E_k[(C_{k+1}E_k)^T(C_{k+1}E_k)]^{-1}(C_{k+1}E_k)^T, \quad (8)$$

which exists and is bounded due to Assumptions (i) and (iv). This first component of $\bar{L}_k$ will ensure the unbiasedness of the filter.

In order to build a second component of $\bar{L}_k$ stabilizing the filter error dynamics, consider the *auxiliary* stochastic system

$$\bar{x}_{k+1} = \bar{A}_k\bar{x}_k + \bar{w}_k \quad (9a)$$

$$\bar{y}_k = C_k\bar{x}_k + \bar{v}_k \quad (9b)$$

where $\bar{x}_k \in \mathbb{R}^n$ is the state, $\bar{y}_k \in \mathbb{R}^m$ the output, $\bar{w}_k \in \mathbb{R}^n$ and $\bar{v}_k \in \mathbb{R}^m$ are Gaussian white noises with

$$\mathsf{Cov}(\bar{w}_k) = I_n \quad (10a)$$

$$\mathsf{Cov}(\bar{v}_k) = I_m. \quad (10b)$$

The state transition matrix in (9a) is defined as

$$\bar{A}_k \triangleq (I_n - G_{k+1}C_{k+1})A_k, \quad (11)$$

which is bounded, since G_{k+1}, C_{k+1}, A_k are all bounded.

Note that the matrices A_k, C_k, E_k appearing in (8),(9),(11) are the same as in (1).

Apply the *classical* Kalman filter to system (9), resulting in

$$\hat{x}_{k+1|k} = \bar{A}_k \hat{x}_{k|k} \tag{12a}$$
$$\tilde{y}_{k+1} = y_{k+1} - C_{k+1}\hat{x}_{k+1|k} \tag{12b}$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\tilde{y}_{k+1}. \tag{12c}$$

The Kalman gain K_k is recursively computed by

$$\bar{P}_{k+1|k} = \bar{A}_k \bar{P}_{k|k} \bar{A}_k^T + I_n \tag{13a}$$
$$\bar{\Sigma}_{k+1} = C_{k+1}\bar{P}_{k+1|k}C_{k+1}^T + I_m \tag{13b}$$
$$K_{k+1} = \bar{P}_{k+1|k}C_{k+1}^T \bar{\Sigma}_{k+1}^{-1} \tag{13c}$$
$$\bar{P}_{k+1|k+1} = (I_n - K_{k+1}C_{k+1})\bar{P}_{k+1|k} \tag{13d}$$

A *non-optima*l gain $L_k = \bar{L}_k$ of the linear filter (2) is then built, for $k = 0, 1, 2 \ldots,$ as

$$\bar{L}_k \triangleq G_k + K_k - K_k C_k G_k. \tag{14}$$

3.3 Analysis of the non-optimal filter

One more assumption is needed in this analysis.

Observability assumption.

(v) The matrix sequence pair $[\bar{A}_k, C_k]$ is uniformly completely observable.

The uniform complete observability is defined with the aid of Gramian matrix. See (Kalman, 1963; Jazwinski, 1970; Moore and Anderson, 1980).

Notice that this observability assumption involves the matrix $\bar{A}_k$ depending on G_k, which in turn depends on E_k. Like in the classical Kalman filter theory for *time varying* systems, if the involved time varying matrices are known in advance (typically constant or periodical), then this assumption can be checked in advance, otherwise it has to be checked in real time (e.g. for *linear parameter varying* systems).

Proposition 1. The time varying gain matrix $\bar{L}_k$ defined in (14) is bounded for all $k \geq 0$. □

Proof.

The boundedness of the first term G_k is trivially due to Assumptions (i) and (iv), whereas the boundedness of K_k relies on the classical Kalman filter theory, as explained in the following.

Under Assumption (i), the boundedness of G_k implies that the matrix $\bar{A}_k$ defined in (11) is also bounded.

The matrix K_k is the gain of the *classical* Kalman filter applied to system (9), as expressed in (12) and (13). According to (Moore and Anderson, 1980), the boundedness of the Kalman filter is ensured by the boundedness of the involved matrices, under uniform complete observability and controllability (regarding the state noise) conditions. In the present case of the auxiliary system (9), the uniform complete observability of the matrix pair $[\bar{A}_k, C_k]$ is ensured by Assumption (v), whereas the uniform complete controllability of the matrix pair $[\bar{A}_k, I_n]$ holds trivially with the state noise covariance $\mathsf{Cov}(\bar{w}_k) = I_n$. In particular, the Kalman gain K_k is bounded, according to (Moore and Anderson, 1980). The matrix $\bar{L}_k$ defined in (14) is then also bounded, by simply reminding that C_k is assumed bounded in Assumption (i). □

Remark that, in the proof above, the boundedness of K_k is based on the results of (Moore and Anderson, 1980), which do not require an invertible state transition matrix, in contrary to those of (Deyst Jr and Price, 1968; Jazwinski, 1970). Indeed, in the case of the auxiliary system (9), the state transition matrix $\bar{A}_k$ defined in (11) is always singular.

Proposition 2. The gain matrix $\bar{L}_k$ defined in (14) satisfies

$$(I_n - \bar{L}_{k+1}C_{k+1})E_k = 0. \tag{15}$$

□

Proof.

It is straightforward to check that

$$\begin{aligned}
&(I_n - \bar{L}_{k+1}C_{k+1})E_k \\
&= E_k - (G_{k+1} + K_{k+1} - K_{k+1}C_{k+1}G_{k+1})C_{k+1}E_k \\
&= E_k - K_{k+1}C_{k+1}E_k - (I_n - K_{k+1}C_{k+1})G_{k+1}(C_{k+1}E_k) \\
&= E_k - K_{k+1}C_{k+1}E_k - (I_n - K_{k+1}C_{k+1})E_k \times \\
&\quad [(C_{k+1}E_k)^T(C_{k+1}E_k)]^{-1}(C_{k+1}E_k)^T(C_{k+1}E_k) \\
&= E_k - K_{k+1}C_{k+1}E_k - (I_n - K_{k+1}C_{k+1})E_k \\
&= 0
\end{aligned}$$

□

This property of $\bar{L}_k$ is essential for the unbiasedness of the linear filter (2) with the gain matrix $L_k = \bar{L}_k$, as shown in the following proposition.

Proposition 3. In the linear filter (2), let the gain matrix $L_k = \bar{L}_k$ as defined in (14). Then this filter applied to system (1), with the initialization $\hat{x}_0 = \mathsf{E}(x_0)$, is unbiased, despite the arbitrary disturbance d_k (or unknown input) affecting system (1). □

Proof.

First consider the linear filter (2) with any gain matrix $L_k \in \mathbb{R}^{n\times m}$. It is straightforward to check that the filter error $\tilde{x}_k$, as defined in (5), satisfies

$$\begin{aligned}\tilde{x}_{k+1} &= (I_n - L_{k+1}C_{k+1})A_k\tilde{x}_k + (I_n - L_{k+1}C_{k+1})E_k d_k \\ &\quad + (I_n - L_{k+1}C_{k+1})w_k - L_{k+1}v_{k+1}.\end{aligned} \tag{16}$$

With the particular gain $L_k = \bar{L}_k$ satisfying (15), the term involving the disturbance d_k disappears from (16). In this case,

$$\begin{aligned}\tilde{x}_{k+1} &= (I_n - \bar{L}_{k+1}C_{k+1})A_k\tilde{x}_k \\ &\quad + (I_n - \bar{L}_{k+1}C_{k+1})w_k - \bar{L}_{k+1}v_{k+1}.\end{aligned} \tag{17}$$

Assumption (iii) then leads to

$$\mathsf{E}(\tilde{x}_{k+1}) = (I_n - \bar{L}_{k+1}C_{k+1})A_k\mathsf{E}(\tilde{x}_k). \tag{18}$$

The initialization $\hat{x}_0 = \mathsf{E}(x_0)$ implies

$$\mathsf{E}(\tilde{x}_0) = \mathsf{E}(x_0 - \hat{x}_0) = \mathsf{E}(x_0) - \hat{x}_0 = 0. \tag{19}$$

It then recursively follows from (18) that $\mathsf{E}(\tilde{x}_k) = 0$ for all $k \geq 0$.

It is thus established that the linear filter (2) with $L_k = \bar{L}_k$ is unbiased. $\square$

Proposition 4. In the linear filter (2), let the gain matrix $L_k = \bar{L}_k$ as defined in (14). Then this filter applied to system (1) has a bounded error covariance matrix, *i.e.*, there exists some positive constant ρ such that, for all $k \geq 0$,

$$\|\mathsf{Cov}(\tilde{x}_k|\bar{L}_k)\| \leq \rho. \tag{20}$$

$\square$

Proof.

Let us first study the error system of the *classical* Kalman filter (12) applied to the auxiliary system (9). More specifically, let

$$\tilde{x}_{k|k} \triangleq \bar{x}_k - \hat{x}_{k|k} \tag{21}$$

where $\bar{x}_k$ is governed by (9a) and $\hat{x}_{k|k}$ computed with (12), both recursively. After some computations combining (9) and (12), the recursive equation governing $\tilde{x}_{k|k}$ writes:

$$\begin{aligned}\tilde{x}_{k+1|k+1} &= (I_n - K_{k+1}C_{k+1})\bar{A}_k\tilde{x}_{k|k} \\ &\quad + (I_n - K_{k+1}C_{k+1})\bar{w}_k - K_{k+1}\bar{v}_{k+1}.\end{aligned} \tag{22}$$

According to (Moore and Anderson, 1980), under the uniform complete observability of $[\bar{A}_k, C_k]$ (Assumption (v)) and the uniform complete controllability of $[\bar{A}_k, I_n]$ (trivially satisfied with $\mathsf{Cov}(\bar{w}_k) = I_n$), the error system (22) is exponentially stable, in the sense that there exist two positive constants α and β such that

$$\|\breve{A}_{l-1}\breve{A}_{l-2}\cdots\breve{A}_k\| \leq \alpha e^{-\beta(l-k)}, \tag{23}$$

with the notation

$$\breve{A}_k \triangleq (I_n - K_{k+1}C_{k+1})\bar{A}_k. \tag{24}$$

Now consider the error system of the linear filter (2) with $L_k = \bar{L}_k$. Its equation was already written in (17). Let us copy it here for ease of reading:

$$\begin{aligned}\tilde{x}_{k+1} &= (I_n - \bar{L}_{k+1}C_{k+1})A_k\tilde{x}_k \\ &\quad + (I_n - \bar{L}_{k+1}C_{k+1})w_k - \bar{L}_{k+1}v_{k+1}.\end{aligned} \tag{25}$$

It turns out that the state transition matrix $(I_n - \bar{L}_{k+1}C_{k+1})A_k$ of this error system is equal to the matrix $\breve{A}_k$ defined in (24). To check this fact, compute, on the one hand

$$\begin{aligned}&(I_n - \bar{L}_{k+1}C_{k+1})A_k \\ &= [I_n - (G_{k+1} + K_{k+1} - K_{k+1}C_{k+1}G_{k+1})C_{k+1}]A_k \quad (26)\\ &= (I_n - G_{k+1}C_{k+1} - K_{k+1}C_{k+1} \\ &\qquad + K_{k+1}C_{k+1}G_{k+1}C_{k+1})A_k \quad (27)\end{aligned}$$

and on the other hand,

$$\begin{aligned}\breve{A}_k &= (I_n - K_{k+1}C_{k+1})\bar{A}_k \quad (28)\\ &= (I_n - K_{k+1}C_{k+1})(I_n - G_{k+1}C_{k+1})A_k \quad (29)\\ &= (I_n - G_{k+1}C_{k+1} - K_{k+1}C_{k+1} \\ &\qquad + K_{k+1}C_{k+1}G_{k+1}C_{k+1})A_k. \quad (30)\end{aligned}$$

Hence indeed $(I_n - \bar{L}_{k+1}C_{k+1})A_k = \breve{A}_k$.

The error dynamics equation (25) is then rewritten as

$$\tilde{x}_{k+1} = \breve{A}_k\tilde{x}_k + \mu_k, \tag{31}$$

with

$$\mu_k \triangleq (I_n - \bar{L}_{k+1}C_{k+1})w_k - \bar{L}_{k+1}v_{k+1}. \tag{32}$$

As w_k and v_k are both zero mean white noises independent of each other with bounded covariance matrices,

$$\mathsf{E}(\mu_k\mu_l^T) = 0 \tag{33}$$

for any pair of positive integers $k \neq l$, though μ_k involves w_k and v_{k+1} corresponding to two different time instants. Hence μ_k is a zero mean white noise. The covariance matrix $\mathsf{Cov}(\mu_k)$ is bounded, since in (32) $\bar{L}_{k+1}$ is bounded according to Proposition 1.

The error dynamics system (31) is exponentially stable, according to (23). It is driven by a zero mean white noise μ_k of bounded covariance. Therefore, the covariance matrix of the state of system (31), namely $\mathsf{Cov}(\tilde{x}_k)$, is bounded by applying Lemma 1 (see the appendix) to (31). $\square$

4. BOUNDEDNESS OF THE KITANIDIS FILTER

The main result of this paper is stated in the following proposition.

Proposition 5. Under Assumptions (i)-(v), the covariance matrix $P_{k|k} = \mathsf{Cov}(\tilde{x}_k)$ of the Kitanidis filter (6) is bounded for all $k \geq 0$, so is the Kitanidis gain matrix L_k^*. $\square$

Proof.

The Kitanidis filter (6) is a particular case of the general filter (2) with the optimal gain matrix $L_k = L_k^*$. By the definition of L_k^* in (3),

$$\mathsf{Trace}\,\mathsf{Cov}(\tilde{x}_k|L_k^*) \leq \mathsf{Trace}\,\mathsf{Cov}(\tilde{x}_k|L_k) \tag{34}$$

for any gain $L_k \in \mathbb{R}^{n\times m}$ corresponding to an unbiased filter (2). Though the Kitanidis filter has been designed by one-step optimization of the trace criterion, as expressed in (3), it is also optimal in the sense of the whole gain sequence (Delyon and Zhang, 2021). In particular for $L_k = \bar{L}_k$:

$$\mathsf{Trace}\,\mathsf{Cov}(\tilde{x}_k|L_k^*) \leq \mathsf{Trace}\,\mathsf{Cov}(\tilde{x}_k|\bar{L}_k). \tag{35}$$

For any symmetric positive (semi)-definite matrix $M \in \mathbb{R}^{n\times n}$, the matrix norm (induced by the Euclidean vector norm) $\|M\|$ is equal to the largest eigenvalue of M, whereas $\mathsf{Trace}(M)$ is equal to the sum of the n eigenvalues of M, which are all positive or zero. Therefore

$$\mathsf{Trace}(M) \leq n\|M\| \tag{36}$$

$$\|M\| \leq \mathsf{Trace}(M). \tag{37}$$

Then,

$$\begin{aligned}\mathsf{Trace}\,\mathsf{Cov}(\tilde{x}_k|L_k^*) &\leq \mathsf{Trace}\,\mathsf{Cov}(\tilde{x}_k|\bar{L}_k) \quad (38)\\ &\leq n\|\mathsf{Cov}(\tilde{x}_k|\bar{L}_k)\| \quad (39)\\ &\leq n\rho, \quad (40)\end{aligned}$$

where ρ is an upper bound of $\|\mathsf{Cov}(\tilde{x}_k|\bar{L}_k)\|$, according to Proposition 4.

Applying (37) then yields

$$\begin{aligned}\|\mathsf{Cov}(\tilde{x}_k|L_k^*)\| &\leq \mathsf{Trace}\,\mathsf{Cov}(\tilde{x}_k|L_k^*) \quad (41)\\ &\leq n\rho. \quad (42)\end{aligned}$$

It is thus established that the trace and the matrix norm of the error covariance of the Kitanidis filter $P_{k|k} = \mathsf{Cov}(\tilde{x}_k|L_k^*)$ are both bounded.

Then it is straightforward to check that, under Assumptions (i)-(v), the Kitanidis gain L_k^* and the auxiliary variables involved in the filter recursions (7), namely, $P_{k+1|k}, \Sigma_{k+1}, \Gamma_{k+1}, \Xi_{k+1}, \Lambda_{k+1}$, are all bounded.

5. CONCLUSION

It has been established in this paper that the error covariance of the Kitanidis filter is bounded, so are the Kitanidis gain matrix and all the auxiliary variables involved in the filter recursive computations. Algorithm boundedness is of prime importance for real time applications.

APPENDIX

Lemma 1. Consider a stochastic system

$$z_{k+1} = F_k z_k + e_k \tag{43}$$

with $z_k \in \mathbb{R}^n$ and $F_k \in \mathbb{R}^{n\times n}$, initialized such that $\mathsf{E}(z_0) = 0$ and driven by a zero mean white noise $e_k \in \mathbb{R}^n$ independent of the initial state z_0. Assume that

- system (43) is exponentially stable, in the sense that there exist two positive constants α and β such that, for any integer pair $l \geq k \geq 0$,

$$\|F_{l-1}F_{l-2}\cdots F_k\| \leq \alpha e^{-\beta(l-k)}, \tag{44}$$

- the noise e_k has a bounded covariance R_k, *i.e.*, there exists a positive constant γ such that, for all $k \geq 0$,

$$\|R_k\| = \|\mathsf{E}(e_k e_k^T)\| \leq \gamma. \tag{45}$$

Then the covariance matrix $\mathsf{Cov}(z_k)$ is bounded for all $k \geq 0$. □

Proof of Lemma 1.

It is assumed that $\mathsf{E}(z_0) = 0$ and $\mathsf{E}(e_k) = 0$ for all $k \geq 0$, then recursively $\mathsf{E}(z_k) = 0$ for all $k \geq 0$, and therefore $\mathsf{Cov}(z_k) = \mathsf{E}(z_k z_k^T)$.

In (43), z_k depends on $e_0, e_1, \ldots, e_{k-1}$, but not on e_k. Therefore, $\mathsf{E}(z_k e_k^T) = 0$ and

$$\begin{aligned}
\mathsf{Cov}(z_{k+1}) &= \mathsf{E}(z_{k+1}z_{k+1}^T) && (46)\\
&= \mathsf{E}[(F_k z_k + e_k)(F_k z_k + e_k)^T] && (47)\\
&= F_k \mathsf{E}(z_k z_k^T)F_k^T + \mathsf{E}(e_k e_k^T) && (48)\\
&= F_k \mathsf{Cov}(z_k) F_k^T + R_k. && (49)
\end{aligned}$$

Recursively applying this result yields

$$\begin{aligned}
&\mathsf{Cov}(z_{k+1})\\
&\quad = F_k F_{k-1}\cdots F_0 \mathsf{Cov}(z_0) F_0^T \cdots F_{k-1}^T F_k^T\\
&\qquad + \sum_{i=0}^{k-1} F_k F_{k-1}\cdots F_{i+1} R_i F_i^T \cdots F_{k-1}^T F_k^T + R_k. \quad (50)
\end{aligned}$$

Then

$$\begin{aligned}
&\|\mathsf{Cov}(z_{k+1})\| \leq \|F_k F_{k-1}\cdots F_0\| \|\mathsf{Cov}(z_0)\| \|F_0^T \cdots F_{k-1}^T F_k^T\|\\
&\quad + \sum_{i=0}^{k-1} \|F_k F_{k-1}\cdots F_{i+1}\| \|R_i\| \|F_i^T \cdots F_{k-1}^T F_k^T\| + \|R_k\|.
\end{aligned}$$

Based on this result, it then follows from (44) and (45) that

$$\begin{aligned}
\|\mathsf{Cov}(z_{k+1})\| &\leq \alpha^2 e^{-2\beta(k+1)} \|\mathsf{Cov}(z_0)\|\\
&\quad + \sum_{i=0}^{k-1} \alpha^2 e^{-2\beta(k-i)}\gamma + \gamma\\
&= \alpha^2 e^{-2\beta(k+1)} \|\mathsf{Cov}(z_0)\|\\
&\quad + \alpha^2\gamma \frac{1-e^{-2\beta k}}{e^{2\beta}-1} + \gamma\\
&\leq \alpha^2 e^{-2\beta(k+1)} \|\mathsf{Cov}(z_0)\|\\
&\quad + \alpha^2\gamma \frac{1}{e^{2\beta}-1} + \gamma.
\end{aligned}$$

An upper bound of $\mathsf{Cov}(z_k)$ is thus established.

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