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Konstantin Zimenko *, **Andrey Polyakov** **,*, **Denis Efimov** **,*, **Artem Kremlev** *

* *Faculty of Control Systems and Robotics, ITMO University, 49 Kronverkskiy av., 197101 Saint Petersburg, Russia. (e-mail: kostyazimenko@gmail.com, kremlev_artem@mail.ru).*
** *Inria, Univ. Lille, CNRS, UMR 9189 - CRIStAL, F-59000 Lille, France (e-mail: andrey.polyakov@inria.fr, denis.efimov@inria.fr).*

Abstract: The paper is devoted to the problem of state observation (particularly, in finite time) of linear MIMO systems. The presented nonlinear observer does not require system transformation to a canonical form and guarantees finite-time (asymptotic with a fixed-time attraction of any compact set containing the origin) stability of observation error equation if homogeneity degree is negative (positive). The proposed observer is robust in input-to-state sense with respect to disturbances and measurement noises. Performance of the observer is illustrated by a numerical example.

Keywords: State estimation, nonlinear observer, homogeneity, robustness

1. INTRODUCTION

The problem of nonlinear observer design has been widely investigated over the past decades (see, for example, Afri et al., 2017; Kazantzis and Kravaris, 1998; Andrieu et al., 2009; Ortega et al., 2018; Perruquetti et al., 2008; Mazenc et al., 2015; Shen and Xia, 2008). It is worth noting that a finite/fixed-time convergence of observed states to the real ones is preferable specially for the cases when observers are used for systems with processes strongly restricted by time or for fault detection. To achieve finite-time stability of the observation error equation, sliding-mode differentiators (Angulo et al., 2013; Cruz-Zavala et al., 2011; Levant, 2003) and homogeneity based observers (Lopez-Ramirez et al., 2018; Andrieu et al., 2009; Perruquetti et al., 2008) may be used.

The present paper addresses the problem of homogeneous observer design for linear MIMO plants. Depending on the sign of degree of homogeneity the presented approach guarantees different types of observation: observation error equation is finite-time stable if degree of homogeneity is negative; it is asymptotically stable with a fixed-time attraction of any compact set containing the origin if degree of homogeneity is positive.

In comparison with existing approaches the presented approach has the following advantages:

- it is applicable for high-order systems;
- it does not require special canonical forms of the system and output matrices or block decomposition as in Lopez-Ramirez et al., 2018 (in some cases block decomposition can be accompanied by significant computational errors);
- parameter tuning is based on solution of a system of linear matrix equations and inequalities.

Moreover, due to homogeneity property the observer is robust (in an input-to-state sense) in the presence of disturbances and measurement noises.

The paper is organized as follows. The problem statement is introduced in Section 2. Section 3 considers preliminaries used in the paper. The results on homogeneous observer design are presented in Section 4. Finally, considered example and conclusions are given in Sections 5 and 6, respectively.

Notation: $\mathbb{R}_+ = \{x \in \mathbb{R} \colon x > 0\}$, where $\mathbb{R}$ is the field of real numbers; the order relation $P > 0$ for $P \in \mathbb{R}^{n\times n}$ means that P is symmetric and positive definite; $|\cdot|$ denotes Euclidean norm and $\|\cdot\|$ denotes weighted Euclidean norm (i.e. $\|x\| = \sqrt{x^T P x}$ with $x \in \mathbb{R}^n$ and $P > 0$); $\|A\|_{\mathbb{A}} = \sup_{x\in\mathbb{R}^n} \frac{\|Ax\|}{\|x\|}$ for $A \in \mathbb{R}^{n\times n}$; $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix; the minimal and maximal eigenvalues of a symmetric matrix $P = P^T$ are denoted by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$, respectively; $\Re(\lambda)$ denotes the real part of the complex number λ; $\mathcal{L}_\infty(\mathbb{R}^p)$ denotes the set of essentially bounded measurable functions $f : \mathbb{R}_+ \to \mathbb{R}^p$; a continuous function $\sigma : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$. It belongs to class $\mathcal{K}_\infty$ if it is also radially unbounded; a continuous function $\beta : \mathbb{R}_+ \cup \{0\} \times \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ belongs to class $\mathcal{KL}$ if $\beta(\cdot, r) \in \mathcal{K}$ and $\beta(r, \cdot)$ is decreasing to zero for any fixed $r \in \mathbb{R}_+$.

⋆ This work was partially supported by the Government of Russian Federation (Grant 08-08) and by the Ministry of Science and Higher Education of Russian Federation, passport of goszadanie no. 2019-0898.

2. PROBLEM STATEMENT

Let us consider the system in the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + d_x(t), \\ y(t) = Cx(t) + d_y(t), \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$ is the state variable, $y \in \mathbb{R}^k$ is the measured output, $u : \mathbb{R} \to \mathbb{R}^s$ is the control input, d_x, d_y denote disturbances and measurement noises, respectively, $A \in \mathbb{R}^{n\times n}$ is the system matrix, $B \in \mathbb{R}^{n\times s}$ is the matrix of input gains and the matrix $C \in \mathbb{R}^{k\times n}$ is the output matrix which links the measured outputs to the state variables. The matrix pair (A, C) is assumed to be observable and $\text{rank}(C) = k$.

The main goal of the paper is to propose a homogeneity-based dynamic observer for the system (1) with constructive tuning rules. For non-perturbed case the observer must guarantee finite-time stability (asymptotic stability with a fixed-time attraction of any compact set containing the origin) of the error equation, depending on the degree of homogeneity. In the presence of $\mathcal{L}_\infty$-bounded disturbances and measurement noises the observer has to ensure robustness property in the sense of input-to-state stability.

3. PRELIMINARIES

3.1 Stability Notions

Consider the following system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geq 0, \quad (2)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field, $f(0) = 0$.

According to Filippov (1988) an absolutely continuous function $x(t, x_0)$ is called a solution of the Cauchy problem associated to (2) if $x(0, x_0) = x_0$ and for almost all $t > 0$ it satisfies the following differential inclusion

$$\dot{x} \in K[f](t, x) = co \bigcap_{\varepsilon>0} \bigcap_{\mu(N)=0} f(t, B(x, \varepsilon)\backslash N), \quad (3)$$

where $co(M)$ defines the convex closure of the set M, $B(x, \varepsilon)$ is the ball with the center at $x \in \mathbb{R}^n$ and the radius ε, the equality $\mu(N) = 0$ means that the measure of $N \subset \mathbb{R}^n$ is zero.

Note that the system (2) may have non-unique solutions for a given $x_0 \in \mathbb{R}^n$ and may admit both weak (a property holds for a solution) and strong stability (a property holds for ***all solutions***) (see, for example, Filippov, 1988; Roxin, 1966). This paper deals only with the strong stability properties, which ask for stable behavior of all solutions of the system (2).

Definition 1 (Bhat and Bernstein, 2000; Orlov, 2004) *The origin of (2) is said to be globally finite-time stable if it is globally asymptotically stable and any solution $x(t, x_0)$ of the system (2) reaches the equilibrium point at some finite time moment, i.e. $x(t, x_0) = 0$ $\forall t \geq T(x_0)$ and $x(t, x_0) \neq 0$ $\forall t \in [0, T(x_0))$, $x_0 \neq 0$, where $T\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}_+$ is the settling-time function.*

Note that in the case the system (2) has non-unique solutions for some $x_0 \in \mathbb{R}^n$ then $T(x_0)$ should be considered as a set-valued map (since for the same $x_0 \in \mathbb{R}^n$ there may be solutions that reach the equilibrium point at different time moments). However, for the sake of simplicity, further in such cases the designation $T(x_0)$ will be understood as the greatest value of the settling time for a given $x_0 \in \mathbb{R}^n$.

Definition 2 (Polyakov, 2012) *A set $M \subset \mathbb{R}^n$ is said to be globally finite-time attractive for (2) if any solution $x(t, x_0)$ of (2) reaches M in some finite time moment $t = T(x_0)$ and remains there $\forall t \geq T(x_0)$, $T\colon \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is the settling-time function. It is fixed-time attractive if in addition the settling-time function $T(x_0)$ is globally bounded by some number $T_{\max} > 0$.*

Theorem 1. (Bhat and Bernstein, 2000). Suppose there exists a positive definite C^1 function V defined on an open neighborhood of the origin $D \subset \mathbb{R}^n$ and real numbers $C > 0$ and $\sigma \geq 0$, such that the following condition is true for the system (2)

$$\dot{V}(x) \leq -CV^\sigma(x), \quad x \in D \setminus \{0\}.$$

Then depending on the value σ the origin is stable with different types of convergence:

- if $\sigma = 1$, the origin is exponentially stable;
- if $0 \leq \sigma < 1$, the origin is finite-time stable and
$$T(x_0) \leq \frac{1}{C(1-\sigma)} V_0^{1-\sigma},$$
where $V_0 = V(x_0)$;
- if $\sigma > 1$ the origin is asymptotically stable and, for every $\varepsilon \in \mathbb{R}_+$, the set $B = \{x \in D\colon V(x) < \varepsilon\}$ is fixed-time (independent on the initial values) attractive with
$$T_{\max} = \frac{1}{C(\sigma-1)\varepsilon^{\sigma-1}}.$$

If $D = \mathbb{R}^n$ and function V is radially unbounded, then the system (2) admits these properties globally.

Let us give definitions of input-to-state stability notions that are widely used for robustness analysis of nonlinear systems. Rewrite (2) for the perturbed case:

$$\dot{x}(t) = \tilde{f}(x(t), d(t)), \quad x(0) = x_0, \quad t \geq 0, \quad (4)$$

where $x \in \mathbb{R}^n$, $d \in \mathcal{L}_\infty(\mathbb{R}^p)$ is a disturbance and $\tilde{f} \in \mathbb{R}^{n+p} \to \mathbb{R}^n$ is a continuous or discontinuous vector field that satisfies Filippov conditions (Filippov, 1988).

Definition 3 (Dashkovskiy et al., 2011) *The system (2) is called*

- ***Input-to-state stable*** *(ISS), if there exist functions $\zeta \in \mathcal{KL}$ and $\vartheta \in \mathcal{K}$ such that for any $d \in \mathcal{L}_\infty(\mathbb{R}^p)$ and any $x_0 \in \mathbb{R}^n$*
$$\|x(t, t_0, d)\| \leq \zeta(\|x_0\|, t) + \vartheta(\|d\|_{[0,t)}), \ \forall t \geq 0.$$
- ***Integral Input-to-state stable*** *(iISS), if there are some functions $\vartheta_1, \vartheta_2 \in \mathcal{K}_\infty$ and $\zeta \in \mathcal{KL}$ such that for any $d \in \mathcal{L}_\infty(\mathbb{R}^p)$ and any $x_0 \in \mathbb{R}^n$ the following estimate holds:*
$$\vartheta_1(\|x(t, t_0, d)\|) \leq \zeta(\|x_0\|, t) + \int_0^t \vartheta_2(\|d(s)\|) ds, \ \forall t \geq 0.$$

3.2 Generalized Homogeneity

The homogeneity is a property that specifies sort of symmetry of an object with respect to a group of transformations (dilation operation). The type of homogeneity, dealing with linear transformations, is called generalized homogeneity.

Definition 4 (Polyakov et al., 2016a,b) *A map* $\mathbf{d}\colon \mathbb{R} \to \mathbb{R}^{n\times n}$ *is called dilation in the space* $\mathbb{R}^n$ *if it satisfies:*

- ***group property:*** $\mathbf{d}(0) = I_n$ *and* $\mathbf{d}(t+s) = \mathbf{d}(t)\mathbf{d}(s) = \mathbf{d}(s)\mathbf{d}(t)$ *for* $t, s \in \mathbb{R}$*;*
- ***continuity property:*** $\mathbf{d}$ *is a continuous map;*
- ***limit property:*** $\lim_{s\to-\infty} \|\mathbf{d}(s)x\| = 0$ *and* $\lim_{s\to+\infty} \|\mathbf{d}(s)x\| = +\infty$ *uniformly on the unit sphere* S*.*

The dilation $\mathbf{d}$ is a uniformly continuous group. Its generator is a matrix $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ defined by $G_{\mathbf{d}} = \lim_{s\to 0} \frac{\mathbf{d}(s)-I_n}{s}$ (Pazy, 1983). The generator $G_{\mathbf{d}}$ satisfies the following properties

$$\frac{d}{ds}\mathbf{d}(s) = G_{\mathbf{d}}\mathbf{d}(s) = \mathbf{d}(s)G_{\mathbf{d}},$$
$$\mathbf{d}(s) = e^{G_{\mathbf{d}}s} = \sum_{i=0}^{+\infty} \frac{s^i G_{\mathbf{d}}^i}{i!},$$

where $s \in \mathbb{R}$.

Definition 5 (Polyakov et al., 2016a; Polyakov, 2019) *The dilation* $\mathbf{d}$ *is said to be strictly monotone if* $\exists \beta$ *such that* $\|\mathbf{d}(s)\|_{\mathbb{A}} \le e^{\beta s}$ *for* $s \le 0$*.*

Thus, monotonicity means that $\mathbf{d}(s)$ is a strong contraction for $s < 0$ (strong expansion for $s > 0$) and implies that for any $x \in \mathbb{R}\backslash\{0\}$ there exists a unique pair $(s_0, x_0) \in \mathbb{R}\times S$ such that $x = \mathbf{d}(s_0)x_0$.

Theorem 2. (Polyakov, 2019). If $\mathbf{d}$ is a dilation in $\mathbb{R}^n$, then

- the generator matrix $G_{\mathbf{d}}$ is anti-Hurwitz, i.e.
  $$\Re(\lambda_i(G_{\mathbf{d}})) > 0, \quad i = 1, ..., n$$
  and there exists a matrix $P \in \mathbb{R}^{n\times n}$ such that
  $$PG_{\mathbf{d}} + G_{\mathbf{d}}^T P > 0, \quad P > 0. \tag{5}$$
- the dilation $\mathbf{d}$ is strictly monotone with respect to the weighted Euclidean norm $\|x\| = \sqrt{x^T P x}$ for $x \in \mathbb{R}^n$ and P satisfying (5):
  $$\begin{array}{lll} e^{\alpha s} \le \|\mathbf{d}(s)\|_{\mathbb{A}} \le e^{\beta s} & if & s \le 0, \\ e^{\beta s} \le \|\mathbf{d}(s)\|_{\mathbb{A}} \le e^{\alpha s} & if & s \ge 0, \end{array} \tag{6}$$
  where $\alpha = \frac{1}{2}\lambda_{\max}\left(P^{\frac{1}{2}}G_{\mathbf{d}}P^{-\frac{1}{2}} + P^{-\frac{1}{2}}G_{\mathbf{d}}^T P^{\frac{1}{2}}\right)$, $\beta = \frac{1}{2}\lambda_{\min}\left(P^{\frac{1}{2}}G_{\mathbf{d}}P^{-\frac{1}{2}} + P^{-\frac{1}{2}}G_{\mathbf{d}}^T P^{\frac{1}{2}}\right)$.

Definition 6 (Polyakov et al., 2016b) *A vector field* $f\colon \mathbb{R}^n \to \mathbb{R}^n$ *(a function* $g\colon \mathbb{R}^n \to \mathbb{R}$*) is said to be* $\mathbf{d}$*-homogeneous of degree* $\nu \in \mathbb{R}$ *if*

$$\begin{array}{l} f(\mathbf{d}(s)x) = e^{\nu s}\mathbf{d}(s)f(x), \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \forall s \in \mathbb{R}. \\ (resp.\ g(\mathbf{d}(s)x) = e^{\nu s}g(x), \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \forall s \in \mathbb{R}.) \end{array} \tag{7}$$

A special case of homogeneous function is a homogeneous norm (Kawski, 1995): a continuous positive definite $\mathbf{d}$-homogeneous function of degree 1. Define the canonical homogeneous norm $\|\cdot\|_{\mathbf{d}}\colon \mathbb{R}^n \to \mathbb{R}_+$ as $\|x\|_{\mathbf{d}} = e^{s_x}$, where $s_x \in \mathbb{R}$ such that $\|\mathbf{d}(-s_x)x\| = 1$. Note that $\|\mathbf{d}(s)x\|_{\mathbf{d}} = e^s\|x\|_{\mathbf{d}}$ and

$$\|\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x\| = 1. \tag{8}$$

Define the unit sphere in the homogeneous norm by $S = \{x \in \mathbb{R}^n : \|x\|_{\mathbf{d}} = 1\}$.

The following lemma gives the necessary and sufficient condition of $\mathbf{d}$-homogeneity of linear systems.

Lemma 3. (Zimenko et al., 2020). Let $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ be a generator of the dilation $\mathbf{d}(s) = e^{G_{\mathbf{d}}s}$, $s \in \mathbb{R}$. Then the linear system $\dot{x} = Cx$, $x \in \mathbb{R}^n$, $C \in \mathbb{R}^{n\times n}$ is $\mathbf{d}$-homogeneous of degree $\nu \in \mathbb{R}$ if and only if

$$CG_{\mathbf{d}} - G_{\mathbf{d}}C = \nu C. \tag{9}$$

The sign of homogeneity degree of stable systems determines the type of convergence.

Theorem 4. (Polyakov, 2019). An asymptotically stable $\mathbf{d}$-homogeneous system $\dot{x} = f(x)$, $f\colon \mathbb{R}^n \to \mathbb{R}^n$ of degree $\nu \in \mathbb{R}$ is uniformly finite-time stable if and only if $\nu < 0$.

The homogeneity theory provides many other advantages to analysis and design of nonlinear control system. For instance, some results about ISS of homogeneous systems can be found in (Bernuau et al., 2018, 2013; Ryan, 1995). Let us develop the result of Bernuau et al. (2013) about ISS stability for generalized homogeneity case.

Let

$$\left|\tilde{f}(y,d) - \tilde{f}(y,0)\right| \le \sigma(|d|), \quad \sigma(s) = \begin{cases} cs^{\varrho_{\max}} & \text{if } s \ge 1, \\ cs^{\varrho_{\min}} & \text{if } s < 1, \end{cases}$$
$$\forall y \in S, \quad d \in \mathbb{R}^p \tag{10}$$

be satisfied for some $c \in \mathbb{R}_+$, $\varrho_{\max} \ge \varrho_{\min} > 0$. Then following Bernuau et al. (2013) one may obtain:

Theorem 5. Define $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$, $\Re(\lambda_i(G_{\mathbf{d}})) > 0$, $i = 1, ..., n$ and $G_{\tilde{\mathbf{d}}} \in \mathbb{R}^{p\times p}$, $\Re(\lambda_j(G_{\tilde{\mathbf{d}}})) \ge 0$, $j = 1, ..., p$. Let the vector field $\left[\tilde{f}(x,d)^T\ 0_p\right]^T \in \mathbb{R}^{n+p}$ be homogeneous with the generators $G_{\mathbf{d}}$, $G_{\tilde{\mathbf{d}}}$ and with degree $\eta \ge -\min_{1\le i\le n}\Re(\lambda_i(G_{\mathbf{d}}))$, i.e. $\tilde{f}(\mathbf{d}(s)x, \tilde{\mathbf{d}}(s)d) = \exp(\eta s)\mathbf{d}(s)\tilde{f}(x,d)$ for all $x \in \mathbb{R}^n$, $d \in \mathbb{R}^p$ and all $s \in \mathbb{R}$. Assume that the system (4) is globally asymptotically stable for $d = 0$, then the system (4) is

$$\textbf{\textit{ISS}}\ if\ \min_{1\le j\le p}\Re(\lambda_j(G_{\tilde{\mathbf{d}}})) > 0$$

$$\textbf{\textit{iISS}}\ if\ \min_{1\le j\le p}\Re(\lambda_j(G_{\tilde{\mathbf{d}}})) = 0\ and\ \eta \le 0.$$

For the case $\min_{1\le j\le p}\Re(\lambda_j(G_{\tilde{\mathbf{d}}})) = 0$ a relaxed constraint $\varrho_{\min} \ge 0$ has to be satisfied, which follows the continuity of f.

In the case f is continuous with respect to both arguments, the condition (10) is satisfied. Note that only continuity with respect to the second argument is not enough. For example, the system

$$\dot{x}_1 = -x_1 + x_1 \arctan\left(d\begin{cases} 1/x_2, & x_2 \ne 0 \\ 0, & x_2 = 0 \end{cases}\right),$$
$$\dot{x}_2 = -x_2$$

is continuous with respect to d, globally asymptotically stable for $d = 0$, homogeneous with $G_{\mathbf{d}} = I_2$, $G_{\tilde{\mathbf{d}}} = 1$, and not ISS.

4. MAIN RESULTS

Let us consider the following observer

$$\frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t) - g_{FT}(y(t) - C\hat{x}(t)), \qquad (11)$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the observer state vector and the function $g_{FT} : \mathbb{R}^k \to \mathbb{R}^n$ is defined as

$$g_{FT}(\varsigma) = L_0\varsigma + |\tilde{P}\varsigma|^{\frac{\rho-\epsilon}{\epsilon-\rho+\nu}}\mathbf{d}\left(\ln|\tilde{P}\varsigma|^{\frac{1}{\epsilon-\rho+\nu}}\right)L_{FT}\varsigma,$$

where $\varsigma \in \mathbb{R}^k$, $\mathbf{d}$ is a dilation with the generator $G_{\mathbf{d}}$, and $\tilde{P} \in \mathbb{R}^{k\times k}$, $L_{FT} \in \mathbb{R}^{n\times k}$, $L_0 \in \mathbb{R}^{n\times k}$, $\nu \geq -1$, ϵ, $\rho \in \mathbb{R}$ are parameters of the observer, to be determined. Then the error equation in the disturbance-free case ($d_x = 0$, $d_y = 0$) has the following form

$$\dot{e} = \left(A + L_0C + |\tilde{P}Ce|^{\frac{\rho-\epsilon}{\epsilon-\rho+\nu}}\mathbf{d}\left(\ln|\tilde{P}Ce|^{\frac{1}{\epsilon-\rho+\nu}}\right)L_{FT}C\right)e, \qquad (12)$$

where $e = x - \hat{x}$.

Theorem 6. Let $d_x = 0$, $d_y = 0$ and

- the matrix $L_0 \in \mathbb{R}^{n\times k}$ is chosen such that the matrix $A + L_0C$ is nilpotent;
- there exists $\nu \geq -1$ such that the system of matrix equations

$$(A + L_0C)H - H(A + L_0C) = A + L_0C, \qquad (13)$$

$$C(\nu H + (\rho - \nu)I_n) = 0, \qquad (14)$$

  has a solution $H \in \mathbb{R}^{n\times n}$, $\rho \in \mathbb{R}$;
- for $\xi > \delta > 0$ the system of matrix inequalities

$$\begin{pmatrix} PA + A^TP + PL_0C + C^TL_0^TP + YC + C^TY^T + \xi P & P \\ P & -Z \end{pmatrix} \leq 0, \qquad (15)$$

$$P > 0, \quad Z > 0, \quad X > 0, \qquad (16)$$

$$\begin{pmatrix} \delta^2 X & Y^T \\ Y & P \end{pmatrix} \geq 0 \qquad (17)$$

$$P \geq C^TXC \qquad (18)$$

$$\Xi^T(\lambda)Z\Xi(\lambda) \leq \frac{1}{\delta}P, \quad \forall\lambda \in [0, 1], \qquad (19)$$

  be feasible for some $P, Z \in \mathbb{R}^{n\times n}$, symmetric $X \in \mathbb{R}^{k\times k}$, $Y \in \mathbb{R}^{n\times k}$ and $\Xi(\lambda) = \exp((\nu H + \rho I_n)\ln\lambda) - I_n$.

Then the error equation (12) with $L_{FT} = P^{-1}Y$, $\tilde{P} = X^{1/2}$, $G_{\mathbf{d}} = \nu H + \epsilon I_n$ and

$$\epsilon > -0.5\lambda_{\min}(\nu P^{1/2}HP^{-1/2} + \nu P^{-1/2}H^TP^{1/2}) \qquad (20)$$

is globally asymptotically (for $\nu > 0$) / exponentially (for $\nu = 0$) / finite-time (for $0 > \nu \geq -1$) stable.

Note that (20) guarantees strict monotonicity of the dilation $\mathbf{d}$ due to Theorem 2.

The nonlinear observer (11) guarantees different types of estimation error convergence depending on the sign of the parameter ν. According to Theorem 1 we have:

- for $0 > \nu \geq -1$ the system (12) is finite-time stable and the settling time function is bounded as follows

$$T(e_0) \leq \frac{2\alpha}{(\xi - \delta)\nu}V_0^{-\nu},$$

  where α is defined in Theorem 2;
- for $\nu > 0$ the system (12) is asymptotically stable and for any $\varepsilon \in \mathbb{R}_+$, the set $B = \{e \in \mathbb{R}^n : V(e) < \varepsilon$ is fixed-time attractive with

$$T(e_0) \leq T_{\max} = \frac{2\alpha}{(\xi - \delta)\nu\varepsilon^{\nu}}.$$

Note that in the case $\nu = 0$ the presented observer becomes the classical Luenberger one.

In order to apply Theorem 6 in practice we need to solve the nonlinear matrix inequality (19) together with the linear ones (15)-(18). Due to the smoothness of $\Xi(\lambda)$ with respect to $\lambda \in (0, 1]$, this can be done on a proper grid constructed over this interval. The following proposition provides sufficient feasibility condition of the inequality (19).

Proposition 7. The parametric inequality (19) holds if there exists $N > 0$ such that

$$\begin{pmatrix} 2(\rho + a)Z + H^TZ + ZH & Z(\rho I_n + H) \\ (\rho I_n + H^T)Z & M \end{pmatrix} \geq 0, \qquad (21)$$

$$P > 0, \quad M > 0, \quad Z > 0, \qquad (22)$$

$$q_i^{2a}\Xi(q_i)Z\Xi(q_i) + q_i^{2a-1}(q_i - q_{i-1})M \leq q_{i-1}^{2a}\frac{1}{\delta}P, \quad i = 1, ..., N, \qquad (23)$$

where $0 = q_0 < q_1 < ... < q_N = 1$, $a \in \mathbb{R}_+$, $H, P, M, Z \in \mathbb{R}^{n\times n}$.

The result of Proposition 7 allows to solve the parametrized system of matrix inequalities (15)-(19) using the following algorithm with fixed ν, ξ, δ, a.

Algorithm 1.
Initialization: $N = 1$, $q_0 = 0$, $q_N = 1$, $\Sigma = \{q_0, q_N\}$.

Loop: While the system of LMI (15)-(18), (21)-(23) is not feasible, do $\Sigma \leftarrow \Sigma \cup \left\{\frac{q_{i-1}+q_i}{2}\right\}_{i=1}^{N}$ and $N \leftarrow 2N$.

Corollary 8. The system of matrix equations (13), (14) and matrix inequalities (15)-(18), (21)-(23) is feasible provided that $|\nu|$ is sufficiently close to zero.

Note, that the system (12) is $\mathbf{d}$-homogeneous of degree ν. Indeed, according to Lemma 3 we have that $\dot{e}(t) = Ae(t) + L_0Ce(t)$ is $\mathbf{d}$-homogeneous of degree ν, then:

$$\left(A + L_0C + |\tilde{P}C\mathbf{d}(s)e|^{\frac{\rho-\epsilon}{\epsilon-\rho+\nu}}\mathbf{d}\left(\ln|\tilde{P}C\mathbf{d}(s)e|^{\frac{1}{\epsilon-\rho+\nu}}\right)L_{FT}C\right)\mathbf{d}(s)e =$$
$$(A + L_0C)\,\mathbf{d}(s)e +$$
$$\left(\exp(\rho s - \epsilon s)|\tilde{P}C(s)e|^{\frac{\rho-\epsilon}{\epsilon-\rho+\nu}}\mathbf{d}(s)\mathbf{d}\left(\ln|\tilde{P}Ce|^{\frac{1}{\epsilon-\rho+\nu}}\right)L_{FT}C\right)\mathbf{d}(s)e =$$
$$\exp(\nu)\mathbf{d}(s)\left(A + L_0C + |\tilde{P}C(s)e|^{\frac{\rho-\epsilon}{\epsilon-\rho+\nu}}\mathbf{d}\left(\ln|\tilde{P}Ce|^{\frac{1}{\epsilon-\rho+\nu}}\right)L_{FT}C\right)e.$$

Basing on homogeneity property a qualitative assessment of robustness can be presented for the observer (11). Consider the system (1) with nonzero $d_x : \mathbb{R}_+ \to \mathcal{L}_\infty(\mathbb{R}^n)$ and $d_y : \mathbb{R}_+ \to \mathcal{L}_\infty(\mathbb{R}^k)$. In this case the perturbed error equation takes the form

$$\dot{e} = (A + L_0C)\,e + L_0d_y + d_x +$$
$$|\tilde{P}(Ce + d_y)|^{\frac{\rho-\epsilon}{\epsilon-\rho+\nu}}\mathbf{d}\left(\ln|\tilde{P}(Ce + d_y)|^{\frac{1}{\epsilon-\rho+\nu}}\right)L_{FT}(Ce + d_y). \qquad (24)$$

Then, basing on homogeneity the following result may be obtained.

Corollary 9. Consider the perturbed error equation (24) and assume that all conditions of Theorem 6 are satisfied and additionally to (20) the parameter ϵ is chosen such that

$$\epsilon \geq -0.5\lambda_{\min}(\nu H + \nu H^T + \nu I_n). \tag{25}$$

Then the system (24) is iISS. If the inequality (25) is satisfied strictly then the system (24) is ISS.

The quantitative analysis is out the scope of this paper and needs further research developments (for example, with the use of results Prasov and Khalil, 2013; Sanfelice and Praly, 2011; Menard et al., 2017).

Remark The proposed observer is an extension of the results presented in (Lopez-Ramirez et al., 2018) and based on the use of weighted homogeneity (the special case of generalized homogeneity). Indeed, consider the system in the block form

$$\tilde{A} = \begin{pmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & A_{m-1\,m} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad \tilde{C} = [I_k \ \ 0], \tag{26}$$

where m is an integer, $A_{j-1\,j} \in \mathbb{R}^{n_{j-1}\times n_j}$, $n_j = \operatorname{rank}(A_{j-1\,j})$, $j = 2, \ldots, m$. Then, applying the presented approach for (26), one obtains an observer with the same structure as in the paper (Lopez-Ramirez et al., 2018). Therefore the main advantage of this paper is not in the better transients or robustness properties, but in the fact the presented approach does not require block decomposition which in some cases may be accompanied by computational errors. It can be viewed also as a nonlinear generalization of Luenberger observer, having non-asymptotic convergence rates for $\nu \neq 0$ and coinciding with it for $\nu = 0$

5. EXAMPLE

Consider the system (1) in the disturbance-free case for $n = 3$, $u(t) = 0$,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Define the observer in the form (11) with the parameter $\nu = -0.2$, where the matrix $L_0 \in \mathbb{R}^{2\times 3}$ is chosen

$$L_0 = \begin{pmatrix} 1 & 0 \\ -1 & -2 \\ 0 & -1 \end{pmatrix}$$

that the matrix $A + L_0 C$ is nilpotent and the matrices $\tilde{P} \in \mathbb{R}^{2\times 2}$, $L_{FT} \in \mathbb{R}^{3\times 2}$, $G_{\mathbf{d}} \in \mathbb{R}^{3\times 3}$ are obtained from the inequalities (13)-(18), (21)-(23):

$$\tilde{P} = \begin{pmatrix} 0.0277 & -0.0001 \\ -0.0001 & 0.0302 \end{pmatrix}, L_{FT} = \begin{pmatrix} -3.1228 & -0.0382 \\ -0.0591 & 0.0483 \\ 0.0153 & -2.6283 \end{pmatrix},$$

$$G_{\mathbf{d}} = \begin{pmatrix} 2.2141 & 0 & 0 \\ -0.2 & 2.0141 & 0 \\ 0 & 0 & 2.2141 \end{pmatrix}.$$

The results of simulation are shown in Fig. 1 with using the logarithmic scale in order to demonstrate fast (finite-time) convergence rate of the observer.

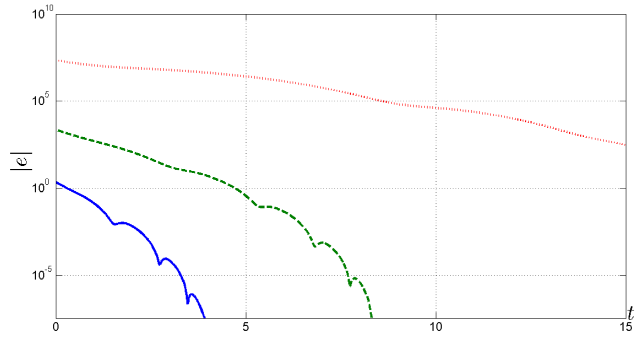

Fig. 1. Simulation plot of $|e|$ for three different initial conditions $x(0) = (0, 1, 2)$, $x(0) = 10^3(0, 1, 2)$, $x(0) = 10^7(0, 1, 2)$

The result for the case of noisy measurements is shown in Fig. 2, where a band-limited white noise of power 10^{-5} has been added to the output signal.

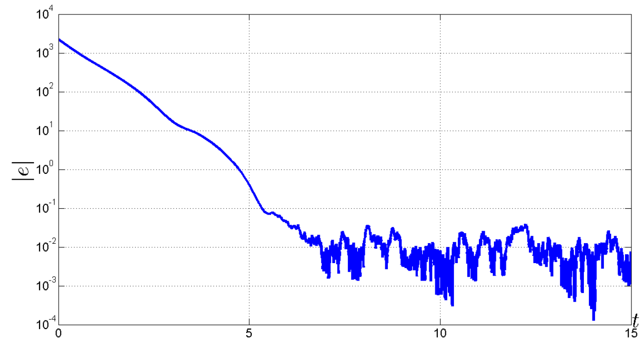

Fig. 2. Simulation plot of $|e|$ for the observer with measurement band limited noise and initial conditions $x(0) = 10^3(0, 1, 2)$

6. CONCLUSIONS

The paper is devoted to homogeneous observer design for linear MIMO systems. The presented observer guarantees finite-time stability (asymptotic stability with a fixed-time attraction of any compact set containing the origin) of observation error equation if homogeneity degree is negative (positive). The key feature of the observer is in the fact that there is no requirements to transform the system to canonical form. The parameters tuning is based on solution of linear matrix equations and inequalities. The qualitative analysis of observers' robustness against bounded measurement noises and disturbances has been studied, namely, it is shown that the observer is iISS/ISS. Numerical example demonstrates effectiveness of the proposed control.

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