Stochastic modeling of mesoscale eddies in oceanic dynamics

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Long Li[1] Werner Bauer[1] Etienne Mémin[1]

[1]INRIA/IRMAR, Fluminance Group
Campus Universitaire de Beaulieu, F–35042 Rennes Cedex, France

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Motivations

- Better small–scale representation : have explicit formulations and interpretations (*v.s. parametrization with ad hoc tuning*)
- Physical consistency : respect a set of conservation laws (*e.g.* energy, circulation) (*v.s. arbitrary Gaussian forcing*)
- Useful in uncertainty quantification (UQ) : provide more reliable ensemble forecasts (EF) and more effiecient spread for ensemble data assimilation (*v.s. perturbations of initial condition (PIC)*)

Outline

1. Location Uncertainty (LU) Principles
2. Stochastic Barotropic Vorticity Equation (SBVE)
3. Parametrizations of Noise
4. Long-term Diagnosis of Time–Statistics
5. Short-term Verification of Ensemble Forecasts

Location Uncertainty (LU) Principles

- Stochastic flow :

$$\mathrm{d}\boldsymbol{X}_t = \underbrace{\boldsymbol{w}(\boldsymbol{X}_t, t)\mathrm{d}t}_{\text{large–scale / resolved}} + \underbrace{\boldsymbol{\sigma}(\boldsymbol{X}_t, t)\mathrm{d}\boldsymbol{B}_t}_{\text{small–scale / unresolved}}$$

- Functional processs :

$$\boldsymbol{\sigma}(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{B}_t = \int_{\Omega} \breve{\boldsymbol{\sigma}}(\boldsymbol{x}, \boldsymbol{y}, t)\mathrm{d}\boldsymbol{B}_t(\boldsymbol{y})\mathrm{d}\boldsymbol{y}$$

$\boldsymbol{B}_t$ is a cyclindrical Wiener process of infinite dimension (in some Hilbert space) and $\breve{\boldsymbol{\sigma}}$ is deterministic symmetric kernel

Location Uncertainty (LU) Principles

- Covariance operator :

$$Q(\boldsymbol{x},\boldsymbol{y},t,s) = \mathbb{E}\Big[\boldsymbol{\sigma}(\boldsymbol{x},t)\mathrm{d}\boldsymbol{B}_t\big(\boldsymbol{\sigma}(\boldsymbol{y},s)\mathrm{d}\boldsymbol{B}_s\big)^T\Big]$$
$$= \delta(t-s)\mathrm{d}t\int_{\Omega}\breve{\boldsymbol{\sigma}}(\boldsymbol{x},\boldsymbol{z},t)\breve{\boldsymbol{\sigma}}^T(\boldsymbol{y},\boldsymbol{z},s)\mathrm{d}\boldsymbol{z}$$

- Variance tensor (per unit of time) :

$$\boldsymbol{a}(\boldsymbol{x},t) = \frac{Q(\boldsymbol{x},t)}{\mathrm{d}t} = \boldsymbol{\sigma}\boldsymbol{\sigma}^T(\boldsymbol{x},t)$$

- Turbulent Kinetic Energy (TKE) :

$$\mathrm{TKE} = \frac{1}{2}\frac{\mathrm{tr}(\boldsymbol{a})}{\mathrm{d}t} \qquad (m^2 \cdot s^{-2})$$

Stochastic Reynolds Transport Theorem (SRTT)

We assume that $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = 0$ in the following.

- Rate of change of a scalar process θ within a volume transported by the stochastic flow :

$$\mathrm{d}\int_{\mathcal{V}(t)} \theta(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} = \int_{\mathcal{V}(t)} (\mathrm{D}_t\theta + \theta\,\boldsymbol{\nabla}\cdot\,\boldsymbol{w}^\star)\mathrm{d}\boldsymbol{x}$$

- Stochastic transport operator :

$$\mathrm{D}_t\theta \triangleq \mathrm{d}_t\theta + (\underbrace{\boldsymbol{w} - \frac{1}{2}\,\boldsymbol{\nabla}\cdot\,\boldsymbol{a}}_{\boldsymbol{w}^\star})\cdot\boldsymbol{\nabla}\,\theta\mathrm{d}t + \underbrace{\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t\cdot\boldsymbol{\nabla}\,\theta}_{\text{multiplicative noise}} - \underbrace{\boldsymbol{\nabla}\cdot(\frac{\boldsymbol{a}}{2}\boldsymbol{\nabla}\theta)\mathrm{d}t}_{\text{subgrid diffusion}}$$

$\boldsymbol{w}^\star$: corrected drift – effect of statistical inhomogeneity of the small-scale flow component; generalization of the Stokes drift

[Bauer, Chandramouli, Chapron, Li & Mémin 2019a]

Conservation Laws

We assume that $\boldsymbol{\nabla}\cdot\boldsymbol{w}^\star = 0$ in the following.

- Conservation of extensive scalar :

$$\mathrm{D}_t\theta = 0$$

- Conservation of tracer energy : *[Resseguier, Memin & Chapron, 2017a]*

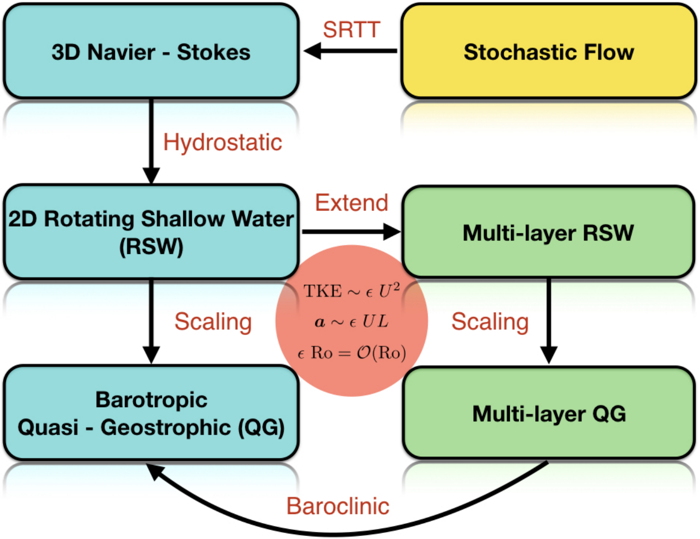
$$\begin{aligned}\mathrm{d}\int_\Omega \frac{1}{2}\theta^2 &= \int_\Omega \theta \mathrm{d}_t\theta + \frac{1}{2}\mathrm{d}\langle\theta\rangle_t \\ &= \int_\Omega \frac{1}{2}\theta^2\, \boldsymbol{\nabla}\cdot\left(\boldsymbol{w}^\star \mathrm{d}t + \boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t\right) = 0\end{aligned}$$

- Energy decomposition of mean and variance fields :

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}(\mathbb{E}[\theta])^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\mathrm{Var}[\theta]$$

Outline

Stochastic Barotropic Vorticity Equation (SBVE)



Stochastic Barotropic Vorticity Equation (SBVE)

- Forced–dissipative advection of the potential vorticity (PV) q with source process :

$$\mathrm{D}_t q = (S_1 + F + D)\mathrm{d}t + S_2 \mathrm{d}\boldsymbol{B}_t$$

$$S_1 = \frac{1}{2} \sum_{i,j=1,2} \partial_{ij}^2 (\boldsymbol{\nabla}^\perp a_{ij} \cdot \boldsymbol{u}), \qquad S_2 \mathrm{d}\boldsymbol{B}_t = -\mathrm{tr}\Big(\boldsymbol{\nabla}^\perp (\boldsymbol{\sigma} \mathrm{d}\boldsymbol{B}_t)^T \boldsymbol{\nabla} \boldsymbol{u}^T\Big)$$

- Kinematic relationship between PV and stream function ψ :

$$q = \nabla^2 \psi + f, \qquad \boldsymbol{u} = \boldsymbol{\nabla}^\perp \psi$$

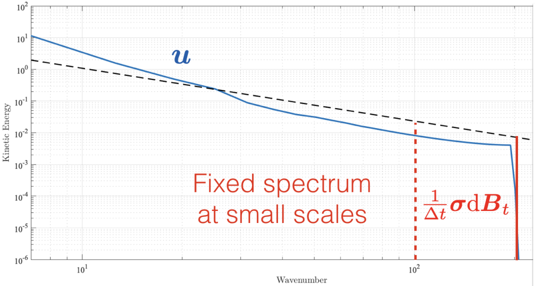
- Conservation of kinetic energy in the absence of D and F : *[Bauer, Chandramouli, Chapron, Li & Mémin 2019a]*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2} \|\boldsymbol{\nabla} \psi\|^2 = 0$$

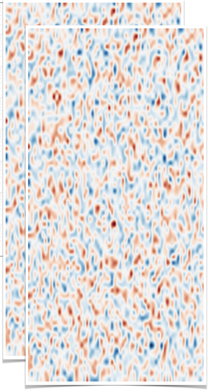
Outline

Parametrizations of Noise

- Homogeneous in space Stationary in time : *[Resseguier, Memin & Chapron, 2017b]*



low-pass band filter $\star$ white noise

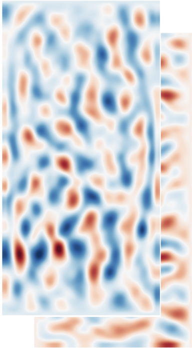


- Homogeneous Non-stationary : *[Resseguier, Li, Jouan, Derian, Mémin & Chapron, 2019]*
  Based on estimation of the absolute diffusivity spectral density (ADSD).

- Heterogeneous Stationary : *[Chandramouli, Mémin, Chapron & Heitz 2019b]*

  Based on snapshot proper orthogonal decomposition (POD) method from data

Off-line simulation → Snapshots $(v_i')_{i=1,\dots,N}$ → SVD → Spatial modes $(\lambda_i, \boldsymbol{\phi}_i(\boldsymbol{x}))_{i=1,\dots,N}$



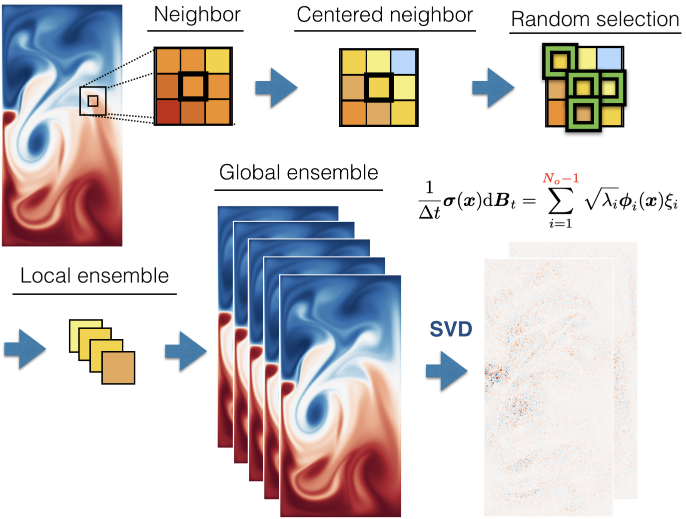
$$\boldsymbol{u}(\boldsymbol{x},t) = \bar{\boldsymbol{u}}(\boldsymbol{x}) + \sum_{i=1}^{n-1} \alpha_i(t)\boldsymbol{\phi}_i(\boldsymbol{x})$$

$$\frac{1}{\Delta t}\boldsymbol{\sigma}(\boldsymbol{x})\mathrm{d}\boldsymbol{B}_t = \sum_{i=n}^{N} \sqrt{\lambda_i}\boldsymbol{\phi}_i(\boldsymbol{x})\xi_i, \;\; \xi_i \sim \mathcal{N}(0,1)$$

$$\frac{1}{\Delta t}\boldsymbol{a}(\boldsymbol{x}) = \sum_{i=n}^{N} \lambda_i \boldsymbol{\phi}_i(\boldsymbol{x})\boldsymbol{\phi}_i^T(\boldsymbol{x})$$

Parametrizations of Noise

- Heterogeneous Non-stationary *[Li, Bauer, & Mémin 2019]* : On-line learning 'Pseudo-observations' from effective resolution



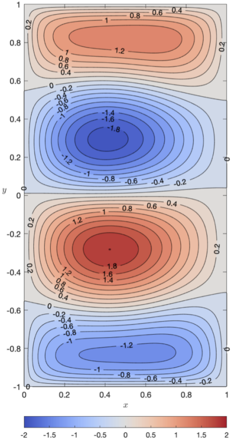
Outline

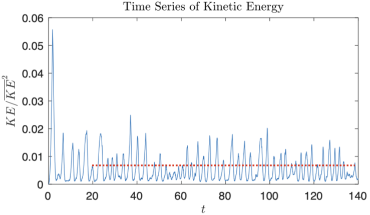
- Results predicted by DNS simulation :

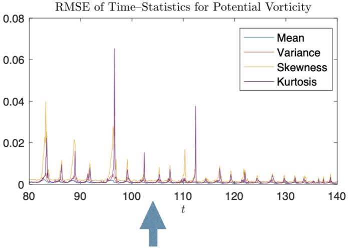
Instantaneous double-gyre circulation

highly variable

Time-Averaged four-gyre circulation

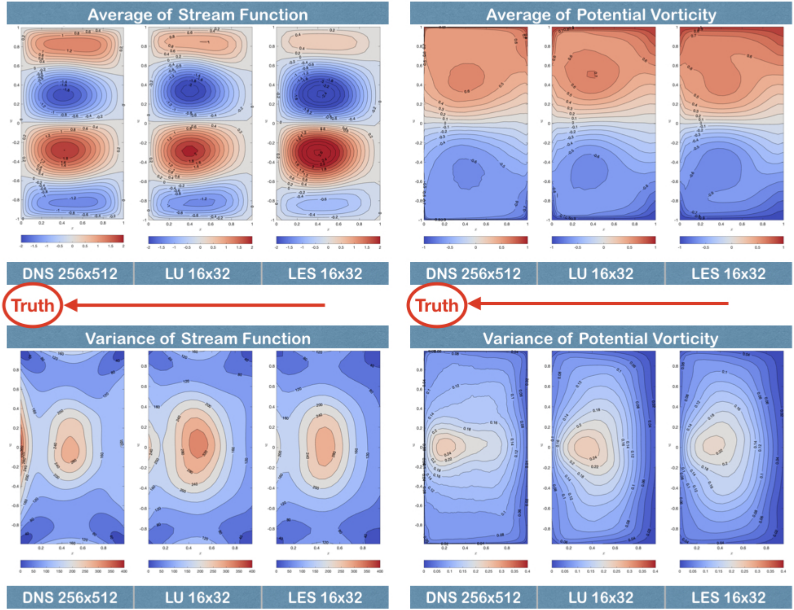




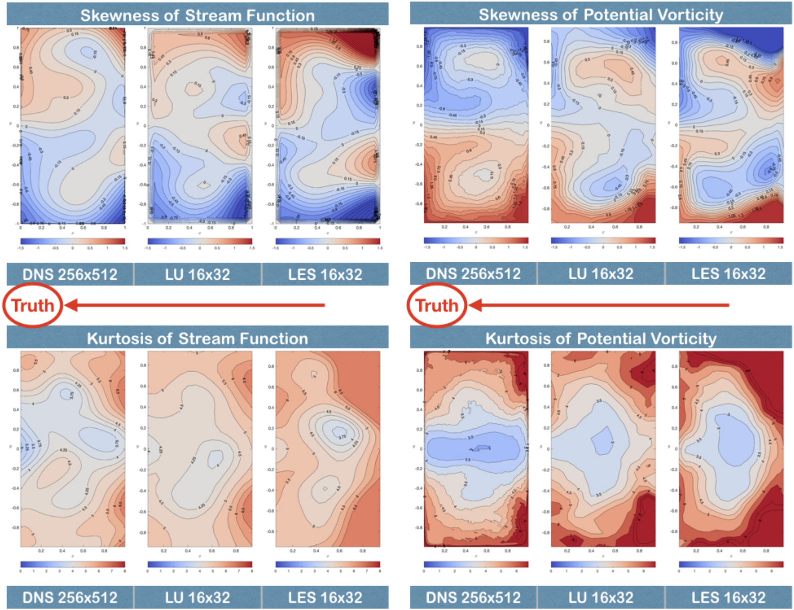


by perturbation of time-intervals

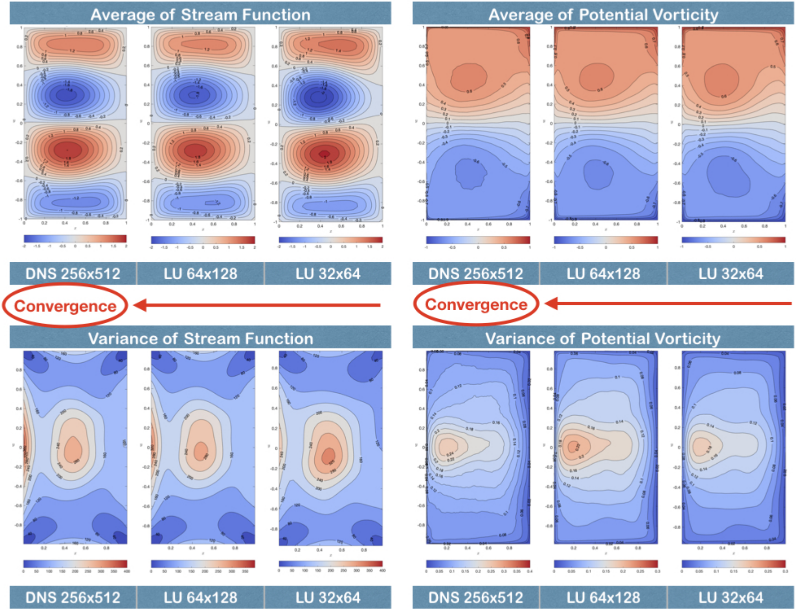
Long-term Diagnosis of Time–Statistics

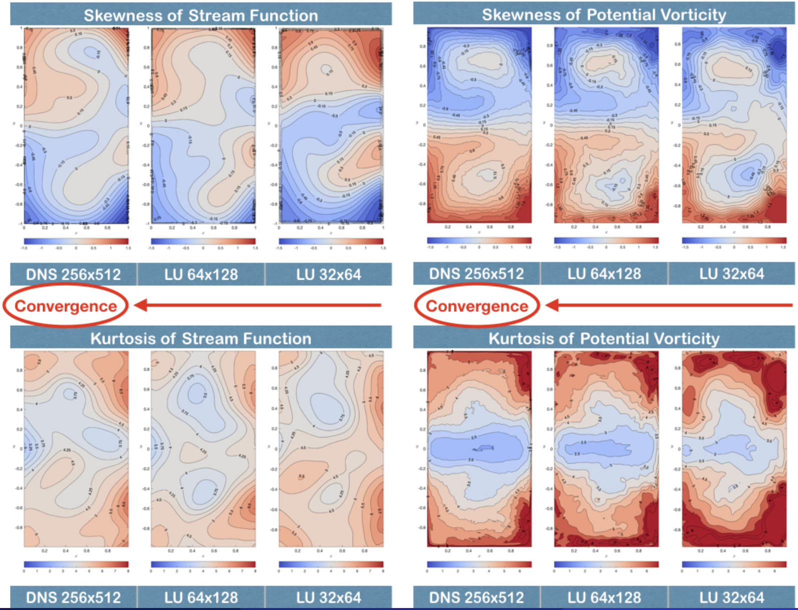


Long-term Diagnosis of Time–Statistics



Long-term Diagnosis of Time–Statistics

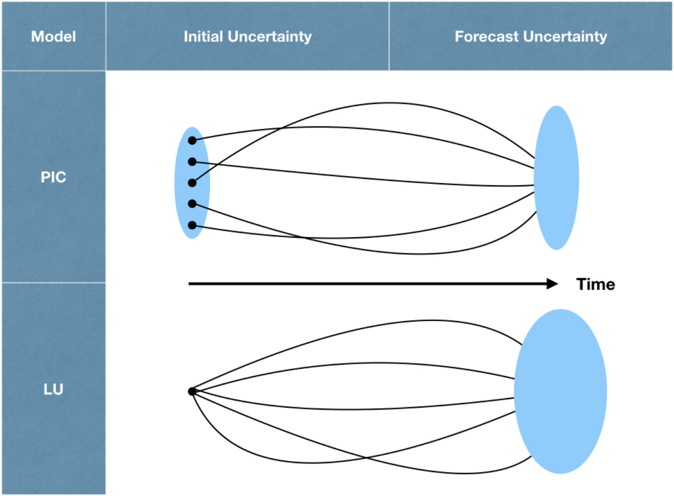



Skewness of Stream Function
Skewness of Potential Vorticity
DNS 256x512
LU 64x128
LU 32x64
DNS 256x512
LU 64x128
LU 32x64
Convergence
Convergence
Kurtosis of Stream Function
Kurtosis of Potential Vorticity
DNS 256x512
LU 64x128
LU 32x64
DNS 256x512
LU 64x128
LU 32x64

Outline

Short-term Verification of Ensemble Forecasts

Observed samples are the filtered DNS PV at each grid point; Ensemble simulations are performed with 30 particles from $t = 0$ to $t = 30$, starting from the (perturbed) filtered DNS.



Ensemble Spreads

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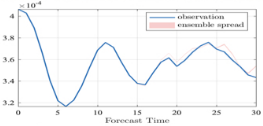
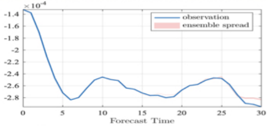
<tr><td>PIC homogeneous</td><td>
</td><td>
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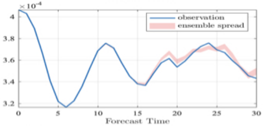
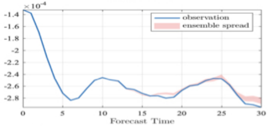
<tr><td>LU homogeneous</td><td>
</td><td>
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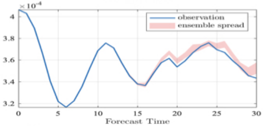
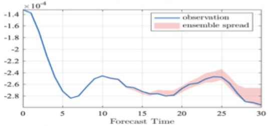
<tr><td>PIC heterogeneous</td><td>
</td><td>
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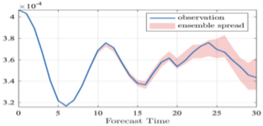
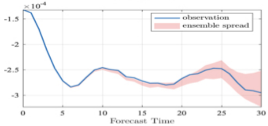
<tr><td>LU heterogeneous</td><td>
</td><td>
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'better'

Reliability of Ensemble Forecasts

- Rank histogram

  (How well does the ensemble spread of the forecast represent the true uncertainty of the observations)

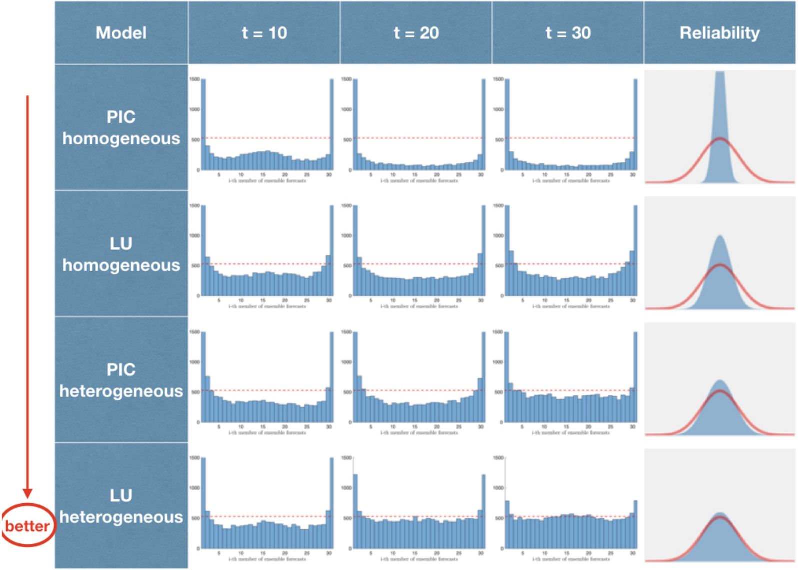
  Flat : ensemble represent well the observed probability distribution;

  U-shaped : ensemble spread too small, many observations falling outside the extremes of the ensemble;

  Dome-shaped : ensemble spread too large, most observations falling near the center of the ensemble;

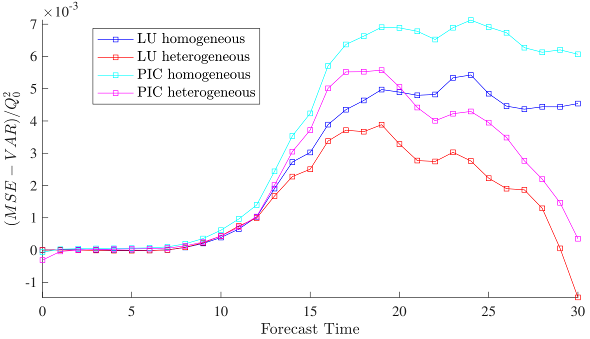
  Asymmetric : ensemble contains bias.

Reliability of Ensemble Forecasts



- The mean squared error (MSE) of the ensemble mean forecasts is identical to the average intra-ensemble sample variance (VAR), multiplied by an ensemble-size N –dependent inflation factor :

$$\overline{(q^o - \hat{\mathbb{E}}[q])^2}^{\boldsymbol{x}} = \frac{N+1}{N} \overline{\hat{\mathrm{Var}}[q]}^{\boldsymbol{x}}$$



- Other UQ metrics *[Resseguier, Li, Jouan, Derian, Mémin & Chapron, 2019]* : Continuous ranked probability score (CRPS), Energy score, Variogram score.

Conclusions

- Better eddy representation based on stochastic tranport;
- Capture better on a coarse mesh the correct long-term time-statistics;
- Spread is more accurate compared to PIC.

Work on ...

Strong interest in ensemble data assimilation with particle filter coupled with LU.

Thanks for Your Attention!