Stability Analysis for Time-Varying Systems with Asynchronous Sampling using Contractivity Approach

Frédéric Mazenc, Michael Malisoff, Miroslav Krstic

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***Abstract*— We provide new sufficient conditions for a generalized exponential input-to-state stability condition to hold. Our proofs use innovative trajectory based contractivity and cooperative systems arguments. Another key ingredient is a new variant of Halanay's inequality, which differs from prior Halanay inequalities because of its use of sampling. Our sufficient conditions allow arbitrarily long intervals between some consecutive sampling times, provided the intervals between consecutive sampling times are frequently small enough. This removes a significant limitation of prior results that relied on versions of Halanay's inequality that did not contain sampling. Unlike standard stability conditions, the upper bounds for the norm of the current state in our less restrictive exponential input-to-state stability conclusions have suprema of disturbances over intervals that include earlier times than the initial times. We apply our work to a broad class of time-varying continuous time systems with sampling.**

***Index Terms*— Sampling, linear systems, stability**

I. Introduction

Dynamics having sampled data and discrete measurements play a fundamental role in current research in systems and controls [1], [5], [10], [12], [14], [15]. However, control and observer design and stability analysis for these systems can be difficult, especially under asynchronous (i.e., nonperiodic) sampling. This is because classical Lyapunov techniques provide conservative stability conditions in terms of upper bounds on the sizes of the sampling intervals.

To address this fundamental case, we propose a new stability analysis for systems with asynchronous sampling when a variant of the celebrated Halanay's inequality [4] is satisfied. The usual Halanay's inequality has the form $\dot{v}(t) \leq -cv(t) + d\sup_{\ell\in[t-T,t]} v(\ell) + \Delta(t)$ for nonnegative valued functions v and Δ and positive constants c (called a decay rate), T, and d (called a gain), where Δ represents a disturbance. By contrast, our relaxed version has an integral instead of the sup, and it removes the usual Halanay's inequality requirement that the decay rate is strictly larger than the gain. Under our relaxed assumptions, we prove generalized exponential input-to-state stability (or ISS) estimates of the form $v(t) \leq \alpha_1 e^{-\alpha_2(t-s)} \sup_{m\in[s-\alpha_3,s]} v(m) + \alpha_4 \sup_{m\in[s-\alpha_3,t]} \Delta(m)$ for suitable constants $\alpha_i > 0$ when $t \geq s \geq \alpha_3$, which is less stringent than traditional exponential ISS estimates that require constants $\mu_i > 0$ such that bounds of the form $v(t) \leq \mu_1 e^{-\mu_2(t-s)}|v(s)| + \mu_3 \sup_{m\in[s,t]} \Delta(m)$ hold for all $s \geq 0$ and $t \geq s$. Our inequalities are also less stringent than those that are commonly used for time delayed ISS systems because our sup of Δ in our estimates includes values $m < s$ for initial times s.

Our proofs use a representation that was proposed in [11, Section 1.5.3], and rely on a novel analog of the trajectory based and contractivity approach that was developed, e.g., in [9]. Our first theorem has a novel variant of Halanay's inequality as its main sufficient condition, and is applied in our second theorem to solve a stability analysis problem for linear time-varying systems with sampled data. Our proof of our second theorem uses an innovative new stability analysis that is based on the theory of cooperative systems [3] and linear Lyapunov functionals. The interval observers in our proof of our second theorem use tools that were developed, e.g., in [8], and [13] and the pioneering paper [2]. A key feature of our results is that they allow arbitrary large time intervals between consecutive sampling instants, provided the intervals between other consecutive sampling times are frequently small enough. Our results complement both Halanay's and the trajectory based approaches, as well as our prior variants of Halanay's inequality from [7] which did not allow the types of sampling that we allow here.

The paper is organized as follows. The main results are stated and proved in Sections II and III. An illustrative example is given in Section IV. Concluding remarks in Section V end the paper. We use standard notation, which is simplified when no confusion would arise. Set $\mathbb{Z}_0 = \{0, 1, 2, ...\}$ and $\mathbb{N} = \mathbb{Z}_0\backslash\{0\}$. The standard Euclidean 2-norm, and the corresponding matrix norm, are denoted by $|\cdot|$, and $|\cdot|_\infty$ is the usual sup norm. Let $M \in \mathbb{R}^{n\times n}$ be a matrix with entry $m_{i,j}$ in its ith row and jth column. We use M^+ to denote the matrix in $\mathbb{R}^{n\times n}$ whose entries are $\max\{0, m_{i,j}\}$, $M^- = M^+ - M$, D_M is the diagonal matrix in $\mathbb{R}^{n\times n}$ whose diagonal entries are $m_{1,1}, \ldots, m_{n,n}$, and $R_M = M - D_M$. For two vectors $\mathcal{V}_1 = (v_{1,1} \ldots v_{1,n})$ and $\mathcal{V}_2 = (v_{2,1} \ldots v_{2,n})$, we write $\mathcal{V}_1 \leq \mathcal{V}_2$ when $v_{1,j} \leq v_{2,j}$ for all $j \in \{1, ..., n\}$. A matrix is called nonnegative (resp., positive) provided all of its entries are nonnegative (resp., positive), and $\mathrm{col}_j(M)$ is the jth column of any matrix M. We let $f(s^-)$ denote the left limit of a function f at a point s, and we use the usual definition of nonnegative systems, e.g., from [6]. We consider sequences t_i such that $t_0 = 0$ that admit two constants $\overline{\nu} > 0$ and $\underline{\nu} > 0$ such that $\underline{\nu} \leq t_{i+1} - t_i \leq \overline{\nu}$ for all $i \in \mathbb{Z}_0$. We then also use the constants $\nu_i = t_{i+1} - t_i$.

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II. Variant of Halanay's Inequality

Let the sequences t_i and ν_i satisfy the requirements from Section I. Consider a nonnegative valued function $v : [0,+\infty) \to [0,+\infty)$ of class C^1 such that there are constants $J \in \mathbb{N}$, $a \in \mathbb{R}$, $b \geq 0$ and $c > 0$ and a piecewise continuous function $\delta : [0,+\infty) \to [0,+\infty)$ such that

$$\begin{array}{rcl} \dot{v}(t) & \leq & -av(t) + (t-t_i)bv(t_i) + c\int_{t_i}^{t} v(\ell)\mathrm{d}\ell \\ & & +\,\delta(t) \text{ for all } t \in [t_i, t_{i+1}) \text{ and } i \geq J. \end{array} \tag{1}$$

We also use $\mathcal{S} = \sqrt{a^2+4c}$, and the function

$$\begin{array}{rcl} \mu(r) & = & \frac{1}{2}\left(1 + \frac{a}{\mathcal{S}} + \frac{b(\mathcal{S}-a)}{c\mathcal{S}}\right) e^{-\frac{a+\mathcal{S}}{2}r} \\ & & +\,\frac{1}{2}\left(1 - \frac{a}{\mathcal{S}} + \frac{b(\mathcal{S}+a)}{c\mathcal{S}}\right) e^{\frac{\mathcal{S}-a}{2}r} - \frac{b}{c}. \end{array} \tag{2}$$

We assume the following, which can be interpreted to mean that $\mu(\nu_l)$ is frequently (but not necessarily always) in $(0,1)$:

Assumption 1: There are $p \in \mathbb{N}$ and $\overline{\mu} \in (0,1)$ such that

$$\textstyle\prod_{l=j}^{j+p} \mu(\nu_l) \leq \overline{\mu} \tag{3}$$

is satisfied for all $j \in \mathbb{N}$. □

See Remark 1 for the motivation for Assumption 1 and our example below, which explain how to check Assumption 1 (and why Assumption 1 is satisfied when $a > 0$). We prove:

Theorem 1: Let v be a function defined above such that Assumption 1 is satisfied. Then there are positive constants α_i for $i = 1$ to 4 such that the inequality

$$v(t) \leq \alpha_1 e^{-\alpha_2(t-s)} \sup_{\ell\in[s-\alpha_3,s]} v(\ell) + \alpha_4 \sup_{m\in[s-\alpha_3,t]} \delta(m) \tag{4}$$

holds for all $s \geq \alpha_3$ and $t \geq s$.

Proof: We can assume that (1) holds with equality for all $t \in [t_i, t_{i+1})$ and all $i \geq J$. To understand why, notice that if (1) did not hold with equality, then it holds with δ replaced by $\delta - \mathcal{D}$ for some nonnegative valued function $\mathcal{D}$ (where $\mathcal{D}$ can depend on v at each time). In that case, we would be able to replace δ by $\delta - \mathcal{D}$ in the proof up through (21), and then we would be able to complete the proof as given below starting in (22) (with δ instead of $\delta - \mathcal{D}$) because of the nonnegative valuedness of the function θ that we use in the proof. This would allow us to to obtain (4) with the original δ, again using the nonnegative valuedness of $\mathcal{D}$.

To adopt a representation which is inspired from [11, Section 1.5.3], we next define

$$w_j(t) = v(t) + \zeta_j \int_{t_i}^{t} [cv(\ell) + bv(t_i)]\mathrm{d}\ell \tag{5}$$

for $j = 1$ and $j = 2$ for all $t \in [t_i, t_{i+1})$ and $i \geq J$, where

$$\zeta_1 = \frac{a-\sqrt{a^2+4c}}{2c} \text{ and } \zeta_2 = \frac{a+\sqrt{a^2+4c}}{2c}. \tag{6}$$

Then

$$\begin{array}{rcl} \dot{w}_1(t) & = & -av(t) + (t-t_i)bv(t_i) + c\int_{t_i}^{t} v(\ell)\mathrm{d}\ell \\ & & +\,\zeta_1[cv(t) + bv(t_i)] + \delta(t) \\ & = & \left(\frac{a-\mathcal{S}}{2} - a\right) v(t) + (t-t_i)bv(t_i) \\ & & +\,c\int_{t_i}^{t} v(\ell)\mathrm{d}\ell + \zeta_1 bv(t_i) + \delta(t) \\ & = & -\frac{a+\mathcal{S}}{2}\left[v(t) - \frac{2}{a+\mathcal{S}}\int_{t_i}^{t}[cv(\ell) + bv(t_i)]\mathrm{d}\ell\right] \\ & & +\,\zeta_1 bw_1(t_i) + \delta(t). \end{array} \tag{7}$$

Here and in the sequel, all equalities and inequalities should be understood to hold for all $t \in [t_i, t_{i+1})$ and all $i \geq J$. Observe that

$$-\frac{2}{a+\mathcal{S}} = -2\frac{a-\sqrt{a^2+4c}}{a^2-(a^2+4c)} = \frac{a-\sqrt{a^2+4c}}{2c} = \zeta_1. \tag{8}$$

Thus,

$$\begin{array}{rcl} \dot{w}_1(t) & = & -\frac{a+\mathcal{S}}{2}\left[v(t) + \zeta_1\int_{t_i}^{t}[cv(\ell) + bv(t_i)]\mathrm{d}\ell\right] \\ & & +\,\zeta_1 bw_1(t_i) + \delta(t) \\ & = & -\frac{a+\mathcal{S}}{2}w_1(t) + \zeta_1 bw_1(t_i) + \delta(t). \end{array} \tag{9}$$

Similarly, by replacing $\mathcal{S}$ by $-\mathcal{S}$ in the preceding argument, one can prove that

$$\dot{w}_2(t) = -\frac{a-\mathcal{S}}{2}w_2(t) + \zeta_2 bw_2(t_i) + \delta(t). \tag{10}$$

By integrating, we obtain

$$\begin{array}{l} w_1(t) = e^{-\frac{a+\mathcal{S}}{2}(t-t_i)}w_1(t_i) \\ +2\frac{1-e^{-\frac{a+\mathcal{S}}{2}(t-t_i)}}{a+\mathcal{S}}\zeta_1 bw_1(t_i) + \int_{t_i}^{t} e^{\frac{a+\mathcal{S}}{2}(\ell-t)}\delta(\ell)\mathrm{d}\ell. \end{array} \tag{11}$$

By reorganizing terms, and noting that

$$\frac{4b}{(a+\mathcal{S})^2} = \frac{4b(a-\mathcal{S})^2}{(a^2-\mathcal{S}^2)^2} = \frac{4b(a-\mathcal{S})^2}{16c^2}, \tag{12}$$

we obtain

$$\begin{array}{rcl} w_1(t) & = & \left[e^{-\frac{a+\mathcal{S}}{2}(t-t_i)} + 2\frac{1-e^{-\frac{a+\mathcal{S}}{2}(t-t_i)}}{a+\mathcal{S}}\zeta_1 b\right] w_1(t_i) \\ & & +\int_{t_i}^{t} e^{\frac{a+\mathcal{S}}{2}(\ell-t)}\delta(\ell)\mathrm{d}\ell \\ & = & \left[e^{-\frac{a+\mathcal{S}}{2}(t-t_i)} + \frac{1-e^{-\frac{a+\mathcal{S}}{2}(t-t_i)}}{a+\mathcal{S}}\frac{a-\mathcal{S}}{c}b\right] w_1(t_i) \\ & & +\int_{t_i}^{t} e^{\frac{a+\mathcal{S}}{2}(\ell-t)}\delta(\ell)\mathrm{d}\ell \\ & = & \left[\left(1+\frac{4b}{(a+\mathcal{S})^2}\right) e^{-\frac{a+\mathcal{S}}{2}(t-t_i)} - \frac{4b}{(a+\mathcal{S})^2}\right]w_1(t_i) \\ & & +\int_{t_i}^{t} e^{\frac{a+\mathcal{S}}{2}(\ell-t)}\delta(\ell)\mathrm{d}\ell \\ & = & \mathcal{G}_1(t-t_i)w_1(t_i) + \int_{t_i}^{t} e^{\frac{a+\mathcal{S}}{2}(\ell-t)}\delta(\ell)\mathrm{d}\ell, \end{array} \tag{13}$$

where

$$\mathcal{G}_1(r) = \left(1 + \frac{b(a-\mathcal{S})^2}{4c^2}\right) e^{-\frac{a+\mathcal{S}}{2}r} - \frac{b(a-\mathcal{S})^2}{4c^2}. \tag{14}$$

Similarly, by replacing $\mathcal{S}$ by $-\mathcal{S}$ in the preceding argument,

$$w_2(t) = \mathcal{G}_2(t-t_i)w_2(t_i) + \int_{t_i}^{t} e^{\frac{a-\mathcal{S}}{2}(\ell-t)}\delta(\ell)\mathrm{d}\ell, \tag{15}$$

where

$$\mathcal{G}_2(r) = \left(1 + \frac{b(a+\mathcal{S})^2}{4c^2}\right) e^{\frac{\mathcal{S}-a}{2}r} - \frac{b(a+\mathcal{S})^2}{4c^2}. \tag{16}$$

As an immediate consequence, we have

$$\begin{array}{l} v(t) + \zeta_1\int_{t_i}^{t}[cv(\ell) + bv(t_i)]\mathrm{d}\ell \\ = \mathcal{G}_1(t-t_i)v(t_i) + \int_{t_i}^{t} e^{\frac{a+\mathcal{S}}{2}(\ell-t)}\delta(\ell)\mathrm{d}\ell \end{array} \tag{17}$$

and

$$\begin{array}{l} v(t) + \zeta_2\int_{t_i}^{t}[cv(\ell) + bv(t_i)]\mathrm{d}\ell \\ = \mathcal{G}_2(t-t_i)v(t_i) + \int_{t_i}^{t} e^{\frac{a-\mathcal{S}}{2}(\ell-t)}\delta(\ell)\mathrm{d}\ell. \end{array} \tag{18}$$

From (17)-(18), we deduce that

$$\begin{array}{l} (\zeta_2 - \zeta_1)v(t) = \zeta_2\mathcal{G}_1(t-t_i)v(t_i) - \zeta_1\mathcal{G}_2(t-t_i)v(t_i) \\ +\int_{t_i}^{t} \zeta_2 e^{\frac{a+\mathcal{S}}{2}(\ell-t)}\delta(\ell)\mathrm{d}\ell - \zeta_1\int_{t_i}^{t} e^{\frac{a-\mathcal{S}}{2}(\ell-t)}\delta(\ell)\mathrm{d}\ell. \end{array} \tag{19}$$

Since $\zeta_2 - \zeta_1 = \mathcal{S}/c$, we have

$$\begin{array}{l} \frac{\mathcal{S}}{c}v(t) = [\zeta_2\mathcal{G}_1(t-t_i) - \zeta_1\mathcal{G}_2(t-t_i)]\, v(t_i) \\ + \int_{t_i}^{t} \left[\zeta_2 e^{\frac{a+\mathcal{S}}{2}(\ell-t)} - \zeta_1 e^{\frac{a-\mathcal{S}}{2}(\ell-t)}\right] \delta(\ell)\mathrm{d}\ell. \end{array} \quad (20)$$

Consequently,

$$\begin{array}{l} v(t) = \mu(t-t_i)v(t_i) + \int_{t_i}^{t} \theta(\ell-t)\delta(\ell)\mathrm{d}\ell, \\ \text{where } \mu(r) = \frac{c}{\mathcal{S}}[\zeta_2\mathcal{G}_1(r) - \zeta_1\mathcal{G}_2(r)] \\ \text{and } \theta(r) = \zeta_2 e^{\frac{a+\mathcal{S}}{2}r} - \zeta_1 e^{\frac{a-\mathcal{S}}{2}r} \end{array} \quad (21)$$

where μ agrees with the function we defined in (2), and θ is positive valued because cosh is positive valued, and because our formulas $\zeta_2 = (a+\mathcal{S})/(2c)$ and $\zeta_1 = (a-\mathcal{S})/(2c)$ from (6) imply that $\theta(r)$ is bounded below by $\frac{1}{c}(\mathcal{S}-|a|)e^{ar/2}(\frac{1}{2}(e^{\mathcal{S}r/2}+e^{-\mathcal{S}r/2})) = \frac{1}{c}(\mathcal{S}-|a|)e^{ar/2}\cosh(\mathcal{S}r/2)$. Setting $\bar{\theta} = |\zeta_2|e^{(a+\mathcal{S})\bar{\nu}/2} + |\zeta_1|$, it follows that $0 \le \theta(r) \le \bar{\theta}$ for all $r \in [0, \bar{\nu}]$ and that

$$v(t_{i+1}) \quad \le \quad \mu(\nu_i)v(t_i) + \bar{\theta}\int_{t_i}^{t_{i+1}} \delta(\ell)\mathrm{d}\ell. \quad (22)$$

Thus, for all $j \ge J - p$, we have

$$\begin{array}{rcl} v(t_{j+p+1}) & \le & \mu(\nu_{j+p})v(t_{j+p}) + \bar{\theta}\mathcal{I}(0) \\ \mu(\nu_{j+p})v(t_{j+p}) & \le & \mu(\nu_{j+p})\mu(\nu_{j+p-1})v(t_{j+p-1}) \\ & & +\mu(\nu_{j+p})\bar{\theta}\mathcal{I}(1) \\ & \vdots & \\ \prod_{l=j+1}^{j+p}\mu(\nu_l)v(t_{j+1}) & \le & \prod_{l=j}^{j+p}\mu(\nu_l)v(t_j) + \prod_{l=j+1}^{j+p}\mu(\nu_l)\bar{\theta}\mathcal{I}(p) \end{array} \quad (23)$$

$$\text{where } \mathcal{I}(q) = \int_{t_{j+p-q}}^{t_{j+p+1-q}} \delta(\ell)\mathrm{d}\ell, \quad (24)$$

and we omit the dependence of $\mathcal{I}(q)$ on j and p to simplify notation. By moving the left side terms to the right side in the last p of the inequalities in (23), then adding the results to the first inequality in (23) and cancelling terms, we get

$$\begin{array}{l} v(t_{j+p+1}) \le \prod_{l=j}^{j+p}\mu(\nu_l)v(t_j) + \bar{\theta}\mathcal{I}(0) \\ \quad +\mu(\nu_{j+p})\bar{\theta}\mathcal{I}(1) + ... + \prod_{l=j+1}^{j+p}\mu(\nu_l)\bar{\theta}\mathcal{I}(p) \\ \le \prod_{l=j}^{j+p}\mu(\nu_l)v(t_j) + \bar{\theta}\sigma(j,p)\int_{t_j}^{t_{j+p+1}} \delta(\ell)\mathrm{d}\ell \end{array} \quad (25)$$

where the function

$$\sigma(j,p) = 1 + \mu(\nu_{j+p}) + ... + \prod_{l=j+1}^{j+p}\mu(\nu_l) \quad (26)$$

admits a bound $\bar{\sigma} > 0$ that is independent of j (because of our bound $\bar{\nu}$ on the ν_i's), and where we use the nonnegative valuedness of μ and the nonnegativity of $\bar{\theta}$ to collect all of the integrals from the $\mathcal{I}(q)$ terms in the last integral in (25) (which justifies the last inequality in (25) and explains why there are no $\mathcal{I}$'s in (26)). This yields a constant $\mathcal{C}$ such that

$$v(t_{j+p+1}) \ = \ \prod_{l=j}^{j+p}\mu(\nu_l)v(t_j) + \mathcal{C}\int_{t_j}^{t_{j+p+1}} \delta(\ell)\mathrm{d}\ell \quad (27)$$

for all j. It follows from Assumption 1 that

$$v(t_{j+p+1}) \le \overline{\mu}v(t_j) + \mathcal{C}\int_{t_j}^{t_{j+p+1}} \delta(s)\mathrm{d}s. \quad (28)$$

Also, we can apply an integrating factor to (1) to conclude that for all $t \in [t_i, t_{i+1}]$ and $i \ge J$, we have

$$\begin{array}{rcl} v(t) & \le & \frac{1}{|a|}\left(b\bar{\nu}v(t_i) + \sup_{\ell\in[t_i,t]}\delta(\ell)\right) + e^{|a|\bar{\nu}}v(t_i) \\ & & +c\bar{\nu}e^{|a|\bar{\nu}}\int_{t_i}^{t} v(\ell)\mathrm{d}\ell, \end{array} \quad (29)$$

so Gronwall's inequality gives a constant $\bar{c} > 0$ such that

$$v(t) \le \bar{c}\left(v(t_i) + \sup_{\ell\in[t_i,t]}\delta(\ell)\right) \quad (30)$$

for all $t \in [t_i, t_{i+1})$ and $i \ge J$.

Let $k \in \mathbb{N}$ be such that $k > -\ln(\bar{c})/\ln(\overline{\mu})$. Then, since $\bar{\mu} \in (0,1)$, the constant $\mathsf{q} = \overline{\mu}^k\bar{c}$ satisfies $\mathsf{q} < 1$. From (28), we deduce that there is a constant $\bar{d} > 0$ such that

$$v(t_{j+k(p+1)}) \le \overline{\mu}^k v(t_j) + \bar{d}\int_{t_j}^{t_{j+k(p+1)}} \delta(s)\mathrm{d}s. \quad (31)$$

Let us take $t \in [t_{j+k(p+1)}, t_{j+k(p+1)+2})$ with $j \ge J$ such that $t_{j+k(p+1)} \ge [k(p+1)+1]\bar{\nu}$. Then (30)-(31) give

$$\begin{array}{rcl} v(t) & \le & \bar{c}\left(v(t_{j+k(p+1)}) + \sup_{\ell\in[t_{j+k(p+1)},t]}\delta(\ell)\right) \\ & \le & \bar{c}\Big(\overline{\mu}^k v(t_j) + \bar{d}\int_{t_j}^{t_{j+k(p+1)}} \delta(s)\mathrm{d}s \\ & & +\sup_{\ell\in[t_{j+k(p+1)},t]}\delta(\ell)\Big) \\ & \le & \mathsf{q}v(t_j) + \bar{c}[k(p+1)\bar{d}\bar{\nu} + 1]\sup_{\ell\in[t_j,t]}\delta(\ell). \end{array} \quad (32)$$

Now, observe that our choice of $\bar{\nu}$ in Section I implies that $t \le t_{j+k(p+1)+2} \le t_j + \bar{\nu}[k(p+1)+2]$ and so also $t_j \ge t - \Omega$ where $\Omega = [k(p+1)+2]\bar{\nu}$. It follows that

$$v(t) \le \mathsf{q}\sup_{\ell\in[t-\Omega,t]} v(\ell) + \bar{c}[k(p+1)\bar{d}\bar{\nu}+1]\sup_{\ell\in[t-\Omega,t]}\delta(\ell). \quad (33)$$

If we now apply [9, Lemma 1] to the functions $w(\ell) = v(s+\ell)$ and $d(\ell) = \sup_{r\in[\ell+s-\Omega,\ell+s]}\delta(r)$, we deduce that

$$\begin{array}{rcl} v(t) & \le & e^{\frac{\ln(\mathsf{q})}{\Omega}(t-s)}\sup_{\ell\in[s-\Omega,s]} v(\ell) \\ & & +\frac{\bar{c}[k(p+1)\bar{d}\bar{\nu}+1]}{1-\mathsf{q}}\sup_{\ell\in[s-\Omega,t]}\delta(r) \end{array} \quad (34)$$

if $t \ge s \ge \Omega$. Since $\mathsf{q} < 1$, we can conclude. ■

Remark 1: Using Taylor's polynomials, we can find a continuous function ϕ such that for all $r \in \mathbb{R}$, we have

$$\mu(r) = \mu(0) + \mu'(0)r + r^2\phi(r). \quad (35)$$

Then simple calculations give $\mu(0) = 1$ (by the relation $\frac{1}{2}(b(\mathcal{S}-a)/(c\mathcal{S})) + \frac{1}{2}(b(\mathcal{S}+a)/(c\mathcal{S})) - b/c = 0$ and cancelling to $a/\mathcal{S}$ terms) and $\mu'(0) = -\frac{\mathcal{S}+a}{2}(\frac{1}{2}(1+a/\mathcal{S}+b(\mathcal{S}-a)/(c\mathcal{S}))) + \frac{\mathcal{S}-a}{2}(\frac{1}{2}(1-a/\mathcal{S}+b(\mathcal{S}+a)/(c\mathcal{S}))) = \frac{1}{4}(-(\mathcal{S}+a)(1+a/\mathcal{S})+(\mathcal{S}-a)(1-a/\mathcal{S})) = \frac{1}{4}(-2a-2a) = -a$. Thus, when $a > 0$, then there are constants $\underline{r} > 0$, $\bar{r} > 0$, and $\mu_0 \in (0,1)$ such that $\mu(r) \in (0, \mu_0]$ when $r \in (\underline{r}, \bar{r}]$ (but Assumption 1 is not satisfied when $a \le 0$). Given any constant $\bar{\nu} > 0$, consider a sequence t_i such that there is a $k \in \mathbb{N}$ such that $k > 2$ and $v_{i+k} = v_i$ for all $i \in \mathbb{Z}_0$ and assume $\nu_i = \bar{r}$ for all $i \in \{0, ..., k-2\}$ and $\nu_{k-1} = \bar{\nu}$. Then

$$\prod_{l=j}^{j+k-1}\mu(\nu_l) \le \mu(\bar{\nu})\mu_0^{k-1} \quad (36)$$

Then Assumption 1 is satisfied provided there is a constant $\omega \in (0,1)$ such that

$$\mu(\overline{\nu})\mu_0^{k-1} \leq \omega, \tag{37}$$

which is equivalent to

$$k \geq 1 + \ln\left(\frac{\omega}{\mu(\overline{\nu})}\right)\frac{1}{\ln(\mu_0)}. \tag{38}$$

For any $\overline{\nu} > 0$, there are $k \in \mathbb{N}$ such that (38) is satisfied, so no constraint on $\bar{\nu}$ is imposed, but larger sampling intervals $[t_i, t_{i+1})$ require more frequent small sampling intervals. $\square$

III. Application

A. Statement of Result

Consider the system

$$\dot{x}(t) = M(t)x(t) + N(t)x(t_i) + \gamma(t) \tag{39}$$

where x is valued in $\mathbb{R}^n$, $M : [0,+\infty) \to \mathbb{R}^{n\times n}$ and $N : [0,+\infty) \to \mathbb{R}^{n\times n}$ are continuous functions that are bounded in norm by the constants $\overline{m} > 0$ and $\overline{n} > 0$ respectively, and $\gamma : [0,+\infty) \to \mathbb{R}^n$ is a piecewise continuous bounded function that can represent uncertainty.

To motivate (39), note that systems of the type (39) are encountered in many circumstances. For instance, a system $\dot{x}(t) = A(t)x(t) + B(t)u(t) + d(t)$ with the discrete output $y(t) = Cx(t_i)$ and the input $u(t) = F(t)y(t)$ gives

$$\dot{x}(t) = A(t)x(t) + B(t)F(t)Cx(t_i) + d(t) \tag{40}$$

Also, if one wants to use an observer of the form $\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) - G(t)[C\hat{x}(t_i) - y(t)]$, then one obtains the error equation

$$\dot{e}(t) = A(t)e(t) - G(t)Ce(t_i) - d(t) \tag{41}$$

for $e = \hat{x} - x$, which also is of the form (39).

Having motivated (39), we next provide conditions ensuring generalized input-to-state stability type properties of the system (39). We introduce the simplifying notation $H(t) = M(t) + N(t)$ and the following assumption, in terms of the matrices D_M, R_M, M^+ and M^- we defined in Section I:

Assumption 2: There exist a C^1 function $P = (P_1 \ldots P_n) : [0,+\infty) \to \mathbb{R}^n$ and positive constants $\underline{p}$, $\bar{p}$, and p such that

$$\begin{array}{l}\dot{P}(t) + [D_H(t) + R_H(t)^+ + R_H(t)^-]^\top P(t) \\ \leq -pP(t) \text{ and}\end{array} \tag{42}$$

$$\underline{p} \leq P_k(t) \leq \overline{p} \text{ for } k = 1, 2, \ldots, n \tag{43}$$

hold for all $t \geq 0$. $\square$

We also choose the positive constants

$$d_1 = \frac{2|P|_\infty \overline{mn}}{\underline{p}} \text{ and } d_2 = \frac{2|P|_\infty \overline{n}^2}{\underline{p}} \tag{44}$$

and $\mathcal{S}_0 = \sqrt{p^2 + 4d_1}$ and the function

$$\begin{array}{rcl}\kappa(r) & = & \frac{1}{2}\left(1 + \frac{p}{\mathcal{S}_0} + \frac{d_2(\mathcal{S}_0 - p)}{d_1\mathcal{S}_0}\right)e^{-\frac{p+\mathcal{S}_0}{2}r} \\ & & + \frac{1}{2}\left(1 - \frac{p}{\mathcal{S}_0} + \frac{d_2(\mathcal{S}_0 + p)}{d_1\mathcal{S}_0}\right)e^{\frac{\mathcal{S}_0 - p}{2}r} - \frac{d_2}{d_1},\end{array} \tag{45}$$

which agrees with the μ formula (2) when $a = p$, $b = d_2$, and $c = d_1$. The following is an analog of Assumption 1:

Assumption 3: There are two constants $q \in \mathbb{N}$ and $\overline{\kappa} \in (0,1)$ such that for all $j \in \mathbb{N}$, the inequality

$$\prod_{l=j}^{j+q} \kappa(\nu_l) \leq \overline{\kappa} \tag{46}$$

is satisfied. $\square$

In the next subsection, we prove:

Theorem 2: Let the system (39) satisfy Assumptions 2-3. Then there are positive constants β_i for $i = 1$ to 4 such that all solutions of the system (39) satisfy

$$\begin{array}{rcl}|x(t)| & \leq & \beta_1 e^{-\beta_2(t-s)} \sup_{m\in[s-\beta_4,s]} |x(m)| \\ & & +\beta_3 \sup_{m\in[s-\beta_4,t]} |\gamma(m)|\end{array} \tag{47}$$

for all $s \geq \beta_4$ and $t \geq s$.

B. Proof of Theorem 2

For all $t \in [t_i, t_{i+1})$, we can rewrite the system (39) as

$$\begin{array}{rcl}\dot{x}(t) & = & H(t)x(t) + N(t)[x(t_i) - x(t)] + \gamma(t) \\ & = & H(t)x(t) - N(t)\int_{t_i}^t \dot{x}(\ell)\mathrm{d}\ell + \gamma(t) \\ & = & H(t)x(t) - N(t)\int_{t_i}^t [M(\ell)x(\ell) \\ & & +N(\ell)x(t_i)]\mathrm{d}\ell - N(t)\int_{t_i}^t \gamma(\ell)\mathrm{d}\ell + \gamma(t).\end{array} \tag{48}$$

Here and in the sequel, statements involving i should be understood to hold for all $i \in \mathbb{Z}_0$. Thus

$$\begin{array}{l}\dot{x}(t) \\ = H(t)x(t) + \int_{t_i}^t Z(t,\ell)x(\ell)\mathrm{d}\ell + K(t)x(t_i) + \varpi(t)\end{array} \tag{49}$$

for all $t \in [t_i, t_{i+1})$, where

$$\begin{array}{l}K(t) = -N(t)\int_{t_i}^t N(\ell)\mathrm{d}\ell,\ Z(t,\ell) = -N(t)M(\ell), \\ \text{and } \varpi(t) = -N(t)\int_{t_i}^t \gamma(\ell)\mathrm{d}\ell + \gamma(t).\end{array} \tag{50}$$

We study the system (49) through the following dynamics, which will play the role of the interval observer in [8]:

$$\left\{\begin{array}{rcl}\dot{\overline{x}}(t) & = & (D_H(t) + R_H(t)^+)\overline{x}(t) - R_H(t)^-\underline{x}(t) \\ & & +\int_{t_i}^t [Z(t,\ell)^+\overline{x}(\ell) - Z(t,\ell)^-\underline{x}(\ell)]\mathrm{d}\ell \\ & & +K(t)^+\overline{x}(t_i) - K(t)^-\underline{x}(t_i) + \varpi(t)^+ \\ \dot{\underline{x}}(t) & = & (D_H(t) + R_H(t)^+)\underline{x}(t) - R_H(t)^-\overline{x}(t) \\ & & +\int_{t_i}^t [Z(t,\ell)^+\underline{x}(\ell) - Z(t,\ell)^-\overline{x}(\ell)]\mathrm{d}\ell \\ & & +K(t)^+\underline{x}(t_i) - K(t)^-\overline{x}(t_i) - \varpi(t)^-\end{array}\right. \tag{51}$$

for all $t \in [t_i, t_{i+1})$. Let $s \geq \overline{\nu}$ and let us consider a solution of (51) with the initial conditions

$$\begin{array}{l}(\overline{x}(m), \underline{x}(m)) = (x(m)^+, -x(m)^-) \\ \text{for all } m \in [s - \overline{\nu}, s].\end{array} \tag{52}$$

Then we prove in the appendix that

$$\overline{x}(t) \geq 0,\ \underline{x}(t) \leq 0,\ \text{and } \underline{x}(t) \leq x(t) \leq \overline{x}(t) \tag{53}$$

hold if $t \geq s$. As a direct consequence, we have $-\tilde{x}(t) = \underline{x}(t) - \overline{x}(t) \leq x(t) \leq \overline{x}(t) - \underline{x}(t) = \tilde{x}(t)$ and so also

$$|x(t)| \leq |\tilde{x}(t)| \tag{54}$$

for all $t \geq s$ where $\tilde{x}(t) = \overline{x}(t) - \underline{x}(t)$.

Direct calculations imply that

$$\begin{array}{rcl}\dot{\tilde{x}}(t) & = & M_H(t)\tilde{x}(t) + \int_{t_i}^{t} N_H(t,\ell)\tilde{x}(\ell)\mathrm{d}\ell \\ & & + K_{\mathcal{L}}(t)\tilde{x}(t_i) + \varrho(t)\end{array} \tag{55}$$

for all $t \geq s$ with $t \in [t_i, t_{i+1})$, where $M_H(t) = D_H(t) + R_H(t)^+ + R_H(t)^-$, $N_H(t,\ell) = Z(t,\ell)^+ + Z(t,\ell)^-$, $K_{\mathcal{L}}(t) = K(t)^+ + K(t)^-$ and $\varrho(t) = \varpi(t)^+ + \varpi(t)^-$. By (53) and Assumption 2, it follows that along all solutions of (55), the time derivative of $V(t,\tilde{x}) = \tilde{x}^\top P(t)$ satisfies

$$\begin{array}{rcl}\dot{V}(t) & \leq & -pV(t,\tilde{x}(t)) + P^\top(t)\int_{t_i}^{t}[Z(t,\ell)^+ \\ & & +Z(t,\ell)^-]\tilde{x}(\ell)\mathrm{d}\ell \\ & & +\tilde{x}(t_i)^\top K_{\mathcal{L}}(t)^\top P(t) + \varrho(t)^\top P(t) \\ & \leq & -pV(t,\tilde{x}(t)) + d_1\int_{t_i}^{t} V(\ell,\tilde{x}(\ell))\mathrm{d}\ell \\ & & + d_2(t-t_i)V(t_i,\tilde{x}(t_i)) + \varrho(t)^\top P(t)\end{array} \tag{56}$$

for all $t \geq s$ with $t \in [t_i, t_{i+1})$, where the formulas for d_1 and d_2 from (44) followed because, for any $n \times n$ matrix valued function $\mathcal{M}(t,\ell)$ having nonnegative valued entries,

$$\begin{array}{l}P^\top(t)\mathcal{M}(t,\ell)\tilde{x}(\ell) = \sum_{j=1}^{n} \frac{P^\top(t)\mathrm{col}_j\mathcal{M}(t,\ell)}{p_j(t)} p_j(t)\tilde{x}_j(\ell) \\ \leq \frac{|P^\top(t)\mathcal{M}(t,\ell)|}{\underline{p}} V(\ell,\tilde{x}(\ell)) \leq \frac{|P(t)||\mathcal{M}(t,\ell)|}{\underline{p}} V(\ell,\tilde{x}(\ell)).\end{array} \tag{57}$$

Given $s \geq 0$ and $\underline{i} \in \mathbb{Z}_0$ such that $t_{\underline{i}} \geq s$, we apply Theorem 1 with the choices $J = \underline{i}$, $a = p$, $b = d_2$, and $c = d_1$ to

$$v(\ell) = V(\ell,\tilde{x}(\ell)) \tag{58}$$

which satisfies

$$\begin{array}{rcl}\dot{v}(t) & \leq & -pv(t) + d_1\int_{t_i}^{t} v(\ell)\mathrm{d}\ell \\ & & + d_2(t-t_i)v(t_i) + \varrho(t+t_{\underline{i}})^\top P(t+t_{\underline{i}})\end{array} \tag{59}$$

for all $t \in [t_i, t_{i+1})$ and $i \geq J$. Then Assumption 3 gives constants $\theta_i > 0$ such that

$$\begin{array}{rcl}V(t,\tilde{x}(t)) & \leq & \theta_1 e^{-\theta_2(t-r)} \sup_{m\in[r-\theta_0,r]} V(m,\tilde{x}(m)) \\ & & +\theta_3 \sup_{m\in[r-\theta_0,t]} \varrho(m)^\top P(m)\end{array} \tag{60}$$

when $t \geq r \geq \theta_0 + s$. Hence Assumption 2 provides constants $\theta_i > 0$ for $i = 4$ to 6 such that

$$\begin{array}{rcl}|\tilde{x}(t)| & \leq & \theta_4 e^{-\theta_5(t-r)} \sup_{m\in[r-\theta_0,r]} |\tilde{x}(m)| \\ & & + \theta_6 \sup_{m\in[r-\theta_0,t]} |\varrho(m)|\end{array} \tag{61}$$

when $t \geq r \geq \theta_0 + s$.

Thus, combining (54) and (61) with $r = \theta_0 + s$, it follows that when $t \geq \theta_0 + s$, we have

$$|x(t)| \leq \theta_7 e^{-\theta_5(t-s)} \sup_{m\in[s,\theta_0+s]} |\tilde{x}(m)| + \theta_6 \sup_{m\in[s,t]} |\varrho(m)| \tag{62}$$

with $\theta_7 = \theta_4 e^{\theta_5\theta_0}$. By applying variation of parameters to (55) on $[t_i, t]$ for each $t \in [t_i, t_{i+1})$ and then applying Gronwall's inequality repeatedly to the result on the intervals $[t_i, t_{i+1})$ for $i \geq \underline{i} - 1$, one finds a constant $C > 0$ such that

$$\begin{array}{l}|\tilde{x}(\ell)| \leq C\left(\sup_{m\in[s-\overline{\nu},s]} |\tilde{x}(m)| + \sup_{m\in[s,\theta_0+s]} |\varrho(m)|\right) \\ \qquad \leq C\left(\sup_{m\in[s-\overline{\nu},s]} |\tilde{x}(m)| + \sup_{m\in[s,t]} |\varrho(m)|\right)\end{array} \tag{63}$$

for all $\ell \in [s, \theta_0 + s]$ and $t \geq \theta_0 + s$ and $s \geq \overline{\nu}$. Combining the last inequality in (63) with (62), we obtain

$$\begin{array}{l}|x(t)| \leq \theta_7 e^{-\theta_5(t-s)} C \sup_{m\in[s-\overline{\nu},s]} |\tilde{x}(m)| \\ + \theta_7 e^{-\theta_5(t-s)} C \sup_{m\in[s,t]} |\varrho(m)| + \theta_6 \sup_{m\in[s,t]} |\varrho(m)|\end{array} \tag{64}$$

when $t \geq \theta_0 + s$. Thus

$$\begin{array}{rcl}|x(t)| & \leq & \theta_8 e^{-\theta_5(t-s)} \sup_{m\in[s-\overline{\nu},s]} |\tilde{x}(m)| \\ & & +\theta_9 \sup_{m\in[s,t]} |\varrho(m)|\end{array} \tag{65}$$

when $t \geq \theta_0 + s$ with $\theta_8 = \theta_7 C$ and $\theta_9 = \theta_7 C + \theta_6$.

It now follows from (54) and (63) that there are constants θ_{10} and θ_{11} such that

$$\begin{array}{rcl}|x(t)| & \leq & \theta_{10} \sup_{m\in[s-\overline{\nu},s]} |\tilde{x}(m)| + \theta_{11} \sup_{m\in[s,t]} |\varrho(m)| \\ & \leq & \theta_{10} e^{\theta_5\theta_0} e^{-\theta_5(t-s)} \sup_{m\in[s-\overline{\nu},s]} |\tilde{x}(m)| \\ & & +\theta_{11} \sup_{m\in[s,t]} |\varrho(m)|\end{array} \tag{66}$$

for all $t \in [s, \theta_0 + s]$. Setting $\theta_{13} = \max\{\theta_8, \theta_{10}e^{\theta_5\theta_0}\}$ and $\theta_{14} = \max\{\theta_9, \theta_{11}\}$, it follows that if $t \geq s \geq \overline{\nu}$, then

$$\begin{array}{rcl}|x(t)| & \leq & \theta_{13} e^{-\theta_5(t-s)} \sup_{m\in[s-\overline{\nu},s]} |\tilde{x}(m)| \\ & & +\theta_{14} \sup_{m\in[s,t]} |\varrho(m)|.\end{array} \tag{67}$$

Since $|\varrho(t)| \leq (2\overline{n\nu} + 2)\sup_{\ell\in[t-\overline{\nu},t]} |\gamma(\ell)|$ for all $t \in [t_i, t_{i+1})$ and $i \in \mathbb{Z}_0$, and since $\tilde{x}(r) = \overline{x}(r) - \underline{x}(r) = x(r)^+ + x(r)^-$ and so also $|\tilde{x}(r)| \leq 2|x(r)|$ for all $r \in [s - \overline{\nu}, s]$, the theorem now follows from (67).

IV. Illustration of Theorem 2

Let t_i be a sequence that satisfies the requirements from Section I. Consider the two dimensional system

$$\left\{\begin{array}{rcl}\dot{x}_1(t) & = & -x_1(t) + x_2(t_i) \\ \dot{x}_2(t) & = & -2x_2(t) + x_1(t) - x_1(t_i).\end{array}\right. \tag{68}$$

Then, with the notation of Section III, we choose

$$\begin{array}{c}M = \left[\begin{array}{cc} -1 & 0 \\ 1 & -2 \end{array}\right],\quad N = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right], \\ H = \left[\begin{array}{cc} -1 & 1 \\ 0 & -2 \end{array}\right],\ \text{and}\ P = \left[\begin{array}{c} 1 \\ 1 \end{array}\right].\end{array} \tag{69}$$

Since $R_H^- = 0$, the equality $H = D_H + R_H^+ + R_H^-$ holds. Therefore, $(D_H + R_H^+ + R_H^-)^\top P = H^\top P = -P$. Thus, Assumption 2 is satisfied and $\overline{p} = \underline{p} = p = 1$, $\overline{m} = 2$, and

$\overline{n} = 1$. Moreover, $d_1 = 4\sqrt{2}$ and $d_2 = 2\sqrt{2}$ and

$$\begin{array}{rcl}\kappa(r) & = & \frac{1}{2}\left(1 + \frac{1}{\mathcal{S}_0} + \frac{d_2(\mathcal{S}_0 - 1)}{d_1\mathcal{S}_0}\right) e^{-\frac{1+\mathcal{S}_0}{2}r} \\ & & +\frac{1}{2}\left(1 - \frac{1}{\mathcal{S}_0} + \frac{d_2(\mathcal{S}_0+1)}{d_1\mathcal{S}_0}\right) e^{\frac{\mathcal{S}_0-1}{2}r} - \frac{d_2}{d_1},\end{array} \tag{70}$$

where $\mathcal{S}_0 = \sqrt{1 + 16\sqrt{2}}$. Let us observe that $\kappa(r)$ is approximated in any neighborhood of zero by $1 - pr$ (by using the approximation from Remark 1 with $a = p$). It follows that there are two constants $\nu_\sharp > 0$ and $\kappa_\sharp \in (0, 1)$ such that $\kappa(\nu_\sharp) = \kappa_\sharp$.

We deduce that for any $\overline{\nu} > 0$, one can satisfy Assumption 3. Indeed, let $h \in \mathbb{N}$ be such that

$$\frac{\ln(2\kappa(\overline{\nu}))}{-\ln(\kappa_\sharp)} + 1 \le h \tag{71}$$

and consider t_i such that $\nu_l = \nu_\sharp$ for all $l \in \{0, .., h-2\}$, $\nu_{h-1} = \overline{\nu}$ and ν_l is periodic of period h. Then for all $j \in \mathbb{Z}_0$,

$$\prod_{l=j}^{j+h-1} \kappa(\nu_l) = \kappa(\overline{\nu})\kappa_\sharp^{h-1}. \tag{72}$$

From (71), if follows that for all $j \in \mathbb{Z}_0$, we have

$$\prod_{l=j}^{j+h-1} \kappa(\nu_l) \le \frac{1}{2}, \tag{73}$$

so (68) satisfies the conclusions of Theorem 2.

V. Conclusion

We proposed new stability analysis results for time-varying systems with sampled data in the case where there may be arbitrary large time intervals between consecutive sampling instants. Our conditions require that other consecutive sampling instants be closer together, to compensate for the ones that are far apart. Our proofs used novel contractivity and cooperative systems arguments. In future work, we may consider systems with several asynchronous sampled data components, pointwise or distributed delays, and extensions that provide stabilization of systems with dynamic output feedback when the measurements are discrete.

Appendix: Technical Result

We prove that our conditions (53) from our proof of Theorem 2 hold for all $t \ge s$, in order to complete the proof of this theorem. Let $z = -\overline{x}$. Then (51) gives

$$\left\{\begin{array}{rcl}\dot{\overline{x}}(t) & = & (D_H(t) + R_H(t)^+)\overline{x}(t) + R_H(t)^- z(t) \\ & & + \int_{t_i}^t [Z(t,\ell)^+\overline{x}(\ell) + Z(t,\ell)^- z(\ell)]\mathrm{d}\ell \\ & & +K(t)^+\overline{x}(t_i) + K(t)^- z(t_i) + \varpi(t)^+ \\ \dot{z}(t) & = & (D_H(t) + R_H(t)^+)z(t) + R_H(t)^-\overline{x}(t) \\ & & + \int_{t_i}^t [Z(t,\ell)^+ z(\ell) + Z(t,\ell)^-\overline{x}(\ell)]\mathrm{d}\ell \\ & & +K(t)^+ z(t_i) + K(t)^-\overline{x}(t_i) + \varpi(t)^-\end{array}\right. \tag{A.1}$$

for all $t \in [t_i, t_{i+1})$. Since $D_H(t) + R_H(t)^+$ is Metzler for all $t \ge 0$ and all of the other matrices in (A.1) are nonnegative valued, we can prove that (A.1) is nonnegative, by writing (A.1) in the form

$$\begin{array}{l}\dot{q}(t) = \\ M_1(t)q(t) + \int_{t_i}^t M_2(\ell)q(\ell)\mathrm{d}\ell + M_3(t)q(t_i) + \rho(t)\end{array} \tag{A.2}$$

for all $t \in [t_i, t_{i+1})$ and $i \in \mathbb{Z}_0$ where M_1 is Metzler for all $t \ge 0$, and where M_2, M_3, and ρ are nonnegative for all $t \ge 0$, then noting that each entry of the state transition matrix Φ_{M_1} for M_1 is positive valued (e.g., by [6, Lemma 2]) to conclude that the positive orthant $(0, \infty)^{2n}$ is forward invariant for (A.2), and finally using a continuous dependence argument to conclude nonnegativity of (A.2). Since $t - t_i \le \overline{\nu}$ for all $t \in [t_i, t_{i+1})$, and since (52) gives

$$\begin{array}{rcl}\overline{x}(m) & = & x(m)^+ \\ z(m) & = & x(m)^-\end{array} \quad \text{for all } m \in [s - \overline{\nu}, s] \tag{A.3}$$

we get $\overline{x}(t) \ge 0$ and $z(t) \ge 0$ for all $t \ge s$. By the same reasoning, we can use (49) and (51) to check that the dynamics for $(\overline{x}(t) - x(t), x(t) + z(t))$ is nonnegative, so (A.3) gives $\overline{x}(t) - x(t) \ge 0$ and $x(t) + z(t) \ge 0$ for $t \ge s$.

References

[1] M. Di Ferdinando, P. Pepe, and A. Borri. On practical stability preservation under fast sampling and accurate quantization of feedbacks for nonlinear time-delay systems. *IEEE Transactions on Automatic Control*, to appear.

[2] J. Gouzé, A. Rapaport, and M. Hadj-Sadok. Interval observers for uncertain biological systems, *Ecological Modelling*, Volume 133, pp. 46-56, 2000.

[3] W. Haddad, V. Chellaboina, and Q. Hui. *Nonnegative and Compartmental Dynamical Systems*. Princeton University Press, Princeton, NJ, 2010.

[4] A. Halanay. *Differential Equations: Stability, Oscillations, Time Lags*. Academic Press, New York, 1966.

[5] M. Mattioni, S. Monaco, and D. Normand-Cyrot. Feedforwarding under sampling. *IEEE Transactions on Automatic Control*, Volume 64, Number 11, pp. 4668-4675, 2019.

[6] F. Mazenc, V. Andrieu, and M. Malisoff. Design of continuous-discrete observers for time-varying nonlinear systems. *Automatica*, Volume 57, Number 7, pp. 135-144, 2015.

[7] F. Mazenc, M. Malisoff, and M. Krstic. Stability analysis using new variant of Halanay's inequality. In *Proceedings of the 24th International Symposium on Mathematical Theory of Networks and Systems (Cambridge, UK, 23-27 August 2021)*, to appear.

[8] F. Mazenc, S.-I. Niculescu, and O. Bernard. Exponentially stable interval observers for linear systems with delay. *SIAM Journal on Control and Optimization*, Volume 50, Number 1, pp. 286-305, 2012

[9] F. Mazenc, S.-I. Niculescu, and M. Malisoff. Stability and control design for time-varying systems with time-varying delays using a trajectory based approach. *SIAM Journal on Control and Optimization*, Volume 55, Number 1, pp. 533-556, 2017.

[10] S. Monaco, D. Normand-Cyrot, and M. Mattioni. Sampled-data stabilization of nonlinear dynamics with input delays through immersion and invariance. *IEEE Transactions on Automatic Control*, Volume 62, Issue 5, pp. 2561-2567, 2017.

[11] H. Omran. *Contribution to the Control of Nonlinear Systems under Aperiodic Sampling*. PhD Thesis, Ecole Centrale de Lille, 2014.

[12] P. Pepe and E. Fridman. On global exponential stability preservation under sampling for globally Lipschitz time-delay systems. *Automatica*, Volume 82, pp. 295-300, 2017

[13] T. Raissi, D. Efimov, and A. Zolghadri. Interval state estimation for a class of nonlinear systems. *IEEE Transactions on Automatic Control*, Volume 57, Issue 1, pp. 260-265, 2012.

[14] A. Selivanov and E. Fridman. Sampled-data implementation of derivative-dependent control using artificial delays. *IEEE Transactions on Automatic Control*, Volume 63, Issue 10, pp. 3594-3600, 2018.

[15] V. Tanasa, S. Monaco, and D. Normand-Cyrot. Backstepping control under multi-rate sampling. *IEEE Transactions on Automatic Control*, Volume 61, Issue 5, pp. 1208-1222, 2016.