# Discrete-Time Homogeneity: Robustness and Approximation

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**Abstract**

In this paper we study robustness properties of discrete-time homogeneous systems. We also analyse stability and robustness properties of a more general class of nonlinear discrete-time systems that can be approximated by homogeneous ones. We apply the results to the investigation of stability properties of discretized continuous-time systems.

*Key words:* Nonlinear systems; Discrete-time systems; Lyapunov methods; Robustness; Homogeneous systems.

## 1 Introduction

A simple strategy for analysis and design of a continuous-time control system is to use its linear approximation. Unfortunately, in many cases, the linear approximation results are insufficient or unsuitable for analysis or control design [17]. On the other hand, the use of the *full* nonlinear model can impose severe complications for analysis and control design purposes. An intermediate approach is to consider a homogeneous system to approximate the original model. This is because homogeneous systems are able to preserve nonlinear features of the original system, and they exhibit useful properties for analysis

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and design, e.g.: scalability of trajectories [33], existence of homogeneous Lyapunov functions [26,23], homogeneous stabilization [17,13,30,10], convergence rates [6,23], intrinsic robustness to exogenous perturbations and delays [5], limit homogeneity [2], and extensions to differential inclusions [23,22]. Unfortunately, for the case of discrete-time systems, a similar homogeneity-based framework has not been developed. The main reason is that, verification of homogeneity developed for continuous-time systems does not provide the same benefits when applied directly to discrete-time systems [12,32,28]. Hence, the concept of $D_{\mathbf{r}}-$homogeneity was introduced in [28,29] for discrete-time systems. This new concept is more suitable for discrete-time dynamic models since it exhibits several benefits analogous to those found in homogeneous continuous-time systems, e.g., scalability of trajectories, homogeneous Lyapunov functions, and convergence rates depending on the homogeneity degree. One of the main properties of $D_{\mathbf{r}}-$homogeneous systems is the simplicity to conclude qualitative stability features directly from the homogeneity degree of the system.

In this paper we first investigate robustness properties of $D_{\mathbf{r}}-$homogeneous discrete-time systems with respect to exogenous disturbances. The main results of this part of the paper consist of simply verifiable conditions to decide robustness properties of $D_{\mathbf{r}}-$homogeneous discrete-time systems. Since, in general, nonlinear discrete-time systems are not $D_{\mathbf{r}}-$homogeneous, we introduce the concept of $D_{\mathbf{r}}-$homogeneous approximation for nonlinear discrete-time systems. The aim is to extend the simple tools for stability and robustness analysis of $D_{\mathbf{r}}-$homogeneous systems to the local analysis of nonlinear discrete-time systems that can be approximated by $D_{\mathbf{r}}-$homogeneous ones. This is in the same fashion as linear systems are used to study local properties of nonlinear models. Indeed, we provide an example for which the linear approximation is not possible, but the $D_{\mathbf{r}}-$homogeneous one is used to establish local stability properties of the system in a very simple way. In fact, the stability and robustness analysis can be reduced to verification of the $D_{\mathbf{r}}-$homogeneity degree of the $D_{\mathbf{r}}-$homogeneous approximating system. In the last part of the paper we show how the results can be specialized to the analysis of discrete-time systems obtained by means of the implicit and explicit Euler discretization of continuous-time systems. This provides useful tools to anticipate qualitative local properties of these discretizations. A version of the results in sections 4 and 5 was announced without proofs in [27]. However, in this paper we have reduced the number of assumptions and have extended such results to a more general class of systems. Moreover, all the results about the robustness properties (presented in this paper) were not provided in [27].

*Paper organization:* In Section 2, the definition and some properties of $D_{\mathbf{r}}-$homogeneity are recalled. In Section 3, we state some robustness properties of disturbed $D_{\mathbf{r}}-$homogeneous systems. Section 4 contains the results about stability and robustness of systems that can be approximated by $D_{\mathbf{r}}-$homogeneous systems. The application of the results to the analysis of discretized continuous-time systems is given in Section 5. Some final remarks are presented in Section 6. Auxiliary concepts and results are collected in the Appendix.

*Notation:* Real and integer numbers are denoted as $\mathbb{R}$ and $\mathbb{Z}$, respectively. $\mathbb{R}_{>0}$ denotes the set $\{x \in \mathbb{R} : x > 0\}$, analogously for the set $\mathbb{Z}$ and the sign $\geq$. For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm and $\|x\|_{\mathbf{r}}$ an $\mathbf{r}$-homogeneous norm (see Definition 31 in the Appendix). The composition of two functions $f$ and $g$ is denoted as $f \circ g$, i.e. $(f \circ g)(x) = f(g(x))$. For a continuous positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ and some $\alpha \in \mathbb{R}_{>0}$, we denote the sets $\mathcal{I}(V, \alpha) = \{x \in \mathbb{R}^n : V(x) \leq \alpha\}$, $\mathcal{E}(V, \alpha) = \{x \in \mathbb{R}^n : V(x) \geq \alpha\}$, and $\mathcal{S}(V, \alpha) = \{x \in \mathbb{R}^n : V(x) = \alpha\}$. The set of bounded sequences $d : J \subset \mathbb{Z}_{\geq 0} \to \mathbb{R}^m$ is denoted by $l^\infty(J)$, with the norm $\|d\|_{l^\infty(J)} = \sup_{k \in J} |d(k)|$, for brevity define $l^\infty = l^\infty(\mathbb{Z}_{\geq 0})$. We use the standard definition of the classes of functions $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$, see [19].

## 2 $D_{\mathbf{r}}-$homogeneity

We consider the following discrete-time system

$$x(k+1) = f(x(k), d(k)) \,, \quad k \in \mathbb{Z}_{\geq 0} \,, \tag{1}$$

where the state $x(k) \in \mathbb{R}^n$, and the disturbance $d(k) \in \mathbb{R}^m$, $d \in l^\infty$. We assume the transition map $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ being well defined and locally bounded for all $[x^\top, d^\top]^\top \in \mathbb{R}^{n+m}$. The solution of (1) with initial condition $x(0)$ and input $d \in l^\infty$ is denoted by

$$F(k; x(0), d) \,, \quad k \in \mathbb{Z}_{\geq 0} \,.$$

The unperturbed version of (1) is given by

$$x(k+1) = f(x(k), 0) \,, \quad k \in \mathbb{Z}_{\geq 0} \,. \tag{2}$$

Now, we recall the definitions of $\mathbf{r}-$homogeneity (or standard weighted homogeneity) and $D_{\mathbf{r}}-$homogeneity.

**Definition 1 ([17])** *Let $\Lambda_\epsilon^{\mathbf{r}}$ denote the family of dilations given by the square diagonal matrix $\Lambda_\epsilon^{\mathbf{r}} = \mathrm{diag}(\epsilon^{\mathrm{r}_1}, \ldots, \epsilon^{\mathrm{r}_n})$, where $\mathbf{r} = [r_1, \ldots, r_n]^\top$, $r_i \in \mathbb{R}_{>0}$, and $\epsilon \in \mathbb{R}_{>0}$. The components of $\mathbf{r}$ are called the* weights *of the coordinates. a) A function $V : \mathbb{R}^n \to \mathbb{R}$ is $\mathbf{r}$-homogeneous of degree $m \in \mathbb{R}$ if $V(\Lambda_\epsilon^{\mathbf{r}} x) = \epsilon^m V(x)$, $\forall x \in \mathbb{R}^n$, $\forall \epsilon \in \mathbb{R}_{>0}$. b) A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is $\mathbf{r}$-homogeneous of degree $\mu \in \mathbb{R}$ if $f(\Lambda_\epsilon^{\mathbf{r}} x) = \epsilon^\mu \Lambda_\epsilon^{\mathbf{r}} f(x)$, $\forall x \in \mathbb{R}^n$, $\forall \epsilon \in \mathbb{R}_{>0}$.*

**Definition 2 ([29])** *Let $\Lambda_\epsilon^{\mathbf{r}}$, $\mathbf{r}$ and $\epsilon$ be as in Definition 1. A map $f : \mathbb{R}^n \to \mathbb{R}^n$, is $D_{\mathbf{r}}$-homogeneous of degree $\nu \in \mathbb{R}_{>0}$ if $f(\Lambda_\epsilon^{\mathbf{r}} x) = (\Lambda_\epsilon^{\mathbf{r}})^\nu f(x) = \Lambda_\epsilon^{\nu\mathbf{r}} f(x) = \Lambda_{\epsilon^\nu}^{\mathbf{r}} f(x)$ $\forall x \in \mathbb{R}^n$, $\forall \epsilon \in \mathbb{R}_{>0}$.*

A system $x(k+1) = f(x(k))$ is said to be $D_{\mathbf{r}}-$homogeneous of degree $\nu$ if its transition map $f$ is $D_{\mathbf{r}}-$homogeneous of degree $\nu$. We have mentioned in the introduction that the concept of $\mathbf{r}-$homogeneity (when applied to discrete-time systems) is not as useful as the concept of $D_{\mathbf{r}}-$homogeneity (the latter one ensures scaling of solutions for (2)). Note that these two concepts coincide for the degrees $\mu = 0$ and $\nu = 1$, respectively. For additional discussion about the comparison between $D_{\mathbf{r}}-$homogeneity and $\mathbf{r}-$homogeneity, see [29, Remark 1].

We recall below some results of analysis of $D_{\mathbf{r}}-$homogeneous systems with the aim to show that verifying their stability properties can be done easily. Additional results, e.g. convergence rates, can be found in [29]. Note that if $f$ is $D_{\mathbf{r}}-$homogeneous, then $x = 0$ is an equilibrium point. Standard Lyapunov theory for discrete-time system can be consulted in [1] and [9].

**Theorem 3 ([29])** *Suppose that* (2) *is $D_{\mathbf{r}}$-homogeneous of degree $\nu > 1$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous positive definite $\mathbf{r}$-homogeneous function of degree $m \in \mathbb{R}_{>0}$. Then $x = 0$ is a locally asymptotically stable equilibrium point of* (2)*, and there exists $\alpha \in \mathbb{R}_{>0}$ such that $V$ is a Lyapunov function for* (2) *on $\mathcal{I}(V, \alpha)$.*

**Definition 4** *The solutions of* (2) *are* ultimately bounded *if there exists a constant $\beta \in \mathbb{R}_{>0}$, and for every $\alpha \in \mathbb{R}_{>0}$, there is $T = T(\alpha, \beta) \in \mathbb{Z}_{\geq 0}$, such that if $|x(0)| \leq \alpha$, then $|F(k; x(0))| \leq \beta$ for all $k \geq T$.*

**Theorem 5 ([29])** *Suppose that* (2) *is $D_{\mathbf{r}}$-homogeneous of degree $\nu \in (0, 1)$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous positive definite $\mathbf{r}$-homogeneous function of degree $m \in \mathbb{R}_{>0}$. Then the solutions of* (2) *are globally ultimately bounded, and there exists $\bar{\alpha} \in \mathbb{R}_{>0}$ such that*

$$\Delta V(x) := V(f(x, 0)) - V(x) < 0 \,, \quad \forall\, x \in \mathcal{E}(V, \bar{\alpha}) \,.$$

For the case of $\nu = 1$ (note that this case contain the set of linear systems) the stability properties cannot be decided directly from the homogeneity degree. Nonetheless, in the case of asymptotic stability, it is guaranteed the existence of homogeneous Lyapunov functions and exponential convergence rates, for more details see [29].

## 3 Robustness of $D_{\mathrm{r}}-$homogeneous systems

We can see in theorems 3 and 5 that the stability properties of (2) can be decided by means of the homogeneity degree. In this section we obtain analogous results for robustness properties of (1).

### *3.1 Preliminary definitions and results*

**Definition 6 ([31,16,14])** *System* (1) *is called input-to-state practically stable (ISpS), if for any input $d \in l^{\infty}$ and any $x(0) \in \mathbb{R}^n$ there exist some functions $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$, and a constant $c \in \mathbb{R}_{\geq 0}$ such that, for all $k \in \mathbb{Z}_{\geq 0}$,*

$$|F(k; x(0), d)| \leq \beta(|x(0)|, k) + \gamma(\|d\|_{l^{\infty}([0,k-1])}) + c\,.$$

*The system is called input-to-state stable (ISS) if $c = 0$. The system is locally ISS (LISS), if the above properties are satisfied only for $c = 0$, $|x(0)| \leq \epsilon^*$ and $\|d\|_{l^{\infty}} \leq \epsilon^*$ for some $\epsilon^* \in \mathbb{R}_{>0}$.*

**Definition 7 ([3])** *System* (1) *is called integral input-to-state stable (iISS), if there exist some functions $\beta \in \mathcal{KL}$, $\sigma, \gamma \in \mathcal{K}_{\infty}$, such that for any input $d$, any $x(0) \in \mathbb{R}^n$, and all $k \in \mathbb{Z}_{\geq 0}$,*

$$\gamma(|F(k; x(0), d)|) \leq \beta(|x(0)|, k) + \sum_{k'=0}^{k-1} \sigma(|d(k')|)\,.$$

**Lemma 8 ([16,21,3,14])** *System* (1) *is ISpS if and only if it admits an ISpS-Lyapunov function, i.e. there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ such that for all $x \in \mathbb{R}^n$, $d \in \mathbb{R}^m$, for some $c \in \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_{\infty}$, and $\theta \in \mathcal{K}$,*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|),\ \ \Delta V(x) \leq -\alpha_3(|x|) + \theta(|d|) + c,$$

*where $\Delta V(x) := V(f(x,d)) - V(x)$. If $c = 0$, $V$ is an ISS-Lyapunov function, and the system is ISS. If $c = 0$ and $\alpha_3$ is a positive definite function, then $V$ is an iISS-Lyapunov function, and the system is iISS. If $c = 0$, and the estimates are satisfied for all $(x, d)$ in a neighbourhood of $0 \in \mathbb{R}^{n+m}$, then $V$ is an LISS-Lyapunov function, and the system is LISS.*

**Remark 9** *In Lemma 8, the condition $\Delta V(x) \leq -\alpha_3(|x|) + \theta(|d|) + c$ is verified if there exist $\beta_1 \in \mathcal{K}_{\infty}$, $\beta_2 \in \mathcal{K}$, and $\bar{c} \in \mathbb{R}_{\geq 0}$ such that if $|x| \geq \beta_2(|d|) + \bar{c}$, then $\Delta V(x) \leq -\beta_1(|x|)$ [15,21].*

**Theorem 10** *If* (2) *is* $D_{\mathbf{r}}$*−homogeneous of degree* $\nu > 1$*, and there exist* $D, X \in \mathbb{R}_{>0}$ *and* $\beta \in \mathcal{K}$ *such that* $|f(x,d) - f(x,0)| \leq \beta(|d|)$ *for all* $d \in \mathcal{I}(|d|, D)$*, for all* $x \in \mathcal{I}(|x|, X)$*, then* (1) *is LISS. Moreover, any function* $V : \mathbb{R}^n \to \mathbb{R}$ *which is locally Lipschitz continuous, positive definite, and* $\mathbf{r}$*-homogeneous of degree* $m \in \mathbb{R}_{>0}$*, is an LISS-Lyapunov function for* (1)*.*

**PROOF.** The proof consists in verifying the assumptions of Lemma 8. First note that $\Delta V(x) = V(f(x,d)) - V(x)$ can be rewritten as follows

$$\Delta V(x) = V(f(x,0)) - V(x) + V_d(x)\,, \tag{3}$$

where $V_d(x) = V(f(x,d)) - V(f(x,0))$. According to Lemma 32 in Appendix, there exists $\gamma_0 \in \mathbb{R}_{>0}$ such that $V(f(x,0)) \leq \gamma_0 V^{\nu}(x)$ for all $x \in \mathbb{R}^n$. Hence $V(f(x,0)) - V(x) \leq -(1 - \gamma_0 V^{\nu-1}(x))V(x)$. Since $\nu > 1$, $V^{\nu-1}(x) \to 0$ as $V(x) \to 0$. Therefore, there exist $\gamma_1, b_1 \in \mathbb{R}_{>0}$ such that [29] $V(f(x,0)) - V(x) \leq -\gamma_1 V(x)$ for all $x \in \mathcal{I}(V, b_1)$. Now, since $V$ is locally Lipschitz and $f$ is locally bounded, we can assure that there exist $L, b_2 \in \mathbb{R}_{>0}$ such that for all $d \in \mathcal{I}(|d|, D)$ we have that $|V_d(x)| \leq L|f(x,d) - f(x,0)| \leq L\beta(|d|)$, for all $x \in \mathcal{I}(V, b_2)$. Therefore, there exists $b_3 \in \mathbb{R}_{>0}$ such that $\Delta V(x) \leq -\gamma_1 V(x) + L\beta(|d|)$, for all $x \in \mathcal{I}(V, b_3)$ and all $d \in \mathcal{I}(|d|, D)$. The result follows from Lemma 8. □

**Theorem 11** *Consider* (1) *and let the following assumptions hold:*

*(1) there exist* $c \in \mathbb{R}_{\geq 0}$*,* $\alpha \in \mathcal{K}$*, and a function* $V : \mathbb{R}^n \to \mathbb{R}$ *which is continuous, positive definite, and* $\mathbf{r}$*-homogeneous of some degree* $m \in \mathbb{R}_{>0}$ *such that* $|V(f(x,d)) - V(f(x,0))| \leq \alpha(|d|) + c$*, for all* $d \in \mathbb{R}^m$*, for all* $x \in \mathcal{S}(\|x\|_{\mathbf{r}}, 1)$*;*
*(2) system* (2) *is* $D_{\mathbf{r}}$*−homogeneous of degree* $\nu \in (0,1)$*, and there exists* $\bar{\mathbf{r}} \in \mathbb{R}^m_{\geq 0}$ *such that the relation* $f(\Lambda^{\mathbf{r}}_{\epsilon} x, \Lambda^{\bar{\mathbf{r}}}_{\epsilon} d) = \Lambda^{\nu\mathbf{r}}_{\epsilon} f(x,d)$ *holds for all* $x \in \mathbb{R}^n$*, for all* $d \in \mathbb{R}^m$*, and for all* $\epsilon \in \mathbb{R}_{>0}$*;*

*then* (1) *is ISpS, and* $V$ *is an ISpS-Lyapunov function for* (1)*.*

**PROOF.** First rewrite $\Delta V$ as in (3). Now, Lemma 32 (see Appendix) guarantees that there exist $\gamma_1, \gamma_2 \in \mathbb{R}_{>0}$ such that $\Delta V(x) \leq -\gamma_1 \|x\|^m_{\mathbf{r}} + \gamma_2 \|x\|^{\nu m}_{\mathbf{r}} + |V_d(x)|$. For $x = 0$ we have that (according to the first assumption of the theorem) $\Delta V(x) = V(f(0,d)) \leq \alpha(|d|) + c$. For $x \neq 0$, let us first find a bound for the term $|V_d(x)|$. Consider the change of coordinates

$$y = \Lambda^{\mathbf{r}}_{\epsilon^{-1}} x, \quad \epsilon = \|x\|_{\mathbf{r}}, \quad x \neq 0\,, \tag{4}$$

and define $\bar{d} = \Lambda_{\epsilon^{-1}}^{\bar{\mathbf{r}}} d$. Observe that, since $f$ is $D_{\mathbf{r}}-$homogeneous and $V$ is $\mathbf{r}-$homogeneous, $V_d(x) = V(f(\Lambda_\epsilon^{\mathbf{r}} y, \Lambda_\epsilon^{\bar{\mathbf{r}}} \bar{d})) - V(f(\Lambda_\epsilon^{\mathbf{r}} y, 0)) = \epsilon^{\nu m}\Big[V(f(y,\bar{d})) - V(f(y,0))\Big]$. Note that $\|y\|_{\mathbf{r}} = 1$. Hence,

$$|V_d(x)| \leq \|x\|_{\mathbf{r}}^{\nu m}(\alpha(|\bar{d}|) + c)\,. \tag{5}$$

Note that in (5), $\bar{d}$ depends on $\|x\|_{\mathbf{r}}$, however, from the definition of $\bar{d}$ we can see that for any $a \in \mathbb{R}_{>0}$, there exists $b \in \mathbb{R}_{>0}$ (which depends only on $a$) such that for all $x \in \mathcal{E}(\|x\|_{\mathbf{r}}, a)$, $\alpha(|\bar{d}|) \leq \alpha(b|d|) := \bar{\alpha}(|d|)$ where $\bar{\alpha} \in \mathcal{K}$. Hence, $\Delta V(x) \leq -\gamma_1\|x\|_{\mathbf{r}}^m + \gamma_2\|x\|_{\mathbf{r}}^{\nu m} + \|x\|_{\mathbf{r}}^{\nu m}\Big[\bar{\alpha}(|d|) + c\Big]$ for all $x \in \mathcal{E}(\|x\|_{\mathbf{r}}, a)$. Note that, $\Delta V(x) \leq -\frac{1}{2}\gamma_1\|x\|_{\mathbf{r}}^m$ if $\|x\|_{\mathbf{r}} \geq \Big[\frac{2}{\gamma_1}\bar{\alpha}(|d|) + \frac{2}{\gamma_1}(\gamma_2 + c)\Big]^{\frac{1}{(1-\nu)m}}$. Hence, there exist (see [19, Eqn. (8)]) $\beta \in \mathcal{K}$ and $\bar{c} \in \mathbb{R}_{>0}$ such that if $\|x\|_{\mathbf{r}} \geq \beta(|d|) + \bar{c}$, then $\Delta V(x) \leq -\frac{1}{2}\gamma_1\|x\|_{\mathbf{r}}^m$. Thus, from Lemma 8 and Remark 9 we conclude the proof. □

**Remark 12** *Note that the second assumption in Theorem 11 is not strong. As it is mentioned in [5], we can modify the dimension or introduce a nonlinear change of coordinates for d, since it is an external input. For example, consider the following usual cases with a $D_{\mathbf{r}}-$homogeneous function $\bar{f}(x)$ of degree $\nu$: (a) Additive disturbances, i.e. $x(k+1) = \bar{f}(x(k)) + B(k)\bar{d}(k)$, where $\bar{d}(k) \in \mathbb{R}^m$ and $B$ is an $n \times m$-matrix (possibly time-varying). Defining $d = B\bar{d}$ we have that $f(x,d) = \bar{f}(x) + d$. Hence, $f(\Lambda_\epsilon^{\mathbf{r}} x, \Lambda_\epsilon^{\bar{\mathbf{r}}} d) = \bar{f}(\Lambda_\epsilon^{\mathbf{r}} x) + \Lambda_\epsilon^{\bar{\mathbf{r}}} d = \Lambda_\epsilon^{\nu\mathbf{r}}\bar{f}(x) + \Lambda_\epsilon^{\bar{\mathbf{r}}} d = \Lambda_\epsilon^{\nu\mathbf{r}} f(x,d)$ with $\bar{\mathbf{r}} = \nu\mathbf{r}$; (b) Noise in the feedback, i.e. $f(x,d) = \bar{f}(x+d)$. Clearly $f(\Lambda_\epsilon^{\mathbf{r}} x, \Lambda_\epsilon^{\bar{\mathbf{r}}} d) = \Lambda_\epsilon^{\nu\mathbf{r}} f(x,d)$ with $\bar{\mathbf{r}} = \mathbf{r}$; (c) Parametric uncertainty by channel, i.e. $f(x,d) = [I + \mathrm{diag}(d_1, \ldots, d_n)]\bar{f}(x)$ where $I$ is the identity matrix. In this case, $f(\Lambda_\epsilon^{\mathbf{r}} x, \Lambda_\epsilon^{\bar{\mathbf{r}}} d) = \Lambda_\epsilon^{\nu\mathbf{r}} f(x,d)$ holds with $\bar{\mathbf{r}} = 0$ (note that in the continuous-time setting with $\mathbf{r}-$homogeneity, the condition $\min \bar{r}_i > 0$ is required for ISS [5], which can be relaxed in the discrete-time case).*

For the following result, we use the definition of *robust global asymptotic stability* as given in [20] (also included in Appendix). Let us stress that if the transition map of (2) is continuous, then global asymptotic stability implies robust global asymptotic stability.

**Theorem 13** *Let* (2) *be $D_{\mathbf{r}}-$homogeneous of degree $\nu = 1$, and let $x = 0$ be robustly globally asymptotically stable. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function, $\mathbf{r}$-homogeneous of some degree $m \in \mathbb{R}_{>0}$, and a Lyapunov function for* (2)*. Then:*

*(1) system* (1) *is iISS;*

*(2) if the following assumptions hold: (a) there exist $c, p \in \mathbb{R}_{>0}$ such that $|V(f(x,d)) - V(f(x,0))| \leq c|d|^p$ for all $d \in \mathbb{R}^m$, and all $x \in \mathcal{S}(\|x\|_{\mathbf{r}}, 1)$; (b) there exists $\bar{\mathbf{r}} \in \mathbb{R}_{>0}^m$ such that the relation $f(\Lambda_\epsilon^{\mathbf{r}} x, \Lambda_\epsilon^{\bar{\mathbf{r}}} d) = \Lambda_\epsilon^{\mathbf{r}} f(x,d)$ holds for all $x \in \mathbb{R}^n$, all $d \in \mathbb{R}^m$, and all $\epsilon \in \mathbb{R}_{>0}$. Then* (1) *is ISS, and*

*$V$ is an ISS-Lyapunov function for (1);*

*(3) if there exist $D, X \in \mathbb{R}_{>0}$ and $\beta \in \mathcal{K}$ such that $|f(x,d) - f(x,0)| \leq \beta(|d|)$ for all $x \in \mathcal{I}(|x|, X)$ and all $d \in \{d \in \mathbb{R}^m : |d| \leq D\}$, then (1) is LISS, and $V$ is an LISS-Lyapunov function for (1).*

**PROOF.** (1) This result is from [3]. Note that, although in [3] $f$ is assumed to be continuous, the result is still valid for our case because robust global asymptotic stability guarantees the existence of a smooth Lyapunov function for (2) for $\nu = 1$ [29, Thm. 5].
(2) The proof of this point is analogous to the the proof of Theorem 11, but adjusting two details: 1) since $V$ is an $\mathbf{r}$-homogeneous Lyapunov function for (2), we know that there exists $\gamma_1 \in \mathbb{R}_{>0}$ such that $V(f(x,0)) - V(x) \leq -\gamma_1 \|x\|_{\mathbf{r}}^m$ [29]; 2) note that in this case, we have that (cf. (5)) $|V_d(x)| \leq \|x\|_{\mathbf{r}}^m c |\bar{d}|^p$ and $|\bar{d}|^p \leq \|x\|_{\mathbf{r}}^{-\rho} m^p |d|^p$ where $\rho = p \max_j \bar{r}_j$ for $\|x\|_{\mathbf{r}} \leq 1$, and $\rho = p \min_j \bar{r}_j$ for $\|x\|_{\mathbf{r}} > 1$. Note that $\rho > 0$. Thus, $|V_d(x)| \leq \|x\|_{\mathbf{r}}^{m-\rho} \bar{c} |d|^p$ for some $\bar{c} \in \mathbb{R}_{>0}$. Hence, $\Delta V(x) \leq -\gamma_1 \|x\|_{\mathbf{r}}^m + \|x\|_{\mathbf{r}}^{m-\rho} \bar{c} |d|^p$. Therefore, if $\|x\|_{\mathbf{r}}^{\rho} \geq \frac{2\bar{c}}{\gamma_1} |d|^p$, then $\Delta V(x) \leq -\frac{1}{2}\gamma_1 \|x\|_{\mathbf{r}}^m$.
(3) The proof of this point is completely analogous to the the proof of Theorem 10. □

## 4 $D_{\mathbf{r}}-$homogeneous approximation

In this section we consider (1), but we do not assume that its undisturbed version (2) is $D_{\mathbf{r}}-$homogeneous. The idea is to verify whether $f(\cdot, 0)$ can be approximated by a $D_{\mathbf{r}}-$homogeneous map $h(\cdot)$, and whether some stability and robustness properties of $f(\cdot, d)$ can be decided through the properties of $h(\cdot)$. First, we define the discrete-time counterpart of local (or limit) homogeneity of continuous-time systems.

**Definition 14** *For a constant $\epsilon_0 \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$, the map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be $D_{\mathbf{r}}-$homogeneous of degree $\nu \in \mathbb{R}_{>0}$ in the $(\epsilon_0, h)-$limit, where $h : \mathbb{R}^n \to \mathbb{R}^n$ is some $D_{\mathbf{r}}-$homogeneous map of degree $\nu$, if*

$$\lim_{\epsilon \to \epsilon_0} \left( \Lambda_\epsilon^{-\nu \mathbf{r}} f(\Lambda_\epsilon^{\mathbf{r}} x) - h(x) \right) = 0 \,,$$

*with the limit computed uniformly for all $x \in \mathcal{S}(\|x\|_{\mathbf{r}}, 1)$.*

Consider (2) and define the map $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ given by $f_0(x) = f(x, 0)$, thus (2) can be rewritten as follows

$$x(k+1) = f_0(x(k)), \quad k \in \mathbb{Z}_{\geq 0}. \tag{6}$$

Note that $f_0$ is well defined and locally bounded for all $x \in \mathbb{R}^n$.

**Theorem 15** *Suppose that the transition map $f_0$ of (6) is $D_{\mathbf{r}}-$homogeneous in the $(\epsilon_0, h)-$limit for some $\epsilon_0 \in \{0, +\infty\}$ with some degree $\nu \in \mathbb{R}_{>0}$. Consider a function $V : \mathbb{R}^n \to \mathbb{R}$ being locally Lipschitz continuous, positive definite, and $\mathbf{r}$-homogeneous of degree $m \in \mathbb{R}_{>0}$.*

*(1) If $\epsilon_0 = 0$, $\nu > 1$, then the origin of (6) is locally asymptotically stable, and there exists $\gamma \in \mathbb{R}_{>0}$ such that $V$ is a Lyapunov function for (6) on $\mathcal{I}(V, \gamma)$.*
*(2) If $\epsilon_0 = +\infty$, and $\nu \in (0, 1)$, then the solutions of (6) are globally ultimately bounded and there exists $\gamma \in \mathbb{R}_{>0}$ such that $\Delta V(x) < 0$ for all $x \in \mathcal{E}(V, \gamma)$.*

**Remark 16** *A function $V$ with the properties required in Theorem 15 does exist for any vector of weights $\mathbf{r}$ [29].*

**PROOF.** Define $\bar{V}(x) = V(f_0(x)) - V(h(x))$, and note that $\Delta V(x) = V(f_0(x)) - V(x)$ can be rewritten as follows

$$\Delta V(x) = V(h(x)) - V(x) + \bar{V}(x). \tag{7}$$

Now, let us analyse the terms $V(h(x))$ and $\bar{V}(x)$ to subsequently find an upper bound for $\Delta V$. According to Lemma 32 given in Appendix, there exists $\bar{\gamma} \in \mathbb{R}_{>0}$ such that $V(h(x)) - V(x) \leq -(1 - \bar{\gamma}V^{\nu-1}(x))V(x)$. Hence, it is easy to see that for $\nu > 1$ (resp., for $\nu \in (0, 1)$) there exist $\gamma_1, \bar{\gamma}_1, \gamma_2 \in \mathbb{R}_{>0}$ such that $V(h(x)) - V(x) \leq -\bar{\gamma}_1 V(x) \leq -\gamma_1 \|x\|_{\mathbf{r}}^m$ for all $x \in \mathcal{I}(V, \gamma_2)$ (resp., for all $x \in \mathcal{E}(V, \gamma_2)$) [29]. For the analysis of $\bar{V}$, consider again the change of coordinates given by (4). Thus, by $D_{\mathbf{r}}-$homogeneity of $h$ and $\mathbf{r}-$homogeneity of $V$ we obtain $\bar{V}(x) = V(f_0(\Lambda_\epsilon^{\mathbf{r}} y)) - V(h(\Lambda_\epsilon^{\mathbf{r}} y)) = \epsilon^{\nu m}\left[V(\Lambda_\epsilon^{-\nu\mathbf{r}} f_0(\Lambda_\epsilon^{\mathbf{r}} y)) - V(h(y))\right]$. Since $h$ is locally bounded and $f_0$ is $D_{\mathbf{r}}-$homogeneous in the $(0, h)-$limit (in the $(+\infty, h)-$limit, resp.), there exists $\bar{\epsilon} \in \mathbb{R}_{>0}$ such that the map given by $\Lambda_\epsilon^{-\nu\mathbf{r}} f_0(\Lambda_\epsilon^{\mathbf{r}} y)$ is locally bounded for all $\epsilon \leq \bar{\epsilon}$ (for all $\epsilon \geq \bar{\epsilon}$, resp.). Thus, by local Lipschitz continuity of $V$ there exists $L \in \mathbb{R}_{>0}$ such that

$$|\bar{V}(x)| \leq \|x\|_{\mathbf{r}}^{\nu m} L |\Lambda_{\|x\|_{\mathbf{r}}}^{-\nu\mathbf{r}} f_0(\Lambda_{\|x\|_{\mathbf{r}}}^{\mathbf{r}} y) - h(y)|, \tag{8}$$

for all $x \in \mathcal{I}(\|x\|_{\mathbf{r}}, \bar{\epsilon})$, (resp., for all $x \in \mathcal{E}(\|x\|_{\mathbf{r}}, \bar{\epsilon})$). Hence (with $\rho = (\nu - 1)m$, and $\epsilon = \|x\|_{\mathbf{r}}$), we have that $\Delta V(x) \leq -\Big(\gamma_1 - \|x\|_{\mathbf{r}}^{\rho} L |\Lambda_{\epsilon}^{-\nu\mathbf{r}} f_0(\Lambda_{\epsilon}^{\mathbf{r}} y) - h(y)|\Big) \|x\|_{\mathbf{r}}^m$. Since $f_0$ is $D_{\mathbf{r}}-$homogeneous of degree $\nu > 1$ in the $(0,h)-$limit (resp., of degree $\nu \in (0,1)$ in the $(+\infty,h)-$limit), $|\Lambda_{\|x\|_{\mathbf{r}}}^{-\nu\mathbf{r}} f_0(\Lambda_{\|x\|_{\mathbf{r}}}^{\mathbf{r}} y) - h(y)| \to 0$ as $\|x\|_{\mathbf{r}} \to 0$ (resp., as $\|x\|_{\mathbf{r}} \to +\infty$), see Definition 14. Hence, there exist $\gamma, \gamma^* \in \mathbb{R}_{>0}$ such that $\Delta V(x) \leq -\gamma^* \|x\|_{\mathbf{r}}^m$ for all $x \in \mathcal{I}(V,\gamma)$ (resp., for all $x \in \mathcal{E}(V,\gamma)$). Therefore, if $\nu > 1$, then $V$ is a Lyapunov function for (6), which proves local asymptotic stability of the origin. For $\nu \in (0,1)$, [1, Corollary 5.14.3] guarantees the existence of $T$ required in Definition 4 to verify ultimate boundedness of the trajectories of (6). □

**Theorem 17** *Suppose that the transition map $f_0$ of (6) is $D_{\mathbf{r}}-$homogeneous in the $(\epsilon_0, h)-$limit for some $\epsilon_0 \in \{0, +\infty\}$ with degree $\nu = 1$. Suppose also that $x = 0$ is a robustly globally asymptotically stable equilibrium point of $x(k+1) = h(x(k))$ with a locally Lipschitz continuous Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$, which is $\mathbf{r}$-homogeneous of degree $m \in \mathbb{R}_{>0}$.*

*(1) If $\epsilon_0 = 0$, then the origin of (6) is locally asymptotically stable and $V$ is a Lyapunov function for (6).*
*(2) If $\epsilon_0 = +\infty$, then the solutions of (6) are globally ultimately bounded. Moreover, there exists $\gamma \in \mathbb{R}_{>0}$ such that $\Delta V(x) < 0$ for all $x \in \mathcal{E}(V,\gamma)$.*

**PROOF.** We analyse $\Delta V$ as given in (7). Since $V$ is an $\mathbf{r}-$homogeneous Lyapunov function for $x(k+1) = h(x(k))$, there exist $\gamma_1, \bar{\gamma}_1 \in \mathbb{R}_{>0}$ such that $V(h(x)) - V(x) \leq -\bar{\gamma}_1 V(x) \leq -\gamma_1 \|x\|_{\mathbf{r}}^m$ [29]. We analyse $\bar{V}$ as in the proof of Theorem 15, thus, from (4) and (8) we obtain $\Delta V(x) \leq -\Big[\gamma_1 - L|\Lambda_{\|x\|_{\mathbf{r}}}^{-\mathbf{r}} f_0(\Lambda_{\|x\|_{\mathbf{r}}}^{\mathbf{r}} y) - h(y)|\Big] \|x\|_{\mathbf{r}}^m$, for all $x \in \mathcal{I}(V,\bar{\gamma}) \setminus \{0\}$ for $\epsilon_0 = 0$ (resp., for all $x \in \mathcal{E}(V,\bar{\gamma})$ for $\epsilon_0 = +\infty$), for some $\bar{\gamma} \in \mathbb{R}_{>0}$. Since $f_0$ is $D_{\mathbf{r}}-$homogeneous in the $(0,h)-$limit (resp., in the $(+\infty,h)-$limit), $|\Lambda_{\|x\|_{\mathbf{r}}}^{-\mathbf{r}} f_0(\Lambda_{\|x\|_{\mathbf{r}}}^{\mathbf{r}} y) - h(y)| \to 0$ as $\|x\|_{\mathbf{r}} \to 0$ (resp., as $\|x\|_{\mathbf{r}} \to +\infty$), see Definition 14. Hence, there exist $c, \gamma \in \mathbb{R}_{>0}$ such that $\Delta V(x) \leq -c\|x\|_{\mathbf{r}}^m$, for all $x \in \mathcal{I}(V,\gamma)$ (resp., for all $x \in \mathcal{E}(V,\gamma)$). □

In [29] it was shown that $D_{\mathbf{r}}-$homogeneous systems exhibit some instability properties. Here, we show that it is also the case for systems that can be approximated by $D_{\mathbf{r}}-$homogeneous ones. We require the following assumption.

**Assumption 18** *The transition map $h : \mathbb{R}^n \to \mathbb{R}^n$ is such that* $\inf_{y \in \mathcal{S}(|y|,1)} |h(y)| > 0$.

**Theorem 19** *Consider (6). Suppose that $f_0$ and $V$ are as in Theorem 15 with $h$ satisfying Assumption 18.*

*(1) If $\epsilon_0 = 0$ and $\nu \in (0,1)$, then the origin of (6) is unstable.*

*(2) If $\epsilon_0 = +\infty$ and $\nu > 1$, then there exists $\gamma \in \mathbb{R}_{>0}$ such that $|F(k; x(0))| \to \infty$ as $k \to \infty$ for all $x_0 \in \mathcal{E}(V,\gamma)$.*

**PROOF.** The proof consists in verifying the conditions of the Lyapunov instability theorem for discrete-time systems, see e.g. [1, Thm. 5.9.3] and [9, Thm. 4.27]. Consider $\Delta V$ as in (7). From the hypotheses of the theorem we have that $V(x) > 0$ and $V(h(x)) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.
(1) From Lemma 32, there exist $c_1, c_2 \in \mathbb{R}_{>0}$ such that $V(h(x)) - V(x) \geq c_1\|x\|_{\mathbf{r}}^{\nu m} - c_2\|x\|_{\mathbf{r}}^{m}$. Since $\Delta V(x) \geq V(h(x)) - V(x) - |\bar{V}(x)|$, we can use (4) and (8) to obtain (with $\rho = (\nu-1)m$, and $\epsilon = \|x\|_{\mathbf{r}}$)

$$\Delta V(x) \geq \left(c_1 - c_2\|x\|_{\mathbf{r}}^{\rho} - L|\Lambda_{\epsilon}^{-\nu\mathbf{r}} f_0(\Lambda_{\epsilon}^{\mathbf{r}} y) - h(y)|\right)\|x\|_{\mathbf{r}}^{\nu m}, \tag{9}$$

for all $x \in \mathcal{I}(V,\gamma_0)\setminus\{0\}$ for some $\gamma_0 \in \mathbb{R}_{>0}$. But, $f_0$ is $D_{\mathbf{r}}-$homogeneous of degree $\nu \in (0,1)$ in the $(0,h)-$limit, thus $\|x\|_{\mathbf{r}}^{(1-\nu)m} \to 0$ and $|\Lambda_{\|x\|_{\mathbf{r}}}^{-\nu\mathbf{r}} f_0(\Lambda_{\|x\|_{\mathbf{r}}}^{\mathbf{r}} y) - h(y)| \to 0$ as $\|x\|_{\mathbf{r}} \to 0$. Hence, there exist $\gamma \in \mathbb{R}_{>0}$ such that $\Delta V(x) > 0$ for all $x \in \mathcal{I}(V,\gamma) \setminus \{0\}$. Thus, from [1, Thm. 5.9.3] we conclude that $x = 0$ is an unstable equilibrium point of (6).
(2) The proof of this part of the theorem is analogous to that for (1), but we have to remark some additional details. First, (8) holds for all $x \in \mathcal{E}(V,\gamma_0)$ for some $\gamma_0 \in \mathbb{R}_{>0}$. Hence, (9) holds for all $x \in \mathcal{E}(V,\gamma_0)$. Since $f_0$ is $D_{\mathbf{r}}-$homogeneous of degree $\nu > 1$ in the $(+\infty,h)-$limit, $\|x\|_{\mathbf{r}}^{-(\nu-1)m} \to 0$ and $|\Lambda_{\|x\|_{\mathbf{r}}}^{-\nu\mathbf{r}} f_0(\Lambda_{\|x\|_{\mathbf{r}}}^{\mathbf{r}} y) - h(y)| \to 0$ as $\|x\|_{\mathbf{r}} \to +\infty$. Therefore, there exist $c, \gamma \in \mathbb{R}_{>0}$ such that $\Delta V(x) > c\|x\|_{\mathbf{r}}^{\nu m}$ for all $x \in \mathcal{E}(V,\gamma)$. Now, to verify that the trajectories of (6) diverge, note that since $\Delta V(x) > c\|x\|_{\mathbf{r}}^{\nu m}$ for all $x \in \mathcal{E}(V,\gamma)$, then $V(F(k+1;x(0))) > V(F(k;x(0)))$ for all $k \in \mathbb{Z}_{\geq 0}$ for all $x(0) \in \mathcal{E}(V,\gamma)$. From Lemma 32, there exist $\bar{c} \in \mathbb{R}_{>0}$ such that $\Delta V(x) > c\|x\|_{\mathbf{r}}^{\nu m} \geq \bar{c}V^{\nu}(x)$. On the other hand, $\sum_{j=0}^{k}[V(F(j+1;x(0))) - V(F(j;x(0)))] = V(F(k+1;x(0))) - V(x_0)$, and also $\sum_{j=0}^{k}[V(F(j+1;x(0))) - V(F(j;x(0)))] \geq \sum_{j=0}^{k} \bar{c}V^{\nu}(F(j;x(0)))$. Thus, $V(F(k+1;x(0))) \geq V(x(0)) + k\bar{c}V^{\nu}(x(0))$ for all $k \in \mathbb{Z}_{\geq 0}$. From this inequality it is clear that $V(F(k;x(0))) \to \infty$ as $k \to \infty$. Since $V$ is continuous, $\mathbf{r}-$homogeneous, and positive definite, it is well defined on $\mathbb{R}^n$ and radially unbounded [6]. Therefore, $|F(k;x(0))| \to \infty$ as $k \to \infty$. □

**Theorem 20** *Suppose that: the transition map $f_0$ of (6) is $D_{\mathbf{r}}-$homogeneous in the $(\epsilon_0,h)-$limit for some $\epsilon_0 \in \{0,+\infty\}$ with degree $\nu = 1$; there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}$, which is positive definite, $\mathbf{r}$-homogeneous of degree $m \in \mathbb{R}_{>0}$, and such that $\inf_{x\in\mathcal{S}(\|x\|_{\mathbf{r}},1)}[V(h(x)) - V(x)] = \delta$ for some $\delta \in \mathbb{R}_{>0}$.*

*(1) If $\epsilon_0 = 0$, then the origin of (6) is unstable.*

*(2) If $\epsilon_0 = +\infty$, then there exists $\gamma \in \mathbb{R}_{>0}$ such that the solutions of (6) satisfy $|F(k;x(0))| \to \infty$ as $k \to \infty$ for all $x(0) \in \mathcal{E}(V,\gamma)$.*

**PROOF.** Consider $\Delta V$ as in (7). First, from $D_{\mathbf{r}}-$homogeneity of $h$ and $\mathbf{r}-$homogeneity of $V$ we can use (4) to see that $V(h(x))-V(x) = V(h(\Lambda^{\mathbf{r}}_{\|x\|_{\mathbf{r}}}y))- V(\Lambda^{\mathbf{r}}_{\|x\|_{\mathbf{r}}}y) = \|x\|^m_{\mathbf{r}}[V(h(y)) - V(y)] \geq \delta\|x\|^m_{\mathbf{r}}$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Thus, by using (8) we have that $\Delta V(x) \geq \left[\delta - L|\Lambda^{-\mathbf{r}}_{\|x\|_{\mathbf{r}}} f_0(\Lambda^{\mathbf{r}}_{\|x\|_{\mathbf{r}}}y) - h(y)|\right]\|x\|^m_{\mathbf{r}}$ for all $x \in \mathcal{I}(V,\gamma_0) \setminus \{0\}$ if $\epsilon_0 = 0$ (resp., for all $x \in \mathcal{E}(V,\gamma_0)$ if $\epsilon_0 = +\infty$) for some $\gamma_0 \in \mathbb{R}_{>0}$. Since $f_0$ is $D_{\mathbf{r}}-$homogeneous in the $(0,h)-$limit (resp., in the $(+\infty,h)-$limit), $|\Lambda^{-\mathbf{r}}_{\|x\|_{\mathbf{r}}} f_0(\Lambda^{\mathbf{r}}_{\|x\|_{\mathbf{r}}}y) - h(y)| \to 0$ as $\|x\|_{\mathbf{r}} \to 0$ (resp., as $\|x\|_{\mathbf{r}} \to +\infty$), see Definition 14. Hence, there exist $c, \gamma \in \mathbb{R}_{>0}$ such that $\Delta V(x) \geq c\|x\|^m_{\mathbf{r}}$, for all $x \in \mathcal{I}(V,\gamma)$ (resp., for all $x \in \mathcal{E}(V,\gamma)$). From [1, Thm. 5.9.3] we conclude instability of $x = 0$ for the first part of the theorem. For the second part of the theorem, the divergence of trajectories can be verified as in the proof of Theorem 19. □

**Example 21** *Let us consider the* Three Oligopolists *model (for details see [25, p. 152]) given by the system*

$$\begin{aligned} x(k+1) &= \sqrt{a|y(k)+z(k)|} - y(k) - z(k)\,, \\ y(k+1) &= \sqrt{b|x(k)+z(k)|} - x(k) - z(k)\,, \\ z(k+1) &= \sqrt{c|x(k)+y(k)|} - x(k) - y(k)\,, \end{aligned} \tag{10}$$

*with* $x(k), y(k), z(k) \in \mathbb{R}$, $k \in \mathbb{Z}_{>0}$, *and* $a, b, c \in \mathbb{R}_{>0}$. *Clearly the origin is an equilibrium of* (10), *but it cannot be analysed by means of a linear approximation. Nonetheless, the transition map of* (10) *is* $D_{\mathbf{r}}-$*homogeneous of degree* $\nu = 1/2$ *in the* $(0,h)-$*limit with* $h = [\sqrt{a|y+z|}\ \sqrt{b|x+z|}\ \sqrt{c|x+y|}]^\top$ *and* $\mathbf{r} = [1\ 1\ 1]^\top$. *Since Assumption 18 is verified, we conclude, from Theorem 19, that the origin of* (10) *is unstable.*

## *4.2 Robustness properties*

We have seen that if a transition map $f$ can be approximated by a $D_{\mathbf{r}}-$homogeneous transition map $h$, then $f$ locally inherits the stability properties of $h$. In this section we show that a similar situation occurs for the robustness properties. First, from theorems 10 and 15 we have the following corollary.

**Corollary 22** *Assume that* (2) *is* $D_{\mathbf{r}}-$*homogeneous of degree* $\nu \in \mathbb{R}_{>1}$ *in the* $(0,h)-$*limit, and let* $V$ *be as in Theorem 15. Suppose that there exist* $\gamma_1, \gamma_2 \in \mathbb{R}_{>0}$, *and* $\beta \in \mathcal{K}$, *such that* $|f(x,d) - f(x,0)| \leq \beta(|d|)$ *for all* $x \in \mathcal{I}(V,\gamma_1)$ *and for all* $d \in \{d \in \mathbb{R}^m : |d| \leq \gamma_2\}$. *Then* (1) *is LISS, and* $V$ *is an LISS-Lyapunov function for* (1).

**PROOF.** This proof follows from the proofs of theorems 10 and 15, by noting that $\Delta V$ along (1) is

$$\Delta V(x) = V(h(x)) - V(x) + \bar{V}(x) + V_d(x)\,, \tag{11}$$

where $\bar{V}(x) = V(f(x,0)) - V(h(x))$ and $V_d(x) = V(f(x,d)) - V(f(x,0))$.

From theorems 11 and 15 we have the following.

**Corollary 23** *Assume that* (2) *is* $D_{\mathbf{r}}-$*homogeneous of degree* $\nu \in (0,1)$ *in the* $(+\infty, h)-$*limit, and let* $V$ *be as in Theorem 15. Suppose that there exists* $\bar{\mathbf{r}} \in \mathbb{R}^m$ *with* $\bar{r}_j \in \mathbb{R}_{\geq 0}$, $j = 1, \ldots, m$, *such that the limit*

$$\bar{h}(x,d) = \lim_{\epsilon \to +\infty} \Lambda_\epsilon^{-\nu\mathbf{r}} f(\Lambda_\epsilon^{\mathbf{r}} x, \Lambda_\epsilon^{\bar{\mathbf{r}}} d)\,, \tag{12}$$

*exists, uniformly for all* $x \in \mathcal{S}(\|x\|_{\mathbf{r}}, 1)$ *and all* $d \in \mathbb{R}^m$. *If the map* $\bar{h} : \mathbb{R}^{n+m} \to \mathbb{R}^n$ *satisfies the following:*

*(1) there exist* $c \in \mathbb{R}_{\geq 0}$, $\alpha \in \mathcal{K}$ *such that* $|V(\bar{h}(x,d)) - V(h(x))| \leq \alpha(|d|) + c$ *for all* $d \in \mathbb{R}^m$, *for all* $x \in \mathcal{S}(\|x\|_{\mathbf{r}}, 1)$;

*(2) the relation* $\bar{h}(\Lambda_\epsilon^{\mathbf{r}} x, \Lambda_\epsilon^{\bar{\mathbf{r}}} d) = \Lambda_\epsilon^{\nu\mathbf{r}} \bar{h}(x,d)$ *holds for all* $x \in \mathbb{R}^n$, *for all* $d \in \mathbb{R}^m$, *and for all* $\epsilon \in \mathbb{R}_{>0}$;

*then* (1) *is ISpS. Moreover,* $V$ *is an ISpS-Lyapunov function for* (1).

**PROOF.** The proof follows the same reasoning of the proofs of theorems 11 and 15, but clarifying the following. $\Delta V$ is as in (11) with $V_d(x) = V(\bar{h}(x,d)) - V(h(x))$ and $\bar{V}(x) = V(f(x,d)) - V(\bar{h}(x,d))$. Thus we must clarify how to bound the term $\bar{V}(x)$.

By following the same procedure of the proof of Theorem 15 we obtain $\bar{V}(x) = \epsilon^{\nu m}\Big[V(\Lambda_\epsilon^{-\nu\mathbf{r}} f(\Lambda_\epsilon^{\mathbf{r}} y, \Lambda_\epsilon^{\bar{\mathbf{r}}} \bar{d})) - V(\bar{h}(y,\bar{d}))\Big]$. Note that for any $\delta \in \mathbb{R}_{>0}$, (12) ensures the existence of $\bar{\epsilon} \in \mathbb{R}_{>0}$ such that $|\bar{h}(y,\bar{d}) - \Lambda_\epsilon^{-\nu\mathbf{r}} f(\Lambda_\epsilon^{\mathbf{r}} y, \Lambda_\epsilon^{\bar{\mathbf{r}}} \bar{d})| \leq \delta$ for all $\epsilon \geq \bar{\epsilon}$. Hence, for $\bar{\epsilon} \in \mathbb{R}_{>0}$ there exists $L_{\bar{\epsilon}} \in \mathbb{R}_{>0}$ such that, for all $\epsilon \geq \bar{\epsilon}$, $|\bar{V}(x)| \leq \epsilon^{\nu m} L_{\bar{\epsilon}} \Big|\Lambda_\epsilon^{-\nu\mathbf{r}} f(\Lambda_\epsilon^{\mathbf{r}} y, \Lambda_\epsilon^{\bar{\mathbf{r}}} \bar{d}) - \bar{h}(y,\bar{d})\Big|$. □

Finally, from theorems 13 and 17 we have the following corollary.

**Corollary 24** *Suppose that* (2) *is* $D_{\mathbf{r}}-$*homogeneous of degree* $\nu = 1$ *in the* $(\epsilon_0, h)-$*limit. Suppose also that* $x = 0$ *is a robustly globally asymptotically stable equilibrium point of* $x(k+1) = h(x(k))$ *with a Lyapunov function* $V$ *as in Theorem 17.*

*(1) If $\epsilon_0 = 0$, and there exists $\bar{\mathbf{r}} \in \mathbb{R}^m$ with $\bar{r}_j \in \mathbb{R}_{\geq 0}$, $j = 1, \dots, m$, such that the limit*

$$\bar{h}(y, d) = \lim_{\epsilon \to 0} \Lambda_\epsilon^{-\mathbf{r}} f(\Lambda_\epsilon^{\mathbf{r}} y, \Lambda_\epsilon^{\bar{\mathbf{r}}} d), \tag{13}$$

*exists, uniformly for all $y \in \mathcal{S}(\|y\|_{\mathbf{r}}, 1)$ and all $d \in \mathbb{R}^m$, and the map $\bar{h} : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is such that the following holds: there exist $D, X \in \mathbb{R}_{>0}$ and $\beta \in \mathcal{K}$ such that $|\bar{h}(x, d) - h(x)| \leq \beta(|d|)$ for all $d \in \mathcal{I}(|d|, D)$, for all $x \in \mathcal{I}(|x|, X)$. Then (1) is LISS, and $V$ is an LISS-Lyapunov function for (1).*

*(2) If $\epsilon_0 = +\infty$, and there exists $\bar{\mathbf{r}} \in \mathbb{R}^m$ with $\bar{r}_j \in \mathbb{R}_{>0}$, $j = 1, \dots, m$, such that the limit*

$$\bar{h}(y, d) = \lim_{\epsilon \to +\infty} \Lambda_\epsilon^{-\mathbf{r}} f(\Lambda_\epsilon^{\mathbf{r}} y, \Lambda_\epsilon^{\bar{\mathbf{r}}} d), \tag{14}$$

*exists, uniformly for all $y \in \mathcal{S}(\|y\|_{\mathbf{r}}, 1)$ and all $d \in \mathbb{R}^m$, and the map $\bar{h} : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is such that the following holds: (a) there exist $c, p \in \mathbb{R}_{>0}$ such that $|V(\bar{h}(y, d)) - V(h(y, 0))| \leq c|d|^p$ for all $d \in \mathbb{R}^m$, for all $y \in \mathcal{S}(\|y\|_{\mathbf{r}}, 1)$; (b) the relation $\bar{h}(\Lambda_\epsilon^{\mathbf{r}} x, \Lambda_\epsilon^{\bar{\mathbf{r}}} d) = \Lambda_\epsilon^{\mathbf{r}} \bar{h}(x, d)$ holds for all $x \in \mathbb{R}^n$, for all $d \in \mathbb{R}^m$, and for all $\epsilon \in \mathbb{R}_{>0}$. Then (1) is ISpS, and $V$ is an ISpS-Lyapunov function for (1).*

**PROOF.** The proof follows the same reasoning of the proofs of theorems 13 and 17. We only have to write $\Delta V$ as in the proof of Corollary 23. Thus, for $\epsilon_0 = 0$ (resp., $\epsilon_0 = +\infty$), the limit (13) (resp., (14)) ensures the existence of $\bar{\epsilon}, L_{\bar{\epsilon}} \in \mathbb{R}_{>0}$ such that, for all $\epsilon \leq \bar{\epsilon}$ (resp., for all $\epsilon \geq \bar{\epsilon}$),

$$|\bar{V}(x)| \leq \epsilon^m L_{\bar{\epsilon}} \left| \Lambda_\epsilon^{-\mathbf{r}} f(\Lambda_\epsilon^{\mathbf{r}} y, \Lambda_\epsilon^{\bar{\mathbf{r}}} \bar{d}) - \bar{h}(y, \bar{d}) \right|. \tag{15}$$

The rest of the proof is a combination of the proofs of theorems 13 and 17 by using the limits (13) and (14). Observe that, for $\epsilon_0 = +\infty$, Theorem 13 guarantees ISS, but Corollary 24 only guarantees ISpS. This is due to (15) is valid for all $x \in \mathbb{R}^n$ such that $\|x\|_{\mathbf{r}} \geq \bar{\epsilon}$. □

## 5 Application to the analysis of discretized continuous-time systems

In this section we consider the continuous-time system

$$\dot{x}(t) = g(x(t)), \quad x(t) \in \mathbb{R}^n, \tag{16}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$ is continuous. If (16) is wanted to be numerically solved, then a discretization method is required. Two of the simplest ones are the Euler methods: the explicit (or forward) and the implicit (or backward), see e.g. [11, Section II.7]. Below we recall these methods, and study some stability properties of the discrete-time system obtained by their application to (16).

### 5.1 Explicit Euler method

The explicit Euler discretization (EED) of (16), with a step $\tau \in \mathbb{R}_{>0}$, is given by (see, e.g. [11])

$$x((k+1)\tau) = G(x(k\tau)), \quad k \in \mathbb{Z}_{\geq 0}, \tag{17}$$

where the map $G : \mathbb{R}^n \to \mathbb{R}^n$ is given by $G(y) = y + \tau g(y)$. From Theorem 19 we can immediately deduce the following properties of (17).

**Corollary 25** *Consider* (16) *and its EED* (17)*. Suppose that $g$ is $D_{\mathbf{r}}-$homogeneous in the $(\epsilon_0, h)-$limit for some $\epsilon_0 \in \{0, +\infty\}$ with some degree $\nu \in \mathbb{R}_{>0}$, and $h$ satisfies Assumption 18. Let $V$ be as in Theorem 15.*

*(1) If $\epsilon_0 = +\infty$ and $\nu > 1$, then there exists $\gamma \in \mathbb{R}_{>0}$ such that the solution $F$ of* (17) *satisfies $|F(k; x(0))| \to \infty$ as $k \to \infty$ for all $x(0) \in \mathcal{E}(V, \gamma)$.*
*(2) If $\epsilon_0 = 0$ and $\nu \in (0, 1)$, then the origin of* (17) *is unstable.*

**Remark 26** *Observe that in Corollary 25, the properties of the EED of* (16) *are deduced directly from the vector field $g$ and not from the map $G$. This is because $G = I + \tau g$, where $I$ is the identity map, and for the computation of the $D_{\mathbf{r}}-$homogeneous approximation of $G$ we have that (see Definition 14) $\Lambda_\epsilon^{-\nu\mathbf{r}} I(\Lambda_\epsilon^{\mathbf{r}} x) = \Lambda_\epsilon^{(1-\nu)\mathbf{r}} x$, thus:*

- *for $\nu \in (0, 1)$, $\Lambda_\epsilon^{(1-\nu)\mathbf{r}} x \to 0$ as $\epsilon \to 0$, and;*
- *for $\nu > 1$, $\Lambda_\epsilon^{(1-\nu)\mathbf{r}} x \to 0$ as $\epsilon \to +\infty$.*

Note that the qualitative instability features of a EED concluded in Corollary 25 do not depend on the stability properties of (16) nor on the size of $\tau$, but only on the homogeneity degree of $h$ which approximates $g$.

### 5.2 Implicit Euler method

The implicit Euler discretization (IED) of (16), with a step $\tau \in \mathbb{R}_{>0}$, is given by $x(k\tau) = G\big(x((k+1)\tau)\big)$ where $k \in \mathbb{Z}_{\geq 0}$ and the map $G : \mathbb{R}^n \to \mathbb{R}^n$ is given by $G(y) = y - \tau g(y)$. If $G$ is invertible with inverse $G^{-1}$, then the explicit representation of the IED of (16) is given by the discrete-time system

$$x((k+1)\tau) = G^{-1}(x(k\tau)), \quad k \in \mathbb{Z}_{\geq 0}. \tag{18}$$

As it was done for the EED, we can use $D_{\mathbf{r}}-$homogeneous approximations to analyse (18). But let us first state the following auxiliary lemma.

**Lemma 27** *Suppose that the map* $g : \mathbb{R}^n \to \mathbb{R}^n$ *is* $D_{\mathbf{r}}-$*homogeneous of degree* $\nu \in \mathbb{R}_{>0}$ *in the* $(\epsilon_0, h)-$*limit, and* $h$ *is invertible.*

*(1) The map* $h^{-1}$ *is* $D_{\mathbf{r}}-$*homogeneous of degree* $\nu^{-1}$*.*
*(2) If* $\epsilon_0 = +\infty$*, and there exists* $a \in \mathbb{R}_{>0}$ *such that* $g$ *is invertible for all* $x \in \mathcal{E}(|x|, a)$*, then the map* $g^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ *is* $D_{\mathbf{r}}-$*homogeneous in the* $(+\infty, h^{-1})-$*limit.*
*(3) If* $\epsilon_0 = 0$*, and there exists* $a \in \mathbb{R}_{>0}$ *such that* $g$ *is invertible for all* $x \in \mathcal{I}(|x|, a)$*, then the map* $g^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ *is* $D_{\mathbf{r}}-$*homogeneous in the* $(0, h^{-1})-$*limit.*

**PROOF.** (1) The proof consists in verifying that $\Lambda_\epsilon^{\frac{1}{\nu}\mathbf{r}} h^{-1}(y) = h^{-1}(\Lambda_\epsilon^{\mathbf{r}} y)$. Since $h$ is $D_{\mathbf{r}}-$homogeneous of degree $\nu$, $h(\Lambda_\delta^{\mathbf{r}} x) = \Lambda_\delta^{\nu\mathbf{r}} h(x)$. Hence, by defining $y = h(x)$, we have that $\Lambda_\delta^{\mathbf{r}} x = h^{-1}(\Lambda_\delta^{\nu\mathbf{r}} y)$ and $\Lambda_\delta^{\mathbf{r}} x = \Lambda_\delta^{\mathbf{r}} h^{-1}(y)$. Thus, by defining $\epsilon = \delta^\nu$, we obtain $\Lambda_\epsilon^{\frac{1}{\nu}\mathbf{r}} h^{-1}(y) = h^{-1}(\Lambda_\epsilon^{\mathbf{r}} y)$.
(2) For this result we have to prove that

$$\lim_{\epsilon \to +\infty} \left( \Lambda_\epsilon^{-\nu\mathbf{r}} g^{-1}(\Lambda_\epsilon^{\mathbf{r}} x) - h^{-1}(x) \right) = 0 \,, \tag{19}$$

uniformly in some homogeneous sphere. Define the functions $g_n : \mathbb{R}^n \to \mathbb{R}^n$ given by $g_n(x) = \Lambda_n^{-\nu\mathbf{r}} g(\Lambda_n^{\mathbf{r}} x)$. Since $\Lambda_n^{\mathbf{r}}$ is invertible for any $n \in \mathbb{Z}_{>0}$, and there is $\bar{n} \in \mathbb{Z}_{>0}$ such that $g(\Lambda_n^{\mathbf{r}} x)$ also has inverse for all $n \geq \bar{n}$, then the functions $g_n$ are invertible for all $n \geq \bar{n}$. Indeed, by denoting $z = g_n(x)$, we can see from the definition of $g_n$ that $x = \Lambda_n^{-\mathbf{r}} g^{-1}(\Lambda_n^{\nu\mathbf{r}} z)$, therefore, $g_n^{-1}(z) = \Lambda_n^{-\mathbf{r}} g^{-1}(\Lambda_n^{\nu\mathbf{r}} z)$. Thus, by denoting $\epsilon = n^\nu$ we have that $g_n^{-1}(z) = \Lambda_\epsilon^{-\mathbf{r}/\nu} g^{-1}(\Lambda_\epsilon^{\mathbf{r}} z)$. Hence, it is clear that, to verify (19) it is sufficient to prove that $g_n^{-1} \to h^{-1}$ uniformly as $n \to \infty$. By hypothesis we know that $g_n \to h$ uniformly as $n \to \infty$. Now, under composition with a uniformly continuous function, a convergent sequence preserves the uniform convergence. Note that since $h$ is a continuous map, $h$ is uniformly continuous in any compact set. Thus, instead of proving that $g_n^{-1} \to h^{-1}$ we will prove that $h \circ g_n^{-1} \to h \circ h^{-1}$. Note that $|h(g_n^{-1}(x)) - h(h^{-1}(x))| = |h(g_n^{-1}(x)) - x| = |h(g_n^{-1}(x)) - g_n(g_n^{-1}(x))|$. Since $g_n \to h$, we have that $\lim_{n\to\infty} |h(g_n^{-1}(x)) - g_n(g_n^{-1}(x))| = 0$. Hence, $g_n^{-1} \to h^{-1}$ as $n \to \infty$.
(3) For this case the proof is analogous to the previous one but by defining the functions $g_n : \mathbb{R}^n \to \mathbb{R}^n$ given by $g_n(x) = \Lambda_{1/n}^{-\nu\mathbf{r}} g(\Lambda_{1/n}^{\mathbf{r}} x)$. □

From Theorem 15 and Lemma 27 we have the following.

**Corollary 28** *Consider* (16) *and its IED* (18)*. Suppose that* $g$ *is* $D_{\mathbf{r}}-$*homogeneous in the* $(\epsilon_0, h)-$*limit for some* $\epsilon_0 \in \{0, +\infty\}$ *with some degree* $\nu \in \mathbb{R}_{>0}$*, and* $h$ *is invertible. Let* $V$ *be as in Theorem 15.*

*(1) If $\epsilon_0 = 0$, $\nu \in (0,1)$, and there exists $\gamma_0 \in \mathbb{R}_{>0}$ such that, for all $x \in \mathcal{I}(V,\gamma_0)\backslash 0$, $G$ is invertible, then the origin of* (18) *is locally asymptotically stable. Moreover, there exists $\gamma \le \gamma_0$ such that $V$ is a Lyapunov function for* (18) *in $\mathcal{I}(V,\gamma)$.*
*(2) If $\epsilon_0 = +\infty$, $\nu > 1$, and there exists $\gamma_0 \in \mathbb{R}_{>0}$ such that for all $x \in \mathcal{E}(V,\gamma_0)$, $G$ is invertible, then the solutions of* (18) *are globally ultimately bounded, and there exists $\gamma \ge \gamma_0$ such that $\Delta V(x) < 0$ for all $x \in \mathcal{E}(V,\gamma)$.*

Notice that, as in Corollary 25, Corollary 28 does not require the verification of a $D_{\mathbf{r}}-$homogeneous approximation for the map $G$, but for the vector field $g$. This situation is clarified in Remark 26.

The results of this section are in accordance to [8], where a thorough stability analysis of IED and EED for continuous-time $\mathbf{r}-$homogeneous systems is presented. Nonetheless, the advantage of $D_{\mathbf{r}}-$homogeneous approximation lies in the facility to verify stability properties by considering only the homogeneity degree and not needing information about the Lyapunov function of the continuous-time system.

**Remark 29** *Note that Corollary 25 and Corollary 28 are useful to detect inconsistencies in the EED or IED of a continuous-time system whose vector field has a $D_{\mathbf{r}}-$homogeneous approximation. This is illustrated in Example 30, where two cases of an $\mathbf{r}-$homogeneous (see Definition 1) continuous-time system are considered (the first one for negative $\mathbf{r}-$homogeneity degree and the second one for positive $\mathbf{r}-$homogeneity degree). It is shown that, although the origin of the continuous-time system is globally asymptotically stable, the origin of its EED cannot be globally asymptotically stable in any case. Indeed, the origin of the EED is unstable for the case of negative degree, and there exist diverging trajectories of the EED for the case of positive degree.*

To finalize this section we exemplify the results obtained in corollaries 25 and 28. We also show in the following example the use of Corollary 23 for robustness analysis. Some additional examples can be found in [29] and [27].

**Example 30** *Consider the following controlled system*

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = u(x)\,. \tag{20}$$

*We analyse two different cases for the feedback controller $u$, namely $u \in \{u_1, u_2\}$ with (we use $\lceil x \rfloor^p := |x|^p \mathrm{sign}(x)$)*

$$u_1 = -a_1\lceil x_1 \rfloor^{\frac{1}{3}} - a_2\lceil x_2 \rfloor^{\frac{1}{2}},\ u_2 = -b_1\lceil x_1 \rfloor^{3} - b_2\lceil x_2 \rfloor^{\frac{3}{2}}\,.$$

Case: $u_1$. *Observe that the closed-loop of* (20) *with $u_1$ is $\bar{\mathbf{r}}-$homogeneous of degree $\kappa = -1$ with $\bar{\mathbf{r}} = [3,2]^\top$. Moreover, its origin is globally finite-time stable [7,24], for all $a_1, a_2 \in \mathbb{R}_{>0}$ [4]. Note that the map $g$ given by*

Fig. 1. States of the EED of (20) with $u_1$.

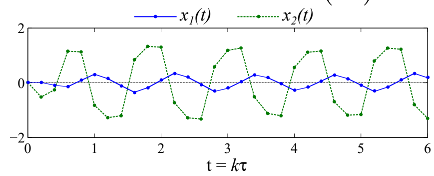



Fig. 2. States of the EED of (20) with $u_2$.

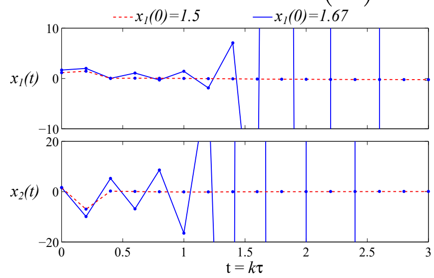
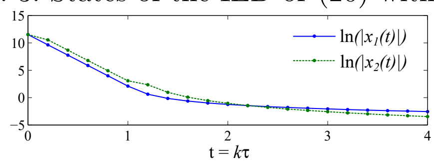


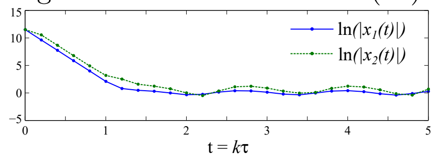

Fig. 3. States of the IED of (20) with $u_2$.




*$g(x) = [x_2,\, u_1(x)]^\top$ is $D_{\mathbf{r}}-$homogeneous in the $(0, h)-$limit with $\nu = 1/2$, $\mathbf{r} = [2, 1]^\top$, and the map $h$ is given by $h(x) = [x_2,\, -a_2\lceil x_2\rfloor^{1/2}]^\top$. Thus, according to Corollary 25 point (2), the origin of the EED of* (20) *with $u_1$ is unstable for any $\tau \in \mathbb{R}_{>0}$. For the simulation we use the parameters: $a_1 = 10$, $a_2 = 5$, and $\tau = 0.2$. Fig. 1 shows the instability of the origin of the EED of* (20) *with $u_1$ and the initial conditions $x_1(0) = 0.01$, $x_2(0) = 0.01$.*

Case: $u_2$. *Now,* (20) *in closed-loop with $u_2$ is $\bar{\mathbf{r}}-$homogeneous of degree $\kappa = 1$ with $\bar{\mathbf{r}} = [1, 2]^\top$. Note that the map $g$ given by $g(x) = [x_2,\, u_2(x)]^\top$ is $D_{\mathbf{r}}-$homogeneous in the $(+\infty, h)-$limit with $\nu = \sqrt{3}$, $\mathbf{r} = [1, \sqrt{3}]^\top$, and $h$ given by $h(x) = [x_2,\, -b_1\lceil x_1\rfloor^3]^\top$. In this example, we consider the parameters $b_1 = 10$ and $b_2 = 5$. For such a case, the origin of* (20) *in closed-loop with $u_2$ is globally asymptotically stable [8]. According to Corollary 25 point (1), for any $\tau \in \mathbb{R}_{>0}$ there is a neighbourhood of the origin such that the solutions of the EED of* (20) *with $u_2$ are unbounded for all initial conditions outside such a neighbourhood. This situation is shown in Fig. 2, where the integration step is $\tau = 0.2$. For the initial conditions $x_1(0) = 1.5$, $x_2(0) = 1.5$, the states converge to the origin, but by increasing the initial condition for $x_1$ to $x_1(0) = 1.67$ the states of the system become unbounded. On the other hand, Corollary 28 point (2) guarantees that the solutions of the IED of* (20) *with $u_2$ are globally ultimately bounded for any $\tau \in \mathbb{R}_{>0}$. Fig. 3 shows the states of the IED of* (20) *with $u_2$, $\tau = 0.2$, and the initial conditions $x_1(0) = 1 \times 10^5$, $x_2(0) = 1 \times 10^5$.*

Fig. 4. States of the IED of (21).




*Now, let us consider the disturbed case, i.e.*

$$\dot{x}_1 = x_2 + d_1 \,, \quad \dot{x}_2 = u_2(x) + d_2 \,, \tag{21}$$

*where the external disturbances $d_i(t) \in \mathbb{R}$ are bounded. The aim of this example is to show how the developed results can be used to verify if the IED preserves some robustness properties of (21). The IED of (21) with step $\tau \in \mathbb{R}_{>0}$ is $x(k\tau) = G\Big(x((k+1)\tau)\Big) - \tau d(k+1)$, where $G(x) = x - \tau g(x)$ with $g(x) = [x_2 \; u_2(x)]^\top$. Since $G$ is invertible outside of the origin we have that $x((k+1)\tau) = G^{-1}\Big(x(k\tau) + \bar{d}\Big)$ where $\bar{d} = \tau d(k+1)$, but we have already seen that $g$ is $D_{\mathbf{r}}-$homogeneous in the $(+\infty, h)-$limit with $\nu = \sqrt{3}$, $\mathbf{r} = [1, \sqrt{3}]^\top$, and $h(x) = [x_2, \, -b_1\lceil x_1 \rfloor^3]^\top$. Hence, from Lemma 27 we have that $G^{-1}$ is $D_{\mathbf{r}}-$homogeneous in the $(+\infty, h^{-1})-$limit with degree $1/\sqrt{3}$. Note that the homogeneity condition on the disturbance $\bar{d}$, required in Theorem 11, is fulfilled with $\bar{\mathbf{r}} = \mathbf{r}$, i.e. $f(\Lambda_\epsilon^{\mathbf{r}} x, \Lambda_\epsilon^{\bar{\mathbf{r}}} \bar{d})$ is (in this example) given by $h^{-1}(\Lambda_\epsilon^{\mathbf{r}} x + \Lambda_\epsilon^{\bar{\mathbf{r}}} \bar{d}) = \Lambda_\epsilon^{\nu \mathbf{r}} h^{-1}(x + \bar{d})$. Thus, from Corollary 23, the IED of (21) is ISpS. For the simulation we use the disturbances $d_1 = 2(1 + \cos(5t))$ and $d_2 = 1.5(1 + \sin(7t))$, the step $\tau = 0.2$, and the initial conditions $x_1(0) = 1 \times 10^5$, $x_2(0) = 1 \times 10^5$. The states of (21) are shown in Fig. 4, there we can appreciate the predicted robustness property.*

## 6 Conclusion

In this paper we have verified the robustness properties of $D_{\mathbf{r}}-$homogeneous systems. We have also provided a methodology to study stability and robustness properties of nonlinear discrete-time systems by means of $D_{\mathbf{r}}-$homogeneous approximations. The qualitative stability and robustness features of a system can be decided in a simple way. However, to obtain quantitative estimates (e.g. size of attraction domains) a more detailed analysis is required. Nevertheless, $D_{\mathbf{r}}-$homogeneity guarantees the existence of Lyapunov functions that can be used for such a purpose. We have also shown that the presented methodology can be used to provide criteria to choose suitable discretization techniques for continuous-time systems.

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## A Appendix

We recall some properties of $\mathbf{r}-$homogeneous functions.

**Definition 31 ([17])** *Given a vector of weights* $\mathbf{r}$*, a* $\mathbf{r}-$*homogeneous norm is defined as a function from* $\mathbb{R}^n$ *to* $\mathbb{R}$*, and given by* $\|x\|_{\mathbf{r},p} = \left(\sum_{i=1}^{n} |x_i|^{p/r_i}\right)^{1/p}$*, for all* $x \in \mathbb{R}^n$*, for any* $p \geq \max_i\{r_i\}$.

Note that any $\mathbf{r}-$homogeneous norm is an $\mathbf{r}-$homogeneous function of degree $m = 1$. Since, for a given $\mathbf{r}$, the $\mathbf{r}-$homogeneous norms are equivalent [17], they are usually denoted as $\|\cdot\|_{\mathbf{r}}$, without the specification of $p$. The following lemma is a direct consequence of lemmas 1 and 6 from [29], and Lemma 4.2 from [6].

**Lemma 32 ([29])** *If the function* $V : \mathbb{R}^n \to \mathbb{R}$ *is continuous, positive definite, and* $\mathbf{r}-$*homogeneous of degree* $m \in \mathbb{R}_{>0}$*, and the map* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is well defined and locally bounded for all* $x \in \mathbb{R}^n$*, and it is* $D_{\mathbf{r}}-$*homogeneous of degree* $\nu \in \mathbb{R}_{>0}$*, then: 1) there exists* $\overline{\gamma} \in \mathbb{R}_{>0}$ *(resp.* $\overline{\delta} \in \mathbb{R}_{>0}$*) such that* $(V \circ f)(x) \leq \overline{\gamma} V^{\nu}(x)$ *(resp.* $(V \circ f)(x) \leq \overline{\delta}\|x\|_{\mathbf{r}}^{m\nu}$*), for all* $x \in \mathbb{R}^n$*; 2) if, additionally,* $f$ *satisfies Assumption 18, then there exists* $\underline{\gamma} \in \mathbb{R}_{>0}$ *(resp.* $\underline{\delta} \in \mathbb{R}_{>0}$*) such that* $(V \circ f)(x) \geq \underline{\gamma} V^{\nu}(x)$ *(resp.* $(V \circ f)(x) \geq \underline{\delta}\|x\|_{\mathbf{r}}^{m\nu}$*), for all* $x \in \mathbb{R}^n$.

To recall the concept of robust global asymptotic stability, consider (6) and its associated difference inclusion

$$x(k+1) \in \Phi(x(k)) \,, \quad x(k) \in \mathbb{R}^n \,, \tag{A.1}$$

where the set-valued map $\Phi(x) \subset \mathbb{R}^n$ is given by

$$\Phi(x) = \bigcap_{\rho \in \mathbb{R}_{>0}} \mathrm{cl}\{ \bigcup_{y \in B(x,\rho)} f_0(y)\} \,,$$

where $B(x,\rho)$ is an open ball in $\mathbb{R}^n$ centred at $x$ with radius $\rho$, and $\mathrm{cl}\{A\}$ denotes the closure of the set $A$ [20].

**Definition 33 ([20],[18])** *(a) The origin of* (A.1) *is* strongly globally asymptotically stable *(strongly GAS) if there exists a class-$KL$ function $\beta$ such that, for every $x(0) \in \mathbb{R}^n$, all the solutions $F$ with initial condition $x(0)$ satisfy $|F(k;x(0))| \leq \beta(|x(0)|,k)$ for all $k \in \mathbb{Z}_{\geq 0}$. (b) The origin of* (A.1) *is* robustly strongly GAS *if there exists a continuous and positive definite function $\delta : \mathbb{R}^n \to \mathbb{R}$ such that the origin of $x(k+1) \in \Phi_\delta(x(k))$ is strongly GAS with $\Phi_\delta(x) = \{y \in \mathbb{R}^n : y \in \Phi(w) + \mathrm{cl}\{B(0,\delta(x))\}, w \in \mathrm{cl}\{B(x,\delta(x))\}\}$. (c) The origin of* (6) *is* robustly GAS *if the origin of* (A.1) *is robustly strongly GAS.*

**Theorem 34 ([20])** *The origin of* (6) *is robustly GAS if and only if* (6) *admits a smooth Lyapunov function.*