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THE MINIMAL FRIED AVERAGE ENTROPY FOR HIGHER-RANK CARTAN ACTIONS

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Abstract. We find the minimal value of the Fried average entropy by proving new lower bounds for regulators of totally real number fields.

1. Introduction

In 2014 Katok, Katok and Rodriguez Hertz [KKR, p. 1216] published the following

Conjecture. *The Cartan action α corresponding to the quartic totally real number field of discriminant 725 and the defining polynomial $x^4 - x^3 - 3x^2 + x + 1$ minimizes the Fried average entropy $h^*(\alpha)$ among all Cartan actions α. For that action $h^*(\alpha) = 0.330027\ldots = h_{\min}$.*

The KKR conjecture applies to actions of $\mathbb{Z}^{n-1}$ by hyperbolic automorphisms on n-tori for $n \geq 3$ (Cartan actions). We refer the reader to their paper for motivation and for the definitions involved in the above conjecture, only noting that they were able to reduce the proof of their conjecture to a purely number-theoretic problem. Namely, they showed [KKR, Prop. 3.2 and (3.8)] that associated to the action α on an n-torus there is a totally real field K of degree $n := [K : \mathbb{Q}]$ for which the Fried average entropy $h^*(\alpha)$ satisfies $h^*(\alpha) = mR_K 2^{n-1}\big((n-1)!\big)^2/(2n-2)!$, where R_K is the regulator of K and $m \geq 1$ is an integer.

We prove the KKR conjecture and give the first six minima of $h^*(\alpha)$.

Theorem 1. *The above conjecture holds. Moreover, except for the six Cartan actions corresponding to the fields in Table 1, all other Cartan actions α satisfy $h^*(\alpha) > 0.49$.*

Table 1. The first six minima of the Fried average entropy (to six decimals).

$h^*(\alpha)$	n	Polynomial	Discriminant	Regulator
0.330027	4	$x^4 - x^3 - 3x^2 + x + 1$	725	0.825068
0.350303	3	$x^3 - x^2 - 2x + 1$	49	0.525454
0.373872	5	$x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1$	14641	1.635694
0.416198	6	$x^6 - x^5 - 7x^4 + 2x^3 + 7x^2 - 2x - 1$	300125	3.277562
0.466182	4	$x^4 - x^3 - 4x^2 + 4x + 1$	1125	1.165455
0.479301	6	$x^6 - x^5 - 5x^4 + 4x^3 + 6x^2 - 3x - 1$	371293	3.774500

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Katok, Katok and Rodriguez Hertz proved the lower bound $h_{\min} \geq 0.089$, and showed that their conjecture held for Cartan actions, except possibly if $8 \leq n \leq 16$. This range was later narrowed to $10 \leq n \leq 16$ [ADF, p. 234]. We find the first six minima in Theorem 1 by proving new lower bounds for regulators of totally real fields in degrees 10 to 18.

Theorem 2. *Let K be a totally real field with degree in the range $10 \leq [K:\mathbb{Q}] \leq 18$. Then its regulator R_K satisfies the lower bounds given in Table 2.*

TABLE 2. New and old lower bounds for regulators of totally real fields.

$[K:\mathbb{Q}]$	$R_K \geq$	Old lower bound	γ
10	47.2298	10.099	0.577
11	111.142237	21.807	0.54
12	266.819052	48.333	0.509
13	611.881830	109.56	0.482
14	1291.090237	253.26	0.46
15	2686.034353	595.65	0.439
16	5600.694261	1422.64	0.4214
17	11769.783217	3445.02	0.4054
18	24936.817837	8447.18	0.391

The third column gives the best previously known lower bound, due to Zimmert [Zi, Satz 3]. The corresponding value of his parameter γ appears in the fourth column. We prove Theorem 2 by applying a variant of Zimmert's techniques [Zi] [Fr, §4].

2. PROOFS

We first show that Theorem 1 follows from Theorem 2 and from results in [ADF] and [KKR]. Thus, we assume $h^*(\alpha) \leq 0.49$ for the Cartan action α and we assume the regulator lower bounds in Table 2. We must show that α is one of the actions associated to the six fields in Table 1.

The Fried average entropy $h^*(\alpha)$ satisfies [KKR, Prop. 3.2 and (3.8)]

$$h^*(\alpha) \geq R_K 2^{n-1}\big((n-1)!\big)^2/(2n-2)!\,. \tag{1}$$

As $h^*(\alpha) \leq 0.49$, we have $R_K \leq 0.49(2n-2)!/(2^{n-1}(n-1)!^2)$. This bound is shown in Table 3 for $3 \leq n \leq 18$.

TABLE 3. **Upper bounds for R_K implied by $h^*(\alpha) \leq 0.49$**

n	3	4	5	6	7	8	9	10
$R_K \leq$	0.735	1.225	2.144	3.86	7.075	13.139	24.634	46.531

n	11	12	13	14	15	16	17	18
$R_K \leq$	88.42	168.79	323.5	622.11	1199.8	2319.6	4494.2	8724

Table 3 and Table 2 are in contradiction for $10 \leq n \leq 18$. Furthermore, Table 3 contradicts the lower bounds for R_K given in [ADF, p. 234] for $n = 7$, 8 and 9. Thus either $n \leq 6$ or $n \geq 19$.

We first deal with $n \geq 19$. Using the inequality $h^*(\alpha) \geq 0.000752 \exp(0.243n)$ [KKR, (3.15)], we find that $h^*(\alpha) \leq 0.49$ implies $n \leq 26$. From Zimmert's lower bound $R_K \geq 0.000376 \exp(0.9371n)$ [KKR, (3.13)] and (1) for $19 \leq n \leq 26$, we find that $n \geq 19$ is impossible. Thus, $3 \leq n \leq 6$.

Fortunately, all fields in degrees 3 to 6 with regulators in the range of Table 3 are listed in [ADF, Theorems 7, 10, 8, 11]. These are exactly the fields in Table 1. For example, according to [ADF, Theorem 11], all but three totally real fields of degree 6 satisfy $R_K > 4.39$. The exceptions have regulators $3.277\ldots$, $3.774\ldots$ and $4.187\ldots$. The first two appear in Table 1, as they are associated to actions with $h^*(\alpha) \leq 0.49$. This concludes the proof that Theorem 1 follows from Theorem 2.

We now turn to the proof of Theorem 2. A version of Zimmert's regulator lower bounds goes as follows [Fr, p. 619].

Lemma 3. *Let K be a totally real field of degree n, let $0 < \beta < \gamma < \kappa$ and define*

$$T(s) := \left(\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1+s+2\gamma}{2}\right)} \right)^n, \qquad R(s) := \frac{s}{(s+\beta)(s+\kappa)^2}. \tag{2}$$

Assume that for some $y, \delta \in \mathbb{R}$ with $y > 0$, $0 < \delta < \beta$, and some $M \in \mathbb{N}$ we have

$$\frac{1}{2\pi i} \int_{-\delta - i\infty}^{-\delta + i\infty} (my)^{s-1} T(s) R(s)\, ds \leq 0 \qquad (m = 1, 2, \ldots, M-1), \tag{3}$$

and

$$\frac{\beta T(-\beta)}{(\kappa - \beta)^2} \geq \frac{(My)^{\beta - \gamma}}{2\pi} \int_{-\infty}^{\infty} |R(-\gamma + it)|\, dt. \tag{4}$$

Then

$$R_K \geq \frac{-\big(\Gamma(1+\gamma)\big)^n}{R(1) 2^n \pi i} \int_{-\delta - i\infty}^{-\delta + i\infty} y^{s-1} T(s) R(s)\, ds. \tag{5}$$

We note that the integral in (3) is independent of δ, as long as $0 < \delta < \beta$. To estimate it we shall need the following result.

Lemma 4. *Let $b > a > 0$, and let $f(t) := |\Gamma(a+it)/\Gamma(b+it)|$. Then f assumes its maximum value for $t \in \mathbb{R}$ at $t = 0$.*

Proof. To find the sign of $f'(t)$, we use the classical series [AAR, p. 13]

$$\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \sum_{n=0}^{\infty} \Big(\frac{1}{z+n} - \frac{1}{n+1} \Big)$$

to calculate

$$\begin{aligned} \psi(b+it) - \psi(a+it) &= \sum_{n=0}^{\infty} \Big(\frac{1}{n+a+it} - \frac{1}{n+b+it} \Big) = \sum_{n=0}^{\infty} \frac{(b-a)}{(n+a+it)(n+b+it)} \\ &= (b-a) \sum_{n=0}^{\infty} \frac{(n+a-it)(n+b-it)}{\big((n+a)^2+t^2\big)\big((n+b)^2+t^2\big)} \\ &= (b-a) \sum_{n=0}^{\infty} \frac{(n+a)(n+b) - t^2 - it(2n+a+b)}{\big((n+a)^2+t^2\big)\big((n+b)^2+t^2\big)}. \end{aligned} \tag{6}$$

Since

$$\big(f(t)\big)^2 = \Gamma(a+it)\Gamma(a-it)/\big(\Gamma(b+it)\Gamma(b-it)\big),$$

we have $f'(t)/f(t) = \mathrm{Im}\big(\psi(b+it) - \psi(a+it)\big)$. From (6) we see that $f'(t) < 0$ for $t > 0$ and $f'(t) > 0$ for $t < 0$. Thus f assumes its maximum at $t = 0$. $\square$

Computing numerically around 10^7 integrals of the type appearing in Lemma 3, as we will need to do below, seems difficult as the integrand oscillates like $\mathrm{e}^{itn\log(t)+it\log(y)}$ when $t := |\mathrm{Im}(s)| \gg 0$. Instead, we will approximate the integral by a sum of residues.

Lemma 5. *Assume $0 < \delta < \beta < \gamma < \kappa$, and let T and R be as in Lemma 3 for some $n \in \mathbb{N}$. Then, for any $q = 2k \in 2\mathbb{N}$ and any $x > 0$ we have*

$$\left|\frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty} x^{s-1}T(s)R(s)\,ds + \sum_{r=1}^{k}\mathrm{Res}_{s=2r-1}\big(x^{s-1}T(s)R(s)\big)\right|$$
$$\leq \frac{x^{q-1}}{2(q+\kappa)}\left(\frac{\Gamma(\frac{1+q}{2})2^q}{\Gamma(\frac{1+q+2\gamma}{2})\prod_{j=0}^{q-1}|q-1-2j|}\right)^n. \quad (7)$$

Proof. The Stirling estimate $|\Gamma(\sigma+iT)| = \sqrt{2\pi}|T|^{\sigma-1/2}\mathrm{e}^{-\pi|T|/2}(1+\mathrm{O}(1/|T|))$, uniform in a vertical strip as $|T| \to \infty$ [AAR, Cor. 1.4.4], allows us to replace the line $\mathrm{Re}(s) = -\delta$ in (7) by $\mathrm{Re}(s) = q$, subtracting the residues. Using $\Gamma(z) = \Gamma(z+1)/z$ successively, we find for $t \in \mathbb{R}$,

$$|T(q+it)| = \left|\frac{\Gamma(\frac{1+q-it}{2})}{\Gamma(\frac{1+q+2\gamma+it}{2})\prod_{j=0}^{q-1}(\frac{1-q}{2}+j-i\frac{t}{2})}\right|^n \leq \left|\frac{\Gamma(\frac{1+q}{2})}{\Gamma(\frac{1+q+2\gamma}{2})\prod_{j=0}^{q-1}(\frac{1-q}{2}+j)}\right|^n,$$

where we used Lemma 4 to obtain the inequality. Lastly,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|R(q+it)|\,dt \leq \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(q+\kappa)^2+t^2}\,dt = \frac{1}{2(q+\kappa)}. \quad (8)$$

$\square$

We are now ready to prove Theorem 2. The idea is to fix for each degree in the range $10 \leq n \leq 18$ a choice of the parameters y, β, γ and κ giving a good regulator lower bound in inequality (5) of Lemma 3. These parameters are given in Table 4.

To fulfill the hypotheses of Lemma 3, we first need to find a value of M satisfying (4). The only difficulty here is finding a good upper bound for $\int_{-\infty}^{\infty}|R(-\gamma+it)|\,dt$. Indeed, the rest of the terms in (4) only involve elementary functions and the Γ-function. These we evaluate using the interval arithmetic software package Arb, which can compute many classical functions with rigorous error bounds [Jo1]. We note that a coarse bound for $\int_{-\infty}^{\infty}|R(-\gamma+it)|\,dt$, as in (8), leads to useless values of $M > 10^{14}$.

Fortunately, the function $t \to |R(-\gamma+it)|$ is even for $t \in \mathbb{R}$ and decreasing for $t \in [0,\infty)$. Since this also holds for $t \to |-\gamma+it|/|-\gamma+\beta+it|$, we can use the left endpoint value of the integrand over N successive intervals of length h to estimate

$$\begin{aligned}\int_0^\infty |R(-\gamma+it)|\,dt &\le h\sum_{j=1}^{N}|R(-\gamma+i(j-1)h)| + \int_{Nh}^\infty \frac{\left|\frac{-\gamma+it}{-\gamma+\beta+it}\right|}{|\kappa-\gamma+it|^2}\,dt\\ &\le h\sum_{j=1}^{N}|R(-\gamma+i(j-1)h)| + \frac{|-\gamma+iNh|}{|-\gamma+\beta+iNh|}\int_{Nh}^\infty \frac{1}{(\kappa-\gamma)^2+t^2}\,dt\\ &= h\sum_{j=1}^{N}|R(-\gamma+i(j-1)h)| + \frac{|-\gamma+iNh|}{|-\gamma+\beta+iNh|}\,\frac{1}{\kappa-\gamma}\big(\frac{\pi}{2}-\arctan(\tfrac{Nh}{\kappa-\gamma})\big). \quad (9)\end{aligned}$$

Table 4 below gives the values we chose for each field degree $[K:\mathbb{Q}]$ for the parameters free y,γ,β and κ in Lemmas 3 and 5, as well as the value of M obtained using $h=10^{-3},\ N=10^4$ in (9).[1]

TABLE 4. Lower bounds for R_K and parameters used.

$[K:\mathbb{Q}]$	$R_K \ge$	γ	y	β	κ	M
10	47.2298	0.46	0.00021	0.37	3.8	9 996 901
11	111.142237	0.42	0.00005	0.33	3.46	9 090 491
12	266.819052	0.375	0.00001	0.285	3.1	9 331 881
13	611.881830	0.545	0.00001	0.465	3.5	9 389 187
14	1291.090237	0.72	0.00001	0.64	4.1	9 813 891
15	2686.034353	0.885	0.00001	0.805	4.4	9 432 795
16	5600.694261	1.055	0.00001	0.965	5.3	10 050 899
17	11769.783217	1.205	0.00001	1.115	5.2	9 647 888
18	24936.817837	1.355	0.00001	1.255	5.7	10 039 075

To complete the proof of Theorem 2, we must still show how the regulator lower bound in Table 4 can be rigorously computed. The idea is to use Lemma 5 to calculate the right-hand side of (5) in Lemma 3, with $n=[K:\mathbb{Q}],\gamma,y,\beta$ and κ as in Table 4. We must also verify the positivity hypothesis (3) of Lemma 3.

To implement Lemma 5 we must calculate (using interval arithmetic) each of the residues in (7) at $x:=my$ for $1\le m\le M-1$. Since it turns out that we need to calculate at most $k=7$ residues to achieve the required precision, it suffices to describe how to compute each residue.

An initial difficulty is that Arb at present can only deal with series expansions of the Γ-function at regular points. This is easily solved by noting that the residue of some function f at a pole s_0 of order n coincides with the Taylor coefficient of order $n-1$ of the regular function $(s-s_0)^n f(s)$. Using an affine change of variables, it suffices to compute the Taylor expansion at a non-positive integer $-\ell$ of $\big((s+\ell)\Gamma(s)\big)^n$ in order to obtain the Taylor expansion at $s=2r-1$ of $\big((s-(2r-1))\Gamma(\frac{1-s}{2})\big)^n$. Using $\Gamma(s)=\Gamma(s+1)/s$ repeatedly (*i. e.* $\ell+1$ times), this is readily computed from the first n terms of the Taylor expansion of $\big(\Gamma(s)\big)^n$ around $s=1$, which is

[1] The parameters y,γ,β and κ were found by trial and error, allowing only a moderate $M\approx 10^7$.

implemented in Arb [Jo2]. All the other functions needed for the residue calculation in (7) are regular at $s = 2r - 1$, so the computation of (the first n terms of) their Taylor series is available in Arb. Since the error bound, *i. e.* the right-hand side of (7), is easily calculated with Arb, we were able to verify the inequalities (3) in Lemma 3 and could calculate the lower bound (5) shown in Table 4.

In carrying out the verification of (3), one improves running time by a factor of about 20 by computing and storing only once (for each r and n) the first n coefficients of the Taylor expansion around $s = 2r - 1$ of $(s - (2r - 1))^n T(s)R(s)$. Indeed, as a function of x the sum of residues in (7) is a polynomial in x and $\log x$, so it can be rapidly calculated. For the whole of Table 4, the sign verification in (3) took 150 minutes on a 1.9 GHz i5-4300U CPU. Our program has been deposited at https://github.com/fredrik-johansson/fried-entropy.

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