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1 **DECOMPOSITION OF HIGH DIMENSIONAL AGGREGATIVE**
2 **STOCHASTIC CONTROL PROBLEMS**

3 *

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6 **Abstract.** We consider the framework of high dimensional stochastic control problem, in which
7 the controls are aggregated in the cost function. As first contribution we introduce a modified
8 problem, whose optimal control is under some reasonable assumptions an ε -optimal solution of the
9 original problem. As second contribution, we present a decentralized algorithm whose convergence
10 to the solution of the modified problem is established. Finally, we study the application to a problem
11 of coordination of energy production and consumption of domestic appliances.

12 **Key words.** Stochastic optimization, Lagrangian decomposition, Uzawa’s algorithm, stochastic
13 gradient.

14 **AMS subject classifications.** 93E20,65K10, 90C25, 90C39, 90C15.

15 **1. Introduction.** The present article aims at solving a high dimensional
16 stochastic control problem (P_1) involving a large number n of agents indexed by
17 $i \in \{1, \dots, n\}$, of the form:

18 (1.1) (P_1)
$$\begin{cases} \text{Min}_{u \in \mathcal{U}} J(u) \\ J(u) := \mathbb{E} \left(F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i(\omega^i, \omega^{-i}) \right) + \frac{1}{n} \sum_{i=1}^n G_i(u^i(\cdot, \omega^{-i}), \omega^i) \right). \end{cases}$$

19 Here the noise $\omega := (\omega^1, \dots, \omega^n)$ belongs to $\Omega := \prod_{i=1}^n \Omega^i$, $(\Omega^i, \mathcal{F}^i, \mu^i)$ is a proba-
20 bility space, and $(\Omega, \mathcal{F}, \mu)$ is the corresponding product probability space. Let $\omega^{-i} :=$
21 $(\omega^1, \dots, \omega^{i-1}, \omega^{i+1}, \dots, \omega^n)$ denotes an element of the space $\Omega^{-i} := \prod_{j=1, j \neq i}^n \Omega^j$. The
22 associated product probability space is $(\Omega^{-i}, \mathcal{F}^{-i}, \mu^{-i})$, where $\mathcal{F}^{-i} := \otimes_{j=1, j \neq i}^n \mathcal{F}^j$
23 and $\mu^{-i} := \prod_{j=1, j \neq i}^n \mu^j$. Each decision variable u^i is a random variable (i.e. is
24 \mathcal{F} -measurable), square summable with value in a Hilbert space \mathbb{U} so that $u :=$
25 (u^1, \dots, u^n) belongs to $L^2(\Omega, (\mathbb{U})^n)$. The function $\omega^i \mapsto u^i(\omega^i, \omega^{-i})$ is denoted by
26 $u^i(\cdot, \omega^{-i})$ and is a.s. (in ω^{-i}) \mathcal{F}^i -measurable and belongs to $L^2(\Omega^i, \mathbb{U})$. Also, $\mathcal{U} :=$
27 $\prod_{i=1}^n \mathcal{U}_i$ where \mathcal{U}_i is, for $i = 1$ to n , a closed convex subset of $L^2(\Omega^i, \mathbb{U})$. (In the ap-
28 plication to dynamical problems, the constraint $u^i \in \mathcal{U}_i$ includes the constraint of

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29 adaptation of u^i to some filtration.) If each u^i is a random variable of ω^i , for $i = 1$
 30 to n , we say that u is a decentralized decision variable.

31 The cost function is the sum of a coupling term $F_0 : \mathbb{U} \rightarrow \mathbb{R}$, function of the
 32 *aggregate strategies* $\frac{1}{n} \sum_{i=1}^n u^i$, and "local terms" functions of the local decision u^i and
 33 local noise ω^i with $G_i : L^2(\Omega^i, \mathbb{U}) \times \Omega_i \rightarrow \mathbb{R}$. This framework aims at containing
 34 stochastic optimal control problems, where the states of the agents are driven by
 35 independent noises (see equations (5.5) and (5.2) developed in Section 5).

36 **1.1. Motivations.** This work is motivated by its potential applications for large-
 37 scale coordination of flexible appliances, to support power system operation in a con-
 38 text of increasing penetration of renewables. One type of appliances that has been
 39 consistently investigated in the last few years, for its intrinsic flexibility and potential
 40 for network support, includes thermostatically controlled loads (TCLs) such as re-
 41 frigerators or air conditioners. Several papers have already investigated the potential
 42 of dynamic demand control and frequency response services of TCLs [22] and how
 43 the population recovers from significant perturbations [4]. The coordination of TCLs
 44 can be performed in a centralized way, like in [9]. However this approach raises chal-
 45 lenging problems in terms of communication requirements and customer privacy. A
 46 common objective can be reached in a fully distributed approach, like in [26], where
 47 each TCL is able to calculate its own actions (ON/OFF switching) to pursue a com-
 48 mon objective. This paper is related to the work of De Paola *et al.* [5], where each
 49 agent represents a flexible TCL device. In [5] a distributed solution is presented for
 50 the operation of a population of $n = 2 \times 10^7$ refrigerators providing frequency sup-
 51 port and load shifting. They adopt a game-theory framework, modelling the TCLs as
 52 price-responsive rational agents that schedule their energy consumption and allocate
 53 their frequency response provision in order to minimize their operational costs. The
 54 potential practical application of our work also considers a large population of TCLS
 55 which, contrarily to [5], have stochastic dynamics. The proposed approach is able to
 56 minimize the overall system costs in a distributed way, with each TCL determining
 57 its optimal power consumption profile in response to price signals.

58 **1.2. Related literature.** The considered problem belongs to the class of
 59 stochastic control: looking for strategies minimizing the expectation of an objective
 60 function under specific constraints. One of the main approaches proposed in the litera-
 61 ture to tackle this problem is to use random trees: this consists in replacing the almost
 62 sure constraints, induced by non-anticipativity, by a finite number of constraints to
 63 get a finite set of scenarios (see. [10] and [20]). Once the tree structure is built, the
 64 problem is solved by different decomposition methods such as scenario decomposition
 65 [19] or dynamic splitting [21]. The main objective of the scenario method is reducing
 66 the problem to an approximated deterministic one. The paper focuses on high dimen-
 67 sional noise problems with large number of time steps, for which this approach is not
 68 feasible. The idea of reducing a single high dimensional problem to a large number
 69 with low dimension has been widely studied in the deterministic case. In determinis-
 70 tic and stochastic problems a possibility is to use time decomposition thanks to the
 71 Dynamic Programming Principle [1] taking advantage of Markov property of the sys-
 72 tem. However, this method requires a specific time structure of the cost function and
 73 fails when applied to problems for which the state space dimension is greater than
 74 five. One can deal with the curse of dimensionality, under continuous linear-convex
 75 assumptions, by using the Stochastic Dual Dynamic Programming algorithm (SDDP)

76 [16] to get upper and lower bounds of the value function, using polyhedral approxi-
 77 mations. Though the almost-sure convergence of a broad class of SDDP algorithms
 78 has been proved [18], there is no guarantee on the speed of the convergence and there
 79 is no good stopping test. In [15], a stopping criteria based on a dual version of SDDP,
 80 which gives a deterministic upper-bound for the primal problem, is proposed. SDDP
 81 is well-adapted for medium sized population problems ($n \leq 30$), whereas it fails for
 82 problems with magnitude similar to one of the present paper ($n > 1000$). It is natural
 83 for this type of high dimensional problem to investigate decomposition techniques in
 84 the spirit of the Dual Approximation Dynamic Programming (DADP). DADP has
 85 been developed in PhD theses (see [8], [13]). This approach is characterized by a
 86 price decomposition of the problem, where the stochastic constraints are projected
 87 on subspaces such that the associated Lagrangian multiplier is adapted for dynamic
 88 programming. Then the optimal multiplier is estimated by implementing Uzawa's
 89 algorithm. To this end in [13], the Uzawa's algorithm, formulated in a Hilbert set-
 90 ting, is extended to a Banach space. DADP has been applied in different cases, such
 91 as storage management problem for electrical production in [8, chapter 4] and hydro
 92 valley management [2]. In the proposed paper, in the same vein as DADP we propose
 93 a price decomposition approach restricted to deterministic prices. This new approach
 94 takes advantage of the large population number in order to introduce an auxiliary
 95 problem where the coupling term is purely deterministic.

96 **1.3. Contributions.** We consider the following approximation of problem (P_1) :

$$97 \quad (1.2) \quad (P_2) \quad \begin{cases} \text{Min}_{u \in \mathcal{U}} \tilde{J}(u) \\ \tilde{J}(u) := F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i) \right) + \frac{1}{n} \mathbb{E} \left(\sum_{i=1}^n G_i(u^i(\cdot, \omega^{-i}), \omega^i) \right). \end{cases}$$

98 Let $\hat{\mathcal{U}}$ be the set of decentralized controls, defined by:

$$99 \quad (1.3) \quad \hat{\mathcal{U}} := \prod_{i=1}^n \hat{\mathcal{U}}_i, \text{ where } \hat{\mathcal{U}}_i := \{u^i \in \mathcal{U}_i \mid u^i \text{ is } \mathcal{F}^i - \text{measurable}\}.$$

100 The decentralized version of problem (P_2) (i.e. \tilde{J} is optimized over the set $\hat{\mathcal{U}}$) can
 101 be written as:

$$102 \quad (1.4) \quad (P'_2) \quad \begin{cases} \text{Min}_{u \in \hat{\mathcal{U}}, v \in \mathbb{U}} \bar{J}(u, v), \\ \bar{J}(u, v) := F_0(v) + \frac{1}{n} \mathbb{E} \left(\sum_{i=1}^n G_i(u^i, \omega^i) \right), \\ \text{s.t } g(u, v) = 0, \end{cases}$$

where $g : \mathbb{U}^n \times \mathbb{U} \rightarrow \mathbb{U}$ is defined by

$$g(u, v) := \frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i) - v.$$

103 Observe that for any $u^i \in \hat{\mathcal{U}}_i$, $G_i(u^i, \cdot)$ is independent of \mathcal{F}^{-i} . As a first contribution,
 104 this paper shows that under some convexity and regularity assumptions on F_0 and
 105 $(G_i)_{i \in \{1, \dots, n\}}$, any solution of problem (P_2) is an ε_n -solution of (P_1) , with $\varepsilon_n \rightarrow 0$

106 when $n \rightarrow \infty$. In addition, an approach of price decomposition for (P_2) , based on the
 107 formulation (P'_2) , is easier than for (P_1) , since the Lagrange multiplier is deterministic
 108 for (P'_2) , whereas it is stochastic for (P_1) . Since computing the dual cost of (P_2) is
 109 expensive, we propose *Stochastic Uzawa* and *Sampled Stochastic Uzawa* algorithms
 110 relying on Robbins Monroe algorithm in the spirit of the stochastic gradient. Its
 111 convergence is established, relying on the proof provided by [7] for the convergence of
 112 the stochastic gradient in a Hilbert space. We check the effectiveness of the *Stochastic*
 113 *Uzawa* algorithm on a linear quadratic Gaussian framework, and we apply the *Sampled*
 114 *Stochastic Uzawa* algorithm to a model of power system, inspired by the work of A.
 115 De Paola *et al.* [5].

116 **2. Approximating the optimization problem.** In this section, the link be-
 117 tween the values of problems (P_1) and (P_2) is analyzed.

118 *Assumption 2.1.* (i) Each set \mathcal{U}_i is bounded, i.e. there exists $M > 0$ such
 119 that $\mathbb{E}\|u_i\|_{\mathbb{U}_i}^2 \leq M^2$, for $i \in \{1, \dots, n\}$.

120 (ii) The function $u^i \mapsto G_i(u^i(\cdot, \omega^{-i}), \omega^i)$ is a.s. non negative, convex and l.s.c.

121 (iii) Problem (P_1) is feasible.

122 From now on, Assumption 2.1 is supposed to hold.

LEMMA 2.2. *Suppose that F_0 is proper, l.s.c. convex. Then Problem (P_1) has a*
 123 *solution, i.e. J reaches its minimum over \mathcal{U} .*

124 *Proof.* The existence and uniqueness of a minimum is proved by considering a
 125 minimizing sequence (which exists since (P_1) is feasible) $\{u_k\}$ of J over \mathcal{U} . The set
 126 \mathcal{U} being bounded and weakly closed, there exists a subsequence $\{u_{k_\ell}\}$ which weakly
 127 converges to a certain $u^* \in \mathcal{U}$. Using Assumptions 2.1.(ii) and convexity of F_0 , it
 128 follows that $\liminf J(u_{k_\ell}) \geq J(u^*)$ and thus u^* is a solution of (P_1) . \square

129 We have the following key result.

130 THEOREM 2.3. *The decentralized problem in the l.h.s. of the following equality*
 131 *has the same value as the centralized problem in the r.h.s. equality i.e.*

$$132 \quad (2.1) \quad \inf_{u \in \hat{\mathcal{U}}} \tilde{J}(u) = \inf_{u \in \mathcal{U}} \tilde{J}(u).$$

133 *Proof.* Since $\hat{\mathcal{U}} \subset \mathcal{U}$, it is immediate that $\inf_{u \in \hat{\mathcal{U}}} \tilde{J}(u) \leq \inf_{u \in \mathcal{U}} \tilde{J}(u)$.

134 Fix $i \in \{1, \dots, n\}$, using the definition of conditional expectation, we define $\tilde{u}^i \in$
 135 $L^2(\Omega^i, \mathbb{U})$ for any $u \in \mathcal{U}$ by:

$$136 \quad \tilde{u}^i(\omega^i) := \mathbb{E}[u^i(\omega^i, \omega^{-i}) | \omega^i] = \int_{\Omega^{-i}} u^i(\omega^i, \omega^{-i}) d\mu^{-i}(\omega^{-i}) \quad \text{for any } \omega^i \in \Omega^i.$$

137 Since G_i is a.s. convex w.r.t. the first variable, the Jensen inequality gives:

$$138 \quad G_i(\tilde{u}^i, \omega^i) \leq \int_{\Omega^{-i}} G_i(u^i(\cdot, \omega^{-i}), \omega^i) d\mu^{-i}(\omega^{-i}) = \mathbb{E}[G_i(u^i(\cdot, \omega^{-i}), \omega^i) | \omega^i] \quad \text{a.s.}$$

139 On the other hand $(u^1, \dots, u^n) \mapsto F_0\left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i)\right)$ is invariant when taking the con-
 140 ditional expectation, thus:

$$141 \quad F_0\left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i)\right) = F_0\left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(\tilde{u}^i)\right).$$

142 Taking the expectation of (2), we have $\inf_{u \in \hat{\mathcal{U}}} \tilde{J}(u) \leq \inf_{u \in \mathcal{U}} \tilde{J}(u)$, and the conclusion
 143 follows. \square

144 *Remark 2.4.* In the applications to stochastic control problems (in discrete and
 145 continuous time) we have the constraint of having progressively measurable control
 146 policies. Since the set of progressively measurable policies is closed and convex, this
 147 enters in the above framework. In particular, the decentralized policy \tilde{u}^i constructed
 148 in the above proof is progressively measurable.

149 *Remark 2.5.* By Theorem 2.3, for any $\varepsilon > 0$ there exists an ε -optimal solution of
 150 problem (P_2) that is a decentralized control.

151 **PROPOSITION 2.6.** *If F_0 is Lipschitz with constant γ , then an optimal solution in*
 152 *$\hat{\mathcal{U}}$ of problem (P_2) (resp. (P_1)) is an ε -optimal solution in $\hat{\mathcal{U}}$ of problem (P_1) (resp.*
 153 *(P_2)), with $\varepsilon = \gamma M / \sqrt{n}$.*

154 *Proof.* Since F_0 is Lipschitz continuous with Lipschitz constant γ , it holds for any
 155 $x, y \in \mathbb{U}$: $|F_0(x) - F_0(y)| \leq \gamma \|x - y\|_{\mathbb{U}}$. We set for any $u \in \mathcal{U}$:

$$156 \quad (2.2) \quad \hat{u}^i := u^i - \mathbb{E}(u^i).$$

157 Using the Jensen and Hölder inequalities, $(\mathbb{E}|Y|) \leq (\mathbb{E}|Y|^2)^{\frac{1}{2}}$, the fact that for any
 158 $j \neq i$, u_i and u_j are mutually independent, and that $\|u_i\|_{\mathbb{U}}$ is bounded a.s. by M , we
 159 have $\forall u \in \hat{\mathcal{U}}$:

$$\begin{aligned} 160 \quad (2.3) \quad \left| \mathbb{E} \left(F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i \right) - F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i) \right) \right) \right| &\leq \mathbb{E} \left| F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i \right) - F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i) \right) \right| \\ &\leq \frac{\gamma}{n} \mathbb{E} \left(\left\| \sum_{i=1}^n \hat{u}^i \right\|_{\mathbb{U}} \right) \\ &\leq \frac{\gamma}{n} \mathbb{E} \left(\left\| \sum_{i=1}^n \hat{u}^i \right\|_{\mathbb{U}}^2 \right)^{\frac{1}{2}} \\ &\leq \frac{\gamma}{n^{\frac{1}{2}}} M. \end{aligned}$$

161 Let \tilde{u}^* denote a minimizer of \tilde{J} on $\hat{\mathcal{U}}$, then using (2.3) for any $u \in \hat{\mathcal{U}}$ it holds:

$$162 \quad (2.4) \quad J(\tilde{u}^*) \leq \tilde{J}(\tilde{u}^*) + \frac{\gamma}{n^{\frac{1}{2}}} M \leq \tilde{J}(u) + \frac{\gamma}{n^{\frac{1}{2}}} M \leq J(u) + \frac{\gamma}{n^{\frac{1}{2}}} M. \quad \square$$

163 If F_0 is convex, using Jensen inequality we have for any centralized control $u \in \mathcal{U}$:

$$164 \quad (2.5) \quad F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i) \right) \leq \mathbb{E} \left(F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i \right) \right).$$

165 Assumption 2.1.(iii) and convexity of F_0 implies that (P_2) is feasible. By using the
 166 same techniques as for Lemma 2.2, one can prove that (P_2) admits a solution and
 167 from (2.5) that $\min_{u \in \hat{\mathcal{U}}} \tilde{J}(u) \leq \min_{u \in \mathcal{U}} J(u)$, when F_0 is convex.

168 *Assumption 2.7.* F_0 is Gâteaux differentiable with c -Lipschitz derivative.

169 **THEOREM 2.8.** *Suppose F_0 is convex and Assumption 2.7 holds, then any decen-*
 170 *tralized optimal solution of problem (P_2) is an ε -optimal solution (where $\varepsilon = cM^2/n$)*
 171 *of problem (P_1) .*

172 *Remark 2.9.* Observe that the centralized problem (P_1) on the l.h.s. of the below
 173 inequality is bounded by the following decentralized problem on the r.h.s of this
 174 inequality i.e.

$$175 \quad \inf_{u \in \mathcal{U}} J(u) \leq \inf_{u \in \hat{\mathcal{U}}} J(u).$$

176 The article by [3] proposes an upper bound for the decentralized problem and a lower
 177 bound for the centralized problem. The upper bound is provided by a resource decom-
 178 position approach (with deterministic quantities) while the lower bound is provided
 179 by a price decomposition approach with deterministic prices (see equation (28) of [3]).
 180 Theorem 2.8 provides an upper bound for problem (P_1) with an a priori quantification
 181 of the deviation from the optimal value which vanishes when the number of agents
 182 grows to infinity. Moreover, in Section 4 we provide an original algorithm that allows
 183 to approach the solution of the decentralized problem.

184 *Proof.* Since F_0 is convex, differentiable, with a c -Lipschitz differential, one can
 185 derive for any $u \in \hat{\mathcal{U}}$ and a.s.:

$$\begin{aligned} & F_0\left(\frac{1}{n} \sum_{i=1}^n u^i\right) - F_0\left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}[u^i]\right) \\ & \leq \frac{1}{n} \langle \nabla F_0\left(\frac{1}{n} \sum_{i=1}^n u^i\right), \sum_{i=1}^n \hat{u}^i \rangle_{\mathbb{U}} \\ 186 \quad (2.6) \quad & = \frac{1}{n} \langle (\nabla F_0\left(\frac{1}{n} \sum_{i=1}^n u^i\right) - \nabla F_0\left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}[u^i]\right)), \sum_{i=1}^n \hat{u}^i \rangle_{\mathbb{U}} \\ & \quad + \frac{1}{n} \langle \nabla F_0\left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}[u^i]\right), \sum_{i=1}^n \hat{u}^i \rangle_{\mathbb{U}} \\ & \leq \frac{c}{n^2} \left\| \sum_{i=1}^n \hat{u}^i \right\|_{\mathbb{U}}^2 + \frac{1}{n} \langle \nabla F_0\left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}[u^i]\right), \sum_{i=1}^n \hat{u}^i \rangle_{\mathbb{U}}, \end{aligned}$$

where \hat{u}^i is defined in (2.2). Taking the expectation of (2.6),

$$\mathbb{E} \left(\langle \nabla F_0\left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}[u^i]\right), \sum_{i=1}^n \hat{u}^i \rangle_{\mathbb{U}} \right) = 0,$$

187 and using the mutual independence of the controls and their boundedness we get as
 188 in (2.3):

$$189 \quad (2.7) \quad \frac{c}{n^2} \mathbb{E} \left(\left\| \sum_{i=1}^n \hat{u}^i \right\|_{\mathbb{U}}^2 \right) \leq \frac{c}{n} M^2.$$

190 Let \tilde{u}^* denote a minimizer of \tilde{J} on $\hat{\mathcal{U}}$, then using (2.1), (2.7) and (2.5), for any $u \in \mathcal{U}$
 191 we have:

$$192 \quad (2.8) \quad J(\tilde{u}^*) \leq \tilde{J}(\tilde{u}^*) + \frac{c}{n} M^2 \leq \tilde{J}(u) + \frac{c}{n} M^2 \leq J(u) + \frac{c}{n} M^2.$$

193 Thus for $\varepsilon = cM^2/n$, \tilde{u}^* constitutes an ε -optimal solution to the stochastic control
 194 problem (P_1) . \square

195 *Remark 2.10.* Let \tilde{u}^* and u^* be respectively the optimal controls of problems (P_2)
 196 and (P_1) . From Jensen inequality and by definition of \tilde{u}^* we have:

$$197 \quad J(u^*) \geq \tilde{J}(u^*) \geq \tilde{J}(\tilde{u}^*).$$

198 Adding $J(\tilde{u}^*)$, one has:

$$199 \quad (2.9) \quad J(\tilde{u}^*) - \tilde{J}(\tilde{u}^*) \geq J(\tilde{u}^*) - J(u^*) \geq 0.$$

200 An approximation scheme to compute \tilde{u}^* is provided in Section 4. The practical
 201 interest of inequality (2.9) is that one can compute an upper bound for the error
 202 $J(\tilde{u}^*) - J(u^*)$, that can be automatically derived from this approximation.

203 **3. Dualization and Decentralization of problem (P_2) .** From now on, the
 204 assumption that F_0 is convex is in force in the sequel. The *Lagrangian* function
 205 associated to the constrained optimization problem (P'_2) , defined in (1.4), is: $L : \hat{\mathcal{U}} \times \mathbb{U} \times \mathbb{U} \rightarrow \bar{\mathbb{R}}$ defined by:

$$207 \quad (3.1) \quad L(u, v, \lambda) := \bar{J}(u, v) + \langle \lambda, \frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i) - v \rangle_{\mathbb{U}}.$$

208 The dual problem (D) associated with (P'_2) is:

$$209 \quad (3.2) \quad (D) \quad \max_{\lambda \in \mathbb{U}} \mathcal{W}(\lambda), \quad \text{where } \mathcal{W}(\lambda) := \text{Min}_{u \in \hat{\mathcal{U}}, v \in \mathbb{U}} L(u, v, \lambda).$$

210 For any $\lambda \in \mathbb{U}$, it holds:

$$211 \quad (3.3) \quad \mathcal{W}(\lambda) = -F_0^*(\lambda) + \frac{1}{n} \sum_{i=1}^n \text{Min}_{u^i \in \hat{\mathcal{U}}_i} \mathbb{E}(G_i(u^i, \omega^i)) + \langle \lambda, \mathbb{E}(u^i) \rangle_{\mathbb{U}},$$

212 where $F_0^*(\lambda) := \sup_{v \in \mathbb{U}} \langle \lambda, v \rangle_{\mathbb{U}} - F_0(v)$.

213 The problem is said to be qualified if it is still feasible after a small perturbation
 214 of the constraint, in the following sense:

$$215 \quad (3.4) \quad \text{There exists } \varepsilon > 0 \text{ such that } \mathcal{B}_{\mathbb{U}}(0, \varepsilon) \subset g(\hat{\mathcal{U}}, \mathbb{U}),$$

216 where $\mathcal{B}_{\mathbb{U}}(0, \varepsilon)$ is the open ball of radius ε in \mathbb{U} , g has been defined in (1.4) and $g(\hat{\mathcal{U}}, \mathbb{U})$
 217 is the image by g of $\hat{\mathcal{U}} \times \mathbb{U}$.

218 **LEMMA 3.1.** *Problem (P'_2) is qualified.*

219 *Proof.* By Assumption 2.1.(iii), there exists \hat{u} feasible for problem (P_1) . Then

$$220 \quad (3.5) \quad \mathcal{B}_{\mathbb{U}}(0, \varepsilon) \subset \mathbb{U} = g(\hat{u}, \mathbb{U}) \subset g(\hat{\mathcal{U}}, \mathbb{U}).$$

221 The conclusion follows. \square

By Assumption 5.2, Lemma 3.1 and the convexity of F_0 , the strong duality holds. Let us denote the set of solutions of the dual problem by S . Since the primal problem is qualified, the primal and dual values are equal, and the set of dual solutions S is nonempty and bounded. In addition, taking $\lambda^* \in S$, any primal solution u^* satisfies both $W(\lambda^*) = \tilde{J}(u^*)$ and $(u^*, v^*) \in \arg \min_{u \in \hat{\mathcal{U}}, v \in \mathbb{U}} L(\lambda^*, u, v)$. Since the set of admissible

controls $\hat{\mathcal{U}} = \hat{\mathcal{U}}_1 \times \dots \times \hat{\mathcal{U}}_n$ is a Cartesian product, if G_i is strictly convex the first variable, then each component u^{*i} can be uniquely determined by solving the following sub problem:

$$u^{*i} = \arg \min_{u^i \in \hat{\mathcal{U}}_i} \{ \mathbb{E} (G_i(u^i, \omega^i) + \langle \lambda^*, u^i \rangle_{\mathbb{U}}) \}.$$

222 *Remark 3.2.* By using the same argument as in Theorem 2.3, one can prove:

$$\begin{aligned} 223 \quad (3.6) \quad & \min_{u^i \in \hat{\mathcal{U}}_i} \{ \mathbb{E} (G_i(u^i, \omega^i) + \langle \lambda^*, u^i \rangle_{\mathbb{U}}) \} \\ & = \min_{u^i \in \mathcal{U}_i} \{ \mathbb{E} (G_i(u^i(\cdot, \omega^{-i}), \omega^i) + \langle \lambda^*, u^i \rangle_{\mathbb{U}}) \}. \end{aligned}$$

224 **4. Stochastic Uzawa and Sampled Stochastic Uzawa algorithms.** We
225 recall that Assumption 2.1 is in force, as well as convexity of F_0 .

226 This section aims at proposing an algorithm to find a solution of the dual problem
227 (3.2).

228 *Assumption 4.1.* (i) The function $u^i \mapsto G_i(u^i, \omega^i)$ is for a.a. $\omega^i \in \Omega^i$ strictly
229 convex on $\hat{\mathcal{U}}_i$.

230 (ii) The function F_0 has quadratic growth.

231 For all $i \in \{1, \dots, n\}$, and $\lambda \in \mathbb{U}$, we define the optimal control $u^i(\lambda)$:

$$232 \quad (4.1) \quad u^i(\lambda) := \arg \min_{u^i \in \hat{\mathcal{U}}_i} \{ \mathbb{E} (G_i(u^i, \omega^i) + \langle \lambda, u^i \rangle_{\mathbb{U}}) \},$$

233 which is well defined since $u^i \rightarrow \mathbb{E}(G_i(u^i, \omega^i))$ is strictly convex.

234 For any $\lambda \in \mathbb{U}$, the subset $V(\lambda)$ is defined by:

$$235 \quad (4.2) \quad V(\lambda) := \arg \min_{v \in \mathbb{U}} \{ F_0(v) - \langle \lambda, v \rangle_{\mathbb{U}} \}.$$

236 Since F_0 is convex and has at least quadratic growth, $V(\lambda)$ is a non empty subset of
237 \mathcal{V} and is reduced to a singleton if F_0 is strictly convex. For any $\lambda \in \mathbb{U}$, we denote by
238 $v(\lambda)$ a selection of $V(\lambda)$, and for any $v(\lambda) \in V(\lambda)$, one has $v(\lambda) \in \partial F_0^*(\lambda)$.

239 Uzawa's algorithm seems particularly adapted for this problem. However at each
240 dual iteration k and any $i \in \{1, \dots, n\}$, for the update of λ^{k+1} , one would have to
241 compute the quantities $\mathbb{E}[u^i(\lambda^k)]$, which is hard in practice. Therefore two algorithms
242 are proposed where at each iteration k , λ^{k+1} is updated thanks to a realization of
243 $u^i(\lambda^k)$.

244 For any real valued function F defined on \mathbb{U} , F^* stands for its Fenchel conjugate.

245 **LEMMA 4.2.** *Assumption 2.7 holds iff F_0^* is proper and strongly convex.*

246 *Proof.* (i) Let Assumption 2.7 hold. Since F_0 is proper, convex and l.s.c., F_0^*
247 is l.s.c. proper. From the Lipschitz property of the gradient of F_0 , it holds that
248 $\text{dom}(F_0) = \mathbb{U}$.

249 Let $s, \tilde{s} \in \text{dom}(F_0^*)$ such that there exist $\lambda_s \in \partial F_0^*(s)$ and $\mu_{\tilde{s}} \in \partial F_0^*(\tilde{s})$. From
250 the differentiability, l.s.c. and convexity of F_0 , it follows that: $s = \nabla F_0(\lambda_s)$ and
251 $\tilde{s} = \nabla F_0(\mu_{\tilde{s}})$. By Assumption 2.7 and the extended Baillon-Haddad theorem [17,

252 Theorem 3.1], ∇F_0 is cocoercive. In other words:

$$\begin{aligned}
 \langle s - \tilde{s}, \lambda_s - \mu_{\tilde{s}} \rangle_{\mathbb{U}} &= \langle \nabla F_0(\lambda_s) - \nabla F_0(\mu_{\tilde{s}}), \lambda_s - \mu_{\tilde{s}} \rangle_{\mathbb{U}} \\
 (4.3) \quad &\geq \frac{1}{c} \|\nabla F_0(\lambda_s) - \nabla F_0(\mu_{\tilde{s}})\|_{\mathbb{U}}^2 \\
 &= \frac{1}{c} \|s - \tilde{s}\|_{\mathbb{U}}^2.
 \end{aligned}$$

254 Therefore ∂F_0^* is strongly monotone, which implies the strong convexity of F_0^* .

255 (ii) Conversely, assume that F_0^* is proper and strongly convex. Then there exist
 256 $\alpha, \beta > 0$ and $\gamma \in \mathbb{U}$ such that for any $s \in \text{dom}(F_0^*)$: $F_0^*(s) \geq \alpha \|s\|_{\mathbb{U}}^2 + \langle \gamma, \alpha \rangle_{\mathbb{U}} - \beta$, and
 257 F_0 being convex, l.s.c. and proper, for any $\lambda \in \mathbb{U}$ it holds:

$$(4.4) \quad F_0(\lambda) \leq \sup_{s \in \mathbb{U}} \langle s, \lambda - \gamma \rangle_{\mathbb{U}} - \alpha \|s\|_{\mathbb{U}}^2 + \beta = \|\lambda - \gamma\|^2 / (4\alpha) + \beta.$$

259 Thus F_0 is proper and uniformly upper bounded over bounded sets and therefore is
 260 locally Lipschitz. In addition, from the strong convexity of F_0^* and the convexity of F_0 ,
 261 for any $\lambda \in \mathbb{U}$, $\partial F_0(\lambda)$ is a singleton. Thus F_0 is everywhere Gâteaux differentiable.

262 Let $\lambda, \mu \in \mathbb{U}$. Since F_0^* is strongly convex, the functions $F_0^*(s) - \langle \lambda, s \rangle_{\mathbb{U}}$ (resp.
 263 $F_0^*(s) - \langle \mu, s \rangle_{\mathbb{U}}$) has a unique minimum point s_λ (resp. s_μ), characterized by: $\lambda \in$
 264 $\partial F_0^*(s_\lambda)$ and $\mu \in \partial F_0^*(s_\mu)$. From the strong convexity of F_0^* , the strong mono-
 265 tonicity of ∂F_0^* holds: $\langle \mu - \lambda, s_\mu - s_\lambda \rangle_{\mathbb{U}} \geq \frac{1}{c} \|s_\mu - s_\lambda\|_{\mathbb{U}}^2$, where $c > 0$ is a constant
 266 related to the strong convexity of F_0^* . Using that $s_\lambda = \nabla F_0(\lambda)$ and $s_\mu = \nabla F_0(\mu)$, it
 267 holds:

$$(4.5) \quad \langle \mu - \lambda, \nabla F_0(\mu) - \nabla F_0(\lambda) \rangle_{L^2(0,T)} \geq \frac{1}{c} \|\nabla F_0(\mu) - \nabla F_0(\lambda)\|_{L^2(0,T)}^2,$$

269 meaning that ∇F_0 is cocoercive. Applying the Cauchy–Schwarz inequality to the left
 270 hand side of the previous inequality, the Lipschitz property of ∇F_0 follows. \square

271 LEMMA 4.3. *If Assumption 2.7 holds, then \mathcal{W} is strongly concave.*

272 *Proof.* For any $\lambda \in \mathbb{U}$, the expression of $\mathcal{W}(\lambda)$ is given by 3.3, where for any
 273 $i \in \{1, \dots, n\}$, $\lambda \mapsto \inf_{u^i \in \tilde{\mathcal{U}}_i} \mathbb{E}(G_i(u^i, \omega^i)) + \langle \lambda, E(u^i) \rangle_{\mathbb{U}}$ is concave and from Lemma
 274 4.2 $-F_0^*$ is strongly concave. Since the sum of a concave function and of a strongly
 275 concave function is strongly concave, the result follows. \square

276 We introduce the function $f : \mathbb{U} \rightarrow \mathbb{U}$ defined by:

$$(4.6) \quad f(\lambda) := g(u(\lambda), v(\lambda)) = \frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i(\lambda)) - v(\lambda).$$

278 Since F_0 has at least quadratic growth, one deduces that F_0^* has at most quadratic
 279 growth. Using the boundedness of \mathcal{U} and Lemma A.1 in Appendix A, there exist
 280 $M_1, M_2 > 0$ such that for any $\lambda \in \mathbb{U}$ one has:

$$(4.7) \quad \|f(\lambda)\|_{\mathbb{U}}^2 \leq M_1 + M_2 \|\lambda\|_{\mathbb{U}}^2.$$

282 For any $\lambda \in \mathbb{U}$, we denote by $\partial(-\mathcal{W}(\lambda))$ the subgradient of $-\mathcal{W}$ at λ . Therefore for
 283 any $\lambda \in \mathbb{U}$:

$$(4.8) \quad \partial(-\mathcal{W}(\lambda)) \ni -f(\lambda).$$

285 The iterative algorithm, proposed as an approximation scheme for $\lambda^* \in \arg \max_{\lambda} \mathcal{W}(\lambda)$,
 286 is summarized in the *Stochastic Uzawa* Algorithm 4.1. Some assumptions on the step
 287 size are introduced.

288 *Assumption 4.4.* The sequence $(\rho_k)_k$ is such that: $\rho_k > 0$, $\sum_{k=1}^{\infty} \rho_k = \infty$ and
 289 $\sum_{k=1}^{\infty} (\rho_k)^2 < \infty$.

290 Note that a sequence of the form $\rho_k := \frac{a}{b+k}$, with $(a, b) \in \mathbb{R}_+^* \times \mathbb{R}_+$, satisfies
 Assumption 4.4.

Algorithm 4.1 Stochastic Uzawa

- 1: Initialization $\lambda^0 \in \mathbb{U}$, set $\{\rho_k\}$ satisfying Assumption 4.4.
 - 2: $k \leftarrow 0$.
 - 3: **for** $k = 0, 1, \dots$ **do**
 - 4: $v^k \leftarrow v(\lambda^k)$ where $v(\lambda^k) \in V(\lambda^k)$, this set being defined in (4.2).
 - 5: $u^{i,k} \leftarrow u^i(\lambda^k)$ where $u^i(\lambda^k)$ is defined in (4.1) for any $i \in \{1, \dots, n\}$.
 - 6: Generate n independent noises $(\omega^{1,k+1}, \dots, \omega^{n,k+1})$, independent also of $\{\omega^{i,p} : 1 \leq i \leq n, p \leq k\}$.
 - 7: Compute the associated control realization $(u^1(\lambda^k)(\omega^{1,k+1}), \dots, u^n(\lambda^k)(\omega^{n,k+1}))$.
 - 8: $Y^{k+1} \leftarrow \frac{1}{n} \sum_{i=1}^n u^i(\lambda^k)(\omega^{i,k+1}) - v(\lambda^k)$.
 - 9: $\lambda^{k+1} \leftarrow \lambda^k + \rho_k Y^{k+1}$.
-

291 At any dual iteration k of Algorithm 4.1, Y^{k+1} is an estimator of
 292 $\mathbb{E}(\frac{1}{n} \sum_{i=1}^n u^i(\lambda^k)(\omega^{i,k+1}) - v(\lambda^k))$. Therefore an alternative approach proposed in the
 293 *Sampled Stochastic Uzawa* Algorithm 4.2 consists in performing less simulations at
 294 each iteration, by taking $m < n$, at the risk of performing more dual iterations, to
 295 estimate the quantity $\mathbb{E}(\frac{1}{n} \sum_{i=1}^n u^i(\lambda^k)(\omega^{i,k+1}) - v(\lambda^k))$.

297 The complexity of the *Stochastic Uzawa* Algorithm 4.2 is proportional to $m \times K$,
 298 where K is the total number of dual iterations and m the number of simulations
 299 performed at each iteration. The error $\mathbb{E}(\|\lambda^{k+1} - \lambda^*\|_{\mathbb{U}}^2)$ for $\lambda^* \in S$ is the sum of
 300 the square of the bias (which only depends on K and not on m) and the variance
 301 (which both depends on K and m). Therefore this algorithm enables a bias variance
 302 trade-off for a given complexity. Similarly for a given error it enables to optimize the
 303 complexity of the algorithm.

304 We recall that S is defined by $S := \arg \max_{\lambda \in \mathbb{U}} \mathcal{W}(\lambda)$ and that S is non empty due
 305 to strong duality. The following result establishes the convergence of the *Stochastic*
 306 *Uzawa* Algorithm 4.1:

Algorithm 4.2 Sampled Stochastic Uzawa

- 1: Initialization of m a positive integer and $\check{\lambda}^0 \in \mathbb{U}$, set $\{\rho_k\}$ satisfying Assumption 4.4.
 - 2: $k \leftarrow 0$.
 - 3: **for** $k = 0, 1, \dots$ **do**
 - 4: $v^k \leftarrow v(\check{\lambda}^k)$ where $v(\check{\lambda}^k) \in V(\check{\lambda}^k)$, this set being defined in (4.2).
 - 5: Generate m i.i.d. discrete random variables I_1^k, \dots, I_m^k uniformly in $\{1, \dots, n\}$.
 - 6: $u_j^{I_j^k, k} \leftarrow u_j^{I_j^k}(\check{\lambda}^k)$ where $u_j^{I_j^k}(\check{\lambda}^k)$ is defined in (4.1) for any $j \in \{1, \dots, m\}$.
 - 7: Generate m independent noises $(\omega^{1, k+1}, \dots, W^{m, k+1})$, independent also of $\{\omega^{i, p} : 1 \leq i \leq m, p \leq k\}$.
 - 8: Compute the associated control realization $(u^{I_1^k}(\check{\lambda}^k)(\omega^{1, k+1}), \dots, u^{I_m^k}(\check{\lambda}^k)(\omega^{m, k+1}))$.
 - 9: $\check{Y}^{k+1} \leftarrow \frac{1}{m} \sum_{j=1}^m u_j^{I_j^k}(\check{\lambda}^k)(\omega^{I_j^k, k+1}) - v(\check{\lambda}^k)$
 - 10: $\check{\lambda}^{k+1} \leftarrow \check{\lambda}^k + \rho_k \check{Y}^{k+1}$.
-

LEMMA 4.5. *Let Assumption 4.4 hold, then:*

- (i) $\{\|\lambda^k - \lambda\|_{\mathbb{U}}^2\}$ converges a.s., for all $\lambda \in S$.
- (ii) $\mathcal{W}(\lambda^k) \xrightarrow[k \rightarrow \infty]{} \max_{\lambda \in \mathbb{U}} \mathcal{W}(\lambda)$ a.s.
- (iii) $\{\lambda^k\}$ weakly converges to some $\bar{\lambda} \in S$ in \mathbb{U} a.s.
- (iv) If Assumption 2.7 holds, then a.s. $\{\lambda^k\}$ converges to $\bar{\lambda}$ in \mathbb{U} , with $S := \{\bar{\lambda}\}$.

307

308 The proof follows [7, Theorem 3.6]. That reference considers (changing minimiza-
 309 tion in maximization) the framework of maximization a function $\mathcal{W}(\lambda) = \mathbb{E}(\mathbf{W}(\lambda, \omega))$
 310 where $\mathbf{W}(\cdot, \omega)$ is a.s. concave. Although our setting does not enter in this framework,
 311 due to the minimization of the Lagrangian w.r.t. the variable u , the proof of Lemma
 312 4.5 follows from an obvious adaptation of the one in [7, Theorem 3.6]. It is enough to
 313 provide the first steps of the proof.

314 *Proof of Lemma 4.5.* First consider point (i). Let $\lambda \in S$. For any k , \mathcal{G}_{k+1} is the
 315 filtration defined by:

$$316 \quad (4.9) \quad \mathcal{G}_{k+1} := \sigma(\{W^{i, p}\} : 1 \leq i \leq n, p \leq k+1).$$

317 Using the definition of $Y^{k+1} \in \mathbb{U}$ line 8 in the *Stochastic Uzawa* Algorithm 4.1, we
 318 have:

$$319 \quad (4.10) \quad \begin{aligned} \|\lambda^{k+1} - \lambda\|_{\mathbb{U}}^2 &= \|\lambda^k + \rho_k Y^{k+1} - \lambda\|_{\mathbb{U}}^2 \\ &= \|\lambda^k - \lambda\|_{\mathbb{U}}^2 + 2\rho_k \langle \lambda^k - \lambda, Y^{k+1} \rangle_{\mathbb{U}} \\ &\quad + (\rho_k)^2 \|Y^{k+1}\|_{\mathbb{U}}^2. \end{aligned}$$

320 Since Y^{k+1} is independent from \mathcal{G}_k , it follows that:

$$321 \quad (4.11) \quad \mathbb{E}(\|Y^{k+1}\|_{\mathbb{U}}^2 | \mathcal{G}_k) = \mathbb{E} \left(\left\| \frac{1}{n} \sum_{i=1}^n u^i(\lambda^k)(W^{i, k+1}) - v(\lambda^k) \right\|_{\mathbb{U}}^2 \right).$$

322 Using previous equality and the inequality (4.7), one can easily show that there exists
 323 $M_3, M_4 > 0$ such that for any $k \in \mathbb{N}$ one has:

$$324 \quad (4.12) \quad \mathbb{E}(\|Y^{k+1}\|_{\mathbb{U}}^2 | \mathcal{G}_k) \leq M_1 + M_2 \|\lambda^k\|_{\mathbb{U}}^2 \leq M_3 + M_4 \|\lambda^k - \lambda\|_{\mathbb{U}}^2$$

325 Since λ^k is \mathcal{G}_k -measurable and that $\mathbb{E}[Y^{k+1}|\mathcal{G}_k] = f(\lambda^k)$, we have that:

$$\begin{aligned}
& \mathbb{E}[\|\lambda^{k+1} - \lambda\|_{\mathbb{U}}^2 | \mathcal{G}_k] \\
&= \|\lambda^k - \lambda\|_{\mathbb{U}}^2 + 2\rho_k \mathbb{E}(\langle \lambda^k - \lambda, Y^{k+1} \rangle_{\mathbb{U}} | \mathcal{G}_k) + (\rho_k)^2 \mathbb{E}[\|Y^{k+1}\|_{\mathbb{U}}^2 | \mathcal{G}_k] \\
326 \quad (4.13) \quad &\leq \|\lambda^k - \lambda\|_{\mathbb{U}}^2 + 2\rho_k \langle \lambda^k - \lambda, f(\lambda^k) \rangle_{\mathbb{U}} + (\rho_k)^2 (M_3 + M_4 \|\lambda^k - \lambda\|_{\mathbb{U}}^2) \\
&\leq \|\lambda^k - \lambda\|_{\mathbb{U}}^2 (1 + M_4 \rho_k^2) + (\rho_k)^2 M_3 - 2\rho_k (\mathcal{W}(\lambda) - \mathcal{W}(\lambda^k)).
\end{aligned}$$

327 In the last inequality, we used the concavity of \mathcal{W} and (4.8). The rest of the proof
328 follows [7, Theorem 3.6]. \square

329 Recalling the definition of $\bar{J}(u, v)$ in (1.4), we define \bar{u} :

$$330 \quad (4.14) \quad \bar{u} := \arg \min_{u \in \mathcal{U}} \left\{ \mathbb{E} \left(\sum_{i=1}^n G_i(u^i, \omega^i) + \langle \bar{\lambda}, u^i \rangle_{\mathbb{U}} \right) \right\}.$$

331 Since G_i is strictly convex w.r.t. the first variable, \bar{u} is well defined. If F_0 is strictly
332 convex, then $V(\bar{\lambda})$ is a singleton and we can write:

$$333 \quad (4.15) \quad \bar{v} := \arg \min_{v \in \mathbb{U}} \{F_0(v) + \langle \bar{\lambda}, v \rangle_{\mathbb{U}}\}.$$

334

335 **THEOREM 4.6.** *Let the Assumptions 2.7 and 4.4 hold, then we have:*

- 336 (i) $\{u(\lambda^k)\}$ weakly converges a.s. to \bar{u} .
337 If furthermore F_0 is strictly convex, then $(\bar{u}, \bar{v}, \bar{\lambda})$ is the unique saddle point \mathcal{L} , there-
338 fore \bar{u} is the unique minimizer of \bar{J} in \mathcal{U} and:
339 (ii) $\bar{J}(u(\lambda^k)) \xrightarrow[k \rightarrow \infty]{} \bar{J}(\bar{u})$ a.s.
340 (iii) $\limsup_{k \rightarrow \infty} J(u(\lambda^k)) \leq \inf_{u \in \mathcal{U}} J(u) + 2\varepsilon$ a.s. where $\varepsilon = cM^2/n$.

341 *Proof.* Proof of point (i). Since the sequence $\{(u(\lambda^k), v(\lambda^k))\}$ is bounded in $\mathbb{U} \times$
342 $L^2(0, T)$, there exists a weakly convergent subsequence $\{(u(\lambda^{\theta_k}), v(\lambda^{\theta_k}))\}$ such that:

$$343 \quad (4.16) \quad (u(\lambda^{\theta_k}), v(\lambda^{\theta_k})) \rightharpoonup (u^\theta, v^\theta) \in \mathcal{U} \times \mathbb{U}.$$

344 Using the definition of $\lambda \mapsto u(\lambda)$ in (4.1), it holds for any $k > 0$:

$$\begin{aligned}
345 \quad (4.17) \quad & \mathbb{E} (G_i(\bar{u}^i, \omega^i) + \langle \lambda^{\theta_k}, \bar{u}^i \rangle_{\mathbb{U}}) \\
& \geq \mathbb{E} (G_i(u^i(\lambda^{\theta_k}), \omega^i) + \langle \lambda^{\theta_k}, u^i(\lambda^{\theta_k}) \rangle_{\mathbb{U}}).
\end{aligned}$$

346 Using that $u^i \mapsto G_i(u^i, \omega^i)$ is a.s. w.l.s.c. on $\hat{\mathcal{U}}_i$ and the a.s. convergence of $\{\lambda^k\}$,
347 resulting from Lemma 4.5.(iv), we have from (4.17) when $k \rightarrow \infty$:

$$348 \quad (4.18) \quad \mathbb{E} (G_i(\bar{u}^i, \omega^i) + \langle \bar{\lambda}, \bar{u}^i \rangle_{\mathbb{U}}) \geq \mathbb{E} (G_i(u^{i,\theta}, \omega^i) + \langle \bar{\lambda}, u^{i,\theta} \rangle_{\mathbb{U}}).$$

349 Since \bar{u} is unique, it follows $u^\theta = \bar{u}$ and (4.18) is an equality. Using that every weakly
350 convergent subsequence of $\{u(\lambda^k)\}$ has the same weak limit \bar{u} , (i) is deduced.

351 Proof of point (ii).

352 From point (i) and (4.18), it follows for any $i \in \{1, \dots, n\}$:

$$353 \quad (4.19) \quad \lim_{k \rightarrow \infty} \mathbb{E} (G_i(u^i(\lambda^k), \omega^i)) = \mathbb{E} (G_i(\bar{u}^i, \omega^i)).$$

354 Using 4.16, the w.l.s.c. of F_0 , equation (4.15), and applying the same previous argu-
 355 ment to $\{v(\lambda^{\theta^k})\}$, it holds that:

$$356 \quad (4.20) \quad \lim_{k \rightarrow \infty} F_0(v(\lambda^k)) - \langle \lambda^k, v(\lambda^k) \rangle_{\mathbb{U}} = F_0(\bar{v}) - \langle \bar{\lambda}, \bar{v} \rangle_{\mathbb{U}},$$

357 and $v(\lambda^k) \xrightarrow[k \rightarrow \infty]{} \bar{v}$.

358 From the two previous equalities and the a.s. convergence of $\{\lambda^k\}$, it follows:

$$359 \quad (4.21) \quad \lim_{k \rightarrow \infty} F_0(v(\lambda^k)) = F_0(\bar{v}).$$

360 Using that $(\bar{u}, \bar{v}, \bar{\lambda})$ is a saddle point, it follows:

$$361 \quad (4.22) \quad \frac{1}{n} \sum_{i=1}^n \mathbb{E}(\bar{u}^i) = \bar{v}.$$

362 From (4.21) and (4.22), it holds:

$$363 \quad (4.23) \quad \lim_{k \rightarrow \infty} F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i(\lambda^k)) \right) = F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(\bar{u}^i) \right).$$

364 Then adding (4.19) and (4.23): $\lim_{k \rightarrow \infty} \tilde{J}(u(\lambda^k)) = \tilde{J}(\bar{u})$.

365 Proof of point (iii). From point (ii), inequality (2.8) and Theorem 2.8, it holds:

$$366 \quad (4.24) \quad \limsup_{k \rightarrow \infty} J(u(\lambda^k)) \leq \limsup_{k \rightarrow \infty} \tilde{J}(u(\lambda^k)) + \varepsilon = \inf_{u \in \mathcal{U}} \tilde{J}(u) + \varepsilon \leq \inf_{u \in \mathcal{U}} J(u) + 2\varepsilon. \quad \square$$

367 *Assumption 4.7.* (i) F_0 is strongly convex.

368 (ii) For any $i \in \{1, \dots, n\}$ and $\omega \in \Omega$, the function $u^i \mapsto \mathbb{E}(G_i(u^i, \omega))$ is strongly
 369 convex.

370 LEMMA 4.8. *Let Assumption 4.7.(i) hold, then the function $\lambda \mapsto v(\lambda)$ is Lipschitz on \mathbb{U} .*

371 *Proof.* From the definition of v in (4.2), we have for any $\lambda \in \mathbb{U}$: $\lambda \in \partial F_0(v(\lambda))$.

372 Thus for any $\lambda, \mu \in \mathbb{U}$, we have from the strong convexity of F_0 :

$$373 \quad (4.25) \quad \begin{cases} F_0(v(\mu)) & \geq F_0(v(\lambda)) + \langle \lambda, v(\mu) - v(\lambda) \rangle_{\mathbb{U}} + \alpha \|v(\mu) - v(\lambda)\|_{\mathbb{U}}^2 \\ F_0(v(\lambda)) & \geq F_0(v(\mu)) + \langle \mu, v(\lambda) - v(\mu) \rangle_{\mathbb{U}} + \alpha \|v(\lambda) - v(\mu)\|_{\mathbb{U}}^2. \end{cases}$$

374 Adding the two previous inequalities, after simplifications, we get:

$$375 \quad (4.26) \quad \langle \lambda - \mu, v(\lambda) - v(\mu) \rangle_{\mathbb{U}} \geq 2\alpha \|v(\lambda) - v(\mu)\|_{\mathbb{U}}^2.$$

376 Applying Cauchy-Schwarz inequality and simplifying by $\|v(\lambda) - v(\mu)\|_{\mathbb{U}}$, we get the
 377 desired Lipschitz inequality. \square

378 LEMMA 4.9. *Let Assumption 4.7.(ii) hold, thus the function $\lambda \mapsto u(\lambda)$ is Lipschitz on \mathbb{U} .*

379 *Proof.* The proof is similar to the proof of Lemma 4.8. \square

380 THEOREM 4.10. *Let the Assumption 2.7, 4.4, and 4.7 hold, then: $u(\lambda^k) \xrightarrow[k \rightarrow \infty]{} u(\bar{\lambda})$ a.s.*

381 $u(\bar{\lambda})$ a.s.

382 *Proof.* The convergence follows from the Lipschitz property of $\lambda \mapsto u(\lambda)$ (as a
383 result of assumption 4.7) associated with the a.s. convergence of $\{\lambda^k\}$. \square

384 *Remark 4.11.* Note that Lemma 4.5 and Theorems 4.6 and 4.10 still hold when
385 replacing λ^k by $\check{\lambda}^k$ and Y^k by \check{Y}^k (defined resp. line 9 and 10 in the *Sampled Stochastic*
386 *Uzawa* Algorithm 4.2). This can be proved by same argument, using that \check{Y}^k is
387 bounded a.s. and $\mathbb{E}(\check{Y}^k | \check{\mathcal{G}}_k) = f(\check{\lambda}^k)$ for any k , where:

$$388 \quad (4.27) \quad \check{\mathcal{G}}_k = \sigma \left(\{W^{\ell,p}\} : 1 \leq \ell \leq m, p \leq k \right) \vee \sigma \left(\{I_\ell^p\} : 1 \leq \ell \leq m, p \leq k \right),$$

389 with $W^{\ell,p}$ and I_ℓ^k defined respectively at lines 7 and 5 of the *Sampled Stochastic*
390 *Uzawa* Algorithm 4.2.

391 *Remark 4.12.* From a practical point of view, this algorithm can be implemented
392 in a decentralized way, where the system operator sends the signal λ , which can
393 be assimilated to a price, to the domestic appliances, which compute their optimal
394 solution $u(\lambda)$, depending on their local parameters.

395 In (5.2), the states and controls of the agents are described in a general framework.
396 To illustrate the results, we consider in the next section stochastic control problems
397 in both continuous and discrete time settings.

398 5. Application to stochastic control.

399 **5.1. Continuous time setting.** Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered proba-
400 bility space on which $W = (W^i)_{i=1,\dots,n}$ is a $n \times d$ -dimensional Brownian motion,
401 such that for any $t \in [0, T]$ and $i \in \{1, \dots, n\}$, W_t^i takes value in \mathbb{R}^d , and generates
402 the filtration $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$. \mathbb{P} stands for the Wiener measure associated with this
403 filtration and \mathbb{F} for the augmented filtration by all \mathbb{P} -null sets. The following notations
404 are used:

$$\mathbb{X} := \{ \varphi : \Omega \rightarrow \mathcal{C}([0, T], \mathbb{R}^d) \mid \varphi(\cdot) \text{ is } \mathbb{F} \text{- adapted, } \|\varphi\|_{\infty, 2} := \mathbb{E} \left(\sup_{\substack{1 \leq k \leq d \\ s \in [0, T]}} |\varphi_k(s)|^2 \right) < \infty \},$$

$$405 \quad \mathbb{U} := L^2((0, T), \mathbb{R}^p) := \{ \varphi : [0, T] \rightarrow \mathbb{R}^p \mid \int_0^T \sum_{k=1}^p |\varphi_k(t)|^2 dt < \infty \},$$

406 and for any $i \in \{1, \dots, n\}$, the feasible set of controls is defined by:

$$407 \quad (5.1) \quad \mathcal{U}_i := \{ v : \Omega \times [0, T] \rightarrow \mathbb{R}, v(\cdot) \text{ is } \mathbb{F} \text{- prog. measurable,} \\ v(\omega) \in \mathbb{U} \text{ and } v_t(\omega) \in [-M_i, M_i]^p, \text{ for a.a. } (t, \omega) \in [0, T] \times \Omega \},$$

408 and we set $M := \max_{i \in \{1, \dots, n\}} M_i$, where $M_i > 0$.

409 Each local agent $i = 1, \dots, n$ is supposed to control its state variable through the
410 control process $u^i \in \mathcal{U}_i$ and suffers from independent uncertainties. More specifically,
411 the state process of each agent, $X^{i, u^i} = (X_t^{i, u^i})_{t \in [0, T]}$, for $i = 1, \dots, n$ takes values in
412 \mathbb{R}^d and follows the dynamics for $i \in \{1, \dots, n\}$:

$$413 \quad (5.2) \quad \begin{cases} dX_t^{i, u^i} &= \mu_i(t, u_t^i(\cdot), W^i), X_t^{i, u^i}) dt + \sigma_i(t, X_t^{i, u^i}) dW_t^i, \text{ for } t \in [0, T], \\ X_0^{i, u^i} &= x_0^i \in \mathbb{R}^d; \end{cases}$$

414 We assume that for any i there exist five functions $\alpha_i \in L^\infty([0, T], \mathbb{R}^{d \times p})$, $\beta_i, \theta_i \in$
 415 $L^\infty([0, T], \mathbb{R}^{d \times d})$, $\gamma_i \in L^\infty([0, T], \mathbb{R}^d)$ and $\xi_i \in L^\infty([0, T], \mathbb{R}^{d \times d \times d})$ such that for any
 416 $(t, \nu, x) \in [0, T] \times [-M, M]^p \times \mathbb{R}^d$:

$$417 \quad (5.3) \quad \mu_i(t, \nu, x) = \alpha_i(t)\nu + \beta_i(t)x + \gamma_i(t) \quad \text{and} \quad \sigma_i(x, t) = \xi_i(t)x + \theta_i(t).$$

418 Without loss of generality, the initial states x_0^i are supposed to be deterministic.
 419 The process X^{i, u^i} is \mathbb{F} -progressively measurable. For all i , \mathcal{F}^i stands for the natural
 420 filtration of the Brownian motion W^i .

421 **5.1.1. On the well-posedness of (P_1) .** In this section, the assumptions needed
 422 for (P_1) to be well posed are studied.

LEMMA 5.1. *Let $i \in \{1, \dots, n\}$ and $v \in \mathcal{U}_i$ be a control process. The map $v^i \mapsto X^{i, v}$ is
 linear continuous from \mathcal{U}_i to \mathbb{X} and there exists a unique process $X^{i, v} \in \mathbb{X}$ satisfying
 (5.2) (in the strong sense) such that for any $p \in [1, \infty)$:*

$$423 \quad (5.4) \quad \mathbb{E} \left(\sup_{\substack{0 \leq t \leq T \\ 1 \leq k \leq d}} |X_{k,t}^{i, v}|^r \right) < C(r, T, x_0, K) < \infty .$$

424 *Proof.* The proof for the existence and uniqueness of a solution of (5.2) relies on
 425 [14, Theorem 3.6, Chapter 2]. The inequality is a result of [14, Theorem 4.4, Chapter
 426 2]. \square

427 Let $F_0 : \mathbb{U} \rightarrow \mathbb{R}$ be proper, convex and lower semi continuous function, satisfying
 428 Assumptions 2.7 and 4.1.(ii). For any $i \in \{1, \dots, n\}$, we assume that there exists F_i
 429 such that the local cost G_i is of the form:

$$430 \quad (5.5) \quad u^i \mapsto G_i(u^i(\cdot, \omega^{-i}), \omega^i) = F_i(u^i(\omega^i, \omega^{-i}), X^{i, u^i}(\omega^i)),$$

431 where $F_i : \mathbb{U} \times \mathcal{C}([0, T] \times \mathbb{R}^d) \rightarrow \mathbb{R}$ is a proper and lower semi continuous function.
 432 Additional assumptions are formulated below.

433 *Assumption 5.2.* For any $i \in \{1, \dots, n\}$:

- 434 (i) F_i is jointly convex w.r.t. to both variables and strictly convex w.r.t first
 435 variable.
 436 (ii) there exists a positive integer r such that F_i has r -polynomial growth, i.e there
 437 exists $K > 0$ such that for any $x^i \in \mathcal{C}([0, T], \mathbb{R}^d)$ and $u^i \in \mathbb{U}$: $|F_i(u^i, x^i)| \leq$
 438 $K(1 + \sup_{\substack{0 \leq t \leq T \\ 0 \leq k \leq n}} |x_{k,t}^i|^r)$.

439 *Remark 5.3.* 1. Assumption 5.2.(i) is satisfied if there exist
 440 $g_i : L^2((0, T), \mathbb{R}^p) \rightarrow \mathbb{R}$ strictly convex and $h_i : \mathcal{C}([0, T], \mathbb{R}^d) \rightarrow \mathbb{R}$ convex,
 441 such that $F_i(v, X) = g_i(v) + h_i(X)$.

442 2. Observe that Assumption 5.2 satisfies Assumptions 2.1.(ii) and 4.1.(i)

443 From now on, Assumption 5.2 is in force in the sequel. Now the optimization
 444 problems (P_1^c) and (P_2^c) can be clearly defined:

$$445 \quad (5.6) \quad (P_1^c) \quad \left\{ \begin{array}{l} \inf_{u \in \mathcal{U}} J^c(u) \\ J^c(u) := \mathbb{E} \left(F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i(\omega) \right) + \frac{1}{n} \sum_{i=1}^n F_i(u^i(\omega), X^{i, u^i}(\omega^i)) \right) \end{array} \right\},$$

446 and

$$447 \quad (5.7) \quad (P_2^c) \quad \begin{cases} \inf_{u \in \mathcal{U}} \tilde{J}^c(u) \\ \tilde{J}^c(u) := F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i) \right) + \frac{1}{n} \mathbb{E} \left(\sum_{i=1}^n F_i(u^i(\omega), X^{i,u^i}(\omega^i)) \right), \end{cases}$$

448 Using the results of Section 2, we can state the following Corollary.

449 **COROLLARY 5.4.** (i) Problems (P_1^c) and (P_2^c) admit both a unique solution.
 450 (ii) Any optimal solution of problem (P_2^c) is an ε -optimal solution, where $\varepsilon =$
 451 cM^2/n , of problem (P_1^c) .

452 *Proof.* The proof of point (i) is a specific case of Lemma 2.2. Similarly, point (ii)
 453 is a particular case of Theorem 2.8. \square

454 *Remark 5.5.* This kind of stochastic optimization problem is illustrated in Section
 455 7 with a problem of coordination of a large population of domestic appliances, where
 456 a system operator has to meet the demand while producing at low cost.

457 **5.2. Discrete time setting.** The main results of the paper are instantiated to
 458 the discrete time setting in this subsection. The following notations are used:

- 459 • Let $n \in \mathbb{N}^*$ be the number of agents, $d, p \in \mathbb{N}^*$ the dimension respectively of
 460 their state and control variables at any time step, and $T \in \mathbb{N}^*$ the finite time
 461 horizon.
- 462 • For any matrix M , M^\top denotes its transpose
- 463 • We consider a global noise process as a sequence of independent random
 464 variables (W_1, \dots, W_T) , where for any $t \in \{1, \dots, T\}$, W_t is a vector of d -
 465 dimensional centered, reduced and independent Gaussian variables, defined
 466 on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$: $W_t := (W_t^1, \dots, W_t^n)$, with $W_t^i \in \mathbb{R}^d$. For
 467 any $i \in \{1, \dots, n\}$ and $t \in \{1, \dots, T\}$ we define $\mathcal{F}_t^i := \sigma(W_1^i, \dots, W_t^i)$ and
 468 $\mathcal{F}_t := \otimes_{i=1}^n \mathcal{F}_t^i$.
- 469 • The space \mathbb{X} is defined by:

$$470 \quad (5.8) \quad \mathbb{X} := \{x = (x_0, \dots, x_T) \mid \forall k \in \{0, \dots, T\}, \mathbb{R}^d \ni x_k \text{ is} \\ \mathcal{F}_k \text{-measurable and } \mathbb{E}\|x_k\|_2^2 < \infty\}.$$

- 471 • For any $i \in \{1, \dots, n\}$, $X^{i,u^i} := (x_0^i, \dots, x_T^i) \in \mathbb{X}$ is the state trajectory
 472 of agent i controlled by $u^i := (u_0^i, \dots, u_{T-1}^i) \in \mathbb{R}^{p \times T}$. Similarly, for any
 473 $t \in \{0, \dots, T\}$ $X_t^u := (x_t^1, \dots, x_t^n) \in \mathbb{R}^{d \times n}$ is the state vector of all the agents
 474 controlled by $u_j := (u_t^1, \dots, u_t^n) \in \mathbb{R}^{p \times n}$. We have the following dynamics:

$$475 \quad (5.9) \quad \begin{cases} X_{t+1}^{i,u^i} &= A^i X_t^{i,u^i} + B^i u_t^i + C^i W_{t+1}^i, \quad \text{for } t \in \{0, \dots, T-1\}, \\ X_0^{i,u^i} &= x_0 \in \mathbb{R}^d, \end{cases}$$

476 where $A \in \mathbb{R}^{d \times d}$, $B \in \mathbb{R}^{d \times p}$ and $C \in \mathbb{R}_+^{d \times d}$.

- 477 • For any $i \in \{1, \dots, n\}$, we define the space of control \mathcal{U}^i of agent i by:

$$478 \quad (5.10) \quad \mathcal{U}^i := \{u = (u_0, \dots, u_{T-1}) \mid \forall k \in \{0, \dots, T-1\}, \mathbb{R}^p \ni u_k \text{ is} \\ \mathcal{F}_k \text{-measurable and } u_k(\omega) \in [-M, M]^p \text{ } \mathbb{P}\text{-a.s.}\},$$

479 where $M > 0$. We finally set $\mathcal{U} := \prod_{i=1}^n \mathcal{U}^i$.

480 Let $F_0 : \mathbb{R}^{p \times T} \rightarrow \bar{\mathbb{R}}$ be proper, lower semi continuous, convex and satisfy Assumptions
 481 2.7 and 4.1.(ii). Similarly to the previous subsection, we assume that there exists for
 482 any i a function $F_i : \mathbb{R}^{p \times T} \times \mathbb{R}^{d \times T} \rightarrow \mathbb{R}$ such that G_i and F_i satisfy (5.5), and F_i
 483 satisfies Assumption 5.2.(i).

484 Now for any $n \in \mathbb{T}^*$ the optimization problems (P_1^d) and (P_2^d) can be clearly
 485 defined:

$$486 \quad (5.11) \quad (P_1^d) \begin{cases} \inf_{u \in \mathcal{U}} J^d(u) \\ J^d(u) := \mathbb{E} \left(F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i \right) + \frac{1}{n} \sum_{i=1}^n F_i(u^i, X^{i,u^i}) \right), \end{cases}$$

487 and

$$488 \quad (5.12) \quad (P_2^d) \begin{cases} \inf_{u \in \mathcal{U}} \tilde{J}^d(u) \\ \tilde{J}^d(u) := F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i) \right) + \frac{1}{n} \mathbb{E} \left(\sum_{i=1}^n F_i(u^i, X^{i,u^i}) \right). \end{cases}$$

489 In the same spirit as in the previous subsection, we have the following results, which
 490 will be useful for the next section.

491 **COROLLARY 5.6.** (i) Problems (P_1^d) and (P_2^d) admit both a unique solution.
 492 (ii) Any optimal solution of problem (P_2^d) is an ε -optimal solution, where $\varepsilon =$
 493 cM^2/n , of problem (P_1^d) .

494 *Proof.* The proof of point (i) is analogous to the one of Lemma 2.2. Similarly,
 495 proof of point (ii) is analogous to the one of Theorem 2.8. \square

496 One can implement the *Stochastic Uzawa* (Algo 4.1) and the *Sampled Stochastic*
 497 *Uzawa* (Algo 4.2) in this discrete time setting with Lemma 4.5 and Theorems 4.6
 498 and 4.10 still ensuring the algorithm convergence.

499 **6. A numerical example: the LQG (Linear Quadratic Gaussian) prob-**
 500 **lem.** This sections aims at illustrating numerically the convergence of the *Stochastic*
 501 *Uzawa* (Algo 4.1) on a simple example. The algorithm speed of convergence is stud-
 502 ied, depending on the number of dual iterations and of agents. A linear quadratic
 503 formulation is considered, with n agents in a discrete setting problem (P_2^{LQG}) . We
 504 use the notations of Section 5.2.

505 This framework constitutes a simple test case, since the (deterministic) Uzawa's
 506 algorithm can be performed, and one can compare the resulting multiplier estimate
 507 with the one provided by the *Stochastic Uzawa* algorithm. Besides all the assump-
 508 tions required for the convergence of the *Stochastic Uzawa* (Algo 4.1) are satisfied
 509 for problem (P_2^{LQG}) . In addition the local problems (line 5 of this algorithm) can be
 510 resolved analytically.

511 Problem (P_2^{LQG}) is similar to (P_2^d) defined in (5.12), but in this specific case, the
 512 function F_0 is a quadratic function of the aggregate strategies of the agents

$$513 \quad (6.1) \quad F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i) \right) := \frac{\nu}{2} \sum_{t=0}^T \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u_t^i) - r_t \right)^2,$$

514 where $\nu > 0$, $\{r_t\}$ is a deterministic target sequence. Similarly, the cost functions F_i
515 of the agents is expressed in a quadratic form of its state X^{i,u^i} and control u^i .

$$516 \quad (6.2) \quad F_i(u^i, X^{i,u^i}) := \frac{1}{2} \left(\sum_{t=0}^T d_i (X_t^{i,u^i})^2 + q_i (u_t^i)^2 \right) + \frac{d_i^f}{2} (X_T^{i,u^i})^2,$$

517 where for any $i \in \{1, \dots, n\}$, $q_i > 0$ and $d_i > 0$. Defining the matrices $D =$
518 $\text{diag}(d_1, \dots, d_n)$, $Q = \text{diag}(q_1, \dots, q_n)$ and $D^f = \text{diag}(d_1^f, \dots, d_n^f)$, we get:

$$519 \quad (6.3) \quad \sum_{i=1}^n F_i(u^i, X^{i,u^i}) = \frac{1}{2} \left(\sum_{t=0}^T X_t^{u^\top} D X_t^u + u_t^\top Q u_t \right) + \frac{1}{2} X_T^{u^\top} D^f X_T^u.$$

520 Now the optimization problem (P_2^{LQG}) is clearly defined.

521 To find the optimal multiplier and control of (P_2^{LQG}) , the *Stochastic Uzawa* Al-
522 gorithm 4.1 is applied where in this specific case the lines 4 and 6 take respectively
523 the following form at any dual iteration k :

$$524 \quad (6.4) \quad u^i(\lambda^k) := \arg \min_{u^i \in \bar{U}^i} \left\{ \mathbb{E} \left(\frac{1}{2} \left(\sum_{t=0}^T d_i (X_t^{i,u^i})^2 + q_i (u_t^i)^2 + \lambda_t^k u_t^i \right) + \frac{d_i^f}{2} (X_T^{i,u^i})^2 \right) \right\},$$

525

$$526 \quad (6.5) \quad v(\lambda^k) := \arg \min_{v \in \mathbb{R}^T} \left\{ \left(\sum_{t=0}^T \nu (v_t - r_t)^2 - \lambda_t^k v_t \right) \right\}.$$

527 The optimization problem (6.4) solved by each local agent is also in the LQG frame-
528 work. One can solve these problems using the results of [24]. The resolution via
529 Riccati equations of (6.4) shows that $u^i(\lambda^k)$ is a linear function of the state X^{i,u^i}
530 and of the price λ^k . Therefore, in this specific example, for any t one can explicitly
531 compute $\mathbb{E}(u_t^i(\lambda^k) | \mathcal{G}_k)$, where \mathcal{G}_k is defined in (4.9). It allows us to implement the
532 (deterministic) Uzawa's algorithm as a reference to evaluate the performances of the
533 *Stochastic Uzawa* algorithm.

534 Different population sizes n are considered, with n ranging between 1 and 10^4 .
535 Similarly the algorithm is stopped for different numbers of dual iteration k , ranging
536 between 1 and 10^4 . In order to evaluate the bias and variance of the *Stochastic Uzawa*
537 algorithm, we have performed $J = 1000$ runs of the *Stochastic Uzawa* algorithm.

538 For any n , given the strong convexity of the dual function associated with (P_2^{LQG}) ,
539 there exists a unique optimal multiplier $\bar{\lambda}^n$. For any n , $\lambda^{k,n,j}$ denotes the dual price
540 computed during the j^{th} simulations ($j = 1, \dots, J$) of the *Stochastic Uzawa* algorithm,
541 after k dual iterations.

542 For any n , the deterministic multiplier $\bar{\lambda}^n$ is obtained by applying Uzawa's al-
543 gorithm, after 10^4 dual iterations. To this end, we applied the *Stochastic Uzawa*
544 Algorithm 4.1 where we ignored the line 8 and we replaced the update of λ^k line 9

$$545 \quad \text{by: } \bar{\lambda}^{k+1} \leftarrow \bar{\lambda}^k + \rho_k \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i(\bar{\lambda}^k)) - v(\bar{\lambda}^k) \right).$$

546 At each dual iteration k , the computation of $\mathbb{E}(u^i(\lambda^k))$ is easy in this specific case,
547 $u^i(\lambda^k)$ being a linear function of X^{i,u^i} and λ^k as explained in the previous subsection.

548 The following results compare the multipliers $\lambda^{k,n,j}$ and $\bar{\lambda}^n$, obtained respectively
549 by applying the *Stochastic Uzawa* and Uzawa algorithms.

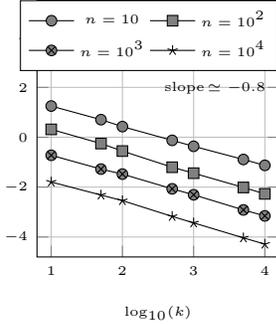


Figure 6.0.1: $\log_{10}(v_{k,n})$ function of k , given the number of agents n

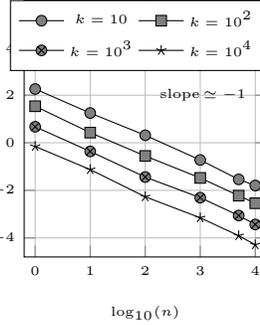


Figure 6.0.2: $\log_{10}(v_{k,n})$ function of n , given the number of agents k

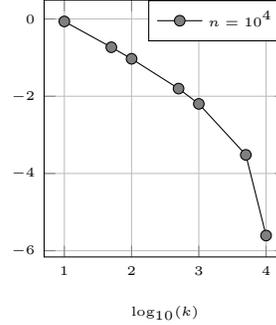


Figure 6.0.3: $\log_{10}(\|b_{k,n}\|_2^2)$ function of k , given the number of agents $n = 10^4$

550 For any k and n , $b_{k,n}$, $v_{k,n}$ and $\ell_{k,n}$ denotes respectively an estimation of the bias,
 551 the variance and the L2 norm of the error, via Monte Carlo method with J simulations.

552 Thus we have for any k and n : $b_{k,n} = \frac{1}{J} \sum_{j=1}^J \lambda^{k,n,j} - \bar{\lambda}^n$, $v_{k,n} = \frac{1}{J} \sum_{j=1}^J \|\lambda^{k,n,j} - \bar{\lambda}^n -$

553 $b_{k,n}\|_2^2$, $\ell_{k,n} = v_{k,n} + \|b_{k,n}\|_2^2$

554 On Figure 6.0.1, we observe a behavior in $1/k^\alpha$ (with $\alpha \simeq 0.8$) of the variance
 555 $v_{k,n}$ w.r.t. the number of iterations k . This rate of convergence is consistent with [6,
 556 Theorem 2.2.12, Chapter 2] for Robbins Monro algorithm where the convergence is
 557 proved to be of order at most in $1/k$.

558 On Figure 6.0.2 we observe a behavior in $1/n^\beta$ (with $\beta \simeq 1$) of the variance $v_{k,n}$
 559 w.r.t. the number of agents n . This is expected, see [6, Theorem 2.2.12, Chapter 2]
 560 and observing that the variance of Y^{k+1} is of order $1/n$ for any iteration k .

561 On Figure 6.0.3 we observe a faster behavior than $1/k$ of the bias $\|b_{k,n}\|_2^2$ w.r.t. the
 562 number of iterations k . Thus for a large number of iterations ($k > 0$), the dominant
 563 term impacting the error $\ell_{k,n}$ is the variance $v_{k,n}$.

564 **7. Price-based coordination of a large population of thermostatically**
 565 **controlled loads.** The goal of this section is to demonstrate the applicability of the
 566 presented approach for the coordination of thermostatic loads in a smart grid context.
 567 The problem analyses the daily operation of a power system with a large penetration
 568 of price-responsive demand, adopting a modelling framework similar to [5]. Two distinct
 569 elements are considered: i) a system operator, that must schedule a portfolio
 570 of generation assets in order to satisfy the energy demand at a minimum cost, and
 571 ii) a population of price-responsive loads (TCLs) that individually determine their
 572 ON/OFF power consumption profile in response to energy prices with the objective
 573 of minimizing their operating cost while fulfilling users' requirements. Note that the
 574 operations of the two elements are interconnected, since the aggregate power consumption
 575 of the TCLs will modify the demand profile that needs to be accommodated by
 576 the system operator.

577 **7.1. Formulation of the problem.** In the considered problem, the function F_0
 578 represents the minimized power production cost and corresponds to the resolution of
 579 an Unit Commitment (UC) problem. The UC determines generation scheduling decisions
 580 (in terms of energy production and frequency response (FR) provision) in order

581 to minimize the short term operating cost of the system while matching generation
 582 and demand. The latter is the sum of an inflexible deterministic component (denoted
 583 for any instant $t \in [0, T]$ by $\bar{D}(t)$) and of a stochastic part, which corresponds to the
 584 total TCL demand profile $nU_{TCL}(t)$.

585 For simplicity, a Quadratic Programming (QP) formulation in a discrete time
 586 setting is adopted for the UC problem. The central planner disposes of Z genera-
 587 tion technologies (gas, nuclear, wind) and schedules their production and allocated
 588 response by slot of 30 min every day. For any $j \in \{1, \dots, Z\}$ and $\ell \in \{1, \dots, 48\}$,
 589 $H_j(t_\ell)$, $G_j(t_\ell)$ and $R_j(t_\ell)$ are respectively the commitment, the power production
 590 and response [MWh] from unit j during the time interval $[t_\ell, t_{\ell+1}]$. The associated
 591 vectors are denoted by $H(t_\ell) = [H_1(t_\ell), \dots, H_Z(t_\ell)]$, $G(t_\ell) = [G_1(t_\ell), \dots, G_Z(t_\ell)]$ and
 592 $R(t_\ell) = [R_1(t_\ell), \dots, R_Z(t_\ell)]$.

593 The cost sustained at time t_ℓ by unit j is linear with respect to the commit-
 594 ment $H_j(t_\ell)$ and quadratic with respect to generation $G_j(t_\ell)$ and can be expressed
 595 as $c_{1,j}H_j(t_\ell)G_j^{Max}(t_\ell) + c_{2,j}G_j(t_\ell) + c_{3,j}G_j(t_\ell)^2$, with G_j^{Max} as the limit of produc-
 596 tion allocated by each generation technology, $c_{1,j}$ [€/MWh] as no-load cost and $c_{2,j}$
 597 [€/MWh] and $c_{3,j}$ [€/MW²h] as production cost of the generation technology j . The
 598 optimization of F_0 must satisfy the following constraints for all $\ell \in \{1, \dots, 48\}$ and
 599 $\ell \in \{1, \dots, 48\}$:

$$600 \quad (7.1) \quad \sum_{j=1}^Z G_j(t_\ell) - \int_{t_\ell}^{t_{\ell+1}} (\bar{D}(t) + nU_{TCL}(t))dt = 0,$$

601

$$602 \quad (7.2) \quad 0 \leq H_j(t_\ell) \leq 1,$$

603

$$604 \quad (7.3) \quad R_j(t_\ell) - r_j H_j(t_\ell) G_j^{max}(t_\ell) \leq 0,$$

605

$$606 \quad (7.4) \quad R_j(t_\ell) - s_j (H_j(t_\ell) G_j^{max}(t_\ell) - G_j(t_\ell)) \leq 0,$$

607

$$608 \quad (7.5) \quad \Delta G_L - \Lambda (\bar{D}(t_\ell) + n(\bar{U}_{TCL}(t_\ell) - \bar{R}_{TCL}(t_\ell))) \Delta f_{qss}^{max} - \hat{R}(t_\ell) \leq 0,$$

609

$$610 \quad (7.6) \quad 2\Delta G_L t_{ref} t_d - t_{ref}^2 \hat{R}(t_\ell) - 4\Delta f_{ref} t_d \hat{H}(t_\ell) \leq 0,$$

611

$$612 \quad (7.7) \quad \bar{q}(t) - \hat{H}(t) \hat{R}(t) \leq 0$$

613

$$614 \quad (7.8) \quad \mu r_j H_j(t_\ell) G_j^{max}(t_\ell) - G_j(t_\ell) \leq 0,$$

615 where (7.1) equals production and aggregated demand (i.e. the system inelastic de-
 616 mand \bar{D} and the TCL flexible demand nU_{TCL}). The quantities \hat{R} and \hat{H} denote
 617 the total reserve and inertia of the system, respectively, and are defined for any
 618 $\ell \in \{1, \dots, 48\}$ as:

$$619 \quad \hat{R}(t_\ell) = \sum_{j=1}^Z R_j(t_\ell) + nR_{TCL}(t_\ell) \quad \text{and} \quad \hat{H}(t_\ell) = \sum_{j=1}^Z \frac{h_j H_j(t_\ell) G_j^{max} - h_L \Delta G_L}{f_0}.$$

620 Assuming that for any generic generation technology j , the size of single plants
 621 included in j is quite smaller than the aggregate installed capacity of j , inequality

622 (7.2) sets that commitment decisions can be extended to the fleet and expressed by
 623 continuous variables $H_j(t_\ell) \in [0, 1]$.

624 The amount of response allocated by each generation technology is limited by
 625 the headroom $r_j H_j(t_\ell) G_j^{max}(t_\ell)$ in (7.3) and the slope s_j linking the FR with the
 626 dispatch level (7.4). Constraints (7.5) to (7.8) deal with frequency response provision
 627 and R_{TCL} (the mean of FR allocated by TCLs). They guaranty secure frequency
 628 deviations following sudden generation loss ΔG_L . Inequality (7.5) allocates enough
 629 FR (with delivery time t_d) such that the quasi-steady-state frequency remains above
 630 Δf_{qss}^{max} , with Λ accounting for the damping effect introduced by the loads [12]. Fi-
 631 nally (7.7) constraints the maximum tolerable frequency deviation Δf_{nad} , following
 632 the formulation and methodology presented in [23] and [25]. The rate of change of
 633 frequency is taken into account in (7.6) where at t_{rcf} the frequency deviation remains
 634 above Δf_{ref} . Constraint (7.8) prevents trivial unrealistic solutions that may arise
 635 in the proposed formulation, such as high values of committed generation $H_j(t_\ell)$ in
 636 correspondence with low (even zero) generation dispatch $G_j(t_\ell)$. The reader can refer
 637 to [5] for more details on the UC problem.

638 The solution of the UC problem, corresponding to the function F_0 , can be de-
 639 scribed by the following optimization problem:

$$640 \quad (7.9) \quad F_0(U_{TCL}, R_{TCL}) := \min_{H, G, R} \sum_{\ell=1}^{48} \sum_{j=1}^Z c_{1,j} H_j(t_\ell) G_j^{max}(t_\ell) + c_{2,j} G_j(t_\ell) + c_{3,j} G_j(t_\ell)^2,$$

641 subject to equations (7.1)-(7.8).

642 Note that the formulation of the present problem does not fulfill all the assumption
 643 presented in Section 4. In particular, the function F_0 is not strictly convex, as instead
 644 supposed in Theorem 4.6.(ii).(iii). Nevertheless, the numerical simulations of Section
 645 7.2 shows that the proposed approach is still able to achieve convergence.

646 Regarding the modelling of the individual price-responsive TCLs, each TCL
 647 $i \in \{1, \dots, n\}$ is characterized at any time $t \in [0, T]$ by its temperature X_t^{i, u^i} [$^\circ C$]
 648 controlled by its power consumption u_t^i [W]. The thermal dynamic X_t^{i, u^i} of a single
 649 TCL i is given by:

$$650 \quad (7.10) \quad \begin{cases} dX_t^{i, u^i} &= -\frac{1}{\gamma_i} (X_t^{i, u^i} - X_{OFF}^i + \zeta_i u_t^i) dt + \sigma_i dW_t^i, \quad \text{for } t \in [0, T], \\ X_{0, u^i}^i &= x_0^i \in \mathbb{R}, \end{cases}$$

651 where:

- 652 • γ_i is its thermal time constant [s].
- 653 • X_{OFF}^i is the ambient temperature [$^\circ C$].
- 654 • ζ_i is the heat exchange parameter [$^\circ C/W$].
- 655 • σ_i is a positive constant [$(^\circ C)s^{\frac{1}{2}}$],
- 656 • W^i is a Brownian Motion [$s^{\frac{1}{2}}$], independent from W^j for any $j \neq i$.

657 For any $i \in \{1, \dots, n\}$, the set of control \mathcal{U}_i is defined by:

$$658 \quad (7.11) \quad \mathcal{U}_i := \{v : \Omega \times [0, T] \rightarrow \mathbb{R}, v(\cdot) \text{ is } \mathbb{F} - \text{prog. measurable}, \\ v(\omega) \in \mathbb{U} \text{ and } v_t(\omega) \in \{0, P_{ON, i}\}, \text{ for a.a. } (t, \omega) \in [0, T] \times \Omega\},$$

659 The TCLs dynamics in (7.10) have been derived according to [11], with the addition
 660 of the stochastic term $\sigma_i dW_t^i$ to account for the influence of the environment (open-

661 ing/closing of the fridge, environment temperature etc) on the evolution of the TCL
662 temperature.

663 By combining the objective functions of the systems, the system operator has to
664 solve the following optimization problem:

(7.12)

$$665 \quad (P_1^{TCL}) \quad \begin{cases} \inf_{u \in \mathcal{U}} J(u) \\ J(u) := \mathbb{E} \left(F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i, \frac{1}{n} \sum_{i=1}^n r_i(u^i, X^{i,u^i}) \right) \right) \\ \quad + \mathbb{E} \left(\frac{1}{n} \sum_{i=1}^n \int_0^T f_i(u_s^i, X_s^{i,u^i}) ds + \gamma_i (X_T^{i,u^i} - \bar{X}^i)^2 \right), \end{cases}$$

666 where, for any $i \in \{1, \dots, n\}$ and any $s \in [0, T]$:

667 • $r_i(u^i, X^{i,u^i})(s)$ is the maximum amount of FR allocated by TCL i at time s :

$$668 \quad (7.13) \quad r_i(u^i, X^{i,u^i})(s) := u_s^i \frac{X_s^{i,u^i} - X_{min}^i}{X_{max}^i - X_{min}^i}.$$

669 • $f_i(u_s^i, X_s^{i,u^i})$ is the individual discomfort term of the TCL i at time s :

$$670 \quad (7.14) \quad f_i(u_s^i, X_s^{i,u^i}) := \alpha_i (X_s^{i,u^i} - \bar{X}^i)^2 + \beta_i ((X_{min}^i - X_s^{i,u^i})_+^2 + (X_s^{i,u^i} - X_{max}^i)_+^2),$$

671 where:

672 – $\alpha_i (X_s^{i,u^i} - \bar{X}^i)^2$ is a discomfort term penalizing temperature deviation
673 from some comfort target \bar{X} [$^{\circ}C$], with α_i a discomfort term parameter
674 [$\pounds/h(^{\circ}C)^2$].

675 – $\beta_i ((X_s^{i,u^i} - X_{min}^i)_+^2 + (X_{max}^i - X_s^{i,u^i})_+^2)$ is a penalization term to keep
676 the temperature in the interval $[X_{min}^i, X_{max}^i]$, with β_i a target term pa-
677 rameter [$\pounds/s(^{\circ}C)^2$] and for any $x \in \mathbb{R}$, $(a)_+ = \max(0, a)$.

678 • $\gamma_i (X_T^{i,u^i} - \bar{X}^i)^2$ is a terminal cost imposing periodic constraints, with γ a
679 target term parameter [$\pounds/s(^{\circ}C)^2$].

680 Note that the control set \mathcal{U} is not convex. We can mention a possible relaxation
681 of the problem by taking the control in the interval $[0, P_{ON,i}]$.

682 The modified problem (P_2^{TCL}) is studied to solve (P_1^{TCL}) .

(7.15)

$$683 \quad (P_2^{TCL}) \quad \begin{cases} \inf_{u \in \mathcal{U}} \tilde{J}(u) \\ \tilde{J}(u) := F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E}(u^i), \frac{1}{n} \sum_{i=1}^n \mathbb{E}(r_i(u^i, X^{i,u^i})) \right) \\ \quad + \mathbb{E} \left(\frac{1}{n} \sum_{i=1}^n \int_0^T f_i(u_s^i, X_s^{i,u^i}) ds + \gamma_i (X_T^{i,u^i} - \bar{X}^i)^2 \right). \end{cases}$$

684 **7.2. Decentralized implementation.** The *Sampled Stochastic Uzawa* Algo-
685 rithm 4.2 is applied to solve (P_2^{TCL}) , with $m = 317$ simulations per iteration. At each
686 iteration k , the lines 4 and 6 correspond respectively to the solution of a deterministic
687 UC problem and of an Hamilton Jacobi Bellman (HJB) equation. The time steps
688 $\Delta t = 7.6$ s and temperature steps $\Delta T = 0.15^{\circ}C$ are chosen for the discretization of
689 the HJB equation. Let us note that at line 6, each TCL solves its own local problem

690 on the basis of the received price signal $\lambda^k = (p^k, \rho^k)$:

$$691 \quad (7.16) \quad \inf_{u^i \in \mathcal{U}_i} \int_0^T f_i(u_s^i, X_s^{i,u^i}) + u_s^i p_s^k - r_i(u^i, X^{i,u^i})(s) \rho_s^k ds,$$

692 where $f_i(u_s^i, X_s^{i,u^i})$ is a discomfort term defined in (7.14), $u_s^i p_s^k$ can be interpreted
 693 as consumption cost and $r_i(u^i, X^{i,u^i})(s) \rho_s^k$ as fee awarded for FR provision. This
 694 implementation has a practical sense: each TCL uses local information and a price
 695 that is communicated to them to schedule its power consumption on the time interval
 696 $[0, T]$. It follows that, with the proposed approach, it is possible to optimize the overall
 697 system costs in (P_1^{TCL}) in a distributed manner, with each TCL acting independently
 698 and pursuing the minimization of its own costs.

699 **7.3. Results.** The generation technologies available in the system are nuclear,
 700 combined cycle gas turbines (CCGT), open cycle gas turbines (OCGT) and wind.
 701 The characteristics and parameters of the UC in this simulation are the same as in
 702 [5].

703 It is assumed that a population of $n = 2 \times 10^7$ fridges with built-in freeze compart-
 704 ment operates in the system according to the proposed price-based control scheme.
 705 For any agent i we set the consumption parameter $P_{ON,i} = 180W$. The values of the
 706 TCL dynamic parameters γ_i and X_{OFF}^i of (7.10) are equal to the ones taken in [5].
 707 Note that it is possible to take a population of heterogeneous TCLs with different
 708 parameter values. The initial temperature are picked randomly uniformly between
 709 $-21^\circ C$ and $-14^\circ C$. For any agent i , the parameters of the individual cost function
 710 f_i , defined in (7.14), are: $\alpha_i = 0.2 \times 10^{-4} \text{ £/s}(\text{ }^\circ C)^2$, $\beta_i = 50 \text{ £/s}(\text{ }^\circ C)^2$, $\bar{X}^i = -17.5^\circ C$
 711 and $X_{max} = -14^\circ C$, $X_{min} = -21^\circ C$. The parameter β_i is taken intentionally very
 712 large to make the temperature stay in the interval $[X_{max}^i, X_{min}^i]$. Note that the indi-
 713 vidual problems solved by the TCLs are distinct than the ones in [5] (different terms
 714 and parameters).

715 Simulations are performed for different values of volatility $\sigma_i := 0, 1, 2$ (all the
 716 TCLs have the same volatility in the simulations), where σ_i is defined in (7.10). The
 717 *Sampled Stochastic Uzawa* Algorithm is stopped after 75 iterations.

718 The resulting profile of total power consumption nU_{TCL} and total allocated re-
 719 sponse nR_{TCL} by the TCLs population are reported on figure 7.3.1. in three "flexibil-
 720 ity scenario" each corresponding to a case where TCL flexibility is enabled with three
 721 different volatilities $\sigma = 0$; $\sigma = 1$ and $\sigma = 2$. The electricity prices p and response
 722 availability prices ρ are shown in Figure 7.3.2. As observed in [5], the total con-
 723 sumption nU_{TCL} is higher when the price p is lower and inversely the total allocated
 724 response nR_{TCL} is higher when the price signal ρ is also higher. This can be observed
 725 during the first hours of the day, between 0 and 6h. The power U_{TCL} then oscillates
 726 during the day in order to maintain feasible levels of the internal temperature of the
 727 TCLs. Though the prices seem not to be sensitive to the values taken by σ , the
 728 average consumption U_{TCL} and response R_{TCL} are highly correlated to the volatility
 729 of the temperature of the TCLs. The less noisy their temperature are, the more price
 730 sensitive and flexible their consumption profiles are. The TCLs impact on system
 731 commitment decisions and consequent energy/FR dispatch levels is also analyzed and
 732 displayed in Figure 7.3.3 and 7.3.4. The production and reserve in the "flexibility sce-
 733 nario" minus the production and reserve in the "no-flexibility scenario" are plotted,
 734 for different volatilities σ . In the no-flexibility scenario we impose $R_{TCL}(t) = 0$ and
 735 we consider that the TCLs operate exclusively according to their internal tempera-
 736 ture X^{i,u^i} . They switch ON ($u^i(t) = P_{ON,i}$) when they reach their maximum feasible

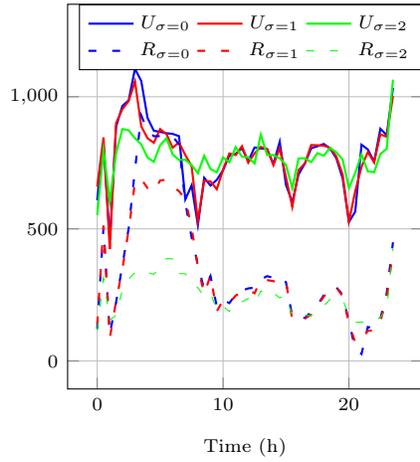


Figure 7.3.1: Total power consumption U and allocated response R (MW) of TCLs after 75 iterations of the algorithm.

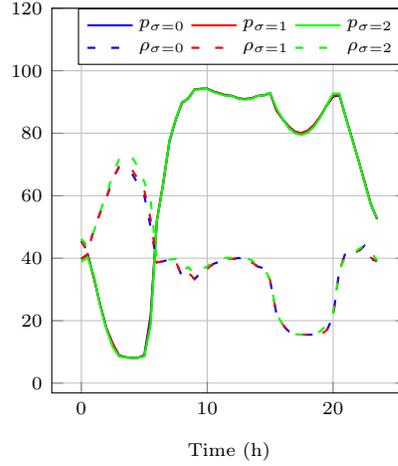


Figure 7.3.2: Electricity price p and response availability price ρ (£/MWh) after 75 iterations of the algorithm.

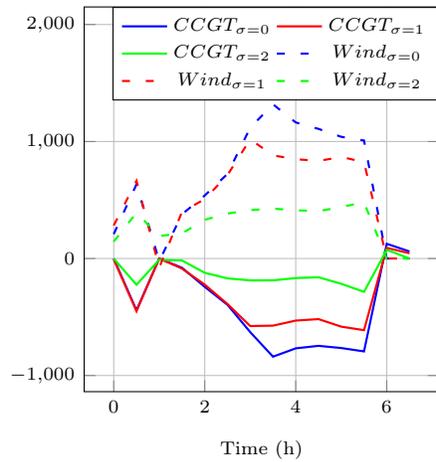


Figure 7.3.3: Deviation of generation profiles (MW) from the "no-flexibility scenario" for three different "flexibility scenario" corresponding to three temperature volatilities.

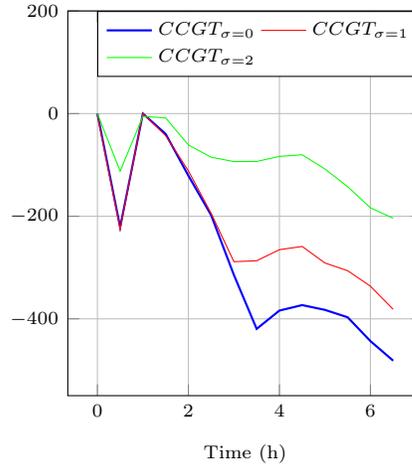


Figure 7.3.4: Deviation of Frequency Response (MW) allocated by CCGT from the "no-flexibility scenario" for three different "flexibility scenario" corresponding to three temperature volatilities.

737 temperature X_{max}^i and they switch back OFF again ($u^i(t) = 0$) when they reach the
 738 minimum temperature X_{min}^i . In figure 7.3.3, we can clearly observe that TCL's flexi-
 739 bility allows to increase the contribution of wind generation (reducing curtailment) to
 740 the energy balance of the system while decreasing the contribution of CCGT both in
 741 energy and frequency response. Without TCL support, the optimal solution envisages
 742 a further curtailment of wind output in favor of an increase in CCGT generation, as
 743 wind does not provide FR. As expected, the influence of the TCL on the system is
 744 larger when the temperature volatility is lower.

745

	$\sigma = 0$	$\sigma = 1$	$\sigma = 2$
BAU	2.770×10^7	2.770×10^7	2.772×10^7
FS	2.719×10^7	2.725×10^7	2.740×10^7

Table 1: Minimized system costs in (£)

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The system costs (i.e. UC solution) obtained with the flexibility scenario (FS) are now compared with the Business-as-usual (BAU) framework ones (the TCLs do not exploit their flexibility and they operate exclusively according to their internal temperature as previously explained) in Tab. 1. As expected the costs are lower in the CF where TCLs participate in reducing the system generation costs. The reduction is higher for $\sigma = 0$, where the reduction is about 1.9%, than for $\sigma = 1$ or $\sigma = 2$, where the the reduction is respectively about 1.6% and 1.2%. This relies on the tendency of the TCLs to be more flexible when their volatility is low. The reduction observed in the CF scenario is due to the smaller use of OCGT and CCGT generation technologies for the benefit of wind.

757

Appendix A. Appendix.

758

LEMMA A.1. *Let H be a Hilbert space and $f : H \mapsto \mathbb{R}$ be l.s.c. and convex. The function f has subquadratic growth if and only if its subgradient has linear growth.*

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Proof. We suppose for all $x \in H$ and $q \in \partial f(x)$ that we have $\|q\|_H \leq C(1 + \|x\|_H)$. For all $q \in \partial f(x)$, we have that $f(0) \geq f(x) - \langle q, x \rangle_H$, so that $f(x) \leq f(0) + \|q\|_H \|x\|_H$, and $f(x) \leq C(1 + \|x\|_H^2)$. So the subquadratic growth property holds if ∂f has linear growth, at points where $\partial f(x)$ is non empty. Since the subdifferential is nonempty in the interior of the domain, the subquadratic growth property holds everywhere.

Conversely let the subquadratic growth property holds. Then for all $x, y \in H$ and $q \in \partial f(x)$:

$$C(1 + \|y\|_H^2) \geq f(y) \geq f(x) + \langle q, y - x \rangle_H \geq -C(1 + \|x\|_H^2) + \langle q, y - x \rangle_H.$$

Take $y = x + \alpha q$, we get

$$C(1 + \|x\|_H^2 + \alpha^2 \|q\|_H^2) \geq -C(1 + \|x\|_H^2) + \alpha \|q\|_H^2.$$

We deduce that:

$$C(2 + 2\|x\|_H^2 + \alpha^2 \|q\|_H^2) \geq \alpha \|q\|_H^2.$$

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Suppose $\|x_k\|_H$ tends to infinity, if we can take q_k in $\partial f(x^k)$ and $\|x_k\|_H / \|q_k\|_H$ converging to 0, then there exists $\alpha = \alpha_k$ converging to 0, such that $2 + 2\|x_k\|_H^2 + \alpha_k^2 \|q_k\|_H^2 \leq 2\alpha_k^2 \|q_k\|_H^2$ so that $1 \leq 2C\alpha_k$, and this gives a contradiction. \square

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