Homogeneous Lyapunov Functions for Homogeneous Infinite Dimensional Systems with Unbounded Nonlinear Operators

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Abstract

The existence of a locally Lipschitz continuous homogeneous Lyapunov function is proven for a class of asymptotically stable homogeneous infinite dimensional systems with unbounded nonlinear operators.

Keywords: stability of distributed parameter systems; homogeneous systems; semigroup and operator theory.

1. Introduction

Homogeneity is a dilation symmetry known since 18th century, when Leonhard Euler studied functions $x \to f(x)$ which are symmetric with respect to the uniform dilation $x \to \lambda x$ of argument: $f(\lambda x) = \lambda^{\nu} f(x), \forall \lambda > 0, \forall x$, where ν is a real number. Such functions were called homogeneous and the number ν was referred as the homogeneity degree. It seems that a generalized homogeneity (the symmetry with respect to a non-uniform dilation) was originally studied by Vladimir Zubov in [32], where the homogeneity was shown to be useful for analysis of nonlinear finite-dimensional dynamical systems (see also [12], [15]. [3]). Indeed, it simplifies stability and robustness analysis of control systems (see e.g. [16], [28], [1]) as well as non-linear controllers/observers design (see [15], [5], [11], [1], [18]). Homogeneity degree specifies a convergence rate of any asymptotically stable homogeneous system (see e.g. [20]). For example, in [31] it is shown that homogeneous systems can be finite-time stable [4] provided that the homogeneity degree satisfies certain restriction. Some applications of the finite-dimensional homogeneity in the theory of differential operators and Lie algebras can be found in [10], [9]. Elements of the homogeneity theory of infinite dimensional dynamical systems are introduced in [25], [24], [22], where it is shown that many infinite dimensional nonlinear models of mathematical physics are homogeneous in the generalized sense (e.g. Saint-Venant, Navier–Stokes, Burgers and KdV equations).

The existence of homogeneous Lyapunov function for a stable homogeneous ordinary differential equation (ODE) is proven by Vladimir Zubov in 1958 (see

[32]) and refined by Lionel Rosier in 1992 (see [26]). The key difference between Zubov's and Rosier's theorems is the regularity of the designed homogeneous Lyapunov function. V. Zubov proved only the existence of a continuous homogeneous Lyapunov function, while L. Rosier constructed a smooth one. The regularity (at least local Lipschitz continuity) of the Lyapunov function is important for robustness (Input-to-State Stability=ISS) analysis (see e.g. [29], [14], [19]). Indeed, the smooth homogeneous Lyapunov function allows rather simple robustness analysis of a homogeneous ODE to be developed [1], [2]. Namely, any asymptotically stable homogeneous system is ISS with respect homogeneously involved perturbations.

In [23] the existence of a locally Lipschitz continuous Lyapunov function was proven for homogeneous infinite dimensional systems with locally Lipschitz nonlinearities. The mentioned class of systems is restrictive. The aim of this paper is to extend the latter result to a class of evolution equations with unbounded nonlinear operators such as Burgers or KdV equations. The well-posedness as well as stability analysis of such systems is more tricky since solutions may exist only in a linear subspace of the infinite dimensional state space. This papers shows that the methodology of homogeneous Lyapunov function design introduced in [23] can be successfully extended to the considered class of systems. As an example, the nearly fixed-time stability of a viscous Burgers equation is analyzed by means of the homogeneous Lyapunov function.

Notation. $\mathbb{R}$ is the field of real numbers; $\mathbb{R}_+ = [0,+\infty)$; $x \cdot y = \sum_{i=1}^{n} x_i y_i$ denotes the dot product of $x = (x_1, ..., x_n)^\top, y = (y_1, ..., y_n)^\top$; $\|\cdot\|_{\mathbb{B}}$ denotes a norm in a real Banach space $\mathbb{B}$; $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ denotes an inner product in a real Hilbert space $\mathbb{H}$; $\mathbf{0}$ is the zero element of a Banach space; I (resp. O) denotes the identity (resp. the zero) operator; $\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$ denotes the space of linear bounded operators $\mathbb{B}_1 \to \mathbb{B}_2$ with the norm $\|A\|_{\mathbb{B}_1,\mathbb{B}_2} = \sup\limits_{x\neq 0} \frac{\|Ax\|_{\mathbb{B}_2}}{\|x\|_{\mathbb{B}_1}}$; if $\mathbb{B}_1 = \mathbb{B}_2 = \mathbb{B}$ we write shortly $\|A\|_{\mathbb{B}}$; $S_{\mathbb{B}}$ is the unit sphere in $\mathbb{B}$; $B_{\mathbb{B}}(r)$ is the ball in $\mathbb{B}$ of the radius $r > 0$ centered at $\mathbf{0}$; for $r > 1$ the set $K_{\mathbb{B}}(r) \subset \mathbb{B}$ is defined as follows

$$K_{\mathbb{B}}(r) := \{x \in \mathbb{B} : 1/r < \|x\|_{\mathbb{B}} < r\};$$

$\mathcal{D}(A)$ denotes the domain of an operator A; $C([t_1, t_2], \mathbb{B})$ is the space of continuous functions $x : [t_1, t_2] \to \mathbb{B}$ with the uniform norm $\|x\|_C = \max_{t\in[t_1,t_2]} \|x(t)\|$ with $-\infty < t_1 < t_2 < +\infty$; $f_1 \circ f_2$ and $f_1 f_2$ denote a composition of operators f_1 and f_2; $C_c^\infty(\Omega, \mathbb{R}^m)$ is a set of infinitely smooth functions $\mathbb{R}^n \to \mathbb{R}^m$ with compact supports in an open connected set $\Omega \subset \mathbb{R}^n$ with a sufficiently smooth boundary (or $\Omega = \mathbb{R}^n$); $L^2(\Omega, \mathbb{R}^m)$ is the Lebesgue space with the inner product $\langle u, v\rangle_{L^2(\Omega,\mathbb{R}^m)} = \int_\Omega u \cdot v$ and the norm $\|u\|_{L^2(\Omega,\mathbb{R}^m)} = \sqrt{\langle u, u\rangle_{L^2(\Omega,\mathbb{R}^m)}}$, where $u, v \in L^2(\Omega, \mathbb{R}^m)$; $\nabla = (\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n})^\top$, where $z = (z_1, ..., z_n)^\top \in \mathbb{R}^n$; $\nabla^1 := \nabla$ and $\nabla^{i+1} := \nabla \cdot \nabla^i, i = 1, 2, ...$; $\Delta = \nabla^2$ is the Laplace operator; $H^p(\Omega, \mathbb{R}^m)$ is the Sobolev space with the inner product $\langle u, v\rangle_{H^p(\Omega,\mathbb{R}^m)} = \sum_{i=0}^{p} \langle \nabla^i u, \nabla^i v\rangle_{L_2(\Omega,\mathbb{R}^m)}$ for $u, v \in H^p(\Omega, \mathbb{R}^m)$ and the norm $\|u\|_{H^p(\Omega,\mathbb{R}^m)} = \sqrt{\langle u, u\rangle_{H^p(\Omega,\mathbb{R}^m)}}$; H_0^p is a completion of C_c^∞ in the norm of H^p; $L^1((t_1, t_2), \mathbb{B})$ is the space of Bochner integrable functions $(t_1, t_2) \to \mathbb{B}$, where $-\infty \leq t_1 < t_2 \leq$

$+\infty$; $\mathcal{K}$ is a set of strictly increasing continuous functions $\sigma : [0, +\infty) \to [0, +\infty)$ such that $\sigma(0) = 0$; $\mathcal{K}^\infty := \{\sigma \in \mathcal{K} : \sigma(s) \to +\infty \text{ as } s \to +\infty\}$; the symbol $\overset{a.e}{=}$(resp. $\overset{a.e}{\leq}$ or $\overset{a.e}{\in}$) means that an identity (resp. inequality or inclusion) holds almost everywhere; $\dot{x}(t) = \lim_{h\to 0} \frac{x(t+h)-x(t)}{h}$ is a time derivative of the function $x : \mathbb{R} \to \mathbb{B}$, where the limit is understood in the strong topology of $\mathbb{B}$; $\overline{D}^+ v(t) = \limsup_{h\to 0^+} \frac{v(t+h)-v(t)}{h}$ denotes the right-hand upper Dini derivative of the function $v : \mathbb{R} \to \mathbb{R}$; $\overline{D}^+ V(x; g) = \limsup_{h\to 0^+} \frac{V(x+hg)-V(x)}{h}$ denotes the right-hand upper directional derivative of the functional $V : \mathbb{B} \to \mathbb{R}$ in the direction $g \in \mathbb{B}$; $DF(x) \in \mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$ denotes the Fréchet derivative of $F : \mathbb{B}_1 \to \mathbb{B}_2$.

2. Model Description and Basic Assumptions

Let us consider the nonlinear system

$$\dot{x} = Ax + f(x), \quad t > 0, \quad x(0) = x_0 \tag{1}$$

where $x(t) \in \mathbb{X}$ is a system state at the time instant t , $\mathbb{X}$ is a real Banach space, $x_0 \in \mathbb{X}$ is an initial state, $A : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ is a linear (possibly unbounded) closed densely defined operator which generates a strongly continuous semigroup Φ of linear bounded operators on $\mathbb{X}$, $f : \mathcal{D}(f) \subset \mathbb{X} \to \mathbb{X}$ is a non-linear (possibly unbounded) closed densely defined operator such that $f(\mathbf{0}) = \mathbf{0}$ and $\mathcal{D}(f)$ is a linear subspace dense in $\mathbb{X}$.

The non-linear evolution equations are well-studied in the literature (see, for example, [21, 8]), where the notion of solution is introduced using the theory of evolution semigroups.

Definition 1. *A continuous function $x : [0, T) \to \mathbb{X}$ is said to be a mild solution of* (1) *if $f(x(\cdot)) \in L^1((0, T), \mathbb{X})$ and*

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t - s)f(x(s))ds, \quad \forall t \in [0, T).$$

If this mild solution satisfies (1) *for (almost) all $t \in (0, T)$ then x is called classical (strong) solution of* (1)*.*

The above integral is understood in the sense of Bochner (see e.g. [7], page 187). The existence of the zero solution for the system (1) follows from $f(\mathbf{0}) = \mathbf{0}$.

Definition 2. *A closed densely defined non-linear operator $f : \mathcal{D}(f) \subset \mathbb{X} \to \mathbb{X}$ is said to be M-regular if there exists a linear closed operator $M : \mathcal{D}(f) \subset \mathbb{X} \to \mathbb{X}$ with a bounded inverse $M^{-1} : \mathbb{X} \to \mathcal{D}(f)$ such that the nonlinear mapping $x \to f(M^{-1}x)$ is locally Lipschitz continuous in $x \in \mathbb{X}\backslash\{\mathbf{0}\}$. More precisely, for any $r > 0$ there exists $L_r > 0$ such that*

$$\|f(M^{-1}x_1) - f(M^{-1}x_2)\|_{\mathbb{X}} \leq L_r\|x_1 - x_2\|_{\mathbb{X}} \tag{2}$$

for all $x_i \in K_{\mathbb{X}}(r)$ and $i = 1, 2$.

For example, if $f(x)=g(Qx,x)$, where $Q:\mathcal{D}(Q)\subset\mathbb{X}\to\mathbb{X}$ is a closed densely defined linear operator and $g:\mathbb{X}\times\mathbb{X}\to\mathbb{X}$ is locally Lipschitz continuous, then f is M-regular with $M=Q-\lambda I$ provided that $\lambda\in\mathbb{R}$ belongs to the resolvent set of Q. For a well-posedness of (1), we assume that f admits some "M-regularization" consistent with A.

Assumption 1. *Let f be M-regular, M commutes with Φ:*

$$\Phi(t)Mx_0=M\Phi(t)x_0 \quad \textit{for all } t\geq 0 \textit{ and all } x_0\in\mathcal{D}(f),$$

the linear operator $M\Phi(t):\mathbb{X}\to\mathbb{X}$ is bounded for any $t>0$ and there exists a continuous function $\omega:(0,+\infty)\to\mathbb{R}_+$ such that

$$\|M\Phi(t)\|_{\mathbb{X}}\leq\omega(t) \quad \textit{and} \quad \int_0^t\omega(\sigma)d\sigma<+\infty,\forall t\in(0,+\infty).$$

If $\mathcal{D}(f)=\mathbb{X}$ (or $A=\mathbf{0}$) then $M=I$ (resp. $\Phi(t)=I,\forall t\geq 0$) and Assumption 1 simply asks the regularity of f on $\mathbb{X}\backslash\{\mathbf{0}\}$ as in [23]. If A is a generator of an analytic semigroup Φ then Assumption 1 is fulfilled for an operator $M=A^{\alpha}$ being a fractional power of the operator A (see, e.g., [21], page 195). The function ω has the form $\omega(t)=t^{-\alpha}C,\alpha\in(0,1)$, $C\geq 1$ in this case.

Since M is a closed linear operator then the linear space $\mathbb{Y}=\mathcal{D}(f)$ with the norm $\|y\|_{\mathbb{Y}}=\|y\|+\|My\|$ is a Banach space as well. Assumption 1 guarantees that for any $x_0\in\mathbb{Y}\backslash\{\mathbf{0}\}$ the evolution equation (1) has a unique mild solution x_{x_0} which depends continuously on the initial condition (see Appendix). In this paper we study the so-called generalized homogeneous evolution equations [22]. The aim of the paper is to proof that the generalized homogeneous system (1) is globally uniformly asymptotically stable in $\mathbb{Y}$ if and only if it admits a locally Lipschitz continuous homogeneous Lyapunov function $V:\mathbb{Y}\to\mathbb{R}$.

3. Homogeneous Systems

3.1. Dilations in Banach Spaces

For more details about theory of dilations in finite-dimensional and infinite-dimensional spaces we refer the reader to [13], [16], [25]. In this paper we deal only with linear dilations [22].

Definition 3. *A mapping $\mathfrak{d}:\mathbb{R}\to\mathcal{L}(\mathbb{B},\mathbb{B})$ is said to be a group of linear dilations (or simply dilation) in a Banach space $\mathbb{B}$ if*

1) (Group property) $\mathfrak{d}(0)=I$, $\mathfrak{d}(t+s)=\mathfrak{d}(t)\mathfrak{d}(s)$, $t,s\in\mathbb{R}$;

2) (Limit property) $\lim\limits_{s\to-\infty}\|\mathfrak{d}(s)u\|_{\mathbb{B}}=0$ and $\lim\limits_{s\to+\infty}\|\mathfrak{d}(s)u\|_{\mathbb{B}}=+\infty$ for any $u\neq\mathbf{0}$.

Obviously, $\mathfrak{d}$ is an one-parameter group of linear bounded invertible operators and $\mathfrak{d}(-s)=(\mathfrak{d}(s))^{-1},\forall s\in\mathbb{R}$. The limit property specifies groups being dilations in $\mathbb{B}$.

Definition 4. *A dilation* $\mathfrak{d}$ *is strongly continuous if* $s \to \mathfrak{d}(s)u$ *is continuous for any* $u \in \mathbb{B}$.

Any continuous linear dilation in $\mathbb{R}^n$ is given by $\mathfrak{d}(s) = e^{sG_{\mathfrak{d}}} = \sum_{i=0}^{\infty} \frac{s^i G_{\mathfrak{d}}^i}{i!}$, where $G_{\mathfrak{d}} \in \mathbb{R}^{n\times n}$ is an anti-Hurwitz matrix. Nonlinear dilations in $\mathbb{R}^n$ are studied in [17], [16], [27]. Examples of linear dilations in Banach spaces are considered below.

Definition 5. *A dilation* $\mathfrak{d}$ *is strictly monotone in* $\mathbb{B}$ *if* $\exists \beta > 0 : \|\mathfrak{d}(s)\|_{\mathbb{B}} \leq e^{\beta s}, \forall s \leq 0$.

Monotonicity of the dilation implies that the function $s \to \|\mathfrak{d}(s)u\|_{\mathbb{B}}$ is monotone, $u \in \mathbb{B}$.

Example 1. *Let us consider the one-parameter group of linear invertible operators in the Sobolev space* $H^p(\mathbb{R}^n, \mathbb{R}^m)$ *given by*

$$(\mathfrak{d}(s)x)(z) = e^{\alpha s} x(e^{\beta s} z), \quad s \in \mathbb{R}, \quad x \in H^p(\mathbb{R}^n, \mathbb{R}^m), \quad z \in \mathbb{R}^n, \tag{3}$$

where $\alpha, \beta \in \mathbb{R}$ *are constant parameters. Since making the change of the variable in the Lebesgue integral we derive*

$$\|\mathfrak{d}(s)x\|_{H^p}^2 = \sum_{i=0}^{p} \|\nabla^i \mathfrak{d}(s)x\|_{L_2}^2 = \sum_{i=0}^{p} \|e^{\beta i s} \mathfrak{d}(s)\nabla^i x\|_{L_2}^2 = \sum_{i=0}^{p} e^{(2\alpha - \beta n + 2\beta i)s} \|\nabla^i x\|_{L_2}^2,$$

then $\mathfrak{d}$ *is a strictly monotone dilation in* $H^p(\mathbb{R}^n, \mathbb{R}^m)$ *provided that* $\alpha > \beta(0.5n/2 - i), i = 0, 1, ..., p$. *Recall that* $L^2 = H^0$. *For more details, about linear dilations in function spaces we refer the reader to [22, Chapter 6], where in particular the strong continuity of the group* $\mathfrak{d}$ *is proven.*

Definition 6. *A set* $\mathcal{D} \subseteq \mathbb{B}$ *is a* $\mathfrak{d}$*-homogeneous cone in* $\mathbb{B}$ *if* $\mathfrak{d}(s)u \in \mathcal{D}, \forall u \in \mathcal{D}, \forall s \in \mathbb{R}$.

Notice that $\mathcal{D}$ becomes the conventional positive cone in $\mathbb{B}$ provided that $\mathfrak{d}$ is the uniform dilation $\mathfrak{d}(s) = e^s I, s \in \mathbb{R}$.

Being a strongly continuous group of linear bounded operators, the linear dilation always has an infinitesimal generator [21] being closed densely defined linear operator $G_{\mathfrak{d}} : \mathcal{D}(G_{\mathfrak{d}}) \subset \mathbb{B} \to \mathbb{B}$ given by $G_{\mathfrak{d}} u = \lim_{s\to 0} \frac{\mathfrak{d}(s)u - u}{s}$.

Example 2. *The generator* $G_{\mathfrak{d}}$ *of the dilation considered in Example 1 is given by (see, for example, [24] or [22], Lemma 6.4)*

$$(G_{\mathfrak{d}}x)(z) = \alpha x(z) + \beta(z \cdot \nabla)x(z), \quad z \in \mathbb{R}^n, \quad x \in \mathcal{D}(G_{\mathfrak{d}}) \subset H^p(\mathbb{R}^n, \mathbb{R}^m), \tag{4}$$

where $z \cdot \nabla = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + ... + z_n \frac{\partial}{\partial z_n}$, *the domain* $\mathcal{D}(G_{\mathfrak{d}})$ *is a completion of* C_c^∞ *with respect to the norm* $\|x\|_{H^p} + \|G_{\mathfrak{d}}x\|_{H^p}$. *All derivatives are understood in the weak sense.*

3.2. The canonical homogeneous norm

The dilation introduces an alternative norm topology in $\mathbb{B}$ using the so-called canonical homogeneous norm.

Definition 7 ([24]). *The functional* $\|\cdot\|_{\mathfrak{d}}:\mathbb{B}\to\mathbb{R}_+$ *given by* $\|\mathbf{0}\|_{\mathfrak{d}}=0$ *and*

$$\|u\|_{\mathfrak{d}}=e^{s_u}, \quad \text{where } s_u\in\mathbb{R}: \|\mathfrak{d}(-s_u)u\|_{\mathbb{B}}=1, \quad u\neq\mathbf{0} \tag{5}$$

is called the canonical homogeneous norm in $\mathbb{B}$*, where* $\mathfrak{d}$ *is a strictly monotone dilation in* $\mathbb{B}$*.*

Obviously, $\|\mathfrak{d}(s)u\|_{\mathfrak{d}}=e^s\|u\|_{\mathfrak{d}}$ and $\|u\|_{\mathfrak{d}}=\|-u\|_{\mathfrak{d}}$ for $\forall u\in\mathbb{B}$ and $\forall s\in\mathbb{R}$. Moreover, $\|u\|_{\mathfrak{d}}=r\Leftrightarrow u\in S_{\mathfrak{d}}(r), r>0$. Notice that $\|\cdot\|_{\mathfrak{d}}=\|\cdot\|_{\mathbb{B}}$ provided that $\mathfrak{d}$ is the uniform (standard) dilation $\mathfrak{d}(s)=e^sI$, $s\in\mathbb{R}$.

Theorem 1 ([24] and [22], Lemmas 7.1, 7.2). *If* $\mathfrak{d}$ *is a strongly continuous strictly monotone dilation then* $\|\cdot\|_{\mathfrak{d}}$ *is single-valued, positive definite, locally Lipschitz continuous on* $\mathbb{B}\backslash\{\mathbf{0}\}$ *and there exist* $\alpha\geq\beta>0$, $C\geq 1$: $\frac{1}{C}\|u\|_{\mathfrak{d}}^{\alpha}\leq\|u\|_{\mathbb{B}}\leq\|u\|_{\mathfrak{d}}^{\beta}$, $u\in B_{\mathfrak{d}}(1)$ *and* $\|u\|_{\mathfrak{d}}^{\beta}\leq\|u\|_{\mathbb{B}}\leq C\|u\|_{\mathfrak{d}}^{\alpha}$, $u\in\mathbb{B}\backslash B_{\mathfrak{d}}(1)$*. Moreover, there exist* $\underline{\sigma},\overline{\sigma}\in\mathcal{K}^{\infty}:\underline{\sigma}(\|u\|_{\mathbb{B}})\leq\|u\|_{\mathfrak{d}}\leq\overline{\sigma}(\|u\|_{\mathbb{B}})$, $\forall u\in\mathbb{B}$.

In [22,Theorem 7.1] it is also proven that $\|\cdot\|_{\mathfrak{d}}$ is a norm in a Banach space $\tilde{\mathbb{B}}$ homeomorphic to $\mathbb{B}$. This justifies the name "norm" for the functional $\|\cdot\|_{\mathfrak{d}}$.

Example 3. *Since the strongly continuous strictly monotone dilation* (3) *in* $H^p(\mathbb{R}^n,\mathbb{R}^m)$ *satisfies*

$$\|\mathfrak{d}(s)x\|_{H^p}^2=\sum_{i=0}^{p}e^{2s(\alpha-\beta n/2+\beta i)}\|\nabla^i x\|_{L^2}^2, \quad x\in H^p(\mathbb{R}^n,\mathbb{R}^m)$$

then the canonical homogeneous norm in H^p *is defined as* $\|x\|_{\mathfrak{d},H^p}=1/V$*, where* $V>0$ *is a unique real positive root of the following fractional polynomial equation*

$$1=\sum_{i=0}^{p}V^{2(\alpha-\beta n/2+\beta i)}a_i,$$

where $a_i=\|\nabla^i x\|_{L^2}$, $x\neq\mathbf{0}$, $\alpha>\beta n/2-\beta i, i=0,1...,p$*. Since the right-hand side of the latter equation is continuously differentiable in* $a_i,V\in(0,+\infty)$ *then, by the implicit function theorem, the function* $(a_0,...,a_p)\to V$ *defined implicitly by this equation is continuously differentiable in* $a_i\in(0,+\infty)$ *as well. In other words, the canonical homogeneous norm* $\|x\|_{\mathfrak{d},H^1}$ *is a continuously differentiable function of* $\|\nabla^i x\|_{L^2}, i=0,1,...,p$ *for* $x\in H^p(\mathbb{R}^n,\mathbb{R}^m)\backslash\{\mathbf{0}\}$*.*

The canonical homogeneous norm can be utilized as a Lyapunov function candidate for some homogeneous systems. The differentiability of the homogeneous norm is important for the corresponding analysis. In the case of a Hilbert space $\mathbb{H}$, the canonical homogeneous norm is always Frechét differentiable at least on the domain of the generator $G_{\mathfrak{d}}$.

Lemma 1 ([24] and [22], Lemma 7.4). *Let $\mathfrak{d}$ be a strongly continuous strictly monotone dilation group in a Hilbert space $\mathbb{H}$ then the homogeneous norm $\|\cdot\|_{\mathfrak{d}}$ is differentiable on $\mathcal{D}(G_{\mathfrak{d}})\backslash\{\mathbf{0}\}$ and the Fréchet derivative of $\|\cdot\|_{\mathfrak{d}}$ at $u \in \mathcal{D}(G_{\mathfrak{d}})\backslash\{\mathbf{0}\}$ is given by*

$$(D\|u\|_{\mathfrak{d}})(\cdot) = \frac{\langle \mathfrak{d}(-\ln\|u\|_{\mathfrak{d}})\cdot\,,\ \mathfrak{d}(-\ln\|u\|_{\mathfrak{d}})u\rangle}{\langle G_{\mathfrak{d}}\mathfrak{d}(-\ln\|u\|_{\mathfrak{d}})u, \mathfrak{d}(-\ln\|u\|_{\mathfrak{d}})u\rangle}\|u\|_{\mathfrak{d}}. \tag{6}$$

Example 4. *Let us consider again the strongly continuous strictly monotone dilation* (3) *in $H^p(\mathbb{R}^n,\mathbb{R}^m)$ with $\alpha > \beta n/2 - \beta i, i = 0, 1, ..., p$. From Example 3 we conclude the Fréchet differentiability of $\|\cdot\|_{\mathfrak{d},H^p}$ on $H^p(\mathbb{R}^n,\mathbb{R}^m)\backslash\{\mathbf{0}\}$. Notice that for $v \in H^1(\mathbb{R}^n,\mathbb{R}^m)$ using integration by parts we derive*

$$\langle G_{\mathfrak{d}}v, v\rangle_{H^1} = \langle \alpha v + \beta(z\cdot\nabla)v, v\rangle_{L_2} + \langle \nabla(\alpha v + (\beta z\cdot\nabla)v), \nabla v\rangle_{L_2} =$$

$$(\alpha - \beta n/2)\langle v, v\rangle_{L_2} + (\alpha + \beta(1 - n/2))\langle \nabla v, \nabla v\rangle_{L_2}$$

and the Fréchet derivative of $\|\cdot\|_{\mathfrak{d},H^1}$ at the point $x \in H^1(\mathbb{R}^n,\mathbb{R}^m)$ can be computed as follows

$$\left(D\|x\|_{\mathfrak{d},H^1}\right)(q) = \frac{\left\langle \mathfrak{d}(-\ln\|x\|_{\mathfrak{d},H^1})q,\, v\right\rangle_{H^1}}{(\alpha-\beta n/2)\langle v,v\rangle_{L_2} + (\alpha+\beta(1-n/2))\langle \nabla v,\nabla v\rangle_{L_2}}\|x\|_{\mathfrak{d},H^1},\ q\in H^1(\mathbb{R}^n,\mathbb{R}^m), \tag{7}$$

where $v := \mathfrak{d}\left(-\ln\|x\|_{\mathfrak{d},H^1}\right)x$.

3.3. Homogeneous operators

Homogeneous functionals and operators on a Banach space $\mathbb{B}$ (see, e.g., [25]) are defined similarly to homogeneous functions and vector fields in $\mathbb{R}^n$ (see e.g. [16], [11]) taking into account their possible unboundedness.

Definition 8. *An operator $f : \mathcal{D}(f) \subset \mathbb{B} \to \mathbb{B}$ (a functional $h : \mathcal{D}(h) \subset \mathbb{B} \to \mathbb{R}$) is said to be $\mathfrak{d}$-homogeneous of a degree $\nu \in \mathbb{R}$ if the domain $\mathcal{D}(f)$ (resp. $\mathcal{D}(h)$) is a $\mathfrak{d}$-homogeneous cone and*

$$\begin{array}{c} e^{\nu s}\mathfrak{d}(s)f(u) = f(\mathfrak{d}(s)u), \quad \forall s\in\mathbb{R}, \quad \forall u \in \mathcal{D}(f), \\ (resp.\ \ h(\mathfrak{d}(s)u) = e^{\nu s}h(u), \quad \forall s\in\mathbb{R}, \quad \forall u\in\mathcal{D}(h)) \end{array} \tag{8}$$

where $\mathfrak{d}$ is a group of linear operators in $\mathbb{B}$.

We say that an evolution equation (inclusion) is $\mathfrak{d}$-homogeneous of a degree $\mu \in \mathbb{R}$ if its right-hand side is a $\mathfrak{d}$-homogeneous operator of the degree μ.

Example 5 ([22], Example 7.6). *The Laplace operator $\Delta : H^2(\mathbb{R}^n,\mathbb{R}) \subset L^2(\mathbb{R}^n,\mathbb{R}) \to L^2(\mathbb{R}^n,\mathbb{R})$ is $\mathfrak{d}$-homogeneous of the degree 2β with respect to the dilation $\mathfrak{d}$ introduced in the Example 1, where $\Delta x = \sum_{i=1}^n \frac{\partial^2 x}{\partial z_i^2}, z_i \in \mathbb{R}, x \in H^2(\mathbb{R}^n,\mathbb{R})$ and the derivatives are understood in the weak sense.*

Homogeneity allows local properties of nonlinear operators (such as regularity) to be extend globally [22, Chapter 7].

3.4. Symmetry of solutions of homogeneous systems

A semigroup generated by a closed densely defined linear homogeneous operator in $\mathbb{X}$ is homogeneous as well [22, Lemma 8.1].

Theorem 2 ([22], Theorem 8.1). *Let $\mathfrak{d}$ be a group of linear bounded invertible operators on $\mathbb{X}$. Let a linear closed densely defined operator $A : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ generate a strongly continuous semigroup Φ of linear bounded operators on $\mathbb{X}$ and $f : \mathcal{D}(f) \subset \mathbb{X} \to \mathbb{X}$. Let A and f be $\mathfrak{d}$-homogeneous operators of a degree $\mu \in \mathbb{R}$. If $x : [0, T) \to \mathbb{X}$ is a mild solution of (1) and $x(t) \overset{a.e.}{\in} \mathcal{D}(f)$ then for any $s \in \mathbb{R}$ the function $x^s : [0, e^{-\mu s}T) \to \mathbb{X}$ given by $x^s(t) := \mathfrak{d}(s)x(e^{\mu s}t)$, $t \in [0, e^{-\mu s}T)$ is a mild solution of the evolution equation (1) and $x^s(t) \overset{a.e.}{\in} \mathcal{D}(f)$.*

The latter theorem proves the symmetry of solutions of (1) with A and f being $\mathfrak{d}$-homogeneous operators.:

$$x_{\mathfrak{d}(s)x_0}(t) = \mathfrak{d}(s)x_{x_0}(e^{\mu s}t), \quad s \in \mathbb{R}, t \geq 0, \tag{9}$$

where x_z denotes a solution of (1) with the initial data $x(0) = z$. This symmetry expands globally any local property of solutions. For example, *if the origin of* (1) *is locally stable then, from* (9) *and the limit property of* $\mathfrak{d}$, *we immediately derive global stability.* Similarly, the existence of solutions for small initial data implies the existence of solutions for large initial data and so on.

Example 6. *The system $\dot{x} = \Delta x - (x \cdot \nabla)x$, $t \geq 0$, $x \in L^2(\mathbb{R}^n, \mathbb{R}^n)$ is $\mathfrak{d}$-homogeneous of the degree 2 with respect to the dilation $(\mathfrak{d}(s)x)(z) = e^s x(e^s z)$, $z \in \mathbb{R}^n, s \in \mathbb{R}$, where $\Delta : H^2(\mathbb{R}^n, \mathbb{R}^n) \subset L^2(\mathbb{R}^n, \mathbb{R}^n) \to L^2(\mathbb{R}^n, \mathbb{R}^n)$ is the Laplace operator and $f : H^1(\mathbb{R}^n, \mathbb{R}^n) \subset L^2(\mathbb{R}^n, \mathbb{R}^n) \to L^2(\mathbb{R}^n, \mathbb{R}^n)$ given by $f(x) = -(x \cdot \nabla)x$ is a non-linear M-regular operator with $M = (-\Delta)^{1/2+n/8}$ being the fractional Laplacian. The M-regularity of f follows from the inequalities*

$$\|\nabla M^{-1}v\|_{L^4} \leq C\|(-\Delta)^{n/8}\nabla(-\Delta)^{-1/2-n/8}v\|_{L^2} \leq \tilde{C}\|v\|_{L^2}, \quad \forall v \in L^2$$

being a consequence of Corollary 6.11 from [21, Chapter 1] and the Hardy-Littlewood-Sobolev fractional integration theorem (see, e.g.,[30, Page 119]) yielding $\|(-\Delta)^{-n/8}y\|_{L^4} \leq C\|y\|_{L^2}$, $\forall y \in L^2$. Assumption 1 is fulfilled for $n \in \{1, 2, 3\}$. By Theorem 2, mild solutions of the considered system satisfy (9).

3.5. Homogeneous Lyapunov function theorem (Main result)

Solutions of (1) with the M-regular operator f are well-defined on $\mathbb{Y}\backslash\{\mathbf{0}\}$ (see Appendix). It is reasonable to analyze the stability and robustness of the system (1) in the norm topology of $\mathbb{Y}$. The origin of the system (1) is said to be

- *globally uniformly Lyapunov stable in* $\mathbb{Y}$ if there exists $\varepsilon \in \mathcal{K}^{\infty}$ such that

$$\|x(t)\|_{\mathbb{Y}} \leq \varepsilon(\|x(t_0)\|_{\mathbb{Y}}), \quad \forall t \geq t_0 \tag{10}$$

  for any mild solution $x : \mathbb{R}_+ \to \mathbb{Y}$ of (1) and any $t_0 \geq 0$;

- *globally uniformly asymptotically stable in* $\mathbb{Y}$ if it is globally uniformly Lyapunov stable in $\mathbb{Y}$ and $\forall \varepsilon > 0$, $\forall R > \varepsilon$ and $\forall t_0 \geq 0$, there exists $\hat{T} = \hat{T}(R, \varepsilon) > 0$ such that $\|x(t_0)\|_{\mathbb{Y}} \leq R$ implies $\|x_{x_0}(t)\|_{\mathbb{Y}} \leq \varepsilon$ for all $t > t_0 + \hat{T}$.

Recall that the homogeneity degree specifies convergence rates of stable homogeneous systems (see e.g. [20], [3] for ODE models and [22, Theorem 8.6] for evolution systems in Banach spaces). For instance, any uniformly asymptotically stable $\mathfrak{d}$-homogeneous system of negative degree is globally uniformly finite-time stable (i.e. the state $x = \mathbf{0}$ is reached by any trajectory of the system is a finite instant of time), but any uniformly asymptotically stable $\mathfrak{d}$-homogeneous system of positive degree is globally uniformly nearly fixed-time stable (i.e. for any neighborhood $U \subset \mathbb{Y}$ of $\mathbf{0}$ there exists an instant of time $T_U \in (0, +\infty)$ such that $x(t) \in U, \forall t \geq T_U$ independently of initial conditions).

Theorem 3. *Let $\mathfrak{d}$ be a strictly monotone strongly continuous dilation in $\mathbb{X}$ and in $\mathbb{Y}$, the evolution equation (1) be $\mathfrak{d}$-homogeneous of a degree $\mu \in \mathbb{R}$, the operators A, f satisfy Assumption 1 and $m > 0$ be an arbitrary real number. The origin of (1) is uniformly asymptotically stable in $\mathbb{Y}$ if and only if there exists a continuous positive definite functional $V : \mathbb{Y} \to \mathbb{R}$ such that*

1) *V is $\mathfrak{d}$-homogeneous of the degree m and locally Lipschitz continuous on $\mathbb{Y}\backslash\{\mathbf{0}\}$;*

2) *there exist $\underline{k}, \overline{k} > 0$ satisfying*

$$\underline{k}\|x\|^m_{\mathfrak{d},\mathbb{Y}} \leq V(x) \leq \overline{k}\|x\|^m_{\mathfrak{d},\mathbb{Y}}, \quad \forall x \in \mathbb{Y}, \tag{11}$$

where $\|\cdot\|_{\mathfrak{d},\mathbb{Y}}$ is the canonical homogeneous norm induced by $\|\cdot\|_{\mathbb{Y}}$;

3) *for any mild solution x_{x_0} of (1) with $x(0) = x_0 \in \mathbb{Y}$ the inequality*

$$\overline{D}^+ V(x_{x_0}(t)) \leq -\|x_{x_0}(t)\|^{\mu+m}_{\mathfrak{d},\mathbb{Y}}, \quad t > 0 \tag{12}$$

holds as long as $x_{x_0}(t) \neq \mathbf{0}$.

Formally, the latter theorem needs a knowledge of trajectories of (1) to check the condition (12). In practice, it is important to analyze the asymptotic stability using the right-hand side of the evolution system only. This can be done under additional restriction on $\mathbb{X}$. The proofs of the above theorem and the following corollary are given in Appendix.

Corollary 1. *Theorem 3 remains true even when the condition 3) is replaced with*

3') $\overline{D}^+ V(x; Ax + f(x)) \leq -\|x\|^{\mu+m}_{\mathfrak{d},\mathbb{Y}}$, $\quad \forall x \in M^{-1}\mathcal{D}(A)$,

provided that $\mathbb{X}$ is a reflexive Banach space and $M^{-1}\mathcal{D}(A)$ is a $\mathfrak{d}$-homogeneous cone dense in $\mathbb{Y}$.

This corollary is applicable, in particular, if $\mathbb{X}$ is a Hilbert space.

Example 7. *Let us consider the nonlinear equation*

$$\dot{x} = \frac{\partial^2 x}{\partial z^2} + x\frac{\partial x}{\partial z} - \frac{1}{3}\|x\|_{L^2}^4 x, \quad t > 0, \quad x(0) = x_0, \tag{13}$$

where $x(t) \in L^2(\mathbb{R}, \mathbb{R})$. This equation admits the representation (1) *with $A = \frac{\partial^2}{\partial z^2}$, $\mathbb{X} = L^2(\mathbb{R}, \mathbb{R})$, $\mathcal{D}(A) = H^2(\mathbb{R}, \mathbb{R})$, $f(x) = x\frac{\partial x}{\partial z} - \frac{1}{3}\|x\|_{L^2}^4 x$ and $\mathcal{D}(f) = \mathbb{Y} = H^1(\mathbb{R}, \mathbb{R})$. Notice that f is unbounded but M-regular nonlinear operator with $M = \frac{\partial}{\partial z} + I$ and $(M^{-1}y)(z) = \int_{-\infty}^{z} e^{-(z-s)} y(s)ds$, $y \in \mathbb{X}, z \in \mathbb{R}$, respectively. Indeed, the inequality* (2)*, obviously, follows from $\|\frac{\partial}{\partial z}M^{-1}\| < \infty$ and*

$$\|M^{-1}y\|_{L^\infty} \leq \sup_{z\in\mathbb{R}} \int_{-\infty}^{z} e^{-(z-s)}|y(s)|ds \leq \|y\|_{L^2} \sup_{z\in\mathbb{R}} \sqrt{\int_{-\infty}^{z} e^{-2(z-s)}ds} = \frac{\|y\|_{L^2}}{\sqrt{2}}, \ \forall y \in \mathbb{X}.$$

The considered system cannot be studied using the results given in [23]. The equation (13) *satisfies Assumption 1 with $(\Phi(t)y)(z) = \frac{1}{\sqrt{4\pi t}} \int_{\infty}^{+\infty} e^{-\frac{(z-s)^2}{4t}} y(s)ds$, so it has a unique mild solution for any $x_0 \in \mathbb{Y}\backslash\{\mathbf{0}\}$.*

The system (13) *is $\mathfrak{d}$-homogeneous of the degree $\mu = 2$ with respect to the dilation $(\mathfrak{d}(s)x)(z) = e^s x(e^s z)$, $z \in \mathbb{R}, x \in \mathbb{X}$. The dilation $\mathfrak{d}$ is strictly monotone and strongly continuous in $\mathbb{X}$ and $\mathbb{Y}$ (see Example 1).*

Let us show that the system (13) *is globally uniformly asymptotically stable and the canonical homogeneous norm $\|\cdot\|_{\mathfrak{d},\mathbb{Y}}$ induced by the norm $\|\cdot\|_{\mathbb{Y}} = \|\cdot\|_{H^1}$ is a $\mathfrak{d}$-homogeneous Lyapunov function of the system. Indeed, for $v \in M^{-1}\mathcal{D}(A) = H^3(\mathbb{R}, \mathbb{R})\backslash\{\mathbf{0}\}$ we have*

$$\langle v, Av + f(v)\rangle_{H^1} =$$

$$\left\langle v, \tfrac{\partial^2 v}{\partial z^2} + v\tfrac{\partial v}{\partial z} - \tfrac{1}{3}\|v\|_{L^2}^4 v \right\rangle_{L^2} + \left\langle \tfrac{\partial v}{\partial z}, \tfrac{\partial}{\partial z}\left(\tfrac{\partial^2 v}{\partial z^2} + v\tfrac{\partial v}{\partial z} - \tfrac{1}{3}\|v\|_{L^2}^4 v\right)\right\rangle_{L^2} =$$

$$-\left\|\tfrac{\partial v}{\partial z}\right\|_{L^2}^2 - \tfrac{1}{3}\|v\|_{L^2}^6 - \left\|\tfrac{\partial^2 v}{\partial z^2}\right\|_{L^2}^2 - \left\langle \tfrac{\partial^2 v}{\partial z^2}, v\tfrac{\partial v}{\partial z}\right\rangle - \tfrac{1}{3}\|v\|_{L^2}^4 \left\|\tfrac{\partial v}{\partial z}\right\|_{L^2}^2 =$$

$$-\left\|\tfrac{\partial v}{\partial z}\right\|_{L^2}^2 - \tfrac{1}{3}\|v\|_{L^2}^6 - \tfrac{1}{2}\left\|\tfrac{\partial^2 v}{\partial z^2}\right\|_{L^2}^2 - \tfrac{1}{2}\left\|\tfrac{\partial^2 v}{\partial z^2} + v\tfrac{\partial v}{\partial z}\right\|_{L^2}^2 + \tfrac{1}{2}\left\|v\tfrac{\partial v}{\partial z}\right\|_{L^2}^2 - \tfrac{1}{3}\|v\|_{L^2}^4\left\|\tfrac{\partial v}{\partial z}\right\|_{L^2}^2 =$$

$$-\left\|\tfrac{\partial v}{\partial z}\right\|_{L^2}^2 - \tfrac{1}{3}\|v\|_{L^2}^6 - \tfrac{1}{2}\left\|\tfrac{\partial^2 v}{\partial z^2}\right\|_{L^2}^2 - \tfrac{1}{2}\left\|\tfrac{\partial^2 v}{\partial z^2} + v\tfrac{\partial v}{\partial z}\right\|_{L^2}^2 - \tfrac{1}{6}\left\langle v^3, \tfrac{\partial^2 v}{\partial z^2}\right\rangle - \tfrac{1}{3}\|v\|_{L^2}^4\left\|\tfrac{\partial v}{\partial z}\right\|_{L^2}^2 =$$

$$-\left\|\tfrac{\partial v}{\partial z}\right\|_{L^2}^2 - \tfrac{1}{3}\|v\|_{L^2}^6 - \tfrac{5}{12}\left\|\tfrac{\partial^2 v}{\partial z^2}\right\|_{L^2}^2 - \tfrac{1}{2}\left\|\tfrac{\partial^2 v}{\partial z^2} + v\tfrac{\partial v}{\partial z}\right\|_{L^2}^2 - \tfrac{1}{12}\left\|v^3 + \tfrac{\partial^2 v}{\partial z^2}\right\|_{L_2}^2 +$$

$$\tfrac{1}{12}\|v\|_{L_6}^6 - \tfrac{1}{3}\|v\|_{L^2}^4\left\|\tfrac{\partial v}{\partial z}\right\|_{L^2}^2.$$

Using the Gagliardo-Nirenberg-Sobolev inequality (see [6]) we derive $\|v\|_{L^6}^6 \leq \sqrt{\frac{\pi}{2}}\|v\|_{L^2}^4 \left\|\frac{\partial v}{\partial z}\right\|_{L^2}^2$ and conclude

$$\langle v, Av + f(v)\rangle_{H^1} \leq -\left\|\tfrac{\partial v}{\partial z}\right\|_{L^2}^2 - \tfrac{1}{3}\|v\|_{L^2}^6.$$

The latter means that $\|\cdot\|_{H_1}$ is a Lyapunov function of the system, but it is not $\mathfrak{d}$-homogeneous. Using $\mathfrak{d}$-homogeneity of operators A, f and the formula (7), for any $x \in M^{-1}\mathcal{D}(A)$ we obtain

$$\left(D\|x\|_{\mathfrak{d},H^1}\right)(Ax+f(x)) = 2\frac{\left\langle \mathfrak{d}(-\ln\|x\|_{\mathfrak{d},H^1})(Ax+f(x)),\, v\right\rangle_{H^1}}{\|v\|^2_{L_2}+\left\|\frac{\partial v}{\partial z}\right\|^2_{L_2}}\|x\|_{\mathfrak{d},H^1} =$$

$$2\frac{\langle Av+f(v),\, v\rangle_{H^1}}{\|v\|^2_{L_2}+\left\|\frac{\partial v}{\partial z}\right\|^2_{L_2}}\|u\|^3_{\mathfrak{d},H^1} \leq 2\frac{-\left\|\frac{\partial v}{\partial z}\right\|^2_{L^2}-\frac{1}{3}\|v\|^6_{L^2}}{\|v\|^2_{L_2}+\left\|\frac{\partial v}{\partial z}\right\|^2_{L_2}}\|x\|^3_{\mathfrak{d},H^1}$$

where $v = \mathfrak{d}(-\ln\|x\|_{\mathfrak{d},H^1})x$. Taking into account $\|v\|_{H^1} = 1$ we conclude that

$$\left(D\|x\|_{\mathfrak{d},H^1}\right)(Ax+f(x)) \leq -\frac{2}{3}\|x\|^{\mu+1}_{\mathfrak{d},H^1}$$

and the functional $V : \mathbb{Y} \to \mathbb{R}$ given by $V(x) = \frac{3}{2}\|x\|_{\mathfrak{d},H^1}$ satisfies Corollary 1. Notice also that the considered system is nearly fixed-time stable in the view of [22, Theorem 8.6].

4. Conclusion

The main contributions of the paper are Theorem 3 and Corollary 1, which provide a characterization (in terms of a locally Lipshcitz continuous Lyapunov function) of the uniform asymptotic stability of homogeneous infinite dimensional systems with unbounded nonlinear operators in Banach and reflexive Banach spaces, respectively. The existence of the regular Lyapunov function allows rather simple ISS analysis of homogeneous systems to be developed [1], [2], [23]. Namely, uniform asymptotic stability of unperturbed homogeneous system implies its ISS with respect to homogeneously involved perturbations. The expansion of this result to homogeneous infinite dimensional systems with unbounded nonlinear operators is an interesting problem for the future investigation.

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5. Appendix

5.1. Auxiliary results

Lemma 2 (Existence of Solutions). *Let $A : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ be closed densely defined linear operator which generates a strongly continuous semigroup*

Φ of linear bounded operators on $\mathbb{X}$ and $f : \mathcal{D}(f) \subset \mathbb{X} \to \mathbb{X}$ satisfy Assumption 1. Let $\delta > 1$ be an arbitrary real number and

$$f_\delta(x) = a_\delta(\|x\|_{\mathbb{Y}})f(x), \quad x \in \mathbb{Y}, q \in \mathbb{V}$$

where $a_\delta \in C_c^\infty(\mathbb{R}_+, \mathbb{R})$ such that $a_\delta(s) = 0$ for $s \notin (1/(2\delta), 2\delta)$ and $a_\delta(s) = 1$ for $s \in [1/\delta, \delta]$. Then for any $x_0 \in \mathbb{Y}$ the initial value problem

$$\dot{x} = Ax + f_\delta(x), \quad t \geq 0 \quad x(0) = x_0 \in \mathbb{Y} \tag{14}$$

has a unique mild solution $x^\delta : [0, +\infty) \to \mathbb{Y}$, which is continuous in $\mathbb{Y}$ and coincides with the mild solution x_{x_0} of (1) as long as $x_{x_0}(t) \in K_{\mathbb{Y}}(\delta)$. Moreover, for any $x_0 \in \mathbb{Y}\backslash\{\mathbf{0}\}$ the system (1) has a mild solution x_{x_0}, which is uniquely defined as long as $0 < \|x_{x_0}(t)\|_{\mathbb{Y}} < +\infty$.

Proof.

For $x_1 \in \mathbb{X}$, $x_2 \in \mathbb{X}$ we have $M^{-1}x_1, M^{-1}x_2 \in \mathbb{Y}$ and

$$\begin{gathered}\|f_\delta(M^{-1}x_1) - f_\delta(M^{-1}x_2)\|_{\mathbb{X}} = \\ \|a_\delta(\|M^{-1}x_1\|_{\mathbb{Y}})f(M^{-1}x_1) - a_\delta(\|M^{-1}x_2\|_{\mathbb{Y}})f(M^{-1}x_2)\|_{\mathbb{X}} \leq \\ \|f(M^{-1}x_1)\|_{\mathbb{X}}|a_\delta(\|M^{-1}x_1\|_{\mathbb{Y}}) - a_\delta(\|M^{-1}x_2\|_{\mathbb{Y}})| + \\ |a_\delta(\|M^{-1}x_2\|_{\mathbb{Y}})|\|f(M^{-1}x_1) - f(M^{-1}x_2)\|_{\mathbb{X}} = \\ \|f(M^{-1}x_1)\| \cdot |a_\delta'(\theta)| \cdot \big|\|M^{-1}x_1\|_{\mathbb{Y}} - \|M^{-1}x_2\|_{\mathbb{Y}}\big| + \\ |a_\delta(\|M^{-1}x_2\|_{\mathbb{Y}})| \cdot \|f(M^{-1}x_1) - f(M^{-1}x_2)\|_{\mathbb{X}}\end{gathered}$$

where the mean value theorem is utilized for $\theta \in [\min \|M^{-1}x_i\|_{\mathbb{Y}}, \max \|M^{-1}x_i\|_{\mathbb{Y}}]$, $i = 1, 2$.

Notice that $M^{-1}x_i \in K_{\mathbb{Y}}(2\delta)$ implies $x_i \in K_{\mathbb{X}}(2\xi\delta)$, where $\xi = 1 + \|M^{-1}\|_{\mathbb{X}} < +\infty$. Local Lipschitz continuity of $f \circ M^{-1}$ yields $\sup_{x \in K_{\mathbb{X}}(2\xi\delta)} \|f(M^{-1}x)\| < +\infty$ and $\exists \tilde{L}_\delta > 0$ such that

$$\|f(M^{-1}x_1) - f(M^{-1}x_2)\|_{\mathbb{X}} \leq \tilde{L}_\delta \|x_1 - x_2\|_{\mathbb{X}} \quad \forall x_1, x_2 \in K_{\mathbb{X}}(2\xi\delta).$$

Since $a_\delta \in C_c^\infty$ then $\sup_{s \in \mathbb{R}} |a_\delta'(s)| < +\infty$. Notice also $M^{-1}x_2 \notin K_{\mathbb{Y}}(2\delta) \Rightarrow a_\delta(\|M^{-1}x_2\|_{\mathbb{Y}}) = 0$. Therefore, taking into account

$$\big|\|M^{-1}x_1\|_{\mathbb{Y}} - \|M^{-1}x_2\|_{\mathbb{Y}}\big| \leq \|M^{-1}(x_1 - x_2)\|_{\mathbb{Y}} \leq (\|M^{-1}\| + 1)\|x_1 - x_2\|_{\mathbb{X}},$$

we conclude that $f_\delta \circ M^{-1}$ satisfy the Lipschitz condition on $\mathbb{X}$ with the Lipschitz constant $L_\delta := \tilde{L}_\delta + (\|M^{-1}\| + 1) \sup_{\theta \in \mathbb{R}} |a_\delta'(\theta)| \sup_{x \in K(2\xi\delta)} \|f(M^{-1}x)\| < +\infty$.

Given $x_0 \in \mathbb{Y}$ let us consider the metric space

$$\mathbf{Z} = \left\{ z \in C([0, T], \mathbb{X}) : z(0) = Mx_0, \sup_{t \in [0, T]} \|z(t)\|_{\mathbb{X}} < +\infty \right\},$$

with $T > 0$ and the metric $\rho(z_1, z_2) = \sup_{t\in[0,T]} \|z_1(t) - z_2(t)\|$. Let the operator F be defined on Z as follows

$$(Fz)(t) = \Phi(t)Mx_0 + \int_0^t M\Phi(t-\tau)f_\delta(M^{-1}z(\tau))d\tau.$$

The function $\tau \to f_\delta(M^{-1}z(\tau))$ is Bochner integrable in $\mathbb{X}$ if and only if $\tau \to \|f_\delta(M^{-1}z(\tau))\|_{\mathbb{X}}$ is Lebesgue integrable. Since $z \in C([0,T], \mathbb{X})$ then M-regularity of f implies $f(M^{-1}x(\cdot)) \in L^1((0,T), \mathbb{X})$ and $F : \mathbf{Z} \to \mathbf{Z}$. Indeed, one has

$$\begin{gathered}\|(Fz)(t)\|_{\mathbb{X}} \le \|\Phi(t)\|_{\mathbb{X}} \cdot \|Mx_0\|_{\mathbb{X}} + \\ \int_0^t \|M\Phi(t-\tau)\|_{\mathbb{X}} \cdot (\|f_\delta(M^{-1}z(\tau)) - f_\delta(\mathbf{0})\|_{\mathbb{X}}\, d\tau \le \\ k\|Mx_0\|_{\mathbb{X}} + L_\delta \int_0^t \omega(t-\tau)\|z(\tau)\|_{\mathbb{X}}\, d\tau\end{gathered}$$

where $k = \sup_{t\in[0,T]} \|\Phi(t)\|_{\mathbb{X}} < +\infty$ and $\omega(t-\tau) \ge \|M\Phi(t-\tau)\|_{\mathbb{X}}$ (see the assumption 1). Taking into account integrability of ω on $[0,T]$ we derive

$$\sup_{t\in[0,T]} \|(Fz)(t)\|_{\mathbb{X}} < +\infty$$

for any $z \in C([0,T], \mathbb{X})$. Let us show that the operator $F : \mathbf{Z} \to \mathbf{Z}$ has the unique fixed point $F(z^*) = z^*$ with $z^* \in \mathbf{Z}$. For any $z_1, z_2 \in \mathbf{Z}$ we have

$$\begin{gathered}\|F(z_1)(t) - F(z_2)(t)\|_{\mathbb{X}} \le \\ \int_0^t \omega(t-\tau)\|f_\delta(M^{-1}z_1(\tau), q(\tau)) - f_\delta(M^{-1}z_2(\tau), q(\tau))\|_{\mathbb{X}} d\tau \le \\ \int_0^t \omega(\sigma)d\sigma \sup_{t\in[0,T]} \|z_1(\tau)) - z_2(\tau))\|_{\mathbb{X}} \le \left(L_{\delta,\bar{q}} \int_0^T \omega(\sigma)d\sigma\right) \rho(z_1, z_2).\end{gathered}$$

Since $\int_0^T \omega(\sigma)d\sigma \to 0$ as $T \to 0$ then for a sufficiently small $T > 0$ we have $\eta = L_{\delta,\bar{q}} \int_0^T \omega(\sigma)d\sigma < 1$ and

$$\|F^k(z_1)(t) - F^k(z_2)(t)\|_{\mathbb{X}} \le \eta^k \rho(x_1, x_2), \quad k = 1, 2, \ldots.$$

Using the contraction principle (Banach Fixed Point Theorem) we deduce the existence of a unique fixed point z^* of the operator F in $\mathbf{Z}$, which corresponds to a mild solution $x^\delta := M^{-1}z^*$ of the nonlinear system (14). Indeed, using Assumption 1 we derive

$$\begin{gathered}x^\delta(t) = M^{-1}z^*(t) = M^{-1}\Phi(t)Mx_0 + \\ M^{-1} \int_0^t M\Phi(t-\tau)f_\delta(M^{-1}z^*(\tau), q(\tau))d\tau = \\ \Phi(t)x_0 + \int_0^t \Phi(t-\tau)f_\delta(x^\delta(\tau), q(\tau))d\tau.\end{gathered}$$

Notice also $x^\delta(t) = M^{-1}z^*(t) \in \mathbb{Y}$ for all $t \in [0,T]$.

The solution x^δ can be prolonged for all $t \ge 0$. Suppose the converse, i.e exists $t_1 \in \mathbb{R}_+$ such that $\|x^\delta(t)\|_{\mathbb{Y}} \to +\infty$ as $t \to t_1$. In this case, $\|M^{-1}z^*(t)\|_{\mathbb{Y}} \to +\infty$ as $t \to t_1$ and there exists $t' \in (0, t_1)$ such that $\|M^{-1}z^*(t)\|_{\mathbb{Y}} \ge 2\delta$ for all $t \in (t', t_1)$. The latter implies that $f_\delta(M^{-1}z^*(t), q(t)) = \mathbf{0}$ for all $t \in (t', t_1)$,

$$z^*(t) = \Phi(t-t')Mx^\delta(t') \quad \text{and} \quad x^\delta(t) = \Phi(t-t')x^\delta(t'), \quad \forall t \ge [t', t_1).$$

Since Φ is a strongly continuous semigroup of linear bounded operators on $\mathbb{B}$ then the inequality $t_1 - t' < +\infty$ implies $\sup_{s\in[0,t_1-t']} \|\Phi(s)\|_{\mathbb{X}} < +\infty$. Using assumption 1 we derive

$$\|x^\delta(t)\|_{\mathbb{Y}} = \|\Phi(t-t')x^\delta(t')\|_{\mathbb{X}} + \|M\Phi(t-t')x^\delta(t')\|_{\mathbb{X}} =$$
$$\|\Phi(t-t')x^\delta(t')\|_{\mathbb{X}} + \|\Phi(t-t')Mx^\delta(t')\|_{\mathbb{X}} \leq \|\Phi(t-t')\|_{\mathbb{X}}\|x^\delta(t')\|_{\mathbb{Y}}$$

and $\limsup_{t\to t_1} \|x^\delta(t)\|_{\mathbb{Y}} < +\infty$.

If $x_0 \in K_{\mathbb{Y}}(\delta)$ then, obviously, the mild solution x^δ is the unique mild solution of (1) as long as $x^\delta(t) = M^{-1}z^*(t) \in K_{\mathbb{Y}}(\delta)$. Since $\|x^\delta(t+h) - x(t)\|_{\mathbb{Y}} = \|M^{-1}(z^*(t+h)-z^*(t))\|_{\mathbb{X}} + \|z^*(t+h)-z^*(t)\|_{\mathbb{X}}, t \geq t_0, h > 0$ then the continuity of z in $\mathbb{X}$ implies the continuity of x^δ in $\mathbb{Y}$. Tending $\delta \to +\infty$ we complete the proof. ■

Corollary 2 (Continuous dependence of initial conditions). *If all conditions of Lemma 2 are fulfilled then for any $T > 0$ there exists $L_{\delta,T} > 0$ such that for any $x_1, x_2 \in \mathbb{Y}$ one has*

$$\|x^\delta_{x_1}(t) - x^\delta_{x_2}(t)\|_{\mathbb{Y}} \leq L_{\delta,T}\|x_1 - x_2\|_{\mathbb{Y}}, \quad \forall t \in [0,T]$$

where $x^\delta_{x_i}(t)$ is a mild solution of (14) with $x(t_0) = x_i$, $i = 1, 2$.

Proof. Since

$$\|x^\delta_{x_1}(t) - x^\delta_{x_2}(t)\|_{\mathbb{X}} \leq \|\Phi(t)\|_{\mathbb{X}} \cdot \|x_1 - x_2\|_{\mathbb{X}} +$$
$$\int_0^t \|\Phi(t-\tau)\|_{\mathbb{X}} \cdot \|f_\delta(x^\delta_{x_1}(\tau)) - f_\delta(x^\delta_{x_2}(\tau))\|_{\mathbb{X}} d\tau$$

then taking into account Lipschitz continuity of $f_\delta \circ M^{-1}$ we derive

$$\|x^\delta_{x_1}(t) - x^\delta_{x_2}(t)\|_{\mathbb{X}} \leq k\|x_1 - x_2\|_{\mathbb{X}} + kL_\delta \int_0^t \|Mx^\delta_{x_1}(\tau) - Mx^\delta_{x_2}(\tau)\|_{\mathbb{X}} d\tau,$$

where $k := \sup_{s\in[0,T]} \|\Phi(s)\|_{\mathbb{X}} < +\infty$. On the other hand, we have

$$Mx^\delta_{x_i}(t) = z^*_{x_i}(t) = \Phi(t-t_0)Mx_0 + \int_0^t M\Phi(t-\tau)f_\delta(M^{-1}z^*_{x_i}(\tau))d\tau, i = 1, 2,$$

where $z^*_{x_i}$ is defined in the proof of Lemma 2. Hence,

$$\|Mx^\delta_{x_1}(t) - Mx^\delta_{x_2}(t)\|_{\mathbb{X}} \leq h\|Mx_1 - Mx_2\|_{\mathbb{X}} +$$
$$\int_0^t \|M\Phi(t-\tau)\|_{\mathbb{X}} \cdot \|f_\delta(M^{-1}z^*_{x_1}(\tau)) - f_\delta(M^{-1}z^*_{x_2}(\tau))\|_{\mathbb{X}} d\tau \leq$$
$$k\|Mx_1 - Mx_2\|_{\mathbb{X}} + L_\delta \int_0^t \omega(t-\tau) \cdot \|M^{-1}z^*_{x_1}(\tau) - M^{-1}z^*_{x_2}(\tau)\|_{\mathbb{X}} d\tau \leq$$

$$k\|Mx_1 - Mx_2\|_{\mathbb{X}} + L_\delta \int_0^t \omega(t-\tau)\cdot \|x^\delta_{x_1}(\tau) - x^\delta_{x_2}(\tau)\|_{\mathbb{X}} d\tau,$$

where $\omega(t-\tau) \geq \|M\Phi(t-\tau)\|_{\mathbb{X}}$ (see Assumption 1). Combining the two obtained estimates we derive

$$\|x^\delta_{x_1}(t) - x^\delta_{x_2}(t)\|_{\mathbb{Y}} \leq k\|x_1 - x_2\|_{\mathbb{Y}} + L_\delta \int_0^t (k + \omega(t-\tau))\|x^\delta_{x_1}(\tau) - x^\delta_{x_2}(\tau)\|_{\mathbb{Y}} d\tau.$$

Hence, using the Grönwall-Bellman inequality we derive

$$\|x^\delta_{x_1}(t) - x^\delta_{x_2}(t)\|_{\mathbb{Y}} \leq k e^{L_\delta \int_0^t (k+\omega(t-\tau))d\tau} \|x_1 - x_2\|_{\mathbb{Y}}$$

for all $t \in [0, T]$. Taking $L_{\delta,T} := k e^{L_\delta \int_0^T (k+\omega(t-\tau))d\tau}$ we complete the proof. ■

Corollary 3. *Let all conditions of Lemma 2 be fulfilled and $x_0 \in M^{-1}\mathcal{D}(A)$. Then the mild solution $x^\delta_{x_0}$ of (14) is*

1) *locally Lipschitz continuous as a function $[0, +\infty) \to \mathbb{X}$;*

2) *locally uniformly continuous as a function $[0, +\infty) \to \mathbb{Y}$;*

3) *a strong solution of (14), $x(t) \overset{a.e.}{\in} M^{-1}\mathcal{D}(A)$ and*

$$M\dot{x}^\delta(t) \overset{a.e.}{=} MAx^\delta(t) + Mf(x^\delta(t)), \quad t > 0,$$

provided that $\mathbb{X}$ is a reflexive Banach space.

Proof. Let us show $x_0 \in M^{-1}\mathcal{D}(A)$ implies $x_0 \in \mathcal{D}(A)$ provided that Assumption 1 holds. Indeed, $x_0 \in M^{-1}\mathcal{D}(A)$ implies $Mx_0 \in \mathcal{D}(A)$ and $x_0 \in \mathbb{Y}$ (since $M^{-1} : \mathbb{B} \to \mathbb{Y}$). In this case, we have

$$\lim_{h\to 0^+} \frac{\Phi(h)Mx_0 - Mx_0}{h} = AMx_0.$$

On the other hand, Assumption 1 implies

$$\lim_{h\to 0^+} \frac{\Phi(h)Mx_0 - Mx_0}{h} = \lim_{h\to 0^+} M\frac{\Phi(h)x_0 - x_0}{h}.$$

Since M by assumption is a closed linear operator and $\mathbb{Y}$ is its domain then the existence of the above limits yields $\frac{\Phi(h)x_0 - x_0}{h} \to y^* \in \mathbb{Y}$ as $h \to 0$. Since A is a closed linear operator as well then the existence of $\lim_{h\to 0^+} \frac{\Phi(h)x_0 - x_0}{h} \in \mathbb{X}$ guarantees $x_0 \in \mathcal{D}(A)$ and

$$MAM^{-1}z = Az, \quad \forall z \in \mathcal{D}(A).$$

1) Recall (see e.g. [21], page 5) that if $z \in \mathcal{D}(A)$ then $A\Phi(s)z = \Phi(s)Az$ and

$$\Phi(s+h)z - \Phi(s)z = A\int_s^{s+h} \Phi(\tau)z d\tau = \int_s^{s+h} \Phi(\tau)Az d\tau$$

for all $s \geq 0$ and all $h > 0$. Let us denote

$$C(T) := \operatorname{ess}\sup_{s\in[0,T]} \|Mx^{\delta}_{x_0}(s)\|_{\mathbb{X}}, \quad \text{where} \quad T > 0.$$

Lemma 2 implies $C(T) < +\infty$ for any $x_0 \in \mathbb{Y}$. Since

$$\|f_\delta(x^{\delta}_{x_0}(\tau))\|_{\mathbb{X}} = \|f_\delta(M^{-1}Mx^{\delta}_{x_0}(\tau)) - f_\delta(0)\|_{\mathbb{X}} \leq L_\delta\|Mx^{\delta}_{x_0}(\tau)\|_{\mathbb{X}}$$

then

$$\|x^{\delta}_{x_0}(t+h)-x^{\delta}_{x_0}(t)\|_{\mathbb{X}} \leq \|\Phi(t+h)x_0-\Phi(t)x_0\|_{\mathbb{X}}+\int_t^{t+h} \|\Phi(t-\tau)f_\delta(x^{\delta}_{x_0}(\tau))\|_{\mathbb{X}}\, d\tau \leq$$

$$\int_t^{t+h} \|\Phi(\tau)\|_{\mathbb{X}}\cdot\|Ax_0\|_{\mathbb{X}}d\tau+\int_t^{t+h} \|\Phi(s)\|\|f_\delta(x^{\delta}_{x_0}(\tau))\|_{\mathbb{X}}\, d\tau \leq hk(\|Ax_0\|_{\mathbb{X}}+L_\delta C(T))$$

for all $h \in [0, T]$, where $k = \sup_{s\in[0,T]} \|\Phi(s)\|$ and the Lipschitz constant L_δ is defined in the proof of Lemma 2.

2) Now let us prove the uniform continuity of the solution in $\mathbb{Y}$. We have

$$\|Mx^{\delta}_{x_0}(t+h) - Mx^{\delta}_{x_0}(t)\|_{\mathbb{X}} \leq \|\Phi(t+h)Mx_0 - \Phi(t)Mx_0\|_{\mathbb{X}}+$$

$$\int_t^{t+h} \omega(t-\tau)\|f_\delta(x^{\delta}_{x_0}(\tau))\|_{\mathbb{X}}\, d\tau \leq$$

$$\int_t^{t+h} \|\Phi(\tau)\|_{\mathbb{X}} \cdot \|AMx_0\|_{\mathbb{X}}d\tau + L_\delta \int_t^{t+h} \omega(t-\tau)\|Mx^{\delta}_{x_0}(\tau)\|_{\mathbb{X}} \leq$$

$$hk\|AMx_0\|_{\mathbb{X}} + L_\delta C(T) \int_0^h \omega(s)ds$$

for all $h \in [0, T]$, where $\omega(s) \geq \|M\Phi(s)\|_{\mathbb{X}}$ (see Assumption 1). Combining the estimate obtained in the case 1), we derive

$$\|x^{\delta}_{x_0}(t+h) - x^{\delta}_{x_0}(t)\|_{\mathbb{Y}} \leq h(k\|Ax_0\|_{\mathbb{Y}} + L_\delta C(T)k) + L_\delta C(T) \int_0^h \omega(s)ds.$$

Taking into account $\int_0^h \omega(s)ds \to 0$ as $h \to 0$, we complete the proof.

3) Let $x^\delta : [t_0, t_0 + T]$ be a mild solution with $x_0 \in M^{-1}\mathcal{D}(A)$. Recall (see the proof of Lemma 2) that $x^\delta(t) = Mz(t)$, where $t \geq t_0$ and

$$z(t) = \Phi(t)Mx_0 + \int_0^t M\Phi(t-s)f_\delta(M^{-1}z(s))ds.$$

Let us show that $z : [p, T] \to \mathbb{X}$ satisfies a Lipschitz condition for each $p \in (0, T)$. Indeed, since

$$z(t+h) - z(t) = (\Phi(t+h) - \Phi(t))Mx_0 + \int_0^{t+h} M\Phi(t+h-s)f_\delta(M^{-1}z(s))ds-$$

$$\int_{t_0}^{t} M\Phi(t-s) f_\delta(M^{-1}z(s))ds = \int_{t}^{t+h} \Phi(s)AMx_0 +$$

$$\int_0^h M\Phi(t+h-s) f_\delta(M^{-1}z(s))ds + \int_0^t M\Phi(t-s)(f_\delta(M^{-1}z(s+h)) - f_\delta(M^{-1}z(s)))ds.$$

Notice that

$$\left\| \int_0^{t+h} M\Phi(t+h-s) f_\delta(M^{-1}z(s))ds \right\|_{\mathbb{X}} \le \int_0^h \omega(t+\sigma)\cdot \|f_\delta(M^{-1}z(h-\sigma))\|_{\mathbb{X}} d\sigma \le$$

$$hC(T)\bar{\omega}(T,p), \quad \text{where} \quad \bar{\omega}(T,p) := \max_{s\in[p,T]} \omega(s).$$

Hence, using Lipschitz continuity of f_δ we derive

$$\|z(t+h) - z(t)\|_{\mathbb{X}} \le \left(k\|AMx_0\|_{\mathbb{X}} + C(T)\bar{\omega}(T,p)\right) h +$$

$$L_\delta \int_0^t \omega(t-s) \|z(s+h) - z(s)\|_{\mathbb{X}} ds.$$

Applying the Grönwall-Bellman inequality we conclude

$$\|z(t+h) - z(t)\|_{\mathbb{X}} \le \left(k\|AMx_0\|_{\mathbb{X}} + C(T)\bar{\omega}(T,p)\right) e^{L_\delta \int_0^t \omega(t-s)ds} h,$$

i.e. $z : [p,T] \to \mathbb{X}$ satisfies Lipschitz condition for each $p \in (0,T)$. Since $z \to f_\delta(M^{-1}z)$ is Lipschitz continuous and $f_\delta(x^\delta(t)) = f_\delta(M^{-1}z(t)), t \in [0,T]$ then the function $t \to f(x^\delta(t))$ is locally Lipschitz continuous. Using [21, Chapter 4, Corollary 2.11] we conclude that x is a strong solution, i.e. $\dot{x}^\delta(t) \overset{a.e.}{=} Ax^\delta(t) + f_\delta(x^\delta(t)), t > 0$. Since $\mathbb{X}$ is a reflexive Banach space then it satisfies the Radon-Nykodym property, i.e. locally Lipschitz continuity of z implies its differentiability almost everywhere. Since M is a closed linear operator then, taking into account $Mx^\delta(t) = z(t)$, we derive $\dot{z}(t) \overset{a.e.}{=} M\dot{x}^\delta(t) \overset{a.e.}{=} MAx^\delta(t) + Mf(x^\delta(t), \mathbf{0}), t \ge 0$ and

$$\dot{z}(t) \overset{a.e.}{=} MAM^{-1}z(t) + Mf(M^{-1}z(t), \mathbf{0}), \quad t \ge 0,$$

i.e. z is a strong solution of the latter equation. Assumption 1 implies that

$$\tilde{\Phi}(t)v := M\Phi(t)M^{-1}v = \Phi(t)v, \quad \forall t \ge 0, \forall v \in \mathbb{X},$$

and $\tilde{\Phi}$ is a strongly continuous semigroup of linear bounded operators on $\mathbb{X}$. Since the operator $MAM^{-1} : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ is, obviously, an infinitesimal generator of $\tilde{\Phi}$ and $\tilde{\Phi} = \Phi$ then $MAM^{-1} = A$ due to the uniqueness of an infinitesimal generator. Using [21, Chapter 4, Theorem 2.9] we derive $z(t) \overset{a.e.}{\in} \mathcal{D}(A)$ and, consequently, $x^\delta(t) = M^{-1}z(t) \overset{a.e.}{\in} M^{-1}\mathcal{D}(A)$. The proof is complete. ■

Corollary 4. *Let all conditions of Lemma 2 be fulfilled. Let the origin of* (1) *be globally uniformly Lyapunov stable in* $\mathbb{Y}$*. Let* $r>1$ *be an arbitrary real number and*

$$\delta > \max\left\{\varepsilon(r), \frac{1}{\varepsilon^{-1}(1/r)}\right\},$$

where $\varepsilon\in\mathcal{K}^{\infty}$ *is given by* (10) *and* ε^{-1} *is the inverse function to* ε*. Then*

1) for any $x_0\in\mathbb{Y} : 1/r\leq \|x_0\|_{\mathbb{Y}}\leq r$ *we have*

$$\|x^{\delta}_{x_0}(t)\|_{\mathbb{Y}}\leq \varepsilon(\|x_0\|_{\mathbb{Y}})\leq \delta, \quad \forall t\geq 0$$

2) if $\exists t_1\geq 0$ *such that* $\|x^{\delta}(t_1)\|_{\mathbb{Y}}\leq 1/\delta$ *then*

$$\|x^{\delta}_{x_0}(t)\|_{\mathbb{Y}}\leq 1/r, \quad \forall t\geq t_1,$$

where $x^{\delta}_{x_0}$ *is the unique mild solution of* (14).

Proof. 1) Notice that the the inequality (10) and $x_{x_0}(0)=x_0$ yield $\varepsilon(r)\geq r$ and $1/\delta<1/r<r<\delta$. Since $x^{\delta}_{x_0}$ coincide with the solution x_{x_0} of (1) as long as $x^{\delta}_{x_0}(t)\in K_{\mathbb{Y}}(\delta)$ then the inequality (10) implies $\|x^{\delta}_{x_0}(t)\|_{\mathbb{Y}}\leq\varepsilon(\|x_0\|_{\mathbb{Y}})\leq\delta$ for all $t\geq 0$ and any $x_0 : \|x_0\|_{\mathbb{Y}}\leq[1/r, r]$. 2) Suppose the contrary, i.e. $\exists t_2>t_1$ such that $\|x^{\delta}_{x_0}(t_2)\|>1/r$ and $\|x^{\delta}_{x_0}(t)\|>1/\delta$ for $t\in(t_1,t_2]$. In this case, we have $x^{\delta}_{x_0}(t)=x_{x_0}(t)$ for all $t\in(t_1,t_2]$ and using the inequality (10) we conclude

$$\|x^{\delta}_{x_0}(t)\|_{\mathbb{Y}}\leq\varepsilon(\|x^{\delta}_{x_0}(t_1)\|_{\mathbb{Y}})=\varepsilon(1/\delta)\leq\varepsilon(\varepsilon^{-1}(1/r))=1/r$$

for all $t\geq t_1$. ■

5.2. The proof of Theorem 3

Notice that the existence of a $\mathfrak{d}$-homogeneous Lyapunov functional $V:\mathbb{Y}\to\mathbb{R}$ of the degree $m>0$ satisfying (11), (12) is equivalent to the existence of a $\mathfrak{d}$-homogeneous Lyapunov functional $\tilde{V}:\mathbb{Y}\to\mathbb{R}$ of the degree 1 satisfying

$$\underline{c}\|x\|_{\mathfrak{d},\mathbb{Y}}\leq\tilde{V}(x)\leq\overline{c}\|x\|_{\mathfrak{d},\mathbb{Y}}, \quad \forall x\in\mathbb{Y}, \quad \underline{c},\overline{c}>0 \tag{15}$$

and

$$\overline{D}^{+}\tilde{V}(x_{x_0}(t))\leq-\|x_{x_0}(t)\|^{\mu+1}_{\mathfrak{d},\mathbb{Y}}, \quad t>0, \tag{16}$$

respectively. Indeed, selecting $\tilde{V}=m\underline{k}^{1-1/m}V^{1/m}$ for $m\in(0,1)$ (resp. $\tilde{V}=m\overline{k}^{1-1/m}V^{1/m}$ for $m>1$) we easily derive the mentioned inequalities with $\underline{c}=m\underline{k}$ and $\overline{c}=m\overline{k}^{1/m}\underline{k}^{1-1/m}$ (resp. $\underline{c}=m\overline{k}^{1-1/m}\underline{k}^{1/m}$and $\overline{c}=m\overline{k}$). This means that without loss of generality we can fix $m=1$ for the proofs of Theorem 3 and Corollary 1.

Sufficiency. Let $\delta>1$ be an arbitrary number. Then by Lemma 2 the solution $x^{\delta}_{x_0}$ of the system (14) with $f_{\delta}(x)=a_{\delta}(\|x\|_{\mathbb{Y}})f(x)$. coincides with the solution x_{x_0} of (1) as long as $x^{\delta}_{x_0}(t)\in K_{\mathbb{Y}}(\delta)$. Using the inequality (12) we conclude that the function $t\to V(x^{\delta}_{x_0}(t))$ is monotone decreasing and

$$\dot{V}(x^{\delta}_{x_0}(t))\overset{a.e.}{=}\overline{D}^{+}V(x^{\delta}_{x_0}(t))\leq-\|x^{\delta}_{x_0}(t)\|^{\mu+1}_{\mathfrak{d}}.$$

Hence, we conclude

$$V(x_{x_0}^{\delta}(t)) \leq V(x_0) - \int_0^t \|x_{x_0}^{\delta}(t)\|_{\mathfrak{d}}^{\mu+1} ds \leq V(x_0) - \frac{1}{\bar{k}} \int_0^t V^{\mu+1}(x_{x_0}^{\delta}(t)) ds$$

as long as $x_{x_0}^{\delta}(t) \in K_{\mathbb{Y}}(\delta)$.

Tending $\delta \to +\infty$ we conclude that for any $x_0 \in \mathbb{Y}\backslash\{\mathbf{0}\}$ the function $t \to V(x_{x_0}(t))$ is strictly decreasing and

$$V(x_{x_0}(t)) \leq V(x_0) - \frac{1}{\bar{k}} \int_0^t V^{\mu+1}(x_{x_0}(t)) ds$$

as long as $0 < \|x_{x_0}(t)\|_{\mathbb{Y}} < +\infty$. Hence, for any $r > 0$ we derive

$$V(x_{x_0}(t)) \leq r, \qquad \forall t \geq \max\{0, \bar{k}(V^{-\mu}(x_0) - r^{-\mu})\}.$$

The latter implies that mild solutions of (1) converges to zero uniformly on the initial data, i.e. for any $\tilde{R} > \tilde{r} > 0$ there exists $\hat{T} = \hat{T}(\tilde{R}, \tilde{r})$ such that $\|x_{x_0}(t)\| \leq \tilde{r}$ for all $x_0 \in B_{\mathbb{Y}}(\tilde{r})$ and for all $t \geq \hat{T}$. Moreover, by assumption, for all $x_0 \in \mathbb{Y}\backslash\{\mathbf{0}\}$ we have

$$\underline{k}\|x_{x_0}(t)\|_{\mathfrak{d},\mathbb{Y}} \leq V(x_{x_0}(t)) \leq V(x_0) \leq \bar{k}\|x_0\|_{\mathfrak{d},\mathbb{Y}}$$

and

$$\|x_{x_0}(t)\|_{\mathbb{Y}} \leq \varepsilon(\|x_0\|) := \underline{\sigma}^{-1}\left(\frac{\bar{k}}{\underline{k}}\bar{\sigma}(\|x_0\|_{\mathbb{Y}})\right) \tag{17}$$

as long as $x_{x_0}(t) \neq \mathbf{0}$, where $\underline{\sigma}^{-1}$ is the inverse function to $\underline{\sigma}$ (see Theorem 1). Since $\underline{\sigma}, \bar{\sigma} \in \mathcal{K}^{\infty}$ then $\varepsilon \in \mathcal{K}^{\infty}$.

Let us prove that if $x_{x_0}(t_0) = \mathbf{0}$ then $x_{x_0}(t) = \mathbf{0}$ for all $t \geq t_0$. Suppose the converse, i.e. $\exists t_1 > t_0 : \epsilon := \|x(t_1)\|_{\mathbb{Y}} > 0$. Then due to continuity of $t \to \|x(t)\|_{\mathbb{Y}}$ there exists $t_2 \in (t_0, t_1)$ such that $\|x(t_2)\|_{\mathbb{Y}} = \delta := \min\{\epsilon/2, \varepsilon^{-1}(\epsilon/2)\}$ and $\|x(t)\|_{\mathbb{Y}} > \delta$ for all $t \in (t_2, t_1]$, where $\varepsilon^{-1} \in \mathcal{K}^{\infty}$ is the inverse function to ε. In this case, from the inequality (17) we derive

$$\|x_{x_0}(t)\|_{\mathbb{Y}} \leq \varepsilon(\|x(t_2)\|_{\mathbb{Y}}) = \varepsilon(\delta) \leq \varepsilon(\varepsilon^{-1}(\epsilon/2)) = \epsilon/2$$

for all $t \in (t_2, t_1]$. This contradicts to the supposition $\|x_{x_0}(t_1)\|_{\mathbb{Y}} = \epsilon$. Therefore, the inequality (17) holds for all $t \geq 0$ and any mild solution of (1) in $\mathbb{Y}$.

Necessity. We design the Lyapunov function in two steps: first, we construct a function $V_r : \mathbb{Y} \to \mathbb{R}_+$ such that $t \to V_r(x_0(t))$ is strictly decreasing as long as $x_{x_0}(t) \in K_{\mathbb{Y}}(r)$ for some $r > 1$; next, using V_r we generate a $\mathfrak{d}$-homogeneous Lyapunov function $V : \mathbb{B} \to \mathbb{R}_+$ by means of an integral transformation inspired by [26].

I. Let $s_0 > 0$,

$$r_0 := \sup_{s \in [-s_0, s_0]} \|\mathfrak{d}(s)\|_{\mathbb{Y}}$$

and $r > r_0^2 > 1$. Let $\delta > r$ and $a_\delta \in C_c^{\infty}$ be defined as in Corollary 4. Then for any $x_0 \in \mathbb{Y}$ the system (14) has the unique mild solution $x_{x_0}^{\delta} : [0, +\infty) \to \mathbb{Y}$

which, for $x_0 \in K_{\mathbb{Y}}(\delta)$, coincides with the unique solution x_{x_0} of (1) as long as $x^{\delta}_{x_0}(t) \in K_{\mathbb{Y}}(\delta)$. Moreover, by Corollary 2, for any $T > 0$ there exists $L_{\delta,T} > 0$ such that

$$\|x^{\delta}_{x_1}(t) - x^{\delta}_{x_2}(t)\|_{\mathbb{Y}} \leq L_{\delta,T}\|x_1 - x_2\|_{\mathbb{Y}},$$

for all $t \in [0, T]$.

Since the origin of (1) is globally uniformly asymptotically stable then there exists $T_r > 0$ such that

$$\|x_0\|_{\mathbb{Y}} < r \quad \Rightarrow \quad \|x_{x_0}(t)\|_{\mathbb{Y}} \leq 1/r, \ \ \forall t \geq T_r.$$

In the view of Lemma 2 and Corollary 4 we have $x^{\delta}_{x_0}(t) = x_{x_0}(t)$ as long as $\|x_{x_0}(t)\|_{\mathbb{Y}} \geq 1/\delta$ provided that $x_0 \in K_{\mathbb{Y}}(r)$. Indeed, the global uniform Lyapunov stability of (1) and the selection of the parameter $\delta > r$ imply that $\|x^{\delta}_{x_0}(t)\|_{\mathbb{Y}} \leq \delta$ for all $t \geq 0$ and all $x_0 \in K_{\mathbb{Y}}(r)$. Moreover, if there exists $t_1 > 0$ such that $\|x^{\delta}_{x_0}(t_1)\|_{\mathbb{Y}} \leq 1/\delta$ (resp. $\|x_{x_0}(t_1)\|_{\mathbb{Y}} \leq 1/\delta$) then $\|x^{\delta}_{x_0}(t)\|_{\mathbb{Y}} \leq 1/r$ (resp. $\|x_{x_0}(t)\|_{\mathbb{Y}} \leq 1/r$) for all $t \geq t_1$. In this case, for the functional $V_0 : \mathbb{Y} \to \mathbb{R}_+$ given by

$$V_0(x_0) := \sup_{t\geq 0} \|x_{x_0}(t)\|_{\mathbb{Y}}$$

we have

$$V_0(x_0) = \sup_{t\in[0,T_r]} \|x^{\delta}_{x_0}(t)\|_{\mathbb{Y}}$$

for all $x_0 \in K(r)$. Hence, using the triangle inequality we derive

$$|V_0(x_1)-V_0(x_2)| \leq \sup_{t\in[0,T_r]} \|x^{\delta}_{x_1}(t)-x^{\delta}_{x_2}(t)\|_{\mathbb{Y}} \leq L_{\delta,T_r}\|x_1-x_2\|_{\mathbb{Y}}, \ \forall x_1, x_2 \in K_{\mathbb{Y}}(r).$$

The latter means that the functional V_0 satisfy the Lipschitz condition on $K_{\mathbb{Y}}(r)$.

The global uniform Lyapunov stability of (1) implies that $\|x\|_{\mathbb{Y}} \leq V_0(x) \leq \varepsilon(\|x\|_{\mathbb{Y}})$, $\forall x \in \mathbb{Y}$, where $\varepsilon \in \mathcal{K}$ is given by (10). Moreover, the function $t \to V_0(x_{x_0}(t))$ is decreasing as long as $x_{x_0}(t) \in K_{\mathbb{Y}}(r)$. Let us consider the functional

$$V_1(x_0) := \int_0^{+\infty} \tilde{a}(\|x_{x_0}(s)\|_{\mathbb{Y}})\|x_{x_0}(s)\|_{\mathbb{Y}}ds,$$

where $\tilde{a} \in C^{\infty}_c$ is such that $\tilde{a}(\rho) = 0$ for $s \notin (1/r, r)$, $0 < \tilde{a}(\rho) \leq 1$ for $\rho \in (1/r, r)$ and $\tilde{a}(\rho) = 1$ for $\rho \in [1/r_0, r_0]$.

Notice also $\tilde{a}(\|x_{x_0}(s)\|_{\mathbb{Y}})\|x_{x_0}(s)\|_{\mathbb{Y}} = 0$ if $x_{x_0}(s) \notin K_{\mathbb{Y}}(r)$. Repeating the above considerations we conclude that

$$V_1(x_0) = \int_0^{T_r} \tilde{a}(\|x_{x_0}(s)\|_{\mathbb{Y}})\|x_{x_0}(s)\|_{\mathbb{Y}}ds = \int_0^{T_r} \tilde{a}(\|x^{\delta}_{x_0}(s)\|_{\mathbb{Y}})\|x^{\delta}_{x_0}(s)\|_{\mathbb{Y}}ds$$

for all $x_0 \in K_{\mathbb{Y}}(r)$. Since $x \to \tilde{a}(\|x\|_{\mathbb{Y}})\|x\|_{\mathbb{Y}}$ satisfy the Lipschitz condition on $\mathbb{Y}$ with some Lipschitz constant $\tilde{L} > 0$ then

$$|V_1(x_1) - V_1(x_2)| \leq \left| \int_0^{T_r} \tilde{a}(\|x^{\delta}_{x_1}(s)\|_{\mathbb{Y}})\|x^{\delta}_{x_1}(s)\|_{\mathbb{Y}} - \tilde{a}(\|x^{\delta}_{x_2}(s)\|_{\mathbb{Y}})\|x^{\delta}_{x_2}(s)\|_{\mathbb{Y}}ds \right| \leq$$

$$\int_0^{T_r} |\tilde{a}(\|x^\delta_{x_1}(s)\|_{\mathbb{Y}})\|x^\delta_{x_1}(s)\|_{\mathbb{Y}} - \tilde{a}(\|x^\delta_{x_2}(s)\|_{\mathbb{Y}})\|x^\delta_{x_2}(s)\|_{\mathbb{Y}}|ds \leq$$

$$\tilde{L}\int_0^{T_r} \|x^\delta_{x_1}(s) - x^\delta_{x_2}(s)\|_{\mathbb{Y}}ds \leq T_r L_{\delta,T_r}\tilde{L}\|x_1 - x_2\|_{\mathbb{Y}}, \quad \forall x_1, x_2 \in K_{\mathbb{Y}}(r),$$

i.e. V_1 also satisfies the Lipschitz condition on $K_{\mathbb{Y}}(r)$.

Moreover, if $h > 0$ and $t \geq 0$ are such that $x_{x_0}(t+\theta) \in K(r)$, $\forall\theta \in [0, h]$ then $x_{x_0}(t+\theta) = x^\delta_{x_{x_0}(t)}(\theta)$, $\forall\theta \in [0, h]$ and

$$V_1(x_{x_0}(t+h)) - V_1(x_{x_0}(t)) =$$

$$\int_0^{T_r} \tilde{a}(\|x^\delta_{x(t+h)}(s)\|_{\mathbb{Y}})\|x^\delta_{x(t+h)}(s)\|_{\mathbb{Y}}ds - \int_0^{T_r} \tilde{a}(\|x^\delta_{x(t)}(s)\|_{\mathbb{Y}})\|x^\delta_{x(t)}(s)\|_{\mathbb{Y}}ds =$$

$$\int_0^{T_r} \tilde{a}(\|x^\delta_{x(t+h)}(s)\|_{\mathbb{Y}})\|x^\delta_{x(t+h)}(s)\|_{\mathbb{Y}}ds - \int_0^{h} \tilde{a}(\|x^\delta_{x(t)}(s)\|_{\mathbb{Y}})\|x^\delta_{x(t)}(s)\|_{\mathbb{Y}}ds -$$

$$\int_h^{T_r+h} \tilde{a}(\|x^\delta_{x(t)}(s)\|_{\mathbb{Y}})\|x^\delta_{x(t)}(s)\|_{\mathbb{Y}}ds =$$

$$\int_0^{T_r} \tilde{a}(\|x^\delta_{x(t+h)}(s)\|_{\mathbb{Y}})\|x^\delta_{x(t+h)}(s)\|_{\mathbb{Y}}ds - \int_0^{h} \tilde{a}(\|x^\delta_{x(t)}(s)\|_{\mathbb{Y}})\|x^\delta_{x(t)}(s)\|_{\mathbb{Y}}ds -$$

$$\int_0^{T_r} \tilde{a}(\|x^\delta_{x(t)}(s+h)\|_{\mathbb{Y}})\|x^\delta_{x(t)}(s+h)\|_{\mathbb{Y}}ds =$$

$$-\int_0^{h} \tilde{a}(\|x^\delta_{x(t)}(s)\|_{\mathbb{Y}})\|x^\delta_{x(t)}(s)\|_{\mathbb{Y}}ds = -\int_0^{h} \tilde{a}(\|x_{x(t)}(s)\|_{\mathbb{Y}})\|x_{x(t)}(s)\|_{\mathbb{Y}}ds =$$

$$-\tilde{a}(\|x_{x_0}(t+\theta_h)\|_{\mathbb{Y}})\|x_{x_0}(t+\theta_h)\|_{\mathbb{Y}}h, \quad \theta_h \in [0, h],$$

where the mean value theorem for integrals is utilized on the last step. The latter means that

$$\limsup_{h\to 0^+} \tfrac{V_1(x_{x_0}(t+h)) - V_1(x_{x_0}(t))}{h} \leq -\tilde{a}(\|x_{x_0}(t)\|_{\mathbb{Y}})\|x_{x_0}(t)\|_{\mathbb{Y}}$$

as long as $x_{x_0}(t) \in K_{\mathbb{Y}}(r)$. Notice that $0 \leq V_1(x_0) \leq T_r\varepsilon(\|x_0\|_{\mathbb{Y}})$.

In this case, the functional $V_r : \mathbb{B} \to \mathbb{R}_+$ given by

$$V_r(x) := V_0(x) + V_1(x)$$

satisfies the Lipschitz condition on $K_{\mathbb{Y}}(r)$ and

$$\|x\|_{\mathbb{Y}} \leq V_r(x) \leq (1 + T_r)\varepsilon(\|x\|_{\mathbb{Y}})$$

for all $x \in \mathbb{Y}$. Moreover, for any $x_0 \in K_{\mathbb{Y}}(r)$ the function $t \to V_r(x_{x_0}(t))$ is strictly decreasing,

$$\limsup_{h\to 0^+} \tfrac{V_r(x_{x_0}(t+h)) - V_r(x_{x_0}(t))}{h} \leq -W_r(\|x_{x_0}(t)\|_{\mathbb{Y}})$$

as long as $x_{x_0}(t) \in K(r)$, where $W_r(\rho) := \tilde{a}(\rho)\rho$ with $\rho \in \mathbb{R}_+$. Due to homogeneity of the system (1) the solutions x_{x_0} are symmetric (see Theorem 2). Hence, if $\mathfrak{d}(-s)x_{x_0}(t+\theta) \in K_{\mathbb{Y}}(r)$ for all $\theta \in [0,h]$ then using the identity $\mathfrak{d}(-s)x_{x_0}(t) = x_{\mathfrak{d}(-s)x_0}(e^{\mu s}t)$, $\forall s \in \mathbb{R}$ we derive

$$V_r(\mathfrak{d}(-s)x_{x_0}(t+h)) - V_r(\mathfrak{d}(-s)x_{x_0}(t)) =$$

$$V_r(x_{\mathfrak{d}(-s)x_0}(e^{\mu s}(t+h))) - V_r(x_{\mathfrak{d}(-s)x_0}(e^{\mu s}t)) \leq$$
$$-\tilde{a}(\|x_{\mathfrak{d}(-s)x_0}(e^{\mu s}t+\theta_h)\|_{\mathbb{Y}})\|x_{\mathfrak{d}(-s)x_0}(e^{\mu s}t+\theta_h)\|_{\mathbb{Y}}e^{\mu s}h,$$

where $\theta_h \in [0, e^{\mu s}h]$, and $\mu \in \mathbb{R}$ is the homogeneity degree of the system (1), i.e. the function $t \to V_r(\mathfrak{d}(-s)x_{x_0}(t))$ is strictly decreasing

$$\limsup_{h\to 0^+} \tfrac{V_r(\mathfrak{d}(-s)x_{x_0}(t+h)) - V_r(\mathfrak{d}(-s)x_{x_0}(t))}{h} \leq -\tilde{a}(\|\mathfrak{d}(-s)x_{x_0}(t)\|_{\mathbb{Y}})\|\mathfrak{d}(-s)x_{x_0}(t)\|_{\mathbb{Y}}e^{\mu s}$$

as long as $\mathfrak{d}(-s)x_{x_0}(t) \in K_{\mathbb{Y}}(r)$.

II. Inspired by [26] let us consider the functional $V : \mathbb{Y} \to \mathbb{R}_+$ given by

$$V(x) := \int_{-\infty}^{+\infty} e^{s-q}\hat{a}(V_r(\mathfrak{d}(-s)x))ds$$

where $\hat{a} \in C^\infty$ is an increasing function such that

- $\hat{a}(\rho) = 1$ for $\rho \geq (1+T_r)\varepsilon(r)$;
- $\hat{a}(\rho) = 0$ for $\rho \leq 1/r$;
- $\hat{a}'(\rho) > 0$ for $1/r < \rho < (1+T_r)\varepsilon(r)$
- $\min\limits_{1/r_0 \leq \rho \leq (1+T_r)\varepsilon(r_0)} \hat{a}'(\rho) > 0$

and

$$q := \ln \frac{\min\{e^{(1+\mu)s_0}, e^{-(1+\mu)s_0}\} \min\limits_{1/r_0 \leq \rho \leq (1+T_r)\varepsilon(r_0)} \hat{a}'(\rho)}{r_0}.$$

For example, the function $\hat{a}$ can be selected as follows

$$\hat{a}(\rho) = e^{-\frac{\rho_{\max}-\rho_{\min}}{\rho-\rho_{\min}}}\left(e^{-\frac{\rho_{\max}-\rho_{\min}}{\rho_{\max}-\rho}} + e\right)$$

for $\rho \in (\rho_{\min}, \rho_{\max})$, where $\rho_{\min} = 1/r, \rho_{\max} = (1+T_r)\varepsilon(r)$.

By construction, the functional $V : \mathbb{Y} \to \mathbb{R}_+$ is $\mathfrak{d}$-homogeneous of the degree 1. Moreover,

$$V(x) \leq \int_{-\infty}^{+\infty} e^{s-q}\hat{a}((1+T_r)\varepsilon(\|\mathfrak{d}(-s)x)\|_{\mathbb{Y}}))ds \leq$$

$$\int_{-\infty}^{\bar{s}(x)} e^{s-q}\hat{a}((1+T_r)\varepsilon(\|\mathfrak{d}(-s)x)\|_{\mathbb{Y}}))ds \leq e^{-q}e^{\bar{s}(x)}$$

where $\bar{s}(x) := \ln\left(\frac{\|x\|_{\mathfrak{d},\mathbb{Y}}}{\underline{\sigma}\left(\varepsilon^{-1}\left(\frac{1}{r(1+T_r)}\right)\right)}\right)$ and $\underline{\sigma} \in \mathcal{K}^\infty$ is defined in Theorem 1. Indeed, if $s \geq s(x)$ then from Theorem 1 we derive

$$\underline{\sigma}\left(\|\mathfrak{d}(-s)x\|_{\mathbb{Y}}\right) \leq \|\mathfrak{d}(-s)x\|_{\mathfrak{d},\mathbb{Y}} = e^{-s}\|x\|_{\mathfrak{d},\mathbb{Y}} \leq e^{-\bar{s}(x)}\|x\|_{\mathfrak{d},\mathbb{Y}} = \underline{\sigma}\left(\varepsilon^{-1}\left(\frac{1}{r(1+T_r)}\right)\right)$$

and taking into account $\underline{\sigma}, \varepsilon \in \mathcal{K}^\infty$ we conclude $\|\mathfrak{d}(-s)x)\|_{\mathbb{Y}} \leq \varepsilon^{-1}\left(\frac{1}{r(1+T_r)}\right)$ or, equivalently, $(1+T_r)\varepsilon(\|\mathfrak{d}(-s)x)\|_{\mathbb{Y}}) \leq 1/r$ for all $s \geq \bar{s}(x)$. The latter means $V_r(\mathfrak{d}(-s)x) \leq 1/r$ and $\hat{a}(V_r(\mathfrak{d}(-s)x))) = 0$ for all $s \geq \bar{s}(x)$.

Similarly, we derive

$$V(x) \geq \int_{-\infty}^{+\infty} e^{s-q}\hat{a}(\|\mathfrak{d}(-s)x)\|_{\mathbb{Y}})ds \geq \int_{-\infty}^{\underline{s}(x)} e^{s-q}ds = e^{-q}e^{\underline{s}(x)},$$

where $\underline{s}(x) := \ln\left(\frac{\|x\|_{\mathfrak{d},\mathbb{Y}}}{\bar{\sigma}((1+T_r)\varepsilon(r))}\right)$. Indeed, if $s \leq s(x)$ then from Theorem 1 we have

$$\bar{\sigma}(\|\mathfrak{d}(-s)x\|_{\mathbb{Y}}) \geq \|\mathfrak{d}(-s)x\|_{\mathfrak{d},\mathbb{Y}} = e^{-s}\|x\|_{\mathfrak{d}} \geq e^{-\underline{s}x}\|x\|_{\mathfrak{d},\mathbb{Y}} = \bar{\sigma}((1+T_r)\varepsilon(r)).$$

Since $\bar{\sigma} \in \mathcal{K}^\infty$ then $\|\mathfrak{d}(-s)x\|_{\mathbb{Y}} \geq (1+T_r)\varepsilon(r)$ and $\hat{a}(\|\mathfrak{d}(-s)x\|_{\mathbb{Y}})) = 1$ for all $s \leq \underline{s}(x)$. Therefore, we have proven that the inequality (11) holds for

$$\underline{k} := \frac{e^{-q}}{\bar{\sigma}\left((1+T_r)\varepsilon(r)\right)}, \quad \bar{k} := \frac{e^{-q}}{\underline{\sigma}\left(\varepsilon^{-1}\left(\frac{1}{r(1+T_r)}\right)\right)}.$$

If $x(t+\theta) \in K_{\mathbb{Y}}(r_0)$ for all $\theta \in [0,h]$ then $\mathfrak{d}(-s)x(t+\theta) \in K_{\mathbb{Y}}(r)$ for all $\theta \in [0,h]$ and for all $s \in [-s_0, s_0]$. Indeed,

$$\|\mathfrak{d}(-s)x(t+\theta)\|_{\mathbb{Y}} \leq \|\mathfrak{d}(-s)\|_{\mathbb{Y}} \cdot \|x(t+\theta)\|_{\mathbb{Y}} \leq r_0^2 = r$$

and

$$1/r_0 \leq \|x(t+\theta)\|_{\mathbb{Y}} = \|\mathfrak{d}(s)\|_{\mathbb{Y}} \cdot \|\mathfrak{d}(-s)x(t+\theta)\|_{\mathbb{Y}}.$$

The latter means that

$$1/r \leq V_r(\mathfrak{d}(-s)x_{x_0}(t+\theta)) \leq (1+T_r)\varepsilon(r)$$

for all $s \in [-s_0, s_0]$ and all $\theta \in [0,h]$. Using

$$V_r(\mathfrak{d}(-s)x_{x_0}(t+h)) - V_r(\mathfrak{d}(-s)x_{x_0}(t)) \leq -W_r(x_{\mathfrak{d}(-s)x_0}(e^{\mu s}t+\theta_h))e^{\mu s}h =$$

$$-W_r(\mathfrak{d}(-s)x_{x_0}(t+e^{-\mu s}\theta_h))e^{\mu s}h$$

we derive

$$V(x_{x_0}(t+h)) - V(x_{x_0}(t)) =$$

$$\int_{-\infty}^{+\infty} e^{s-q}[\hat{a}(V_r(\mathfrak{d}(-s)x_{x_0}(t+h))) - \hat{a}(V_r(\mathfrak{d}(-s)x_{x_0}(t)))]ds =$$

$$\int_{-s_0}^{s_0} e^{s-q}[\hat{a}(V_r(\mathfrak{d}(-s)x_{x_0}(t+h))) - \hat{a}(V_r(\mathfrak{d}(-s)x_{x_0}(t)))]ds =$$

$$\int_{-s_0}^{s_0} e^{s-q}\hat{a}'(\theta)[V_r(\mathfrak{d}(-s)x_{x_0}(t+h)) - V_r(\mathfrak{d}(-s)x_{x_0}(t))]ds \leq$$

$$-h\int_{-s_0}^{s_0} e^{s-q}\hat{a}'(\theta)W_r(\mathfrak{d}(-s)x_{x_0}(t+e^{-\mu s}\theta_h))e^{\mu s}ds,$$

where the mean value theorem is utilized with

$$\theta \in [V_r(\mathfrak{d}(-s)x_{x_0}(t+h)), V_r(\mathfrak{d}(-s)x_{x_0}(t))].$$

Therefore, the function $t \to V(x_{x_0}(t))$ is strictly decreasing and

$$\limsup_{h\to 0^+} \tfrac{V(x_{x_0}(t+h))-V(x_{x_0}(t))}{h} \leq$$

$$-\int_{-\infty}^{+\infty} e^{(1+\mu)s-q}\hat{a}'(V_r(\mathfrak{d}(-s)x_{x_0}(t)))W_r(\|\mathfrak{d}(-s)x_{x_0}(t)\|_{\mathbb{Y}})ds = -W(x_{x_0}(t)),$$

as long as $x_{x_0}(t) \in K_{\mathbb{Y}}(r_0)$, where

$$W(x) := \int_{-\infty}^{+\infty} e^{(1+\mu)s-q}\hat{a}'(V_r(\mathfrak{d}(-s)x))W_r(\|\mathfrak{d}(-s)x\|_{\mathbb{Y}})ds.$$

The functional W is, obviously, nonnegative and $\mathfrak{d}$-homogeneous of the degree $\mu+1$. If $\|x\|_{\mathbb{Y}}=1$ then $\mathfrak{d}(-s)x \in K_{\mathbb{Y}}(r_0)$ and $W_r(\|\mathfrak{d}(-s)x\|_{\mathbb{Y}}) = \|\mathfrak{d}(s)x\|_{\mathbb{Y}}$ for all $s\in[-s_0,s_0]$. Using the mean value theorem for integrals we derive

$$W(x) \geq \int_{-s_0}^{s_0} e^{(1+\mu)s-q}\hat{a}'(V_r(\mathfrak{d}(-s)x))\|\mathfrak{d}(-s)x\|_{\mathbb{Y}}ds \geq \tfrac{1}{r_0}e^{(1+\mu)s^*}\hat{a}'(V_r(\mathfrak{d}(-s^*)x)),$$

where $s^*\in[-s_0,s_0]$. Since $\mathfrak{d}(-s^*)x \in K_{\mathbb{Y}}(r_0)$ then

$$\frac{1}{r} < \frac{1}{r_0} \leq V_r(\mathfrak{d}(-s^*)x) \leq (1+T_r)\varepsilon(r_0) < (1+T_r)\varepsilon(r).$$

Hence, we derive

$$\inf_{\|x\|_{\mathbb{Y}}=1} W(x) \geq \frac{\min\{e^{(1+\mu)s_0}, e^{-(1+\mu)s_0}\} \min\limits_{1/r_0\leq\rho\leq(1+T_r)\varepsilon(r_0)} \hat{a}'(\rho)}{e^q r_0} = 1$$

and using $\mathfrak{d}$-homogeneity of W we conclude

$$W(x) = W(\mathfrak{d}(\ln\|x\|_{\mathfrak{d},\mathbb{Y}})\mathfrak{d}(-\ln\|x\|_{\mathfrak{d},\mathbb{Y}})x) =$$

$$\|x\|_{\mathfrak{d},\mathbb{Y}}^{1+\mu} W(\mathfrak{d}(\ln\|x\|_{\mathfrak{d},\mathbb{Y}})\mathfrak{d}(-\ln\|x\|_{\mathfrak{d},\mathbb{Y}})x) \leq \|x\|_{\mathfrak{d},\mathbb{Y}}^{1+\mu}.$$

Therefore, we have proven that

$$\limsup_{h\to 0^+} \tfrac{V(x_{x_0}(t+h))-V(x_{x_0}(t))}{h} \leq -\|x_{x_0}(t)\|_{\mathfrak{d},\mathbb{Y}}^{1+\mu}$$

as long as $x_{x_0}(t) \in K_{\mathbb{Y}}(r_0)$. If $x_{x_0}(t) \in \mathfrak{d}(\tau)K_{\mathbb{Y}}(r_0)$, where $\tau \in \mathbb{R}$, then $\mathfrak{d}(-\tau)x_{x_0}(t) \in K_{\mathbb{Y}}(r_0)$. Using homogeneity of V we derive

$$V(x_{x_0}(t)) = V(\mathfrak{d}(-\tau)\mathfrak{d}(\tau)x_{x_0}(t)) = e^{-\tau}V(\mathfrak{d}(\tau)x_{x_0}(t)).$$

Since by Theorem 2 we have $x_{\mathfrak{d}(\tau)x_0}(e^{-\mu\tau}t) = \mathfrak{d}(\tau)x_{x_0}(t)$ then the function $t \to V(x_{x_0}(t))$ is strictly decreasing as well and

$$\limsup_{h\to 0^+} \tfrac{V(x_{x_0}(t+h))-V(x_{x_0}(t))}{h} = e^{-\tau s} \limsup_{h\to 0^+} \tfrac{V(x_{\mathfrak{d}(\tau)x_0}(e^{-\mu\tau}(t+h)))-V(x_{\mathfrak{d}(\tau)x_0}(e^{-\mu\tau}t))}{h}$$

$$\leq -e^{-\tau(1+\mu)}\|x_{\mathfrak{d}(\tau)x_0}(e^{-\mu\tau}t)\|_{\mathfrak{d}}^{1+\mu} = -\|\mathfrak{d}(-\tau)x_{\mathfrak{d}(\tau)x_0}(e^{-\mu\tau}t)\|_{\mathfrak{d}}^{1+\mu} = -\|x_{x_0}(t)\|_{\mathfrak{d}}^{1+\mu}$$

as long as $x_{x_0}(t) \in \mathfrak{d}(\tau)K_{\mathbb{Y}}(r_0)$, where $\tau \in \mathbb{R}$ is an arbitrary real number. Taking into account $\bigcup_{\tau\in\mathbb{R}} \mathfrak{d}(\tau)K_{\mathbb{Y}}(r_0) = \mathbb{Y}\backslash\{\mathbf{0}\}$ we complete the proof.

5.3. The proof of Corollary 1

Since $\mathbb{X}$ is a reflexive Banach space then by Corollary 3 (case 3), for any $x_0 \in M^{-1}\mathcal{D}(A)\backslash\{\mathbf{0}\}$ we have a unique mild solution $x : [0, T) \to M^{-1}\mathcal{D}(A)$ of (1) satisfying

$$\dot{x}(t) \overset{a.e.}{=} Ax(t) + f(x(t)) \quad \text{and} \quad M\dot{x}(t) \overset{a.e.}{=} MAx(t) + Mf(x(t)), \quad t \geq 0,$$

as long as $0 < \|x(t)\|_{\mathbb{Y}} < +\infty$. Since

$$\tfrac{V(x(t+h))-V(x(t)))}{h} =$$

$$\tfrac{V(x(t+h))-V(x(t)+h(Ax(t)+f(x(t)))}{h} + \tfrac{V(x(t)+h(Ax(t)+f(x(t)))-V(x(t))}{h}$$

and V is locally Lipschitz continuous in $\mathbb{Y}$ then, taking into account

$$\tfrac{|V(x(t+h))-V(x(t)+h(Ax(t)+f(x(t)))|}{h} \leq \tfrac{L\|x(t+h)-x(t)-h(Ax(t)+f(x(t))\|_{\mathbb{Y}}}{h} =$$

$$L\left\|\tfrac{x(t+h)-x(t)}{h} - \dot{x}(t)\right\|_{\mathbb{Y}} \to 0 \text{ as } h\to 0 \text{ for amost all } t \geq 0,$$

we derive

$$\overline{D}^+V(x(t)) \overset{a.e.}{=} \overline{D}^+V(x(t), Ax(t) + f(x(t)), \ \forall x_0 \in M^{-1}\mathcal{D}(A)\backslash\{\mathbf{0}\}.$$

Sufficiency. The condition 3') imply the fulfillment of the condition 3) of Theorem 3 for all $x_0 \in M^{-1}\mathcal{D}(A)$. This means that all corresponding solutions uniformly converge to $\mathbf{0}$ and satisfy the inequality (10). Since $M^{-1}\mathcal{D}(A)$ in dense in $\mathbb{Y}$ then the continuous dependence of solutions on initial conditions implies the same properties for all solutions with $x_0 \in \mathbb{Y}\backslash\{\mathbf{0}\}$. This guarantees the uniform asymptotic stability of the system (1) in $\mathbb{Y}$.

Necessity. In the proof of Theorem 3 we design a Lyapunov function satisfying 1), 2) and

$$\overline{D}^+V(x(t)) \leq -\|x(t)\|_{\mathfrak{d},\mathbb{Y}}^{1+\mu},$$

as long as $x(t) \in \mathbb{Y}\backslash\{\mathbf{0}\}$.

Let $t_j \to 0, t_j > 0$ as $j \to +\infty$ be an arbitrary a sequence of time instances such that $x(t_j) \in M^{-1}\mathcal{D}(A)$. Since any mild solution is continuous function of time in $\mathbb{Y}$ then for any $h > 0$ there exists $i : \|x(t_i) - x_0\|_{\mathbb{Y}} < h^2$ and $\|x(t_i + h) - x(h)\|_{\mathbb{Y}} < h^2$. Obviously, $i \to +\infty$ as $h \to 0$. Let us consider

$$\frac{V(x_0+h(Ax_0+f(x_0))-V(x_0)}{h} = \frac{V(x_0+h(Ax_0+f(x_0))-V(x_0+h(Ax(t_i)+f(x(t_i))}{h} +$$

$$\frac{V(x_0+h(Ax(t_i)+f(x(t_i)))-V(x(t_i)+h(Ax(t_i)+f(x(t_i))}{h} +$$

$$\frac{V(x(t_i)+h(Ax(t_i)+f(x(t_i)))-V(x(t_i+h))}{h} +$$

$$\frac{V(x(t_i+h))-V(x(h))}{h} + \frac{V(x(h))-V(x_0)}{h}.$$

The third term in the above sum tends to zero as $h \to 0^+$ (see above). Since the operators A and M are closed then $x(t_i) \to x_0, x(t_i) \in M^{-1}\mathcal{D}(A)$ implies $MAx(t_i) \to MAx_0$, $Ax(t_i) \to Ax_0$ and

$$\frac{|V(x_0+h(Ax_0+f(x_0))-V(x_0+h(Ax(t_i)+f(x(t_i))|}{h} \leq$$

$$\|Ax_0 - Ax(t_i)\|_{\mathbb{Y}} + \|f(x_0) - f(x(t_i))\|_{\mathbb{Y}} \to 0 \text{ as } h \to 0.$$

Using the local Lipschitz continuity of V we derive

$$\frac{|V(x_0+h(Ax(t_i)+f(x(t_i)))-V(x(t_i)+h(Ax(t_i)+f(x(t_i))|}{h} \leq \frac{L\|x_0-x(t_i)\|_{\mathbb{Y}}}{h} \leq Lh \to 0 \text{ as } h \to 0.$$

Similarly, we obtain

$$\frac{|V(x(t_i+h))-V(x(h))|}{h} \leq \frac{L_{x_0}\|x_0-x(t_i)\|_{\mathbb{Y}}}{h} \leq Lh \to 0 \text{ as } h \to 0.$$

Therefore, $\forall x_0 \in M^{-1}\mathcal{D}(A)$ we have

$$\limsup_{h \to 0^+} \frac{V(x_0+h(Ax_0+f(x_0))-V(x_0)}{h} = \limsup_{h \to 0^+} \frac{V(x(h))-V(x_0)}{h} \leq -\|x_0\|^{\mu+1}_{\mathfrak{d},\mathbb{Y}}.$$

The proof is complete.