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Abstract. We address the description of the tropicalization of families of rational varieties under parametrizations with prescribed support, via curve valuations. We recover and extend results by Sturmfels, Tevelev and Yu for generic coefficients, considering rational parametrizations with non-trivial denominator. The advantage of our point of view is that it can be generalized to deal with non-generic parametrizations. We provide a detailed analysis of the degree of the closed image, based on combinatorial conditions on the relative positions of the supports of the polynomials defining the parametrization. We obtain a new formula and finer bounds on the degree, when the supports of the polynomials are different. We also present a new formula and bounds for the order at the origin in case the closed image is a hypersurface.

1. Introduction

A classical question that has thrived research in Computational Algebraic Geometry is the problem of *implicitization*. The aim is to describe the prime ideal of polynomial relations among the coordinates of a rational map $\mathbf{f}$. We concentrate on sparse elimination and we consider a family of rational maps of the following form: the input is a field $\mathbb{K}$ of characteristic zero, $n+1$ finite sets $A_0, \dots, A_n$ of lattice points in $\mathbb{Z}^d$ and $n+1$ non-zero Laurent polynomials in d variables $f_0, \dots, f_n \in \mathbb{K}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$ supported on these sets. More precisely,

$$f_i = \sum_{a \in A_i} c_{i,a} x^a, x = (x_1, \dots, x_d),\, c_{i,a} \in \mathbb{K}, \text{ for all } i \in \{0, \dots, n\}, \tag{1.1}$$

and our parametrization is of the form

$$\mathbf{f} \colon (\mathbb{K}^*)^d \dashrightarrow (\mathbb{K}^*)^n \qquad \mathbf{f} = \left(\frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right). \tag{1.2}$$

As the ideal of relations among the coordinates of $\mathbf{f}$ is defined over any subfield containing the coefficients, we will assume in what follows without loss of generality that $\mathbb{K}$ is algebraically closed. We can think of the implicitization problem as a moduli question, as the answer differs depending on the genericity of the coefficients $(c_{i,a})_{i=0}^n$. We will consider $\mathbb{K}$ with the trivial valuation. In [19], Sturmfels, Tevelev and Yu approached this question with tools of tropical geometry, thus exploiting the combinatorial nature of the problem. Instead of finding the ideal of relations, we can compute its tropicalization. This is a rational weighted balanced polyhedral fan in $\mathbb{R}^n$ that captures the combinatorics of this ideal. Tropical implicitization techniques are well suited to study the generic points of this space via the theory of Geometric Tropicalization developed by Hacking, Keel and Tevelev [4, 11]. These techniques are hard to address for special choices of coefficients, since they depend on the process of resolution of singularities.

We approach this question from a different perspective. Namely, we go back to the valuative approach to tropical geometry, manifested in the Fundamental Theorem of Tropical Geometry [6, 7], and we reinterpret tropical implicitization in this elementary language. The main advantage of this approach is that we recover in Theorem 2.5 a straightforward generalization of the description of [19, Theorem 2.1] for generic parametrizations, while developing tools that could be further

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Date: October 5, 2020.

2010 *Mathematics Subject Classification.* 14M25, 14T05, 68W30.

Key words and phrases. implicitization, Newton polytope, tropical geometry, generalized Puiseux series.

MIH and AD were supported by ANPCyT PICT 2016-0398, Argentina. AD was also partially supported by UBACYT 20020170100048BA, CONICET PIP 11220150100473, Argentina.

refined to study the non-generic case. In fact, Theorem 2.5 can be deduced from their result, as we discuss in Section 2.6.

We also study in detail the degree of the closed image of $\mathbf{f}$ (that is, the closure of the image of $\mathbf{f}$) which we denote by S, in case it has dimension d. When S is a hypersurface (that is, when $d = n-1$), one can get the direction and the length of the edges of the Newton polytope $N(H)$ of a defining equation H for S from the description of the cones in the tropicalization of the image of $\mathbf{f}$ and their multiplicities Then, $N(H)$ can be algorithmically reconstructed, see for instance [13]. We could then from this information extract the degree $\deg(H) = \deg(S)$. In the case of lattice polygons, there is a clear description of $N(H)$ in [5]. If $f_0 = 1$ and all supports are equal, $\deg(S)$ is computed in [20] by means of resultants. In case f_0 is a monomial, the description of the Newton polytope of H was also studied in general by Esterov and Khovanskii in [9], where they develop an elimination theory of polytopes via mixed fiber polytopes in the sense of McMullen. This gives the Newton polytope $N(H)$ only for generic coefficients, from which $\deg(S)$ can be read in this case. We give a precise description in Theorem 3.3.

There is a straightforward upper bound for $\deg(S)$, given by the lattice volume of the convex hull $(\mathrm{conv}(\cup_{j=0}^n A_j))$ of the union of the supports (see for instance [8] in case $d = n-1$). The divergence from this upper bound for generic coefficients is related to the following question. Given $A_0, \dots, A_n$ and Laurent polynomials $f_0, \dots, f_n$ in d variables with these supports respectively and generic coefficients, when d generic linear combinations $\ell_{1,0} f_0 + \cdots + \ell_{1,n} f_n, \dots, \ell_{d,0} f_0 + \cdots + \ell_{d,n} f_n$ with generic coefficients $\ell_{i,j} \in \mathbb{K}$, have generic coefficients with respect to the support $\cup_{j=0}^n A_j$? In Theorem 3.3 we give conditions on the relative positions of $A_0, \dots, A_n$ under which this upper bound and some refined ones are attained. The conditions are similar to the conditions of monotonicity of the mixed volume studied in [2] for the case of n sparse polynomials in $n-1$ variables, but they are in fact different as Example 3.8 shows.

When $S = (H = 0)$ is a hypersurface, we also compute by means of curve valuations the order at the origin $\mathrm{ord}_0(S) = \mathrm{ord}_0(H)$. Again, if the Newton polytope $N(H)$ is known, this information can be extracted from it. Under our hypotheses, the origin is never in the image of $\mathbf{f}$, but it could be in its closure. We compute $\mathrm{ord}_0(S)$ in Theorem 4.2 under a condition on the family of supports that holds in many cases. This order can be greater than one (that is, the origin is a singular point of the closure of S in $\mathbb{K}^n$) even for generic coefficients, depending again on the relative positions of the supports $A_0, \dots, A_n$. We also give along the paper examples not satisfying the hypotheses of our results, but were we can still compute degree and order adapting the arguments in our proofs.

In Section 2 we describe the tropicalization of S. We prove the basic Theorem 2.1 and the main Theorem 2.5. We also show in § 2.5 how this result and its proof can be used in generic and non-generic cases. Section 3 deals with the degree computation for any d. Our main result there is Theorem 3.3 and in § 3.1 we give different conditions for equality in the inequalities in its statement, in case $d = n-1$. Section 4 deals with the computation of the order of S at the origin for $d = n-1$. We present our main result Theorem 4.2 and we discuss examples that show how its proof can be used to study cases in which the hypotheses are not satisfied.

Acknowledgements: We are very grateful to M. Angélica Cueto, who worked with us at an earlier stage of this project. In particular, she suggested to us the example we present in Section 2.5 and provided the first picture in Example 2.9.

2. The tropicalization of the image of f

We fix a family of finite integer sets $A_0, \dots, A_n \subset \mathbb{R}^d$ and we take $n+1$ Laurent polynomials $f_0, \dots, f_n$ as in (1.1) with these respective supports and coefficients in $\mathbb{K}$. The goal of this section is to describe the tropicalization of the closure $\overline{\mathrm{im}(\mathbf{f})}$ of the image of $\mathbf{f}$ in (1.2) in case the coefficients of $f_0, \dots, f_n$ are generic. Since the parametrization is invariant under multiplication of all the f_i's by the same monomial, that is under a common translation of the supports, we will assume all along, without loss of generality, that our supports $A_0, \dots, A_n$ lie in the positive orthant $(\mathbb{Z}_{\geq 0})^d$, that is, that $f_0, \dots, f_n \in \mathbb{K}[x_1, \dots, x_d]$.

Our main result in this section is Theorem 2.5, which describes the tropicalization of the closure of the image of $\mathbf{f}$ as a set. It extends [19, Theorem 2.1] to the case of rational functions. Our proof is built from Theorem 2.1 and Lemma 2.7 below using curve valuations. The main advantage of this approach is that it only involves elementary techniques and it hints on how to proceed in case the coefficients fail to be generic. We develop this idea in § 2.5. In Section 2.6 we recover the associated multiplicities from the results in [18].

2.1. **Tropical varieties.** In this section, we present some basic definitions and notation. In particular, we give an introduction to useful statements on tropical geometry from the valuative perspective and recall the connection between initial ideals, tropicalizations and power series solutions to Laurent polynomial equations. Our main result here is Theorem 2.1.

We start by recalling the definition of the tropical variety of an ideal from the point of view of Gröbner theory. We consider an algebraically closed field $\mathbb{K}$ of characteristic zero with trivial valuation. As usual, we set $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$ and we call $(\mathbb{K}^*)^d$ the d-torus. For any nonzero Laurent polynomial $g = \sum g_\alpha x^\alpha \in \mathbb{K}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$ and $w \in \mathbb{R}^n$, we denote by $\mathrm{in}_w(g)$ the subsum of those terms $g_\alpha x^\alpha$ in g for which $g_\alpha \neq 0$ and $\langle w, \alpha \rangle$ is minimum. We denote by $\mathrm{in}_w(I)$ the ideal generated by all the initials $\mathrm{in}_w(h)$ with respect to w of all nonzero polynomials $h \in I$.

Given an ideal $I \subset \mathbb{K}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$, the support of the associated *tropical variety* or *tropicalization* $\mathcal{T}(I)$ of I is the set:

$$\mathcal{T}(I) = \{w \in \mathbb{R}^d \mid \mathrm{in}_w(I) \text{ does not contain any monomial}\}. \tag{2.1}$$

Given an algebraic variety $V \subseteq (\mathbb{K}^*)^d$, we set $\mathcal{T}(V) = \mathcal{T}(I(V))$, where $I(V)$ denotes the ideal of Laurent polynomials vanishing on V.

By Hilbert's Nullstellensatz, $\mathcal{T}(I)$ records all points w such that the associated variety $V_{\mathbb{K}^*}(\mathrm{in}_w(I))$ is nonempty. When the ideal I is principal, generated by a Laurent polynomial h, the set $\mathcal{T}(I)$ consists of all directions ω for which the initial form $\mathrm{in}_\omega(h)$ is not a monomial. Thus, $\mathcal{T}(I)$ consists precisely of the codimension one cones in the inner normal fan of the Newton polytope $N(h)$ of h, that is, of the convex hull of the exponents of the monomials occuring in h with nonzero coefficient.

Tropical geometry can be approached from the perspective of valuations, when we think of them as Bieri-Groves' sets [3]. To state explicitly the link between the previous definition (2.1) and the characterization of the tropicalization of $\mathcal{T}(I)$ via curve valuations, we introduce the algebraically closed field of power series with real exponents and *well-ordered* supports $\mathbb{L} = \mathbb{K}\{\{\varepsilon^{\mathbb{R}}\}\}$, known as *generalized Puiseux series*.

An element $\sigma \in \mathbb{L}^*$ has the form $\sigma = b_0\, \varepsilon^{\alpha_0} + \mathrm{h.o.t.}(\varepsilon)$, where $b_0 \in \mathbb{K}^*$, $\alpha_0 \in \mathbb{R}$ and $\mathrm{h.o.t.}(\varepsilon)$ is a sum of terms of the form $b_i\, \varepsilon^{\alpha_i}$ with $b_i \in \mathbb{K}$ and $\alpha_{i+1} > \alpha_i$ for all $i \geq 0$. We call $\mathrm{in}(\sigma) := b_0\, \varepsilon^{\alpha_0}$ its *initial term* and $\mathrm{val}(\sigma) := \alpha_0$ its *order*. In particular $\mathrm{val}(c) = 0$ for all $c \in \mathbb{K}^*$. We can extend these notions to tuples in $\mathbb{L}^d$ coordinatewise.

The *Fundamental Theorem of Tropical Geometry* [6, 7] asserts that given an ideal I in the Laurent polynomial ring $\mathbb{K}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$, the tropical variety $\mathcal{T}(I)$ is the set of all valuations of nonzero generalized Puiseux series solutions to I:

$$\mathcal{T}(I) = \{\mathrm{val}(\zeta) \mid \zeta \in V_{\mathbb{L}^*}(I)\}. \tag{2.2}$$

Note that by choosing the field of generalized Puiseux series $\mathbb{L}$ which is algebraically closed and complete [16] with value group $\mathbb{R}$, it is not necessary to take closure on the right side [15].

The tropicalization $\mathcal{T}(I)$ is a rational polyhedral set that can be given a (non-unique rational polyhedral) fan structure and any such fan can be refined to a *tropical fan structure*. The multiplicity of a maximal cone σ in a fan structure of $\mathcal{T}(I)$ is defined as the sum over all minimal associated primes P of the initial ideal $\mathrm{in_w(I)}$, of the multiplicities:

$$\mathrm{mult}(\sigma) = \sum \mathrm{mult(P, in_w(I))},$$

where w is any point in the relative interior of σ. These multiplicities satisfy a balancing condition. We refer the reader to [18] for details.

Consider now nonzero Laurent polynomials $f_0, \dots, f_n$ and the map $\mathbf{f}$ from (1.2). Denote by $\mathbf{f}_{\mathbb{L}} : (\mathbb{L}^*)^d \dashrightarrow \mathbb{L}^n$ the extension of $\mathbf{f}$ to $\mathbb{L}$. If $I_{\mathbf{f}} \subset \mathbb{K}[y_1^{\pm 1}, \dots, y_n^{\pm 1}]$ is the ideal that defines the variety $\overline{\mathrm{im}\,\mathbf{f}}$ and $J_{\mathbf{f}} \subset \mathbb{L}[y_1^{\pm 1}, \dots, y_n^{\pm 1}]$, the ideal that defines the variety $\overline{\mathrm{im}\,\mathbf{f}_{\mathbb{L}}}$, then $J_{\mathbf{f}} = I_{\mathbf{f}} \otimes_{\mathbb{K}} \mathbb{L}$. Given a set X, denote by $\mathrm{cl}(X)$ its closure in the Euclidean topology. The next result, which is an easy consequence of (2.2), shows the connection between the tropicalization of the variety parameterized by the rational map $\mathbf{f}$ and the valuations of points in its image.

Theorem 2.1. *Let $f_0, \dots, f_n$ be nonzero Laurent polynomials in d variables with coefficients in $\mathbb{K}$ and let $\mathbf{f} : \mathbb{K}^d \dashrightarrow (\mathbb{K}^*)^n$ be the rational map they define as in (1.2). Then,*

$$\mathcal{T}(I_{\mathbf{f}}) = \mathrm{cl}\{\mathrm{val}(\mathbf{f}_{\mathbb{L}}(\sigma)) \mid \sigma \in (\mathbb{L}^*)^d,\, \mathbf{f}_{\mathbb{L}}(\sigma) \in (\mathbb{L}^*)^n\}. \tag{2.3}$$

Proof. Let $V = \{\mathrm{val}(\mathbf{f}_{\mathbb{L}}(\sigma)) \mid \sigma \in (\mathbb{L}^*)^d,\, \mathbf{f}_{\mathbb{L}}(\sigma) \in (\mathbb{L}^*)^n\}$. The Fundamental Theorem of Tropical Geometry implies that $\mathrm{val}(\mathbf{f}_{\mathbb{L}}(\sigma)) \in \mathcal{T}(I_{\mathbf{f}})$ for every $\sigma \in (\mathbb{L}^*)^d$ in the preimage of the torus $(\mathbb{L}^*)^n$ belonging to the domain of $\mathbf{f}_{\mathbb{L}}$. As tropical varieties are closed in the Euclidean topology, we have the inclusion $\mathrm{cl}(V) \subset \mathcal{T}(I_{\mathbf{f}})$ in (2.3). Moreover, since V is dense in $\mathcal{T}(I_{\mathbf{f}})$, we have equality in (2.3) by Theorem A in [3]. Indeed, Bieri-Groves Theorem A asserts that all maximal cones in $\mathcal{T}(I_{\mathbf{f}})$ have dimension equal to $\dim(\overline{\mathrm{im}\,\mathbf{f}_{\mathbb{L}}})$. Let $Y \subset \overline{\mathrm{im}\,\mathbf{f}_{\mathbb{L}}}$ be an algebraic variety of codimension 1 such that $\overline{\mathrm{im}\,\mathbf{f}_{\mathbb{L}}} \smallsetminus Y \subset \mathrm{im}\,\mathbf{f}_{\mathbb{L}} \subset (\mathbb{L}^*)^n$. Then, all maximal cones in $\mathcal{T}(I(Y))$ have dimension equal to $\dim(Y) < \dim(\overline{\mathrm{im}\,\mathbf{f}_{\mathbb{L}}})$. Consequently, $\mathcal{T}(I_{\mathbf{f}}) \setminus \mathcal{T}(I(Y))$ is dense in $\mathcal{T}(I_{\mathbf{f}})$, as well as V. □

2.2. **Defining the cones and the statement of Theorem 2.5.** We present in Definition 2.4 the cones that occur in the tropicalization of the image of $\mathbf{f}$. We first need some notation concerning supports and polytopes. Let B be a finite set in $\mathbb{Z}^d$ and $h \in \mathbb{L}[x_1, \dots, x_d]$ a polynomial $h = \sum_{q \in B} c_q x^q$ with support B with $c_q \neq 0$ for all $q \in B$. For any $\alpha \in \mathbb{R}^d$, we define

$$m_\alpha(h) = \min_{q \in B}\{\mathrm{val}(c_q) + \langle \alpha, q\rangle\} \text{ and } \mathrm{in}_\alpha(h) = \sum_{m_\alpha(h) = \mathrm{val}(c_q) + \langle \alpha, q\rangle} \mathrm{in}(c_q) x^q. \tag{2.4}$$

In particular, if $h \in \mathbb{K}[x_1, \dots, x_d]$, $m_\alpha(h)$ does not depend on its coefficients and we have

$$m_\alpha(B) = m_\alpha(h) = \min_{q \in B}\{\langle \alpha, q\rangle\} \text{ and } \mathrm{in}_\alpha(h) = \sum_{m_\alpha(B) = \langle \alpha, q\rangle} c_q x^q, \tag{2.5}$$

as at the beginning of § 2.1.

Let $Q \subset \mathbb{R}^d$ be a lattice polytope and $\alpha \in \mathbb{R}^d$. We will call $\mathrm{face}_\alpha(Q)$ the face of Q which has α as an interior normal vector. That is,

$$\mathrm{face}_\alpha(Q) = \{q \in Q \mid \ < \alpha, q >= m_\alpha(Q)\}. \tag{2.6}$$

If $B \subset \mathbb{Z}^d$ is a finite set, $\mathrm{face}_\alpha(B)$ (called a face of B) is obtained as the intersection with B of the face of the convex hull of B defined by α. Thus, for any nonzero Laurent polynomial g, the Newton polytope of $\mathrm{in}_\alpha(g)$ is $\mathrm{face}_\alpha(N(g))$.

Given a lattice polytope $Q \subset \mathbb{R}^d$, we use the notation $\mathrm{vol}(Q)$ for its normalized volume with respect to the lattice $\mathbb{Z}^d$ (so $\mathrm{vol}(Q)$ equals $d!$ times its Euclidean volume $\mathrm{vol}_e(Q)$). Given a family of polytopes $Q_1, \dots, Q_d \in \mathbb{Z}^d$, we denote by $\mathrm{MV(Q_1, \dots, Q_d)}$ their d-dimensional mixed volume

$$\mathrm{MV(Q_1, \dots, Q_d)} = \sum_{\mathrm{I} \subset \{1, \dots, \mathrm{d}\}} (-1)^{\mathrm{n} - \#\mathrm{I}} \mathrm{vol_e}(\sum_{\mathrm{j} \in \mathrm{I}} \mathrm{Q_j}). \tag{2.7}$$

If $B_1, \dots, B_d$ are finite sets, the mixed volume $\mathrm{MV(B_1, \dots, B_d)}$ of $B_1, \dots, B_d$ is defined as the mixed volume of their convex hulls $\mathrm{conv}(B_1), \dots, \mathrm{conv}(B_d)$. Recall that, by Bernstein's Theorem ([1]), this is an upper bound for the number of isolated common zeros in $(\mathbb{L}^*)^d$ of any system of polynomials with support set $B_1, \dots, B_d$. Moreover, equality holds for generic polynomials with those supports.

We denote by $[n]_0$ the set $[n]_0 = \{0, \dots, n\}$ and by $[n]$ the set $[n] = \{1, \dots, n\}$.

Definition 2.2. Given $B = (B_1, \dots, B_r)$ a family of finite sets in $\mathbb{R}^d$ (or convex polytopes) and F_j a face of B_j for all $j \in [r]$, we will say that $F = (F_1, \dots, F_r)$ is a coherent collection of faces of B if there exist $\alpha \in \mathbb{R}^d$ such that $F_j = \mathrm{face}_\alpha(B_j)$ for all $j \in [r]$.

Definition 2.3. A family of finite sets $B_1, \dots, B_r$ in $\mathbb{Z}^d$ is said to be essential if for any subset $J \subset [r]$ of size at most d, it holds that

$$\dim(\sum_{i \in J} \mathrm{conv}(B_i)) \geq |J|. \tag{2.8}$$

With the previous notation, given a set $J \subset [r]$, we will say that a coherent collection $F = (F_1, \dots. F_r)$ of faces of B is adapted to J if $|J| \leq d$ and the family $\{F_j\}_{j \in J}$ is essential.

Consider again the family of supports $A_0, \ldots, A_n \subset \mathbb{Z}^d_{\geq 0}$. Given any $\alpha \in \mathbb{R}^d$, we define

$$\psi_i(\alpha) = m_\alpha(A_i) - m_\alpha(A_0), \quad 1 \in [n], \tag{2.9}$$

and we consider the map $\psi : \mathbb{R}^d \to \mathbb{R}^n$ defined by

$$\psi(\alpha) = (\psi_1(\alpha), \ldots, \psi_n(\alpha)). \tag{2.10}$$

We now define the cones on which the description of $\mathcal{T}(\overline{\operatorname{im}\mathbf{f}})$ is based. We use the notation $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$. Given a subset $J \subset [n]_0$, we use the standard notation $(\mathbb{R}_{\geq 0})^J$ to denote the real vectors $y \in \mathbb{R}^n$ satisfying $y_j \geq 0$ for all $j \in J$ and $y_j = 0$ for $j \notin J$.

Definition 2.4. Given a set $J \subset [n]_0$ and a coherent collection $F = (F_0, \ldots, F_n)$ of faces of $(A_0, \ldots, A_n)$ adapted to J, we define the following cones in $\mathbb{R}^n$:

* if $0 \notin J$,

$$C_J^F := \{\psi(\alpha) + u \mid \alpha = 0 \text{ or } \operatorname{face}_\alpha(A_i) = F_i,\ \forall\ i \in [n]_0 \text{ and } u \in (\mathbb{R}_{\geq 0})^J\}, \tag{2.11}$$

* if $0 \in J$, set $J' = J - \{0\}$ and

$$C_J^F := \{\psi(\alpha) + u - \lambda\,\mathbf{1} \mid \alpha = 0 \text{ or } \operatorname{face}_\alpha(A_i) = F_i,\ \forall\ i \in [n]_0,\ u \in (\mathbb{R}_{\geq 0})^{J'} \text{ and } \lambda \in \mathbb{R}_{\geq 0}\}. \tag{2.12}$$

Note that there is a bijection between faces $\sum_{j=0}^r F_n$ of $\sum_{j=0}^n A_j$, and cones of the inner normal fan of $\operatorname{conv}(\sum_{j=0}^n A_j)$. Denoting by C^F the cone corresponding to a coherent collection F of faces of $(A_0, \ldots, A_n)$, we have that for every $0 \notin J$, the cone C_J^F can be written as $C_J^F = \psi(C^F) + (\mathbb{R}_{\geq 0})^J$ as defined by [19], and for $0 \in J$, $C_J^F = \psi(C^F) + (\mathbb{R}_{\geq 0})^J + \mathbb{R}_{\leq 0}\,\mathbf{1}$.

We now present Theorem 2.5. Our statements are direct generalizations of [19, Theorem 2.1], but the proofs are different and based on curve valuations and give a hint on how to describe with this tool the tropicalization of the image when the coefficients are not generic.

Theorem 2.5. *Consider the rational map* $\mathbf{f}$ *as in* (1.2) *where* $f_0, \ldots, f_n \in \mathbb{K}[x_1, \ldots, x_d]$ *have supports* $A_0, \ldots, A_n$ *and generic coefficients. Then, the following subsets of* $\mathbb{R}^n$ *coincide:*

(i) the tropical variety $\mathcal{T}(\overline{\operatorname{im}\mathbf{f}})$*,*

(ii) the union of the cones C_J^F*, for all* $J \subseteq [n]_0$ *such that* $|J| \leq d$ *and all coherent collections* F *of faces of* $(A_0, \ldots, A_n)$ *adapted to* J*.*

The proof will be given in § 2.4.

Denote by P_i the convex hull of A_i for any $i \in [n]_0$. For instance, in the unmixed case when all P_i are dilations of a fixed polytope $P \in \mathbb{R}^d$, that is, when there exists positive integers $d_0, \ldots, d_n$, such that $P_i = d_i P$ for all $i \in [n]_0$ [19, Section 3.2], we have that

$$\psi(\alpha) = m_\alpha(P)(d_1 - d_0, \ldots, d_n - d_0).$$

The image of ψ varies as in [19] according to whether P contains the origin or not. When P contains the origin, the image of ψ is $\mathbb{R}_{\leq 0}(d_1 - d_0, \ldots, d_n - d_0)$ and the tropical variety $\mathcal{T}(\overline{\operatorname{im}\mathbf{f}})$ is the union of the cones $\mathbb{R}_{\leq 0}(d_1 - d_0, \ldots, d_n - d_0) + (\mathbb{R}_{\geq 0})^J$ over all $|J| = \dim(P) - 1$ such that $0 \notin J$, and $\mathbb{R}_{\leq 0}(d_1 - d_0, \ldots, d_n - d_0) + (\mathbb{R}_{\geq 0})^{J\setminus\{0\}} + \mathbb{R}_{\geq 0}(-1, \ldots, -1)$ over all $|J| = \dim(P) - 1$ such that $0 \in J$. When P does not contain the origin, the image of ψ is $\mathbb{R}(d_1 - d_0, \ldots, d_n - d_0)$ and $\mathcal{T}(\overline{\operatorname{im}\mathbf{f}})$ is the union of the cones $\mathbb{R}(d_1 - d_0, \ldots, d_n - d_0) + (\mathbb{R}_{\geq 0})^J$ over all $|J| = \dim(P) - 1$ such that $0 \notin J$, and $\mathbb{R}(d_1 - d_0, \ldots, d_n - d_0) + (\mathbb{R}_{\geq 0})^{J\setminus\{0\}} + \mathbb{R}_{\geq 0}(-1, \ldots, -1)$ over all $|J| = \dim(P) - 1$ such that $0 \in J$.

We could use Theorem 2.5 to compute the dimension of the variety parameterized by $\mathbf{f}$, which equals the maximal dimension of a cone C_J^F, because the tropical variety $\mathcal{T}(\overline{\operatorname{im}\mathbf{f}})$ and $\overline{\operatorname{im}\mathbf{f}}$ have the same dimension by [3]. For instance, let $d = n - 1$. If there exists a subset $J_0 \subset [n]_0$ with $|J_0| = n - 1$ and $C_{J_0} = \cup_F C_{J_0}^F \neq \emptyset$, then $\mathbf{f}$ parameterizes a hypersurface. Note that the reciprocal is not true. For instance, consider the rational map $\mathbf{f} : (\mathbb{C}^*)^2 \dashrightarrow (\mathbb{C}^*)^3$ defined by $\mathbf{f}(x_1, x_2) = (c_1 x_1, c_2 x_1, x_2)$,with $c_1, c_2 \neq 0$. Then, $\mathbf{f}$ parameterizes a hypersurface but $C_J = \emptyset$ for all $J \neq \emptyset$, in particular those with cardinal 2. This is not a contradiction because in this case $\dim(C_\emptyset) = 2$.

Theorem 2.5 could be seen as a consequence of [19, Theorem 2.1], which applies to the case when f_0 is a monomial (A_0 consists of a single point). Consider again finite sets $A_0, \ldots, A_n \subset \mathbb{Z}^d$

and polynomials $f_0, \dots, f_n$ with these respective supports such that the closure of the image of the associated rational map $\mathbf{f}$ has dimension d.

Let ρ be the homomorphism of tori with its corresponding linear map A of lattices:

$$\begin{array}{cccccc} \rho : (\mathbb{K}^*)^{n+1} & \dashrightarrow & (\mathbb{K}^*)^n & A : \mathbb{Z}^{n+1} & \to & \mathbb{Z}^n \\ (z_0, \dots, z_n) & \to & (\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}) & (w_0, \dots, w_n) & \to & (w_1 - w_0, \dots, w_n - w_0) \end{array}. \tag{2.13}$$

Denote by F the rational map:

$$\mathbf{F} : (\mathbb{K}^*)^d \dashrightarrow (\mathbb{K}^*)^{n+1} \quad \mathbf{F} = (f_0, f_1, \dots, f_n). \tag{2.14}$$

Then, $\mathbf{f} = \rho \circ \mathbf{F}$ and thus $\mathcal{T}(\overline{\operatorname{im} \mathbf{f}}) = A(\mathcal{T}(\overline{\operatorname{im} \mathbf{F}}))$ (see Remark 1.2 in [18]). The cones of the tropical variety $\mathcal{T}(\overline{\operatorname{im} \mathbf{f}})$ can then be deduced from the description of the tropical variety $\mathcal{T}(\overline{\operatorname{im} \mathbf{F}})$. As we show in Section 2.5, our direct and simple approach via curve valuations can be extended to deal with non-generic parametrizations. Also, curve valuations are used in Section 4 to give a formula for the order at the origin of S.

2.3. **Two auxiliary results.** We first recall Hensel's lemma. The valuation ring of $\mathbb{L}$ is $\mathcal{O} = \{\sigma \in \mathbb{L} \mid \operatorname{val}(\sigma) \geq 0\}$, with maximal ideal $\mathfrak{m}$. Here $\mathcal{O}^* = \mathcal{O} \setminus \mathfrak{m}$ denotes the units in $\mathcal{O}$, i.e. those elements with valuation 0. Given a sequence of polynomials $h = (h_1, \dots, h_r) \in \mathbb{L}[x_1, \dots, x_d]$, let us denote by $J_h(x) \in \mathbb{L}(x)^{d \times r}$ its associate Jacobian matrix. Observe that the field $\mathbb{L}$ is algebraically closed, and so a Henselian field. *Hensel's lemma* asserts the following (see, for example, [14]). Let $h = (h_1, \dots, h_d) \in \mathcal{O}[x_1, \dots, x_d]$ be a family of non-zero polynomials. Assume that there exists $b = (b_1, \dots, b_d) \in \mathcal{O}^d$ such that

$$\operatorname{val}(h_i(b)) > 0 \text{ for } 1 \in [d] \text{ and } \operatorname{val}(\det J_h(b)) = 0.$$

Then, there exists a unique $\sigma \in \mathcal{O}^d$ such that $h_i(\sigma) = 0$ and $\operatorname{val}(\sigma_i - b_i) > 0$ for all $i \in [d]$.

The following lemma for polynomials with nonnegative exponents is straightforward.

Lemma 2.6. *Let $B \subset \mathbb{Z}^d_{\geq 0}$ be a finite set and $h \in \mathbb{L}[x_1, \dots, x_d]$ be a non-zero polynomial supported on B. For any $\sigma \in (\mathbb{L}^*)^d$ with $\operatorname{in}(\sigma) = b\,\varepsilon^\alpha = (b_1\varepsilon^{\alpha_1}, \dots, b_d\varepsilon^{\alpha_d})$, $b_i \in \mathbb{L}^*$, $\alpha_i \in \mathbb{R}$, we have*

$$h(\sigma) = \operatorname{in}_\alpha(h)(b)\,\varepsilon^{m_\alpha(h)} + \text{h.o.t.}(\varepsilon). \tag{2.15}$$

Notice that, a priori, we could have $\operatorname{in}_\alpha(h)(b) = 0$ so Lemma 2.6 implies that $\operatorname{val}(h(\sigma)) \geq m_\alpha(h)$, with equality if and only if $\operatorname{in}_\alpha(h)(b) \neq 0$.

Lemma 2.7 below gives conditions which ensure that a vector w can be realized as the (coordinatewise) valuation of a point in $im(\mathbf{f}_\mathbb{L})$.

Lemma 2.7. *Let $g_1, \dots, g_r$ be non-zero polynomials in $\mathbb{K}[x_1, \dots, x_d]$ with $r \leq d$ and call $B_1, \dots B_r$ in $\mathbb{Z}^d_{\geq 0}$ their respective supports. Given $\alpha \in \mathbb{R}^d$, assume the initial polynomials $\operatorname{in}_\alpha(g_1), \dots, \operatorname{in}_\alpha(g_r)$ have a common zero $b \in (\mathbb{K}^*)^d$ such that their Jacobian matrix $J_{\operatorname{in}_\alpha(g)}(b)$ has maximal rank r. Then,*

- *(i) for each $\omega \in \mathbb{R}^r$ verifying that $\omega_j > m_\alpha(B_j)$ for all $j \in [r]$, there exist $\sigma \in (\mathbb{L}^*)^d$ with $\operatorname{in}(\sigma) = b\varepsilon^\alpha$ such that $\operatorname{val}(g_j(\sigma)) = \omega_j$ for all $j \in [r]$.*
- *(ii) if $g_1, \dots, g_r$ have generic coefficients, for each $\omega \in \mathbb{R}^r$ verifying that $\omega_j \geq m_\alpha(B_j)$ for all $j \in [r]$, there exist $\sigma \in (\mathbb{L}^*)^d$ with $\operatorname{val}(\sigma) = \alpha$ such that $\operatorname{val}(g_j(\sigma)) = \omega_j$ for all $j \in [r]$.*

Proof. Write $g_i = \sum_{a \in B_i} c_{i,a} x^a$ and denote $\overline{g}_i(x) = \varepsilon^{-m_\alpha(B_i)} g_i(\varepsilon^\alpha x)$, $\varepsilon^\alpha x = (\varepsilon^{\alpha_1} x_1, \dots, \varepsilon^{\alpha_d} x_d)$. By Lemma 2.6, we know that $\overline{g}_i(x) \in \mathcal{O}[x_1, \dots, x_d]$, $\operatorname{in}_0(\overline{g}_i) = \operatorname{in}_\alpha(g_i)$ for all $i \in [r]$ and $J_{\operatorname{in}_0(\overline{g})} = J_{\operatorname{in}_\alpha(g)}$. Therefore, we can assume that $\alpha = 0$ and $m_\alpha(g_i) = 0$ with $g_i \in \mathcal{O}[x_1, \dots, x_d]$ for all $i \in [r]$.

To prove item (i), under these assumptions on α and the polynomials $g_1, \dots, g_r$, consider the polynomials $h_i = g_i - \varepsilon^{w_i}$ for every $i \in [d]$ and $w \in (\mathbb{R}_{>0})^r$. By Lemma 2.6, $\operatorname{val}(h_i(b)) > 0$ for all $i \in [r]$. Since $\operatorname{in}_0(h_i) = \operatorname{in}_0(\overline{g}_i)$ and $J_{\operatorname{in}_0(\overline{g})}(b)$ has rank r, for any $d - r$ generic affine linear forms $\ell_1, \dots, \ell_{d-r} \in \mathbb{K}[x_1, \dots, x_d]$ that vanish on b, the Jacobian matrix $J_{\operatorname{in}_0(h),l}$ of the family of polynomials $\operatorname{in}_0(h_1), \dots, \operatorname{in}_0(h_r), \ell_1, \dots, \ell_{d-r}$ is invertible when evaluated at b. Consider now the Jacobian matrix $J_{h,l}$ of the family of polynomials $h_1, \dots, h_r, \ell_1, \dots, \ell_{d-r}$. Since $\det(J_{h,l}(b)) = \det(J_{\operatorname{in}_0(h),l}(b)) + \text{h.o.t.}(\varepsilon)$ and $\det(J_{\operatorname{in}_0(h),l}(b)) \in \mathbb{K}^*$, $\operatorname{val}(\det(J_{h,l}(b))) = 0$. Then, by Hensel's Lemma there exist $\sigma \in \mathcal{O}^n$ such that $\operatorname{in}(\sigma) = b$ and $h_i(\sigma) = 0$ for all $i \in [r]$, and so $g_i(\sigma) = \varepsilon^{\omega_i}$.

To prove item (ii), note that for a generic point $c \in (\mathbb{K}^*)^d$ we have $\mathrm{val}(g_i(c)) = 0$ for all $i \in [r]$, thus $\omega = 0 \in \mathbb{R}^r$ satisfies the statement. Consider now any $\omega \in (\mathbb{R}_{\geq 0})^r$. We need to show that there exists a point $\sigma \in (\mathbb{L}^*)^d$ with $\mathrm{val}(\sigma) = 0$ and such that $\mathrm{val}(g_i(\sigma)) = \omega_i$ for all $i \in [r]$. Consider the set $I = \{i \in [r] \mid \omega_i > 0\}$. Since $\mathrm{in}_0(g_1), \dots, \mathrm{in}_0(g_r)$ are generic polynomials and have a common zero $b \in (\mathbb{K}^*)^d$, the solution set in $(\mathbb{K}^*)^d$ is an equidimensional variety of dimension $d - r$. If we denote by $(B_1)_0, \dots, (B_r)_0$ the supports of these initial polynomials, $\Delta_d = \{0, e_1, \dots, e_d\}$ the set with e_i the i-th vector of the canonical basis of $\mathbb{R}^d$ and $\Delta_d^{(d-r)}$ when it is repeated $d-r$ times, then the mixed volume $MV((B_1)_0, \dots, (B_r)_0, \Delta_d^{(d-r)})$ is a positive integer. By the monotony of the mixed volume, $MV(\{(B_i)_0\}_{i \in I}, \{(B_i)_0 \cup \{0\}\}_{i \notin I}, \Delta_d^{(d-r)})$ is also positive. That is, for $\ell_1, \dots, \ell_{d-r}$ generic affine linear forms in $\mathbb{K}[x_1, \dots, x_d]$ and $\{\lambda_i\}_{i \notin I} \subset \mathbb{K}^*$ generic numbers, the polynomials $\mathrm{in}_0(h_1), \dots, \mathrm{in}_0(h_d)$ have an isolated simple common zero $c \in (\mathbb{K}^*)^d$, where

$$h_i(x) = \begin{cases} g_i(x) \text{ for every } i \in I \\ g_i(x) + \lambda_i \text{ for every } i \notin I, i \leq r \\ \ell_{i-r}(x) \text{ for every } r < i \leq d \end{cases}.$$

Consider now the polynomials $h'_1, \dots, h'_d \in \mathcal{O}[x_1, \dots, x_d]$ defined as $h'_i(x) = h_i(x) - \varepsilon^{\omega_i}$ if $i \in I$ and $h'_i(x) = h_i(x)$ otherwise. Since $c \in (\mathbb{K}^*)^d \subset \mathcal{O}^d$ satisfies $\mathrm{val}(h'_i(c)) > 0$ for all $i \in [d]$ and $J_{\mathrm{in}_0(h')}(c) = J_{\mathrm{in}_0(g),l}(c)$, then $\mathrm{val}(J_{h'}(c)) = 0$. Then, by Hensel's Lemma there exists σ such that $\mathrm{in}(\sigma) = c$ and $h'_i(\sigma) = 0$ for all $i \in [r]$, or equivalently, $g_i(\sigma) = \varepsilon^{w_i}$ for all $\omega_i > 0$ and $g_i(\sigma) = -\lambda_i \in \mathbb{K}^*$ for all $\omega_i = 0$. □

2.4. **The proof of Theorem 2.5.** We first show that $\mathcal{T}(I)$ is contained on the union of the cones C_J^F. We consider for any $J \subseteq [n]_0$ the cone $C_J = \cup_F C_J^F$, where the union is over all coherent collections F of faces of $(A_0, \dots, A_n)$ that are adapted to J.[1] Then,

* if $0 \notin J$,
$$C_J := \{\psi(\alpha) + u,\, \alpha \in \tau_J, \text{ and } u \in (\mathbb{R}_{\geq 0})^J\},$$
* if $0 \in J$, set $J' = J - \{0\}$ and
$$C_J := \{\psi(\alpha) + u - \lambda\,\mathbf{1},\, \alpha \in \tau_J, u \in (\mathbb{R}_{\geq 0})^{J'}, \text{ and } \lambda \in \mathbb{R}_{\geq 0}\},$$

where $\tau_J := \{\alpha \in \mathbb{R}^d / \dim(\sum_{j \in K} \mathrm{face}_\alpha(\mathrm{conv}(A_j))) \geq |K|, \forall\, K \subseteq J\}$.

By the Fundamental Theorem of Tropical Geometry, it suffices then to show that $\mathrm{val}(\mathbf{f}_\mathbb{L}(\sigma))$ lies in one of the cones C_J for σ in $(\mathbb{L}^*)^d \backslash \{\prod_{i=0}^n f_i = 0\}$. Let $\alpha = \mathrm{val}(\sigma)$, with $\sigma = (b_1\, \varepsilon^{\alpha_1} + \mathrm{h.o.t.}(\varepsilon), \dots, b_d\, \varepsilon^{\alpha_d} + \mathrm{h.o.t.}(\varepsilon))$ and $b = (b_1, \dots, b_d) \in (\mathbb{K}^*)^d$. Consider the initial forms $\mathrm{in}_\alpha(f_i)$. If $\mathrm{in}_\alpha(f_0)(b) \neq 0$, take J as the subset of $[n]$ of those indices for which $\mathrm{in}_\alpha(f_i)(b) = 0$.

Since $b \in (\mathbb{K}^*)^n$ is a common zero of the generic polynomials $\mathrm{in}_\alpha(f_i))_{i \in J}$, for all $\alpha \in \tau_J$, Lemma 2.6 implies that $\mathrm{val}(f_i(\sigma)) = m_i(\alpha)$ for all $i \in [n]_0 \backslash J$ and $\mathrm{val}(f_i(\sigma)) > m_i(\alpha)$ for all $i \in J$, so $\mathrm{val}(\mathbf{f}_\mathbb{L}(\sigma)) = \psi(\alpha) + u \in C_J$ where $\alpha \in \tau_J$ and $u \in \mathbb{R}^J$. On the contrary, if $\mathrm{in}_\alpha(f_0)(b) = 0$, this means that $\mathrm{val}(f_0(\sigma)) = m_0(\alpha) + \lambda > m_0(\alpha)$. Take J' be the set of indices $j \geq 1$ satisfying $\mathrm{in}_\alpha(f_j)(b) = 0$ and $J = J' \cup \{0\}$, it follows by the same argument that $\mathrm{val}(\mathbf{f}_\mathbb{L}(\sigma)) \in C_J$.

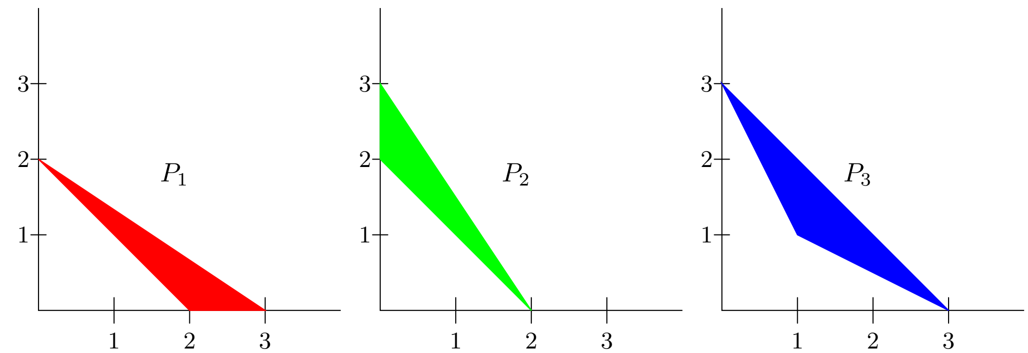
Our next task is to prove that the cones C_J lie in $\mathcal{T}(I)$ if the polynomials f_i have generic coefficients. Let $\alpha \in \tau_J$. Since the polynomials f_i are generic, we may assume that there exists $b \in (\mathbb{K}^*)^d$ with $\mathrm{in}_\alpha(f_j)(b) = 0$ if and only if $j \in J$, with the additional property that the Jacobian matrix $\mathrm{Jac}_{(in_\alpha(f_j))_{j \in J}}(b)$ associated to those initial polynomials has maximal rank. In particular, $\mathrm{in}_\alpha(f_i(b)) \neq 0$ for every $i \in [n]_0 \backslash J$ and, by Lemma 2.6, $\mathrm{val}(f_i(\sigma)) = m_\alpha(P_i)$ for every σ with $\mathrm{in}(\sigma) = b\,\varepsilon^\alpha$.

First, assume $0 \notin J$ and consider $\psi(\alpha) + u \in C_J$ where $u \in (\mathbb{R}_{>0})^J$ and $\omega \in \mathbb{R}^n$ defined as $\omega = \psi(\alpha) + u + m_\alpha(P_0)(1, \dots, 1)$. Applying Lemma 2.7 to the polynomials (and coordinates of ω) indexed by J, there exists $\sigma \in (\mathbb{L}^*)^d$ with $\mathrm{in}(\sigma) = b\varepsilon^\alpha$ and $\mathrm{val}(f_i(\sigma)) = \omega_i$ for every $i \in J$. Hence, $\mathrm{val}(\mathbf{f}_\mathbb{L}(\sigma)) = \omega - m_\alpha(P_0)(1, \dots, 1) = \psi(\alpha) + u$. Since $\mathcal{T}(I)$ is a closed set, we deduce that $C_J \subset \mathcal{T}(I)$. In case $0 \in J$, consider $\psi(\alpha) + u - \lambda(1, \dots, 1) \in C_J$ where $u \in (\mathbb{R}_{>0})^{J'}$ and $\lambda > 0$ and $\omega = (\omega_0, \dots, \omega_n) = (m_\alpha(P_0) + \lambda, m_\alpha(P_1) + u_1, \dots, m_\alpha(P_n) + u_n) \in \mathbb{R}^{n+1}$. Applying Lemma 2.7

[1] Unlike the cones C_J^F, the cones C_J are not necessarily convex.

to the polynomials $\{f_i\}_{i\in J}$ (and the corresponding coordinates of ω), there exists $\sigma \in (\mathbb{L}^*)^d$ with $\mathrm{in}(\sigma) = b\varepsilon^\alpha$ and $\mathrm{val}(f_i(\sigma)) = \omega_i$ for all $i \in J$. Then $\mathrm{val}(\mathbf{f}_{\mathbb{L}}(\sigma)) = \psi(\alpha) + u - \lambda(1,\dots,1)$.

2.5. **An example.** We apply Theorem 2.5 to the simple Example 2.8 with generic polynomials and we then consider in Example 2.9 polynomials with the same supports but non-generic coefficients. These examples are addressed in terms of geometric tropicalization in Examples 4.3 and 5.3 in [4]. In both cases, we consider polynomials $f_0,\dots,f_3$ with the following respective supports $A_0,\dots,A_3 \subset (\mathbb{Z}_{\geq 0})^2$: $A_0 = \{0\}, A_1 = \{(2,0),(3,0),(0,2)\}$, $A_2 = \{(0,2),(0,3),(2,0)\}$, $A_3 = \{(1,1),(0,3),(1,2),(2,1),(3,0)\}$. We depict the corresponding Newton polytopes $P_i = \mathrm{conv}\,(A_i)$ for $i = 1,2,3$:



Example 2.8 (Generic coefficients)**.** Consider the parametrization $\mathbf{f} : (\mathbb{C}^*)^2 \dashrightarrow (\mathbb{C}^*)^3$ defined by the polynomials

$$\begin{aligned} f_0 &= 1 \\ f_1 &= a_1x_1^2 + b_1x_2^2 + x_1^3 \\ f_2 &= a_2x_2^2 + b_2x_1^2 + x_2^3 \\ f_3 &= x_1x_2 + a_3x_1^3 + b_3x_1^2x_2 + c_3x_1x_2^2 + d_3x_2^3, \end{aligned}$$

where $a_1, b_1, a_2, b_2, a_3, b_3, c_3, d_3$ are generic nonzero elements of $\mathbb{K}$. Applying Theorem 2.5, we can check that:

- For $J \subset [3]_0$ with at least 3 elements, there is no possible coherent collection of faces adapted to J.
- For $\#J = 2$ the only coherent collection of faces adapted to J is (A_0, A_1, A_2, A_3) and the associated cones are: $C_1 = \mathbb{R}_{\geq 0}r_1 + \mathbb{R}_{\geq 0}r_2, C_2 = \mathbb{R}_{\geq 0}r_1 + \mathbb{R}_{\geq 0}r_3, C_3 = \mathbb{R}_{\geq 0}r_2 + \mathbb{R}_{\geq 0}r_3$, defined by the rays $r_1 = (1,0,0),\ r_2 = (0,1,0)$ and $r_3 = (0,0,1)$.
- For $\#J = 1$ we obtain the cones $C_4 = \mathbb{R}_{\geq 0}r_3 + \mathbb{R}_{\geq 0}r_4, C_5 = \mathbb{R}_{\geq 0}r_1 + \mathbb{R}_{\geq 0}r_5, C_6 = \mathbb{R}_{\geq 0}r_2 + \mathbb{R}_{\geq 0}r_5$, where we define the rays $r_4 = (-1,-1,-1)$ and $r_5 = (1,1,1)$, and $C_{a,1} = \mathbb{R}_{\geq 0}r_1 + \mathbb{R}_{\geq 0}r_6, C_{a,2} = \mathbb{R}_{\geq 0}r_2 + \mathbb{R}_{\geq 0}r_7, C_{a,3} = \mathbb{R}_{\geq 0}r_3 + \mathbb{R}_{\geq 0}r_8$, where $r_6 = (-2,-3,-3),\ r_7 = (-3,-2,-3)$ and $r_8 = (2,2,3)$.
- Finally, for $J = \emptyset$, the image of ψ give us the cones $C_{b,1} = \mathbb{R}_{\geq 0}r_4 + \mathbb{R}_{\geq 0}r_6, C_{b,2} = \mathbb{R}_{\geq 0}r_4 + \mathbb{R}_{\geq 0}r_7$. Here, the union of cones $C_{a,i} \cup C_{b,i} = C_{i+6}$ for $i = 1,2,3$ yields the cones $C_7 = \mathbb{R}_{\geq 0}r_1 + \mathbb{R}_{\geq 0}r_4, C_8 = \mathbb{R}_{\geq 0}r_2 + \mathbb{R}_{\geq 0}r_4, C_9 = \mathbb{R}_{\geq 0}r_3 + \mathbb{R}_{\geq 0}r_5$.

In fact, the Newton polytope of a (reduced) polynomial H that defines the closure of the image of $\mathbf{f}$ is a truncated simplex with vertices $\{(4,0,0),(0,4,0),(0,0,4),(9,0,0),(0,9,0),(0,0,9)\}$. The cones C_1 to C_9 correspond to the 9 edges of $N(H)$. We explain how to compute a priori the multiplicities (the lengths of the edges) in § 2.6.

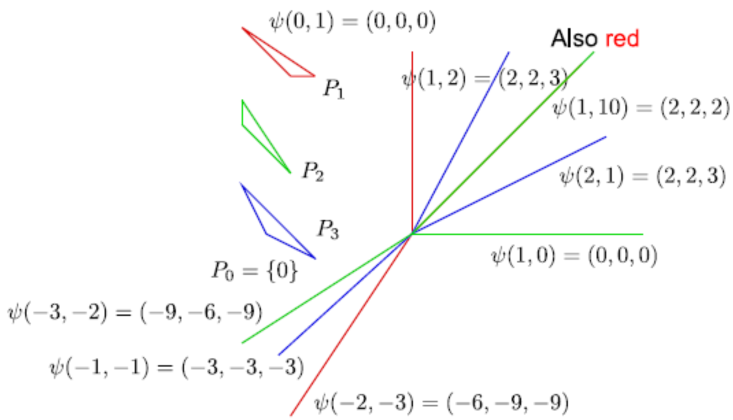
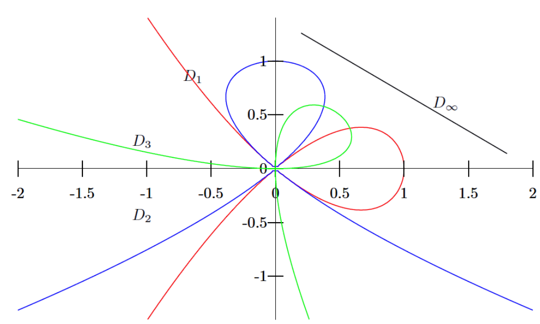


FIGURE 1. The image by ψ of the cones spanned by $\{(-3,2),(0,1)\}$, $\{(0,1),(1,2)\}$, $\{(2,1),(1,0)\}$, $\{(1,0),(-2,-3)\}$ is one dimensional.

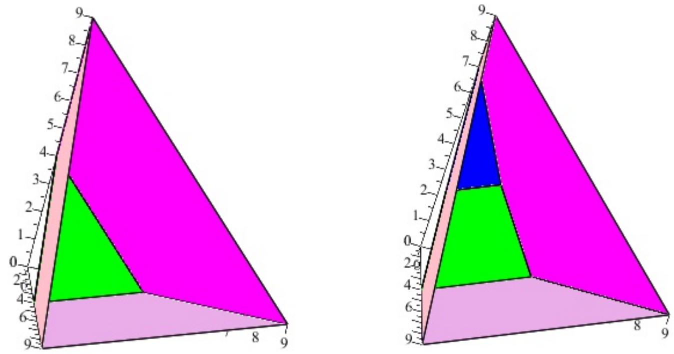
Example 2.9 (Non-generic coefficients)**.** Consider now the rational map $\mathbf{f} : (\mathbb{C}^*)^2 \dashrightarrow (\mathbb{C}^*)^3$ associated to the polynomials

$$\begin{aligned} f_0 &= 1 \\ f_1 &= x_1^2 - x_2^2 - x_1^3 \\ f_2 &= x_2^2 - x_1^2 - x_2^3 \\ f_3 &= 4x_1x_2 - (x_1+x_2)^3. \end{aligned}$$

The union of the zeros of these f_i's (that is, the base locus of $\mathbf{f}$) has a highly singular point at the origin:



Let H be a (reduced) polynomial that defines the hypersurface parameterized by $\mathbf{f}$. Its Newton polytope $N(H)$ is the pyramid in $(\mathbb{Z}_{\geq 0})^3$ with equation $X+Y+Z \leq 9$ cut by the two half-spaces $3\,X+3\,Y+2\,Z \geq 12$ and $2\,X+2\,Y+\,Z \geq 7$. Note that it is contained in the convex hull P of the support in the generic case in Example 2.8. We depict $N(H)$ on the right and P on the left:



The tropicalization $\mathcal{T}(H)$, instead, is not equal to the union of cones of Theorem 2.5, but bigger. However, we can describe some of the cones in $\mathcal{T}(H)$ by considering the approach in the proof of Theorem 2.5 and using it as a recipe to find the tropicalization.

In Theorem 2.5, to prove that a cone C_J lies in $\mathcal{T}(I)$, the genericity condition is used to ensure that there is a common zero $b \in (\mathbb{C}^*)^{n-1}$ of $\mathrm{in}_\alpha(f_j)$ if and only if $j \in J$ and that it satisfies all hypotheses from Lemma 2.7. But this Lemma does not require that the polynomials are generic. Using this idea, we consider for all possible J the common zeros of $\{\mathrm{in}_\alpha(f_j)\}_{j\in J}$ that are not zeros of any polynomial in $\{\mathrm{in}_\alpha(f_j)\}_{j\notin J}$. If either $0 \in J$ or $\#J \geq 3$, then there is no possible α such that $\{\mathrm{in}_\alpha(f_j)\}_{j\in J}$ have common zeros. If $J = \emptyset$, for all $\alpha \in \mathbb{R}^2$ there are points $b \in (\mathbb{C}^*)^2$ that are not zeros of any $\mathrm{in}_\alpha(f_j)$. Then, as in the generic case (see Example 2.8),

$$\psi(\mathbb{R}^2) = C_{b,1} \cup C_{b,2} \cup C_{b,3} \tag{2.16}$$

and this set is included in $\mathcal{T}(I)$. If $J \in \{\{1,3\},\{2,3\}\}$ the only vector $\alpha \in \mathbb{R}^2$ for which there are common zeros of $\{\mathrm{in}_\alpha(f_j)\}_{j\in J}$ that are not zeros of any $\mathrm{in}_\alpha(f_j)$ for all $j \notin J$ or of $\det(\mathcal{J}_{\{\mathrm{in}_\alpha(f_j)\}_{j\in J}})$ is $\alpha = (0,0)$. For $J = \{1,2\}$, a possible vector is $\alpha = (0,0)$ (although is not the only one). Then, $C_1 \cup C_2 \cup C_3 \subset \mathcal{T}(I)$. These 3 are in fact the cones associated to the edges of $N(H)$ lying in the coordinate axes.

When looking at $J{=}\{1,2\}$, we also need to consider $\alpha = (1,1)$. There is no zero of $\mathrm{in}_{(1,1)}(f_1)$ that is not a zero of $\mathrm{in}_{(1,1)}(f_2)$. Consider $\sigma = (\varepsilon^v(1+s_1), \varepsilon^v(1+t_1))$, where $v > 0, (s_1,t_1) \in \mathbb{L}^2$ and $\mathrm{val}(s_1), \mathrm{val}(t_1) > 0$. Then

$$\begin{aligned} f_1(\sigma) &= \varepsilon^{2v}(2s_1 - 2t_1 + s_1^2 - t_1^2 - \varepsilon^v(1+s_1)^3) \\ f_2(\sigma) &= -\varepsilon^{2v}(2s_1 - 2t_1 + s_1^2 - t_1^2 + \varepsilon^v(1+t_1)^3) \\ f_3(\sigma) &= \varepsilon^{2v}(4(1+s_1)(1+t_1) - \varepsilon^v(2+s_1+t_1)^3). \end{aligned}$$

If $2s_1 - 2t_1 + s_1^2 - t_1^2 \neq 0$ and we choose $s_1 = a\varepsilon^\omega + s_2, t_1 = b\varepsilon^\omega + t_2$ with $0 < \omega < v = \omega + v'$ with $v' > 0$, then $\mathrm{val}(f_1(\sigma)) = \mathrm{val}(f_2(\sigma)) = 2v' + 3\omega$ and $\mathrm{val}(f_3(\sigma)) = 2v' + 2\omega$. Thus, the cone $\mathbb{R}_{\geq 0}(1,1,1) + \mathbb{R}_{\geq 0}(3,3,2)$ is contained in $\mathcal{T}(I)$. The union of this cone with $C_9 \subset \mathcal{T}(I)$ gives the entire cone corresponding to the edge $\overline{(4,0,0)(0,4,0)}$. If $J = \{1\}$, we analyze those α such that $\mathrm{face}_\alpha(A_1)$ has dimension 1. When $\alpha = (0,1)$ we obtain the cone $\mathbb{R}_{\geq 0}(1,0,0) \subset C_1$ which we already have. When $\alpha = (-2,-3)$ there are zeros of $\mathrm{in}_{(-2,-3)}(f_1) = x_2^2 + x_1^3$ that are not zeros of $\mathrm{in}_{(-2,-3)}(f_2) = \mathrm{in}_{(-2,-3)}(f_3) = x_2^3$ or $\mathcal{J}_{\mathrm{in}_{(-2,-3)}(f_1)} = (3x_1^2\ 2x_2)$ and we get the cone $C_{a,1}$. Recall that (with the notation of Example 2.8) $C_{a,i} \cup C_{b,i} = C_{6+i}$. Interestingly and as in the generic case, since we already obtained in (2.16) the cone $C_{b,1}$ we have that $C_{a,1} \cup C_{b,1} = C_7 \subset \mathcal{T}(I)$. In the same way, for $J = \{2\}$ we obtain that $C_8 \subset \mathcal{T}(I)$. These cones correspond to the edges $[(9,0,0)(0,0,9)]$ and $[(0,9,0)(0,0,9)]$ of $N(H)$. The set $J = \{3\}$ is a bit more complicated to analyze. If we consider all faces $\mathrm{face}_\alpha(A_1)$ with dimension 1, both $\alpha = (1,2)$ and $\alpha = (2,1)$ gives the cone $C_{a,3}$. Via the cones obtained in (2.16), $C_9 \subset \mathcal{T}(I)$. As we mentioned before, joining C_9 with $\mathbb{R}_{\geq 0}(1,1,1) + \mathbb{R}_{\geq 0}(3,3,2)$ we obtain the cone $\mathbb{R}_{\geq 0}(0,0,1) + \mathbb{R}_{\geq 0}(3,3,2) \subset \mathcal{T}(I)$ associated to the edge $[(4,0,0),(0,4,0)]$.

Finer computations are required to find the other five cones associated to the edges of the two new faces of the Newton polytope of H, as it is necessary to consider higher order terms in σ.

2.6. **The multiplicities.** We complete the description of the tropicalization of the closure of the image of $\mathbf{f}$ with the computation of the multiplicities of the maximal cones in Theorem 2.5.

Recall that as the variety $\overline{\operatorname{im}\mathbf{f}}$ is irreducible of dimension d, $\mathcal{T}(\overline{\operatorname{im}\mathbf{f}})$ is a polyhedral set that can be given the structure of a pure d-dimensional fan. Theorem 2.5 gives a description of $\mathcal{T}(\overline{\operatorname{im}\mathbf{f}})$ as a set, and as in [19], this description does not have a natural fan structure. However, every maximal dimensional cone σ in $\mathcal{T}(\overline{\operatorname{im}\mathbf{f}})$ has a multiplicity $\operatorname{mult}(\sigma) \in \mathbb{Z}_{>0}$. This multiplicity can be computed at any regular point w in the cone, by means of [18, Theorem 3.12] and [18, Theorem 5.1].

Consider again the homomorphism of tori ρ in (2.13). Given a regular point w in a maximal cone σ of $\mathcal{T}(\overline{\operatorname{im}\mathbf{f}})$ such that $f^{-1}(w)$ is a finite set of regular points in $\mathcal{T}(\overline{\operatorname{im}\mathbf{F}})$, $m_w = \operatorname{mult}(\sigma)$ equals the sum over all $v \in \mathbf{f}^{-1}(w)$, of the quantities

$$\frac{1}{\deg(\rho)} m_v \cdot \mathrm{index}(\mathbb{L}_\mathrm{w} \cap \mathbb{Z}^\mathrm{n} : \mathrm{A}(\mathbb{L}_\mathrm{v} \cap \mathbb{Z}^\mathrm{d})),$$

where the multiplicity m_v of the regular point v in $\mathcal{T}(\overline{\operatorname{im}\mathbf{F}})$ is as computed in [18, Theorem 5.1].

Consider the map $\Psi^{ST}(\alpha) = (m_\alpha(A_0), \dots, m_\alpha(A_n))$ defined as in (2.9), but for the rational map $\mathbf{F}$). For every $J \subset [n]_0$ with $|J| \geq d$ and every coherent collection of faces $F = (F_0, \dots, F_n)$ of $(A_0, \dots, A_n)$ adapted to J, define $\operatorname{index}(F, J)$ as the index of the lattice $\Psi^{ST}(C^F \cap \mathbb{Z}^d) + \mathbb{Z}^J$ in its saturated lattice, when both have rank d. Otherwise, $\operatorname{index}(F, J) = 0$. With this notation, the multiplicity m_v of the regular point v in $\mathcal{T}(\overline{\operatorname{im}\mathbf{F}})$ is equal to the sum of

$$\frac{1}{\deg(\mathbf{F})} \mathrm{index}(\mathrm{F}, \mathrm{J}) \cdot \mathrm{MV}(\mathrm{F_i} \mid \mathrm{i} \in \mathrm{J})$$

over all sets $J \subset [n]_0$ and all coherent collection F of faces of $A_0, \dots, A_n$ adapted to J such that $\Psi(C^F) + \mathbb{R}^J_{\geq 0}$ contains v. Here, $\mathrm{MV}(\mathrm{F_i} \mid \mathrm{i} \in \mathrm{J})$ denotes the $|J|$-dimensional mixed volume.

The following is a simple example of the multiplicity computation via this formula.

Example 2.10. Let $f_0, f_1, f_2 \in \mathbb{K}[x]$ be the polynomials

$$f_0 = x^3 + 3x, \quad f_1 = x^5 + 5x^3 \quad \text{and} \quad f_2 = x^5 + 22x^3 + 17x.$$

The rational map $\mathbf{f}$ has degree 2 and the cones in the tropicalization of its image are $C_{\{1\}} = \psi(\mathbb{R}_{\geq 0}) = \mathbb{R}_{\geq 0}(1,0)$; $C_{\{2\}} = \mathbb{R}_{\geq 0}(0,1)$ and $C_{\{0\}} = \psi(\mathbb{R}_{\leq 0}) = \mathbb{R}_{\geq 0}(-1,-1)$. The map $\mathbf{F}$ has degree 1 and the tropicalization of its image has 5 cones. The following table shows the multiplicity m_w of a regular point w in each of them (and the data used to compute it):

w	$v \in \rho^{-1}(w)$	m_v	$\mathrm{index}(\mathbb{L}_\mathrm{w} \cap \mathbb{Z}^2 : \mathrm{A}(\mathbb{L}_\mathrm{v} \cap \mathbb{Z}^3))$	m_w
$(-\lambda, -\lambda)$	$(\lambda, 0, 0)$	2	1	2
	$\frac{-\lambda}{2}(3,5,5)$	1	2	
$(\lambda, 0)$	$(0, \lambda, 0)$	2	1	2
	$\frac{\lambda}{2}(1,3,1)$	1	2	
$(0, \lambda)$	$(0, 0, \lambda)$	4	1	2

.

For example, take a regular point $w = (\lambda, 0) \in \mathbb{R}^2_{\geq 0}$. The set $\mathbf{f}^{-1}(w)$ has two points in $\mathcal{T}(\overline{\operatorname{im}\mathbf{F}})$: the point $v_1 = (0, \lambda, 0)$ which is in a cone of multiplicity 2 and the point $v_2 = (\lambda/2, 3\lambda/2, \lambda/2)\}$ which lies in a cone of multiplicity 1. Since $A(0,1,0) = (1,0)$, the index is 1, but since $A(1,3,1) = 2(1,0)$, the corresponding index is 2. By equation (2.6), $m_w = 2$.

3. Degree of rational varieties under sparse parametrizations

As before, we consider $n+1$ supports $A_0, \dots, A_n$ which lie in the nonnegative orthant $(\mathbb{Z}_{\geq 0})^d$ and generic polynomials $f_0, \dots, f_n$ with respective supports $A_0, \dots, A_n$, such that the rational map from (1.2)

$$\mathbf{f} = \left(\frac{f_1}{f_0}, \dots, \frac{f_n}{f_0}\right)$$

is generically finite and thus rationally parameterizes an irreducible variety S of dimension d. We give in Theorem 3.3 an explicit formula for its degree in terms of the supports when the coefficients are generic, and which is an upper bound in any case. Our sharp bound for the degree of S in Theorem 3.3 for generic coefficients is similar to the bound in Proposition 4 in [12] in case $f_0 = 1$. The upper bound given by the volume of the convex hull of the union of the supports appears in Section 3 in [8]. We give conditions for equalities in the chain (3.5) of inequalities in Theorem 3.9, and we will also show in Example 3.5 that all the inequalities can be strict.

Consider the open set $U = \{x \in (K^*)^d \,:\, \prod_{i=0}^n f_i(x) \neq 0\}$, the incidence variety

$$W = \{(x, y) \in U \times (\mathbb{K}^*) \,:\, f_0(x)\, y_1 - f_1(x) = \cdots = f_0(x)\, y_n - f_n(x) = 0\}, \tag{3.1}$$

and the projection

$$\pi \colon W \times (\mathbb{K}^*)^n \to (\mathbb{K}^*)^n, \qquad \pi(x, y) = y. \tag{3.2}$$

Then, S coincides with the closure of the image $\pi(W)$. Moreover, this map is finite iff $\mathbf{f}$ is finite and both have the same degree. Observe that the polynomials defining W don't have in general generic coefficients with respect to their supports even if $f_0, \dots, f_n$ have generic coefficients (so the results in [9] do not directly apply). In fact, we compute $\deg(S)$ in Theorem 3.3 studying the following similar genericity question: given a subset I of $[n]_0$, with cardinality d and generic coefficients $(c_{ij}, i = 1, \dots d, j = 0, \dots n)$, when do the polynomials $\sum_{j=0}^n c_{ij} f_j$, $i = 1, \dots, d$, have generic coefficients with respect to their support? If not, which is the number of their common zeros in the torus $(\mathbb{K}^*)^d$?

In order to state our main result in this section, we need to introduce two definitions.

Definition 3.1. Given a subset $A \subset \mathbb{Z}^d$ and a natural number j, denote by $A^j \subset \mathbb{Z}^{d+j}$ the finite subset

$$A^j \;=\; \{(\alpha, 0) \in \mathbb{Z}^{d+j} \mid \alpha \in A\} \cup \{e_{d+1}, \dots, e_{d+j}\}, \tag{3.3}$$

where $0 \in \mathbb{Z}^j$ and e_i denotes the i-th canonical basis vector.

We also define new lattice subsets associated to a collection of finite subsets.

Definition 3.2. Let $A_0, \dots, A_n \subset \mathbb{Z}^d$ with $d \leq n$, be a collection of finite lattice subsets. Given $I = \{i_1, \dots, i_d\} \subset [n]_0$ with $|I| = d$, for any $j \in [d]$ we denote

$$A_{i,I} \;=\; A_i \cup (\cup_{j \notin I} A_j) \subset \mathbb{Z}^d, \tag{3.4}$$

and we define d_I as the mixed volume $d_I = \mathrm{MV}(\mathrm{A_{i_1,I}}, \dots, \mathrm{A_{i_d,I}})$.

We then have

Theorem 3.3. *Consider Laurent polynomials $f_0, \dots, f_n$ in d variables and coefficients in $\mathbb{K}$ with respective supports $A_0, \dots, A_s \subset \mathbb{Z}^d$, such that the rational map $\mathbf{f} \colon (\mathbb{K}^*)^d \dashrightarrow (\mathbb{K}^*)^n$ defined as in (1.2) by $\mathbf{f} = \left(\frac{f_1}{f_0}, \dots, \frac{f_n}{f_0}\right)$, is generically finite with $\deg(\mathbf{f}) = \delta$. Then,*

$$\deg(S) \cdot \delta \;\leq\; \mathrm{MV}(\mathrm{A_0^{n+1-d}}, \dots, \mathrm{A_n^{n+1-d}}) \leq \min_{|I|=d} \mathrm{d_I} \leq \max_{|I|=d} \mathrm{d_I} \leq \mathrm{vol}(\mathrm{conv}(\cup_{j=0}^n \mathrm{A_j})). \tag{3.5}$$

Moreover, the left equality holds when $f_0, \dots, f_n$ have generic coefficients.

Proof. As we mentioned, we can assume without loss of generality (possibly after a common translation of the supports) that all exponents are nonnegative. The degree of S equals the number of intersection points of S with the zero set of d generic linear forms in n variables with coefficients in $\mathbb{K}$, which we will write as follows:

$$\ell_i = a_{i,1} y_1 + \cdots + a_{i,n} y_n + a_{i,0}, \quad i \in [d]. \tag{3.6}$$

Since $\ell_1, \dots, \ell_d$ are generic, we can assume that every intersection point lies in $\mathbf{f}(U)$, and it is enough to consider all common zeros of the form $z = \mathbf{f}(\sigma)$ for some σ in the domain of $\mathbf{f}$ (that is, such that $f_i(\sigma) \neq 0$ for any $i \in [n]_0$). Such σ belongs to the variety defined by the following polynomials $g_1, \dots, g_d \in \mathbb{K}[x_1, \dots, x_d]$:

$$g_i = \sum_{j=0}^n a_{i,j}\, f_j, \quad i \in [d].$$

These polynomials have support $\cup_{i=0}^n A_i$ and so, the number of isolated common zeros in the torus is bounded by $\mathrm{vol}(\mathrm{conv}(\cup_{j=0}^n A_j))$ by Bernstein's theorem, with equality in case $g_1,\dots,g_d$ are generic polynomials with this support. But, depending on the relative positions of $A_0,\dots,A_n$, this need not be the case.

A first refinement of this bound is given by the following observation. As the coefficients $(a_{i,j})_{\substack{i\in[n]_0\\ j\in[d]}}$ are generic, for every set $I=\{i_1,\dots,i_d\}\subset[n]_0$ with $|I|=d$, the system $g_1=\dots g_d=0$ is equivalent by means of row operations to a system of the form

$$\begin{array}{rclcl} h_1^I & = & f_{i_1} & + \sum_{j\notin I}\mu_{1j}^I f_j & = & 0\\ & & \ddots & \vdots & & \\ h_d^I & = & f_{i_d} & + \sum_{j\notin I}\mu_{dj}^I f_j & = & 0, \end{array} \tag{3.7}$$

where the coefficients $\mu_{ij}^I\in\mathbb{K}$ are generic. These polynomials h_j^I have supports $A_{i_j,I}$ for any $j\in[d]$. By the BKK bound, the system (3.7) has at most d_I isolated common zeros in $(\mathbb{K}^*)^d$ and by the monotony of the mixed volume, $d_I\le \mathrm{MV}(\cup_{j=0}^n A_j,\dots,\cup_{j=0}^n A_j))=\mathrm{vol}(\mathrm{conv}(\cup_{j=0}^n A_j))$. But still, depending on the original supports $A_0,\dots,A_n$, $h_1^I,\dots,h_d^I$ need not have generic coefficients with those supports, even if f_i are generic for their support A_i.

Take $I=[d]$ and consider the following polynomials in $\mathbb{K}[x_1,\dots,x_{n+1}]$:

$$\begin{array}{rclr} h_i^{[d]} & = & f_i+\sum_{j=d+1}^{n+1}\mu_{ij}^{[d]}x_j, & 1\le i\le d,\\ h_j^{[d]} & = & f_j-x_n, & d+1\le j\le n,\\ h_{n+1}^{[d]} & = & f_0-x_{n+1}. & \end{array} \tag{3.8}$$

Denote by U the complement in $(\mathbb{K}^*)^d$ of the union of the zeros of $f_0,\dots,f_{n+1}$, that is, the domain of definition of $\mathbf{f}$. Clearly, there is a bijection between common zeros in $U\times(\mathbb{K}^*)^{n+1-d}$ of $h_1^{[d]},\dots,h_{n+1}^{[d]}$ and common zeros in U of $g_1,\dots,g_d$, so $\deg(S)\deg(\mathbf{f})$ is bounded above by the mixed volume of their supports. But, for generic $f_0,\dots,f_n$ and generic linear forms $\ell_1,\dots,\ell_d$, these new polynomials $h_1^{[d]},\dots,h_{n+1}^{[d]}$ have generic coefficients. Moreover, they do not have common zeros in common in $(\mathbb{K}^*)^{n+1}$ with any f_i. Indeed, no f_i with $i=d+1,\dots,n$ can vanish if $(x_{d+1},\dots,x_{n+1})\in(\mathbb{K}^*)^{n+1-d}$. Take any $i\in[d]$, for instance $i=1$. Then, if $x=(x_1,\dots,x_{n+1})$ is a solution in the torus of system (3.8) and $f_1(x_1,\dots,x_d)=0$, then x is also a solution of the system of $n+2$ generic polynomials in $n+1$ variables:

$$f_1=\sum_{j=d+1}^{n+1}\mu_{1j}^{[d]}x_j=h_2^{[d]}=\dots=h_{n+1}^{[d]}=0,$$

which is a contradiction. So the number of common zeros in $U\times(\mathbb{K}^*)^{n+1-d}$ is the mixed volume of their supports, $\mathrm{MV}_{n+1}(\mathrm{A}_1^{n+1-d},\dots,\mathrm{A}_d^{n+1-d},(\mathrm{A}_{d+1},0)\cup\{\mathrm{e}_{d+1}\},\dots,(\mathrm{A}_n,0)\cup\{\mathrm{e}_n\},(\mathrm{A}_0,0)\cup\{\mathrm{e}_{n+1}\})$, where $0\in\mathbb{Z}^{n+1-d}$ as in Definition 3.1. Note that replacing $(x_{d+1},\dots,x_{n+1})$ by a generic linear combination, we would get the same number of solutions in the torus, and thus, this mixed volume coincides with the mixed volume $\mathrm{MV}(\mathrm{A}_0^{n+1-d},\dots,\mathrm{A}_n^{n+1-d})$, as stated in (3.5), and the first inequality follows.

To prove that $\mathrm{MV}(\mathrm{A}_0^{n+1-d},\dots,\mathrm{A}_n^{n+1-d})\le\min_{|I|=d}d_I$, observe that $\mathrm{MV}(\mathrm{A}_0^{n+1-d},\dots,\mathrm{A}_n^{n+1-d})$ is the number of solutions of a generic system of Equations (3.8), which coincides with $\deg(S)\ \deg(\mathbf{f})$. As proved using system (3.7), we have $\deg(S)\ \deg(\mathbf{f})\le d_I$ for all $I\subset[n]$ with $|I|=d$, we deduce the second inequality in (3.5), which concludes the proof. ☐

We deduce from the proof of Theorem 3.3 that the generic value $\mathrm{MV}(\mathrm{A}_0^{n+1-d},\dots,\mathrm{A}_n^{n+1-d})$ of $\deg(S)\cdot\deg(\mathbf{f})$ equals $\mathrm{MV}_{n+1}(\mathrm{A}_1^{n+1-d},\dots,\mathrm{A}_d^{n+1-d},(\mathrm{A}_{d+1},0)\cup\{\mathrm{e}_{d+1}\},\dots,(\mathrm{A}_n,0)\cup\{\mathrm{e}_n\},(\mathrm{A}_0,0)\cup\{\mathrm{e}_{n+1}\})$. Moreover, we could replace the choice of indices $[d]$ by any subset I of $[n]_0$ of cardinal d.

Example 3.4. Consider the application $\mathbf{f}$ given by the generic polynomials from Example 2.8. Theorem 3.3 proves the known fact that $\deg(S)=9$. Moreover, in this case the first inequality in (3.5)

$$\deg(S)\cdot\deg(\mathbf{f})\;=\;\mathrm{MV}(\mathrm{A}_0^2,\dots,\mathrm{A}_3^2)\le\;\min_{|I|=2}\mathrm{d}_I$$

in the statement is actually an equality. In fact, we can consider the steps in the proof of Theorem 3.3 to verify that $\deg(S) \cdot \deg(\mathbf{f}) = d_{\{1,2\}}$. We will see below that for general supports the equalities in (3.5) can be strict.

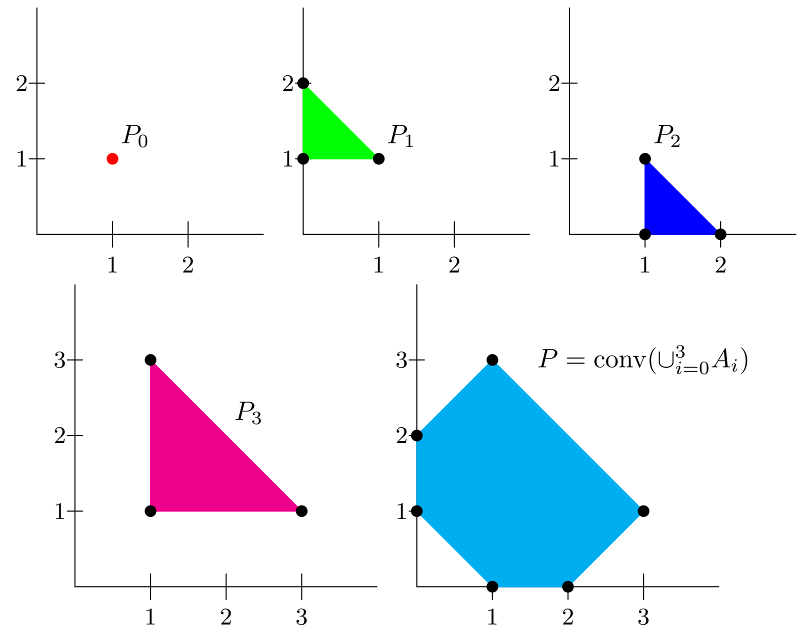
3.1. **Sufficient conditions for equality.** With the notations from Theorem 3.3, if $A_0 = A_1 = \cdots = A_n = A$ and $f_0, \dots, f_n$ are generic, the inequalities in (3.5) are all equalities and $\deg(S) \cdot \deg(\mathbf{f})$ equals $\mathrm{vol}(\mathrm{conv}(\mathrm{A}))$. But in general, even for generic polynomials, the inequalities may be strict, as we show in the next example.

Example 3.5. Let $d = 2, n + 1 = 4$, so S is a hypersurface in dimension 3. Consider the generic polynomials in $\mathbb{K}[x_1, x_2]$:

$$f_0(x_1, x_2) = x_1x_2, \quad f_1(x_1, x_2) = a_1x_2 + b_1x_2^2 + c_1x_1x_2,$$

$$f_2(x_1, x_2) = a_2x_1 + b_2x_1^2 + c_2x_1x_2 \quad \text{and} \quad f_3(x_1, x_2) = a_3x_1x_2 + b_3x_1^3x_2 + c_3x_1x_2^3.$$

Let A_0, A_1, A_2, A_3 be their supports sets. We draw their respective convex hulls P_i, $i \in [3]_0$.



The closure $S \subset \mathbb{K}^3$ of the image of the associated rational map $\mathbf{f} = \left(\frac{f_1}{f_0}, \frac{f_2}{f_0}, \frac{f_3}{f_0}\right)$ is a hypersurface. Let H be a reduced polynomial defining S. We observe that the degree of the map $\mathbf{f}$ is 1, and we can compute $\deg(S) = \deg(H) = \mathrm{MV}(\mathrm{A}_0^2, \dots, \mathrm{A}_3^2) = 5$, $\mathrm{vol}(\mathrm{conv}(\cup_{j=0}^3 A_j)) = 11$ and

$$5 < 6 = d_{\{1,3\}} = d_{\{2,3\}} < d_{\{0,3\}} < d_{\{1,2\}} < d_{\{0,1\}} = d_{\{0,2\}} = 10 < 11.$$

In fact, there exists a unique monomial in H of maximal degree 5.

In what follows we will present conditions on the supports $A_0, \dots, A_n$ for which some of the inequalities in the statement of Theorem 3.3 are actually equalities in case $d = n - 1$. We will need some preliminary results, which include equivalences due to Bihan and Soprunov (see [2]), and the notion of an essential collection of subsets in Definition 2.3, which was originally introduced in [17]. We then have:

Lemma 3.6. *Let $B_1, \dots, B_{n-1}$ and $B'_1, \dots, B'_{n-1}$ be finite sets in $(\mathbb{Z}_{\geq 0})^{n-1}$ such that $B_i \subset B'_i$ for every $i \in [n-1]$. Then, the following statements are equivalent:*

- *(i)* $\mathrm{MV}(\mathrm{B}_1, \dots, \mathrm{B}_{n-1}) < \mathrm{MV}(\mathrm{B}'_1, \dots, \mathrm{B}'_{n-1})$.
- *(ii) There exists a coherent collection of proper faces $F = (F_1, \dots, F_{n-1})$ of the collection $B'_1, \dots, B'_{n-1}$ such that the collection $B'_{1,F}, \dots, B'_{n-1,F}$ is essential, where*

$$B'_{i,F} = \begin{cases} F_i & \text{if } B_i \cap F_i \neq \emptyset \\ B'_i & \text{if } B_i \cap F_i = \emptyset \end{cases}.$$

(iii) There exists a coherent collection of proper faces $F = (F_1, \ldots, F_{n-1})$ of the collection $B'_1, \ldots, B'_{n-1}$ such that the collection $\{B_i \cap F_i \mid B_i \cap F_i \neq \emptyset\}$ is either empty or essential.

Proof. The equivalence between items (i) and (ii) follows from [2, Theorem 3.3]. To see that item (i) is equivalent to item (iii) we apply Bernstein's Theorem. Let $g_1, \ldots, g_{n-1}$ be generic polynomials with support sets $B_1, \ldots, B_{n-1}$. Write $g_i(x) = \sum_{\alpha \in B_i} c_{i\alpha} x^\alpha$. Then $\mathrm{MV}(\mathrm{B}_1, \ldots, \mathrm{B}_{n-1}) < \mathrm{MV}(\mathrm{B}'_1, \ldots, \mathrm{B}'_{n-1})$ if and only if there exists a coherent collection of proper faces $F = (F_1, \ldots, F_{n-1})$ of $(B'_1, \ldots, B'_{n-1})$ such that the polynomials $\{\sum_{\alpha \in B_i \cap F_i} c_{i\alpha} x^\alpha\}_{i=1}^{n-1}$ have a common zero with all non-zero coordinates. But, since $g_1, \ldots, g_{n-1}$ are generic, this is equivalent to the set $\{B_i \cap F_i \mid B_i \cap F_i \neq \emptyset\}$ being empty or essential. ☐

Corollary 3.7. *Let $B_1, \ldots, B_{n-1}$ be finite sets in $(\mathbb{Z}_{\geq 0})^{n-1}$. Then, the following are equivalent:*

(i) $\mathrm{MV}(\mathrm{B}_1, \ldots, \mathrm{B}_{n-1}) = \mathrm{vol}(\mathrm{conv}(\cup_{\mathrm{i}=1}^{\mathrm{n}-1} \mathrm{B}_\mathrm{i}))$.

(ii) For every $r \in [n-1]$ and every face $\mathcal{F}$ of $\cup_{i=1}^{n-1} B_i$ of codimension r, the cardinality of the set $\{i \mid B_i \cap \mathcal{F} \neq \emptyset\}$ is at least $n - r$.

(iii) For every proper face $\mathcal{F}$ of $\cup_{i=1}^{n-1} B_i$ there exists a nonempty subset $J \subset \{i \mid B_i \cap \mathcal{F} \neq \emptyset\}$ such that $\dim(\sum_{j \in J}(B_j \cap \mathcal{F})) < |J|$.

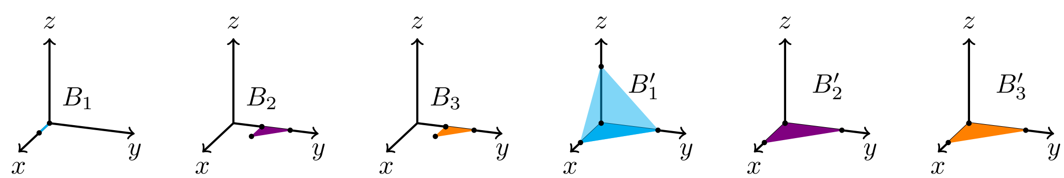
Proof. The equivalence between items (i) and (ii) follows from applying [2, Corollary 3.7] using the polytope $\mathrm{conv}(\cup_{i=1}^{n-1} B_i)$. Item (iii) follows from negation of item (iii) in Lemma 3.6. ☐

Example 3.5, continuation. Recall that by Definition 3.2, for any any $I = \{j, k\} \subset [3]$ with 2 elements and i the index in its complement, $d_I = \mathrm{MV}(\mathrm{A_j} \cup \mathrm{A_0} \cup \mathrm{A_i}, \mathrm{A_k} \cup \mathrm{A_0} \cup \mathrm{A_i})$ and both supports are contained in the union $\cup_{j=0}^3 A_i$. It can be checked that item (iii) in Corollary 3.7 holds and so $d_I < \mathrm{vol}(\mathrm{conv}(\cup_{j=0}^3 A_j))$ for all I.

The following example shows that the conditions given in items (ii) and (iii) in Lemma 3.6 are different.

Example 3.8. Consider the finite sets in $(\mathbb{Z}_{\geq 0})^3$

$$B_1 = \{(0,0,0),(1,0,0)\}, \qquad B'_1 = \{(0,0,0),(2,0,0),(0,2,0),(0,0,2)\},$$
$$B_2 = \{(0,1,0),(1,1,0),(0,2,0)\}, \qquad B'_2 = \{(0,0,0),(2,0,0),(0,2,0)\},$$
$$B_3 = \{(0,1,0),(1,1,0),(0,2,0)\} \quad \text{and} \quad B'_3 = \{(0,0,0),(2,0,0),(0,2,0)\}.$$



Clearly $0 = \mathrm{MV}(\mathrm{B}_1, \mathrm{B}_2, \mathrm{B}_3) < \mathrm{MV}(\mathrm{B}'_1, \mathrm{B}'_2, \mathrm{B}'_3)$. If we take the coherent collection of faces of B'_1, B'_2, B'_3 associated to the interior normal vector $(1,1,0)$, the collection from item (ii) in Lemma 3.6 is essential. However, the collection from item (iii) is not. On the other hand, if we consider the coherent collection of faces associated to the interior normal vector $(0,1,1)$, the collection from item (iii) Lemma 3.6 is essential but the collection from item (ii) is not.

Therefore, it is not straightforward to prove in a direct way the equivalence of items (ii) and (iii) in Lemma 3.6. It would be interesting to see a direct combinatorial proof without going through item (i).

The following theorem provides a necessary and sufficient condition for the equality $d_I = \mathrm{vol}(\mathrm{conv}(\cup_{j=0}^n A_j))$ for some $I \subset [n]_0$ with $|I| = n - 1$. We will denote by I^c the complement of I in $[n]_0$.

Theorem 3.9. *Let $A_0, \ldots, A_n$ be finite sets in $(\mathbb{Z}_{\geq 0})^{n-1}$. Fix $I \subset [n]_0$ with $|I| = n - 1$. The following statements are equivalent:*

(i) For every $r \in [n-1]$ and every face $\mathcal{F}$ of $\cup_{j=0}^n A_j$ of codimension r and every $J \subset [n]_0$ such that $|J| = r + 2$ that contains I^c, we have $(\cup_{j \in J} A_j) \cap \mathcal{F} \neq \emptyset$.

(ii) $\mathrm{vol}(\mathrm{conv}(\cup_{j=0}^n A_j)) = d_I$.

Proof. For every set $J \subset [n]_0$ that contains I^c, we can define $\widetilde{J} = J \backslash I^c \subset I$. Then item (i) happens if and only if for every face $\mathcal{F}$ of $\cup_{j=0}^n A_j$ of codimension $r > 0$ and every $\widetilde{J} \subset I$ with $|\widetilde{J}| = r$, $(\cup_{j \in \widetilde{J}}(A_j \cup (\cup_{i \in I^c} A_i)) \cap \mathcal{F} \neq \emptyset$. Using item (ii) of Corollary 3.7, this is equivalent to item (ii). □

The next theorem provides a sufficient condition for the equality $\deg(S) \cdot \deg(\mathbf{f}) = d_I$ for some $I \subset [n]_0$ with $|I| = n-1$.

Theorem 3.10. *Let $A_0, \ldots, A_n$ be finite sets in $(\mathbb{Z}_{\geq 0})^{n-1}$. Let $f_0, \ldots, f_n$ be generic polynomials with support $A_0, \ldots, A_n$ and coefficients in $\mathbb{K}$ such that the rational map $\mathbf{f}$ from (1.2) is generically finite. Assume $I \subset [n]_0$ with $|I| = n-1$ satisfies that for every coherent collection of proper faces F of $(A_i \cup (\cup_{j \in I^c} A_j))_{i \in I}$, there exists J nonempty subset of $\mathcal{P}_F = \{i \in I \mid A_i \cap F_i \neq \emptyset\}$ such that*

$$dim(\sum_{i \in J} A_i \cap F_i) < |J|.$$

Then, $deg(S) \cdot \deg(\mathbf{f}) = d_I = \mathrm{MV}(\mathrm{A_{i_1}}, \ldots, \mathrm{A_{i_{n-1}}})$.

Proof. Denote $I = \{i_1, \ldots, i_{n-1}\}$ and $I^c = \{i_0, i_n\}$. As in the proof of Theorem 3.3, take generic coefficients $\{\mu_{j,I}, \nu_{j,I}\}_{j=1}^{n-1}$ so that the system $h_{j,I} = f_{i_j} + \mu_{j,I}\, f_{i_0} + \nu_{j,I}\, f_{i_n} = 0$ for $j \in [n-1]$ has $\deg(S)\, \deg(\mathbf{f})$ common isolated zeros, all of them in the open set $\{x \in (\mathbb{K}^*)^{n-1} \mid \prod_{j=0}^n f_j(x) \neq 0\}$.

Consider the homotopy $\{f_{i_j} + t\, \mu_{j,I}\, f_{i_0} + t\, \nu_{j,I}\, f_{i_n}\}_{j=1}^{n-1}$. For all but finitely many values of $t \in \mathbb{K}$, the system given by these polynomials has $\deg(S) \cdot \deg(\mathbf{f})$ isolated common zeros with all non-zero coordinates, and $\mathrm{MV}(\mathrm{A_{i_1}}, \ldots, \mathrm{A_{i_{n-1}}})$ when $t = 0$. Then, using Theorem 3.3, $\mathrm{MV}(\mathrm{A_{i_1}}, \ldots, \mathrm{A_{i_{n-1}}}) \leq \mathrm{deg(S)} \cdot \deg(\mathbf{f}) \leq \mathrm{d_I} = \mathrm{MV}(\mathrm{A_{i_1}} \cup \mathrm{A_{i_0}} \cup \mathrm{A_{i_n}}, \ldots, \mathrm{A_{i_{n-1}}} \cup \mathrm{A_{i_0}} \cup \mathrm{A_{i_n}})$ common zeros with all non-zero coordinates. By Lemma 3.6, $\mathrm{MV}(\mathrm{A_{i_1}}, \ldots, \mathrm{A_{i_{n-1}}}) = \mathrm{d_I}$. □

Example 3.11. Let $d \in \mathbb{N}$ be an even number. Consider the hypersurface S parameterized by the rational map $\mathbf{f}$ as in (1.2), where $f_0, \ldots, f_3$ are generic polynomials with two variables, coefficients in $\mathbb{K}$ and supports

$$A_0 = \{(0,0), (1,0), (0,1), (d/2, d/2)\} = A_3,\ A_1 = \{(0,0), (1,0), (0,d)\},\ A_2 = \{(0,0), (d,0), (0,1)\}.$$

Then the conditions of Theorem 3.10 are satisfied for $I = \{1,2\}$ and we have that $\deg(\mathbf{f}) = 1$, so the degree of the hypersurface equals $\deg(S) = d^2 = MV(A_0 \cup A_1 \cup A_3, A_0 \cup A_2 \cup A_3)$.

4. Orders of rational hypersurfaces under sparse parametrizations

In this section, we assume that $d = n-1$ and S is a hypersurface in $(\mathbb{K}^*)^n$. We will study its order at the origin $\mathrm{ord}_0(S)$, which is defined as follows. Let H be a (reduced) polynomial such that $S = (H = 0)$. Then,

$$\mathrm{ord}_0(S) = \max\{m \in \mathbb{Z}_{\geq 0} \mid \partial^\beta(H)(0) = 0, \text{ for all } \beta \in (\mathbb{Z}_{\geq 0})^n, |\beta| < m\}.$$

We compute $\mathrm{ord}(S)$ in Theorem 4.2 under an additional condition on the family of supports. As we remarked in the Introduction, depending on the relative positions of the supports $A_0, \ldots, A_n$ it could happen that $\mathrm{ord}(S) > 1$ (that is, the origin is a singular point of the closure of S in $\mathbb{K}^n$) even for generic coefficients, We end with two examples where our proofs can be extended to other cases not falling into the hypotheses of Theorem 4.2.

Before stating Theorem 4.2, we need to introduce a definition.

Definition 4.1. Let $A_0, \ldots, A_n$ be a family of finite sets in $(\mathbb{Z}_{\geq 0})^{n-1}$. Given $\alpha \in \mathbb{Q}^{n-1}$, we call $J_\alpha = \{j \in [n]_0 \mid m_\alpha(A_j) < \max_{i \in [n]_0}\{m_\alpha(A_i)\}\}$. We say that the family is tame if for any α such that $J_\alpha \neq \emptyset$ and $|J_\alpha| \neq n$, the family $\{\mathrm{face}_\alpha(A_j)\}_{j \in J_\alpha}$ in $(\mathbb{Z}_{\geq 0})^{n-1}$ is not essential (see Definition 2.3).

Note that in case all A_i are equal for all $i \in [n]_0$ or if all A_i are equal for $i \in [n]$ and A_0 consists of a single point (thus f_i/f_0 are Laurent polynomials with the same support, the case studied in [20]), the family is always tame.

Recall that by Theorem 3.3 the degree of S for generic coefficients satisfies $\deg(S) \cdot \deg(\mathbf{f}) = \mathrm{MV}(\mathrm{A_0^2}, \ldots, \mathrm{A_n^2})$, where the notation $A_i^2 \subset \mathbb{Z}^{n+1}$ is given in Definition 3.1.

Theorem 4.2. *Let $A_0,\ldots,A_n$ be a tame family of finite sets in $(\mathbb{Z}_{\geq 0})^{n-1}$. Let $f_0,\ldots,f_n$ be generic polynomials with respective supports $A_0,\ldots,A_n$ and coefficients in $\mathbb{K}$ such that the map $\mathbf{f}$ from (1.2) is generically finite and parameterizes a hypersurface S in $\mathbb{K}^n$. Then,*

$$ord_0(S)\cdot\deg(\mathbf{f}) = \mathrm{MV}(A_0^2,\ldots,A_n^2)-\mathrm{MV}(A_1^1,\ldots A_n^1) \geq \mathrm{MV}(A_0^2,\ldots,A_n^2)-\min_{j\in[n]}\{\mathrm{MV}(\{A_i\cup A_j\}_{\substack{i\in[n]\\ i\neq j}})\}.$$

Proof. Consider $\ell_i = a_{i,1}y_1+\cdots+a_{i,n}y_n$ generic linear forms with coefficients in $\mathbb{K}$, and generic constants $a_{i,0}\in\mathbb{K}^*$ for all $i\in[n-1]$. As before, consider the open set $U=\{x\in(\mathbb{L}^*)^{n-1} \mid \prod_{j=0}^n f_j(x)\neq 0\}$. We can assume that $H,\ell_1-a_{1,0}\varepsilon,\ldots,\ell_{n-1}-a_{n-1,0}\varepsilon$ have $D=\mathrm{MV}(A_0^2,\ldots,A_n^2)/\deg(\mathbf{f})$ common roots, and that they all lie in $\mathbf{f}_{\mathbb{L}}(U)$.

To compute the order at the origin of H, we will consider the line in $\mathbb{L}^n$ defined by the generic affine linear forms $\ell_1-a_{1,0}\varepsilon,\ldots,\ell_{n-1}-a_{n-1,0}\varepsilon$. Notice that this line can be parameterized as

$$\{x\in\mathbb{L}^n \mid \ell_1(x)=a_{1,0}\varepsilon,\ldots,\ell_{n-1}(x)=a_{n-1,0}\varepsilon\}=\{\lambda\, u+\varepsilon\, v \mid \lambda\in\mathbb{L}\},$$

with $u,v\in\mathbb{K}^n$. The common zeros of the polynomials $H,\ell_1-a_{1,0}\varepsilon,\ldots,\ell_{n-1}-a_{n-1,0}\varepsilon$ in $\mathbf{f}_{\mathbb{L}}(U)$ are exactly the points

$$\omega_k=\left(\frac{f_1(\sigma_k(\varepsilon))}{f_0(\sigma_k(\varepsilon))},\ldots,\frac{f_n(\sigma_k(\varepsilon))}{f_0(\sigma_k(\varepsilon))}\right)=\lambda_k\, u+v\,\varepsilon,\quad k\in[D],$$

for which $H(\lambda_k\, u+v\,\varepsilon)=0$. If we consider the polynomial $H(\lambda\, U+\varepsilon\, V)\in\mathbb{K}[U,V,\varepsilon][\lambda]$, we have

$$H(\lambda\, U+\varepsilon\, V)=H_0(U)\lambda^D+H_1(U,V,\varepsilon)\lambda^{D-1}+\cdots+H_D(V,\varepsilon), \tag{4.1}$$

where for all $j\in[D]_0$, $H_j(U,V,\varepsilon)$ is a polynomial of degree at most j in ε.

By genericity of ℓ_i and $a_{i,0}$, we can assume that $u,v\in(\mathbb{K}^*)^n$, $H_0(u)\neq 0$ and $H_D(v,\varepsilon)=H(v\,\varepsilon)\neq 0$. Note that $\mathrm{val}(H_0(u))=0$ and $\mathrm{val}(H_j(u,v,\varepsilon))\geq 0$ for all $j\in[D-1]_0$. Hence, if we consider the coefficients of $H(u\lambda+v\varepsilon)=\sum_{i=0}^D H_{D-i}(u,v,\varepsilon)\lambda^i\in\mathbb{L}[\lambda]$ as elementary symmetric functions of the roots $\{\lambda_k\}_{k=1}^D\subset\mathbb{L}$, we can easily see that $\mathrm{val}(\lambda_k)\geq 0$ for all $k\in[D]$. Indeed, assume that $\mathrm{val}(\lambda_k)<0$ for some k and consider the non-empty set $J=\{k\in[D] \mid \mathrm{val}(\lambda_k)<0\}$. Then, the elementary symmetric function $H_{\#J}(u,v,\varepsilon)$ has the same negative valuation as $H_0(u)\prod_{k\in J}\lambda_k$, which is a contradiction. Moreover, the minimum power of λ with a coefficient of valuation 0 is $D-\#\{k \mid \mathrm{val}(\lambda_k)=0\}$. On the other hand, if we now consider H as a sum of homogeneous terms of degree $\mathrm{ord}_0(S)$ to D and we evaluate it at $u\lambda+v\varepsilon$, the minimum power of λ with a coefficient of valuation 0 is $\mathrm{ord}_0(S)$. Therefore,

$$\mathrm{ord}_0(S)=D-\#\{k \mid \mathrm{val}(\lambda_k)=0\}. \tag{4.2}$$

We will focus in what follows on bounding the number of points $\{\omega_k\}_{k=1}^D$ for which $\mathrm{val}(\lambda_k)=0$. Let $\omega_k=\left(\frac{f_1(\sigma_k)}{f_0(\sigma_k)},\ldots,\frac{f_n(\sigma_k)}{f_0(\sigma_k)}\right)$ be one of them. Since $m=\mathrm{val}(f_i(\sigma_k))$ is fixed for all $i\in[n]_0$ and the supports are tame, let us prove that this implies that if $\sigma_k=b_k\varepsilon^\alpha+\mathrm{h.o.t.}(\varepsilon)$ with $b_k\in(\mathbb{K}^*)^n$, then necessarily $\alpha=0$. Assume first that $\alpha\neq 0$ and recall the notation in (2.5): $m_\alpha(A_j)=\min_{p\in A_j}\langle\alpha,p\rangle$. Since $\mathrm{in}_\alpha(f_1),\ldots,\mathrm{in}_\alpha(f_n)$ are generic, they do not all vanish on b_k. Also, $m=m_\alpha(A_j)$ if and only if $\mathrm{in}_\alpha(f_j)(b_k)\neq 0$ for all $j\in[n]_0$. However, since the supports are tame, $\{\mathrm{in}_\alpha(f_j)\}_{j\in J_\alpha}$ can only have common zeros with all non-zero coordinates when $J_\alpha=\emptyset$, and so $\mathrm{in}_\alpha(f_j)(b_k)\neq 0$ for all $j\in[n]_0$. As $a_{i,1}f_1(\sigma_k)+\cdots+a_{i,n}f_n(\sigma_k)-a_{i,0}f_0(\sigma_k)\varepsilon=0$ for all $i\in[n-1]$, $b_k\in(\mathbb{K}^*)^{n-1}$ is a common zero of the polynomials $\ell_i(\mathrm{in}_\alpha(f_1(x)),\ldots,\mathrm{in}_\alpha(f_n(x)))\in\mathbb{K}[x_1,\ldots,x_{n-1}]$ for all $i\in[n-1]$. The convex hull of the support of any linear combination of $\mathrm{in}_\alpha(f_1(x))$, $\ldots,\mathrm{in}_\alpha(f_n(x))$ has dimension at most $n-2$ because all supports are contained in the hyperplane with equation $\langle\alpha,p\rangle=m$. Then, the variety $W_\alpha\subset(\mathbb{K}^*)^n$ parameterized for all $x\in U$ by $(\mathrm{in}_\alpha(f_1(x)),\ldots,\mathrm{in}_\alpha(f_n(x))$ has dimension at most $n-2$. The space of lines from a point in W_α through the origin has then codimension at least one in the space of lines in $\mathbb{K}^n$ through the origin. Thus, the intersection of W_α with a generic line $\ell_1=0,\ldots,\ell_{n-1}=0$ is empty. Therefore, $\alpha=0$.

Consider then $\sigma_k=b_k+\mathrm{h.o.t.}(\varepsilon)$ with $b_k\in(\mathbb{K}^*)^{n-1}$. The polynomials

$$g_i=\ell_i(f_1(x),\ldots,f_n(x)),\quad i\in[n-1],$$

have finitely many common zeros in $(\mathbb{K}^*)^{n-1}$, b_k being one of those zeros. As in the proof of Theorem 3.3, by taking linear combinations, we can tranform them into polynomials of the form $f_1+\mu_1 f_n,\ldots,f_{n-1}+\mu_{n-1}f_n$ with generic coefficients $\mu_1,\ldots,\mu_{n-1}$, and thus its number of common zeros is bounded above by $\min_{j\in[n]}\{\mathrm{MV}(\{\mathrm{A_i}\cup\mathrm{A_j}\}_{\mathrm{i}\neq\mathrm{j},0})\}$. Moreover, one can take an extra variable x_n and define an associated generic system $h_1 = \cdots = h_n = 0$ where

$$h_i = f_i + \mu_i\, x_n \in \mathbb{K}[x_1, \ldots, x_n] \text{ for all } i \in [n-1], \text{ and } h_n = f_n - x_n.$$

By the genericity of their coefficients, $f_1, \ldots, f_n$ do not have common zeros, and hence common zeros of $g_1, \ldots, g_{n-1}$ in $(\mathbb{K}^*)^{n-1}$ correspond to common zeros of the polynomials $h_1, \ldots, h_n$ in $(\mathbb{K}^*)^n$. By Bernstein's Theorem, the number of common roots of $g_1, \ldots g_{n-1}$ equals $\mathrm{MV}(\{\mathrm{A_i^1}\}_{\mathrm{i=1}}^{\mathrm{n}})$. Since U is smooth, [10, Corollary 6.7.2] ensures that the Jacobian matrix of $g_1, \ldots, g_{n-1}$ has nonzero determinant when evaluated at a common zero $b \in U \subset (\mathbb{K}^*)^{n-1}$. This implies that the Jacobian matrix of $\ell_i(f_1(x), \ldots, f_n(x)) - a_{i,0} f_0(x)\varepsilon$ has valuation zero when evaluated at the common roots $b \in (\mathbb{K}^*)^{n-1}$ of $g_1, \ldots, g_{n-1}$ and, by Hensel's Lemma, each of these common zeros b can be lifted to a unique common zero σ of $a_{i,1}f_1 + \cdots + a_{i,n}f_n - a_{i,0}f_0\varepsilon$, for all $i \in [n-1]$. We deduce that the number of λ_k with $\mathrm{val}(\lambda_k) = 0$ is $\deg(\mathbf{f})^{-1} \cdot \mathrm{MV}(\{\mathrm{A_i^1}\}_{\mathrm{i=1}}^{\mathrm{n}})$ and the result follows from (4.2). ☐

The following proposition presents a sufficient condition for the inequality in the statement Theorem 4.2 to be an equality and the order can be computed in an easier way.

Proposition 4.3. *Under the hypotheses and notation of Theorem 4.2, if there exists $j_0 \in [n]$ such that for all coherent collection of proper faces F of $(A_i \cup A_{j_0})_{i\neq j_0}$, there exists a nonempty subset J of $\mathcal{P}_F = \{i \in [n]\backslash\{j_0\} \mid A_i \cap F_i \neq \emptyset\}$ such that $\dim(\sum_{i\in J} A_i \cap F_i) < |J|$, then*

$$\mathrm{ord}_0(S) \cdot \deg(\mathbf{f}) = \mathrm{MV}(\mathrm{A_0^2}, \ldots, \mathrm{A_n^2}) - \mathrm{MV}(\mathrm{A_1}, \ldots, \mathrm{A_{j_0-1}}, \mathrm{A_{j_0+1}}, \ldots, \mathrm{A_n}). \tag{4.3}$$

In particular,

$$MV(A_1^1, \ldots A_n^1) = \mathrm{MV}(\{\mathrm{A_i} \cup \mathrm{A_{j_0}}\}_{\substack{\mathrm{i}\in[\mathrm{n}]\\ \mathrm{i}\neq\mathrm{j_0}}}) = \mathrm{MV}(\mathrm{A_1}, \ldots, \mathrm{A_{j_0-1}}, \mathrm{A_{j_0+1}}, \ldots, \mathrm{A_n}). \tag{4.4}$$

Proof. Denote $\mathrm{MV_{j_0}} = \mathrm{MV}(\mathrm{A_1}, \ldots, \mathrm{A_{j_0-1}}, \mathrm{A_{j_0+1}}, \ldots, \mathrm{A_n})$. As the polynomials $f_0, \ldots, f_n$ are generic with respect to their supports, we have by Theorem 3.3 that $\deg(S)\cdot\deg(\mathbf{f}) = \mathrm{MV}(\mathrm{A_0^2}, \ldots, \mathrm{A_n^2})$. Because the supports are tame, we saw in the proof of Theorem 4.2 that $ord_0(S) \cdot \deg(\mathbf{f}) = \mathrm{MV}(\mathrm{A_0^2}, \ldots, \mathrm{A_n^2}) - \#\{\mathrm{k} \mid \mathrm{val}(\lambda_\mathrm{k}) = 0\} \cdot \deg(\mathbf{f})$ and that we can compute $N = \#\{k \mid \mathrm{val}(\lambda_k) = 0\}\cdot\deg(\mathbf{f})$ as the number of common zeros in $(\mathbb{K}^*)^{n-1}$ of $\ell_i(f_1(x), \ldots, f_n(x))$ for $i \in [n-1]$. Taking linear combinations, this system is equivalent to $f_i + \mu_i f_{j_0} = 0$ for all $i \in [n],\ i \neq j_0$. Thus, N is bounded above by $\mathrm{MV}(\{\mathrm{A_i} \cup \mathrm{A_{j_0}}\}_{\substack{\mathrm{i}\in[\mathrm{n}]\\ \mathrm{i}\neq\mathrm{j_0}}})$. Now, as in the proof of Theorem 3.10, we introduce a new variable t and consider the homotopy:

$$f_i + t\mu_i f_{j_0} = 0 \ \text{ for all } i \in [n],\ i \neq j_0. \tag{4.5}$$

For almost all t, its number of common zeros in $(\mathbb{K}^*)^{n-1}$ is N. As $f_1, \ldots, f_n$ are generic polynomials, they have $MV(A_1, \ldots, A_{j_0-1}, A_{j_0+1}, \ldots, A_n)$ common zeros with all non-zero coordinates. Since these zeros are isolated, each is in a germ of curve of zeros of (4.5) not contained in $\{t = 0\}$. This implies that $MV(A_1, \ldots, A_{j_0-1}, A_{j_0+1}, \ldots, A_n) \leq N$. Therefore,

$$\mathrm{MV_{j_0}} \leq \mathrm{N} \leq \mathrm{MV}(\mathrm{A_1} \cup \mathrm{A_{j_0}}, \ldots, \mathrm{A_{j_0-1}} \cup \mathrm{A_{j_0}}, \mathrm{A_{j_0+1}} \cup \mathrm{A_{j_0}}, \ldots, \mathrm{A_n} \cup \mathrm{A_{j_0}}).$$

But now, it follows from Lemma 3.6 that $\mathrm{MV}(\mathrm{A_1}, \ldots, \mathrm{A_{j_0-1}}, \mathrm{A_{j_0+1}}, \ldots, \mathrm{A_n}) = \mathrm{MV_{j_0}}$, and hence $\mathrm{ord}_0(S)\ \deg(\mathbf{f}) = MV(A_0^2, \ldots, A_n^2) - MV_{j_0}$, as claimed. The remaining equality in (4.4) follows from the computation of $\mathrm{ord}_0(S)$ in the statement of Theorem 4.2. ☐

Example 4.4. Consider again the application $\mathbf{f}$ given by the generic polynomials from Example 2.8. Since f_0, f_1, f_2, f_3 are generic, the conditions in Proposition 4.3 are realized for any $j_0 \in [3]$ and we deduce that $\mathrm{ord}_0(S) = 4$, as we mentioned, because $\deg(\mathbf{f}) = 1$.

Remark 4.5. While the hypothesis in the statement of Theorem 4.2 that the supports are tame is sufficient, it is not always necessary. For example, consider the family of lattice sets $A_0 = \{(0,0,0)\}$, $A_1 = \{(2,2,0),(1,1,1)\}$, $A_2 = \{(1,1,0),(1,2,1)\}$, $A_3 = \{(2,1,0),(2,0,1)\}$, $A_4 = \{(2,1,0),(1,0,0)\}$. It is easy to check that this family of supports is not tame: for $\alpha = (-1,1,1)$, the set J_α from Definition 4.1 is $J_\alpha = \{3,4\}$. However, the family $\{\mathrm{face}_\alpha(A_3), \mathrm{face}_\alpha(A_4)\}$ is essential. Even in this case, for generic f_0, f_1, f_2, f_3, f_4 with those supports, we still have (as in the statement of Theorem 4.2) that

$$\mathrm{ord}_0(S)\cdot\deg(\mathbf{f}) = \mathrm{MV}(\mathrm{A_0^2}, \ldots, \mathrm{A_4^2}) - \mathrm{MV}(\mathrm{A_1^1}, \ldots \mathrm{A_4^1}) \geq \mathrm{MV}(\mathrm{A_0^2}, \ldots, \mathrm{A_4^2}) - \min_{\mathrm{j}\in[4]}\{\mathrm{MV}(\{\mathrm{A_i}\cup\mathrm{A_j}\}_{\substack{\mathrm{i}\in[4]\\ \mathrm{i}\neq\mathrm{j}}})\}.$$

Moreover, the inequality above is an equality. It can be computed that for generic polynomials, $\deg(\mathbf{f}) = 1$, $\deg(S) = 9$, $\mathrm{ord}_0(S) = 4$, and $\mathrm{MV}(\{\mathrm{A}_\mathrm{i}^1\}_{\mathrm{i}=1}^4) = \min_{\mathrm{i}\in[4]}\{\mathrm{MV}(\{\mathrm{A}_\mathrm{j} \cup \mathrm{A}_\mathrm{i}\}_{\mathrm{j}\neq\mathrm{i}})\} = 5$.

Example 4.6. Consider again Example 2.9 with non-generic coefficients. As in Example 4.4, by applying Bernstein's Theorem to the polynomial systems in the proof of Theorem 3.3, we can see that all inequalities in Theorem 3.3 are actually equalities and the degree of the surface S is $9 = \mathrm{vol}(\mathrm{conv}(\cup_{j=0}^3 A_j))$. The polytope $\mathrm{conv}(\cup_{j=1}^3 A_j)$ with vertices $(2, 0), (0, 2), (3, 0), (0, 3) \in (\mathbb{Z}_{\geq 0})^2$ has volume $\mathrm{vol}(\mathrm{conv}(\cup_{j=1}^3 A_j)) = 5$. Theorem 4.2 predicts the lower bound 4 for generic systems with those supports, which is also true in this case. To see that, we can follow the steps in its proof, and first verify that there are no $\sigma_k = b_k\varepsilon^\alpha + \mathrm{h.o.t.}(\varepsilon) \in \mathbb{C}\{\{\varepsilon^\mathbb{R}\}\}$ with $\alpha \neq 0$ and $\mathrm{val}(\lambda_k) = 0$. Then, we can check that the polynomials $\ell_i(f_1, f_2, f_3)$ for $i \in \{1, 2\}$ have exactly 5 common zeros in $(\mathbb{K}^*)^2$ by means of the genericity conditions in Bernstein's Theorem. Finally, applying [10, Corollary 6.7.2] and Hensel's Lemma, the order at the origin is exactly $9 - 5 = 4$.

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