Book review ”A Course on Rough Paths: With an Introduction to Regularity Structures”

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BOOK REVIEW: *A COURSE ON ROUGH PATHS — WITH AN INTRODUCTION TO REGULARITY STRUCTURES* (PETER K. FRIZ AND MARTIN HAIRER)

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Abstract. This review concerns the book *A course on rough paths — With an introduction to regularity structures* by P.K. Friz and M Hairer which contains an introduction to both the theory of rough paths and the theory of regularity structures and develop their applications to integration theory, stochastic analysis and mathematical finance, among others.

In a broad sense, the *rough paths theory* can be seen as a natural extension of the theory of ordinary differential equations which plays well in a stochastic context. *Regularity structures* — for which M. Hairer was awarded a Field medal in 2014 — extend the notion of Stochastic Partial Differential Equations (SPDE) in a way well suited to deal with singularities. The book [1] under review is an introduction to both theories which are part of approaches sometimes called "pathwise stochastic calculus".

In relation with arbitrage-free theory, Itô's stochastic calculus is among the core tools that quantitative analysts and academics should master. Therefore, why consider other theories of integration? Actually, there is a full room for pathwise stochastic calculus in mathematical finance as it allows to

- deal with stochastic differential equations driven by a fractional Brownian motion or other Gaussian processes,
- set up "cubature formula" or other numerical Monte Carlo methods for efficient option pricing,
- develop innovative, state-of-art machine learning techniques,
- integrate anticipating stochastic processes,
- solve rough volatility models,
- model the evolution of interest rate with singular SPDE,
- perform stochastic control,
- *etc*

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In addition, rough paths theory reveals insights on some aspects of classical stochastic calculus and allows one to set up other kind of stochastic calculus. Developed initially by T. Lyons at the end of the '90 [2, 3], the rough paths theory is connected to many topics ranging from algebra to machine learning. It combines both theoretical and practical results. This explains its richness and why it is now expanding in many directions.

This book is divided in 15 chapters. Chapters 2-6 cover the basics of the rough paths theory in relation with stochastic calculus, although the former may be developed without any reference to the later. This presentation is however helpful to understand how to define pathwise integrals of type $Y_t = \int_0^t f(B_s)\mathrm{d}B_s$ and differential equations of type $X_t = a + \int_0^t f(X_s)\mathrm{d}B_s$ against α-Hölder continuous paths B like Brownian paths where $\alpha < 1/2$. To do so, the core idea is to extend the path B to a so-called "rough path" $\mathbb{B}$ that lives in a larger, non-commutative space. A rough path is defined by both algebraic and analytic properties. This non-unique extension specifies a choice of the *Lévy area*, the area between the path and its chord. This is exactly the information needed to define integrals in pathwise manners. As a bonus, $\mathbb{B} \mapsto Y$ and $\mathbb{B} \mapsto X$ are continuous in the suitable spaces. These continuity properties explain the practical importance of the rough paths theory and why it naturally extends of smooth calculus.

These chapters allow one to better understand the features of these non-linear and non-trivial extensions. For Brownian rough paths, connections with Itô, Stratonovich and anticipating calculus are also presented. A deterministic version of the Doob-Meyer decomposition is also given. It encompasses the role of the roughness properties of the paths.

Chapters 7-8 contain the statement regarding existence, uniqueness/non-uniqueness and continuity of solutions to Rough Differential Equations. Chapter 9 considers Stochastic Differential Equations driven by Brownian motions, the relation with Wong-Zakai results and how the continuity properties yield short proofs to support and large deviations theorems. Chapters 10-11 deal with differential equations driven by Gaussian rough paths and their features such as Malliavin calculus and concentration of measure properties. Chapter 12 contains some applications of the rough paths theory to first and second-order SPDE. It serves also as a transition toward the theory of regularity structure.

Chapters 13-15 are devoted to the theory of regularity structures. Existence and uniqueness results are given after a thorough presentation of the algebraic foundations. Here, the main idea is to extend distributions through some Taylor development in a way allowing multiplication and integration them against singular kernels. The central piece is the *reconstruction theorem* that defines the solution from these suitable approximations defined in abstract spaces. Chapter 15 illustrates how the theory of regularity structures gives a rigorous meaning to the KPZ equation, a very singular and ill-posed semilinear stochastic PDE. Such a breakthrough result solved a long-standing open problem.

Written by two leading researchers, this book offers an excellent introduction to both rough paths and regularity structures, with an up-to-date presentation of these two major theories. Effectively written, packed with exercises and applications — several of them are related to the aforementioned topics regarding mathematical finance —, it is well fitted for a newcomer in the field and provides a secure base from where to explore alternative formulations of theses theories or other specific developments such as in machine learning (not covered by this book) or in other fields.

References

[1] Friz, P. K., & Hairer, M. (2014). *A Course on Rough Paths: With an Introduction to Regularity Structures*. Springer.

[2] Lyons, T. J. (1998). Differential equations driven by rough signals. *Revista Matemática Iberoamericana*, 14(2), 215-310.

[3] Lyons, T., & Qian, Z. (2002). *System control and rough paths*. Oxford University Press.