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Tonametl Sanchez, Andrey Polyakov, Denis Efimov

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Tonametl Sanchez[1,2], Andrey Polyakov[1], and Denis Efimov[1]

[1]Inria, Univ. Lille, CNRS, UMR 9189 - CRIStAL, France
[2]IPICYT, Mexico

Abstract

In this paper we propose a discretisation scheme for asymptotically stable homogeneous systems. This scheme exploits the information provided by a homogeneous Lyapunov function of the system. The main features of the scheme are: 1) the discretisation method is explicit and; 2) the discrete-time system preserves the asymptotic stability, the convergence rate, and the Lyapunov function of the original continuous-time system.

Keywords: Nonlinear systems; homogeneous systems; Lyapunov-based methods; discrete-time systems.

1 Introduction

Homogeneous systems exhibit useful properties for analysis and design of control systems, for example: they can be used to approximate nonlinear models by preserving important nonlinear features of the original system [51, 25, 3]; for the case of sliding-mode systems, homogeneity allows to develop systematic procedures for control design [35] and stability analysis [12]. Some additional interesting features of homogeneous systems are, the existence of homogeneous Lyapunov functions [44, 39], and homogeneous controllers [30, 26, 48, 20], the direct association of the homogeneity degree with the convergence rates [21, 27, 39], and the intrinsic robustness properties to exogenous perturbations and delays [5, 16].

On the other hand, in the nowadays processes to design control systems, discretisation of continuous-time models has become not only a usual but essential practice in many cases. It is useful, for example, for computer simulation, for implementation in electronic devices, and for the design of digital (or sampled-data) controllers [40, 18]. However, the design of suitable discretisation schemes is a very challenging task, in particular for sliding-mode systems due to their discontinuous nature [13, 49, 34, 32, 2, 28].

Unfortunately, unlike linear systems, nonlinear ones do not have (in general) an exact discretisation. Nevertheless, it is expected that an approximate discretisation preserves the most relevant features of the continuous-time system.

Although, there exist many methods to discretise nonlinear systems [23, 22, 1], it has been proven that they can be generally unsuitable for homogeneous systems since they do not preserve stability and convergence properties [15]. For example: 1) for continuous-time asymptotically stable homogeneous systems of positive degree, standard explicit discretisation methods can produce unbounded trajectories for large initial conditions [15, 37]; 2) for continuous-time homogeneous systems of negative degree, whose origin is unstable, the implicit Euler method can produce a discrete-time system with an asymptotically stable origin [45]. These inconsistencies of the Euler discretisation can occur even when applied to continuous-time linear systems,[1] see, e.g. [9, Section 216]. The standard discretisation techniques become even more inappropriate when the continuous-time system does not match the standard smoothness assumptions, for example for systems with sliding-mode controllers [13, 37]. This has motivated several authors to study the properties of standard discretisation methods applied to homogeneous systems, ranging from particular systems [7, 29] to more general cases [15, 14]. Also, new strategies with improved properties have been designed to discretise homogeneous systems and sliding-mode controllers, e.g.: the implicit discretisation of sliding mode controllers [13, 49]; the discrete-time realization of an arbitrary order robust exact differentiator [31] based on a discrete-time redesign of the differentiator through a nonlinear eigenvalue assignment; the method for finite-time and fixed-time stable systems [42] based on a suitable coordinate transformation and implicit (or semi-implicit) discretisation; the discrete-time realization of sliding-mode algorithms based on implicit discretisation of differential inclusions, see [2, 28] and the references therein.

In this paper we propose a procedure to discretise homogeneous systems whose origin is asymptotically stable.[2] Our method relies on the information provided by a homogeneous Lyapunov function. The main properties of the proposed discretisation scheme are the following:

1. it preserves the Lyapunov function, i.e, the Lyapunov function of the continuous-time system used for the discretisation is also a Lyapunov function for the discrete-time system. Therefore, the origin of the obtained discrete-time system is Lyapunov stable. Note that this feature automatically guarantees *stability*[3] of the method, which is one of the most important properties expected from any numerical procedure;

2. the discretisation is consistent in the sense described in [42], namely, the convergence rate of the continuous-time trajectory is preserved by its discrete-time counterpart, e.g., if the origin of the continuous-time system is finite-time stable, then the origin of its discrete-time approximation is finite-time stable as well. The same is guaranteed for exponential and fixed-time stability;

3. the method can be used to discretise systems with a discontinuity at the origin, e.g., for the high-order sliding mode controllers known as quasi-continuous [36];

[1] Recall that if a continuous-time linear system is stable the explicit Euler discretisation can result unstable (depending on the eigenvalues), while an unstable the implicit Euler discretisation of an unstable linear system can be stable.

[2] Some of the results of this paper were briefly announced without proofs in [47].

[3] Here stability must be understood as the property of a numerical method to generate bounded solutions if the solutions of the differential equation are bounded. This property is known as *A-Stability* for the case of linear methods, see, e.g. [24, Section IV.3].

4. the discretisation is explicit, hence its implementation is simpler and requires less computational power in comparison with implicit methods, e.g. [42];

5. the stability and consistency of the method are independent of the size of the discretisation step.

Paper organization: In Section 2 the definition of homogeneity and some properties of homogeneous systems are recalled. In that section we also state the problem to be solved in the paper. In Section 3 we study the dynamics of a homogeneous system projected on the unitary level set of the Lyapunov function. In Section 4 we describe the proposed discretisation method for homogeneous systems. Some examples of the discretisation technique are presented in Section 5. In Section 6 we state some final remarks.

Notation: Real and integer numbers are denoted as $\mathbb{R}$ and $\mathbb{Z}$, respectively. $\mathbb{R}_+$ denotes the set $\{x \in \mathbb{R} \,:\, x \geq 0\}$ and $\mathbb{R}_+^* = \mathbb{R}_+ \setminus \{0\}$, analogously for the set $\mathbb{Z}$. For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm. For a continuous positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$, we denote the set $\mathcal{S}_V = \{x \in \mathbb{R}^n \,:\, V(x) = 1\}$. The set of strictly increasing continuous functions $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\eta(0) = 0$ is denoted by $\mathcal{K}$.

2 Preliminaries and problem statement

In this section we recall some properties of homogeneous systems and give the statement of the problem to be solved.

We consider the following continuous-time system

$$\dot{x}(t) = f(x(t))\,, \quad x(t) \in \mathbb{R}^n\,, \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on $\mathbb{R}^n \setminus \{0\}$ but can be discontinuous at the origin. In such a case, the solution of the system is understood as the solution of its associated differential inclusion $\dot{x} \in B(x)$, where the set-valued map B is the Filippov regularisation of f given in [17, p. 50], see also [43] and [4, p. 6]. In this context, a solution of (1) is defined as an absolutely continuous function $x : \Gamma \subset \mathbb{R}_+ \to \mathbb{R}^n$ such that it satisfies the inclusion $\dot{x} \in B(x)$ for almost all $t \in \Gamma$ [17].

Under these assumptions on f, the Filippov regularisation satisfies all the conditions required to guarantee the existence of solutions of (1) [17]. Nonetheless, in this paper we assume that for each $x(0) \in \mathbb{R}^n$, the solution of (1) is unique in forward time for all $t \in \mathbb{R}_+$.

2.1 Homogeneity

Now, let us recall the definition of weighted homogeneity.

Definition 1 (Weighted homogeneity [30])**.** *Let $\Lambda_\epsilon^{\mathbf{r}}$ denote the family of dilations given by the square diagonal matrix $\Lambda_\epsilon^{\mathbf{r}} = \mathrm{diag}(\epsilon^{\mathrm{r}_1}, \ldots, \epsilon^{\mathrm{r}_\mathrm{n}})$, where $\mathbf{r} = [r_1, \ldots, r_n]^\top$, $r_i \in \mathbb{R}_+^*$, and $\epsilon \in \mathbb{R}_+^*$. The components of $\mathbf{r}$ are called the* weights *of the coordinates. Thus:*

(a) a function $V : \mathbb{R}^n \to \mathbb{R}$ is $\mathbf{r}$-homogeneous of degree $m \in \mathbb{R}$ if

$$V\left(\Lambda_\epsilon^{\mathbf{r}} x\right) = \epsilon^m V(x)\,, \quad \forall x \in \mathbb{R}^n,\ \forall \epsilon \in \mathbb{R}_+^*\,;$$

(b) a vector field $f:\mathbb{R}^n\to\mathbb{R}^n$, is **r***-homogeneous of degree $\mu\in\mathbb{R}$ if*

$$f\left(\Lambda_\epsilon^{\mathbf{r}}x\right)=\epsilon^{\mu}\Lambda_\epsilon^{\mathbf{r}}f(x)\,,\quad \forall x\in\mathbb{R}^n,\ \forall\epsilon\in\mathbb{R}_+^*\,.$$

System (1) is said to be **r**-homogeneous of degree $\mu\in\mathbb{R}$ if its vector field f is **r**-homogeneous of degree μ. In the case f is **r**−homogeneous and discontinuous, its associated Filippov differential inclusion preserves the homogeneity, see [39] for more details.

Now, we recall some properties of homogeneous systems. Suppose that (1) is **r**−homogeneous of degree μ, and $V:\mathbb{R}^n\to\mathbb{R}$ is a differentiable **r**−homogeneous function of degree m. Hence,

$$\dot{V}=-W(x)\,,\quad W(x):=-\frac{\partial V(x)}{\partial x}f(x)\,,\tag{2}$$

where the function $W:\mathbb{R}^n\to\mathbb{R}$ is **r**−homogeneous of degree $m+\mu$. Note that if the origin of (1) is asymptotically stable, then there exists a continuously differentiable function which is a strict[4] Lyapunov function, and is **r**−homogeneous of some degree $m\in\mathbb{R}_+^*$. Indeed, the existence of such a Lyapunov function is guaranteed if the condition $m>\max_{i=1,\dots,n}r_i$ holds [44, 39]. In such a case, W is positive definite, continuous for all $x\in\mathbb{R}^n\setminus\{0\}$, and there exists $\alpha\in\mathbb{R}_+^*$ such that [27, 39]

$$\dot{V}\le-\alpha V^{\frac{m+\mu}{m}}(x)\,,\tag{3}$$

where the constant α can be given as follows

$$\alpha=\inf_{x\in\mathcal{S}_V}W(x)\,.\tag{4}$$

Note that the degree of homogeneity of W is strictly positive by restricting the degree of V to $m>-\mu$. An interesting consequence of (3) is the estimation of the decreasing rate of V along the solutions of (1) as stated in Lemma 1. Recall that the origin of (1) is said to be *nearly fixed-time stable*[5] if it is Lyapunov stable, and for any $a\in\mathbb{R}_+^*$ there exists $T(a)\in\mathbb{R}_+$ such that any solution $x(t;x_0)$ of (1) satisfies $|x(t;x_0)|\le a$ for all $t\ge T(a)$ independently of the initial condition $x(0)=x_0\in\mathbb{R}^n$ [42].

Lemma 1 (See, e.g. [21, 27, 39])**.** *Let* (1) *be* **r**−*homogeneous of degree μ, with a differentiable strict Lyapunov function $V:\mathbb{R}^n\to\mathbb{R}$ which is* **r**−*homogeneous of degree m. For all $x(0)\in\mathbb{R}^n$ and all $t\in\mathbb{R}_+$, the following holds (with α as given in* (3)*):*

1. if $\mu>0$, then

$$V(x(t))\le\frac{V(x(0))}{\left(1+\frac{\mu}{m}\alpha V^{\frac{\mu}{m}}(x(0))t\right)^{\frac{m}{\mu}}}\,,\tag{5}$$

i.e. the origin of (1) *is nearly fixed-time stable;*

[4]In this paper we consider only strict Lyapunov functions, i.e. those such that $\frac{\partial V(x)}{\partial x}f(x)<0$ for all $x\in\mathbb{R}^n\setminus\{0\}$.

[5]In some publications (see, e.g. [42]), this property was called *practical fixed-time stability* or *fixed-time convergence to a ball*. However, these names are not appropriate for the considered stability property. Indeed, the practical stability (or convergence to a ball) means the existence of some attractive set, while, in our case, any neighbourhood of the origin is fixed-time attractive.

2. if $\mu = 0$, then

$$V(x(t)) \leq V(x(0)) \exp(-\alpha t)\,, \tag{6}$$

i.e. the origin of (1) *is exponentially stable;*

3. if $\mu < 0$, then $V(x(t)) \leq \bar{V}(x(t))$ where

$$\bar{V}(x(t)) = \begin{cases} \left(V^{\frac{-\mu}{m}}(x(0)) - \frac{-\mu}{m}\alpha t\right)^{\frac{m}{-\mu}}, & t < \frac{m}{-\mu\alpha} V^{\frac{-\mu}{m}}(x(0)), \\ 0, & t \geq \frac{m}{-\mu\alpha} V^{\frac{-\mu}{m}}(x(0)), \end{cases} \tag{7}$$

i.e. the origin of (1) *is finite-time stable.*

From Lemma 1, we can see that for $\mu < 0$ the convergence time $T(x(0))$ to the origin is such that $T(x(0)) \leq \frac{m}{-\alpha\mu} V^{\frac{-\mu}{m}}(x(0))$. On the other hand, for the cases $\mu = 0$ and $\mu > 0$, the convergence time to the set $\{x \in \mathbb{R}^n : V(x) \leq c\}$ (with $0 < c < V(x(0))$) is such that $T(x(0)) \leq \frac{1}{\alpha} \ln\left(\frac{V(x(0))}{c}\right)$ and $T(x(0)) \leq \frac{m}{\alpha\mu c^{\frac{\mu}{m}}}$, respectively.

2.2 Problem statement

As we mentioned in the introduction, we cannot have (in general) an exact discretisation for a nonlinear system. However, it is expected that a suitable discretisation preserves important properties of the solutions, for example, the convergence rates described in Lemma 1. It is also expected that if a Lyapunov function is available, then it can be used to improve the discretisation scheme. For the case of homogeneous systems, a homogeneous Lyapunov function provides the information about stability and convergence rates, as stated in Lemma 1. Hence, the problem to be solved in this paper is:

For systems given by (1)*, which are* $\mathbf{r}-$*homogeneous of any degree $\mu \in \mathbb{R}$, and whose origin is asymptotically Lyapunov-stable, to construct explicit discretisation schemes such that the obtained discrete-time system preserves the the asymptotic stability and the convergence rate (in the sense of Lemma 1) of the continuous-time system.*

This problem is solved by exploiting the information provided by the homogeneous Lyapunov function of the system, and by considering a homogeneous projection of the system's dynamics on the unitary level set of the Lyapunov function. Such a projection is uniquely defined (see Section 3 for more details), so the original state of the system can always be recovered from the projected state using the value of the Lyapunov function.

It is important to mention that there exist in the literature some discretisation techniques based on Lyapunov functions and projections. However, even when they are able to preserve the Lyapunov function for the discrete-time system, they cannot guarantee the preservation of the convergence rates, see [19] and the references therein. In fact, an additional advantage of our proposal (compared with the existent projection techniques) is that the projection is explicit, therefore, it does not require that a system of algebraic equations must be solved in each step.

3 Projected dynamics

In this section we develop the fundamentals for the discretisation scheme proposed in Section 4.

Let (1) be $\mathbf{r}-$homogeneous of degree μ, and $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function, which is positive definite and $\mathbf{r}-$homogeneous of degree m. Define the following auxiliary variable

$$y = \Lambda^{\mathbf{r}}_{V^{\frac{-1}{m}}(x)} x\,, \quad \forall x \in \mathbb{R}^n \setminus \{0\}\,. \tag{8}$$

Note that (8) is the homogeneous projection of x over the unitary level set of V, thus, $y \in \mathcal{S}_V$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Let us compute the dynamics of the auxiliary variable y. By differentiating (8) along (1) we obtain

$$\dot{y} = \Lambda^{\mathbf{r}}_{V^{\frac{-1}{m}}(x)} \Big(I - \tfrac{1}{m} V^{-1}(x) G x \tfrac{\partial V(x)}{\partial x} \Big) f(x)\,, \tag{9}$$

where I is the identity matrix, and $G := \operatorname{diag}(r_1, \ldots, r_n)$. From (8) we see that $x = \Lambda^{\mathbf{r}}_{V^{\frac{1}{m}}(x)} y$, thus, we rewrite (9) as follows

$$\begin{aligned}
\dot{y} &= \Lambda^{\mathbf{r}}_{V^{\frac{-1}{m}}(x)} f(x) - \tfrac{1}{m} V^{-1}(x) \Lambda^{\mathbf{r}}_{V^{\frac{-1}{m}}(x)} G \Lambda^{\mathbf{r}}_{V^{\frac{1}{m}}(x)} y \tfrac{\partial V(x)}{\partial x} f(x)\,, \\
&= V^{\frac{\mu}{m}}(x) f(y) + \tfrac{1}{m} \tfrac{W(x)}{V(x)} G y\,, \\
&= V^{\frac{\mu}{m}}(x) f(y) + \tfrac{1}{m} \tfrac{V^{\frac{m+\mu}{m}}(x) W(y)}{V(x)} G y\,,
\end{aligned}$$

therefore,

$$\dot{y} = V^{\frac{\mu}{m}}(x) \big[f(y) + \tfrac{1}{m} W(y) G y \big]\,. \tag{10}$$

Equation (10) describes the dynamics (1) projected on $\mathcal{S}_V$. However, observe that we cannot recover the trajectory of (1) directly from the trajectory of (10) because (8) is not bijective. To overcome this problem, we proceed to study the dynamics of V, i.e. the derivative of V along (1). Thus, from (2) and (8), we obtain

$$\dot{V} = -W\big(\Lambda^{\mathbf{r}}_{V^{\frac{1}{m}}(x)} y \big) = -V^{\frac{m+\mu}{m}}(x) W(y)\,. \tag{11}$$

Now, define the function $v : \mathbb{R}_+ \to \mathbb{R}_+$ such that it satisfies the scalar differential equation (cf. (11))

$$\dot{v}(t) = -v^{\frac{m+\mu}{m}}(t) W(z(t))\,, \tag{12}$$

with the function $z : \mathbb{R}_+ \to \mathbb{R}^n$ being the solution of the system (cf. (10))

$$\dot{z}(t) = v^{\frac{\mu}{m}}(t) \big[f(z(t)) + \tfrac{1}{m} W(z(t)) G z(t) \big]\,. \tag{13}$$

From these developments we are ready to state the main results of this section. First, we verify that $\mathcal{S}_V$ is a positively invariant set for the trajectories of (13).

Lemma 2. *Consider* (13) *with* $z(0) \in \mathcal{S}_V$, *and any continuous function* $v : \mathbb{R}_+ \to \mathbb{R}$.

1. *If* $\mu \geq 0$, *then* $z(t) \in \mathcal{S}_V$ *for all* $t \in \mathbb{R}_+$.

2. *If $\mu < 0$ and $v(t) \neq 0$ for all $t \in [0, T)$ for some $T \in \mathbb{R}_+^*$, then $z(t) \in \mathcal{S}_V$ for all $t \in [0, T)$.*

Proof. To verify the positive invariance of $\mathcal{S}_V$, let us compute the derivative of $V(z)$ along the trajectories of (13), thus

$$\dot{V}(z(t)) = v^{\frac{\mu}{m}}(t)\big[\tfrac{\partial V(z)}{\partial z} f(z) + \tfrac{1}{m} W(z) \tfrac{\partial V(z)}{\partial z} G z\big] .$$

If (for $\mu \geq 0$) $v(t) = 0$ then $\dot{V}(z(t)) = 0$. Thus, without loss of generality, we assume that $v(t) \neq 0$ (analogously we assume $W(z) \neq 0$, since $\frac{\partial V(z)}{\partial z} f(z) = -W(z)$). According to the equality[6]

$$\tfrac{\partial V(z)}{\partial z} G z = m V(z) , \tag{14}$$

we obtain

$$\dot{V}(z(t)) = v^{\frac{\mu}{m}}(t) W(z)\big[-1 + V(z)\big] .$$

Hence $\dot{V}(z(t)) = 0$ if and only if $V(z(t)) = 1$. □

Now, we give a useful result about a solution representation of (12).

Lemma 3. *Let* (1) *and V be as in Lemma 1.*

1. *For all $t \in \mathbb{R}_+$, and any initial condition $v(0) \in \mathbb{R}_+$, the function $v : \mathbb{R}_+ \to \mathbb{R}_+$ given by*

$$v(t) = v(0) \exp\big(-\hat{W}_0(t)\big) , \qquad \textit{for } \mu = 0 , \tag{15a}$$

$$v(t) = \frac{v(0)}{\Big(1 + \frac{\mu}{m} v^{\frac{\mu}{m}}(0) \hat{W}_0(t)\Big)^{\frac{m}{\mu}}} , \quad \textit{for } \mu > 0 , \tag{15b}$$

with $\hat{W}_0(t) := \int_0^t W(z(\tau))\,\mathrm{d}\tau$, satisfies (12).

2. *For any initial condition $v(0) \in \mathbb{R}_+^*$, there exists $\Theta(v(0)) \in \mathbb{R}_+^*$ such that the function $v : \mathbb{R}_+ \to \mathbb{R}_+$ given by*

$$v(t) = \begin{cases} \Big(v(0)^{\frac{-\mu}{m}} - \frac{-\mu}{m} \hat{W}_0(t)\Big)^{\frac{m}{-\mu}}, & \frac{-\mu}{m} \hat{W}_0(t) < v(0)^{\frac{-\mu}{m}} \\ 0, & \frac{-\mu}{m} \hat{W}_0(t) \geq v(0)^{\frac{-\mu}{m}} \end{cases} , \quad \textit{for } \mu < 0 , \tag{15c}$$

satisfies (12) *with $v(t) > 0$ for all $t \in \big[0, \Theta(v(0))\big)$, and $v(t) \to 0$ as $t \to \Theta(v(0))$.*

Proof. The result is obtained by direct integration of (12). Nevertheless, we want to clarify a detail for the second part of the lemma. Since $W(z) > 0$ for all $z \in \mathcal{S}_V$, then $v(\cdot)$ in (15c) is strictly decreasing to zero, hence, for each $v(0) \in \mathbb{R}_+^*$ there exists a maximal $\Theta(v(0)) \in \mathbb{R}_+^*$ such that $\hat{W}_0(t) < v(0)^{\frac{-\mu}{m}}$ for all $t \in \big[0, \Theta(v(0))\big)$. Note that, $\big[0, \Theta(v(0))\big)$ is the time interval for which the right-hand side of (13) is well-defined. □

[6] This equality is known as the Euler's theorem for homogeneous functions, see, e.g. [4, Proposition 5.4].

Note that the function v given in (15a)-(15b) is continuous for all $t \in \mathbb{R}_+$. Also note that v as given in (15c) is continuous and $v(t) \neq 0$ for $v(0) \in \mathbb{R}_+^*$ and for all $t \in [0, \Theta(v(0))$. Therefore, (15a)-(15c) have the properties required for the function v in Lemma 2 for each case of μ. We finalise this section with the statement of the fundamental results for the discretisation method described in Section 4.

Theorem 1. *Let* (1) *be* $\mathbf{r}-$*homogeneous of degree* μ, *and* $V : \mathbb{R}^n \to \mathbb{R}$ *be a continuously differentiable strict Lyapunov function which is* $\mathbf{r}-$*homogeneous of degree* m. *Define* $\zeta = [v, z^\top]^\top \in Z$, $Z = \mathbb{R}_+^* \times \mathcal{S}_V$. *Consider* (1) *on* $\mathbb{R}^n \setminus \{0\}$ *and* (12)-(13) *on* Z. *The solutions of* (1) *and the solutions of* (12)-(13) *are equivalent with the homeomorphism* $\Phi : \mathbb{R}^n \setminus \{0\} \to Z$ *given by*

$$\Phi(x) = \begin{bmatrix} V(x) \\ \Lambda^{\mathbf{r}}_{V^{-\frac{1}{m}}(x)} x \end{bmatrix}. \tag{16}$$

Proof. Since V is continuous, we can see that Φ is continuous with a continuous inverse $\Phi^{-1} : Z \to \mathbb{R}^n \setminus \{0\}$ given by

$$\Phi^{-1}(\zeta) = \Lambda^{\mathbf{r}}_{v^{\frac{1}{m}}} z\,. \tag{17}$$

The rest of the proof is straightforward by noting that $\zeta(t) = \Phi(x(t))$ satisfies (12)-(13) and $x(t) = \Phi^{-1}(\zeta(t))$ satisfies (1). $\square$

Corollary 1. *Let* (1) *be* $\mathbf{r}-$*homogeneous of degree* μ, *and* $V : \mathbb{R}^n \to \mathbb{R}$ *be a continuously differentiable Lyapunov function which is* $\mathbf{r}-$*homogeneous of degree* m. *Let* v *and* z *be solutions of* (12), (13), *respectively, with initial conditions* $v(0) = V(x(0))$, $z(0) = \Lambda^{\mathbf{r}}_{v^{\frac{-1}{m}}(0)} x(0)$, *for any* $x(0) \in \mathbb{R}^n \setminus \{0\}$.

1. *If* $\mu \geq 0$, *then the function* $x : \mathbb{R}_+ \to \mathbb{R}^n$ *given by*

$$x(t) = \Lambda^{\mathbf{r}}_{v^{\frac{1}{m}}(t)} z(t)\,, \tag{18a}$$

is solution of (1) *for all* $t \in \mathbb{R}_+$.

2. *If* $\mu < 0$, *the function* $x : \mathbb{R}_+ \to \mathbb{R}^n$ *given by*

$$x(t) = \begin{cases} \Lambda^{\mathbf{r}}_{v^{\frac{1}{m}}(t)} z(t), & t < \Theta(v(0)), \\ 0, & t \geq \Theta(v(0)), \end{cases} \tag{18b}$$

(with Θ *as given in Lemma 3) is solution of* (1) *for all* $t \in \mathbb{R}_+$.

4 Discretisation scheme

The main idea of the discretisation scheme comes from the developments of Section 3, and it can be intuitively introduced as follows: Since (13) describes the dynamics of (1) projected on $\mathcal{S}_V$, and v (as given in Lemma 3) describes the decreasing behaviour of V along the solutions of (1), the idea is to obtain a numerical solution[7] of (13) on $\mathcal{S}_V$, and expand its values to $\mathbb{R}^n$ by using (18) and a discretisation of v.

[7] We mean by *numerical solution* a sequence $\{z_k\}_{k \in \mathbb{Z}_+}$ such that $z_0 = z(0)$, and for some $h \in \mathbb{R}_+^*$, z_k approximates $z(kh)$.

Remark 1. *Although, a numerical solution of* (13) *can be obtained by different techniques, we restrict ourselves in this paper to the explicit (also known as forward) Euler method.*

To construct the discretisation of v, we see from Lemma 3 that for any $h \in \mathbb{R}_+$,

$$v(t+h) = v(t)\exp\Big(-\hat{W}(t)\Big)\,, \qquad \text{for } \mu = 0\,, \tag{19a}$$

$$v(t+h) = \frac{v(t)}{\Big(1 + \frac{\mu}{m} v^{\frac{\mu}{m}}(t)\hat{W}(t)\Big)^{\frac{m}{\mu}}}\,, \qquad \text{for } \mu > 0\,, \tag{19b}$$

$$v(t+h) = \begin{cases} \Big(v^{\frac{-\mu}{m}}(t) - \frac{-\mu}{m}\hat{W}(t)\Big)^{\frac{m}{-\mu}}, & \frac{-\mu}{m}\hat{W}(t) < v^{\frac{-\mu}{m}}(t), \\ 0, & \frac{-\mu}{m}\hat{W}(t) \geq v^{\frac{-\mu}{m}}(t), \end{cases} \qquad \text{for } \mu < 0\,, \tag{19c}$$

where $\hat{W}(t) := \int_t^{t+h} W(z(\tau))\,\mathrm{d}\tau$. Thus, for the discretisation of v we only have to define a discrete-time approximation of $\hat{W}(t)$. For example, by using the forward Euler method with an integration step h, we obtain the discrete-time approximation $v_k \in \mathbb{R}$ to $v(kh)$ given by

$$v_{k+1} = v_k \exp\Big(-W(z_k)h\Big)\,, \qquad \text{for } \mu = 0\,, \tag{20a}$$

$$v_{k+1} = \frac{v_k}{\Big(1 + \frac{\mu}{m} v_k^{\frac{\mu}{m}} W(z_k)h\Big)^{\frac{m}{\mu}}}\,, \qquad \text{for } \mu > 0\,, \tag{20b}$$

$$v_{k+1} = \begin{cases} \Big(v_k^{\frac{-\mu}{m}} - \frac{-\mu h}{m}W(z_k)\Big)^{\frac{m}{-\mu}}, & \frac{-\mu h}{m}W(z_k) < v_k^{\frac{-\mu}{m}}, \\ 0, & \frac{-\mu h}{m}W(z_k) \geq v_k^{\frac{-\mu}{m}}, \end{cases} \qquad \text{for } \mu < 0\,, \tag{20c}$$

for all $k \in \mathbb{Z}_+$. The vector $z_k \in \mathbb{R}^n$ is the discrete-time approximation to $z(kh)$ given by

$$z_{k+1} = \Lambda^{\mathbf{r}}_{V^{\frac{-1}{m}}(\tilde{z}_{k+1})}\tilde{z}_{k+1}\,, \quad \tilde{z}_{k+1} = \begin{cases} z_k + hv_k^{\frac{\mu}{m}}\big(f(z_k) + \frac{1}{m}W(z_k)Gz_k\big), & v_{k+1} > 0, \\ z_k, & v_{k+1} = 0. \end{cases} \tag{21}$$

Note that in (21) and for the case $\mu \geq 0$, $\tilde{z}_{k+1}$ simplifies to $\tilde{z}_{k+1} = z_k + hv_k^{\frac{\mu}{m}}\big(f(z_k) + \frac{1}{m}W(z_k)Gz_k\big)$. Observe that (21) can be seen as the explicit Euler discretisation of (13), including a scaling factor given by $\Lambda^{\mathbf{r}}_{V^{\frac{-1}{m}}(\tilde{z}_{k+1})}$. This scaling is necessary to guarantee that $z_k \in \mathcal{S}_V$ for all $k \in \mathbb{Z}_+$, nonetheless, it is also necessary that $\tilde{z}_{k+1} \neq 0$, thus we require the following assumption.

Assumption 1. *Consider* (1) *and* V *as in Theorem 1. For all* $z \in \mathcal{S}_V$ *and all* $\tau \in \mathbb{R}_+^*$,

$$z + \tau\Big(f(z) + \tfrac{1}{m}W(z)Gz\Big) \neq 0\,.$$

In the following we state some conditions to verify Assumption 1.

Claim 1. *1. Assumption 1 holds if and only if for all* $z \in \mathcal{S}_V$ *such that* $\frac{\partial V(z)}{\partial z} z = 0$ *we have that* $z^\top F(z) > 0$*, where* $F(z) := f(z) + \frac{1}{m}W(z)Gz$.

2. Assumption 1 holds in any of the following cases:

(a) $\frac{\partial V(z)}{\partial z}z \neq 0$ *for all* $z \in \mathcal{S}_V$*;*

(b) the set $\{z \in \mathbb{R}^n : V(z) \leq 1\}$ *is convex.*

Proof. 1. Observe that, if there exist $\tau \in \mathbb{R}_+^*$ and $z \in \mathcal{S}_V$ such that $z + \tau F(z) = 0$, then the vector $F(z)$ is necessarily collinear to z. This holds if and only if there exists $w \in \mathbb{R}^n \setminus \{0\}$ which is orthogonal to both $F(z)$ and z. Note that the vector w, given by $w^\top = \frac{\partial V(z)}{\partial z}$, is orthogonal to $F(z)$ since $w^\top F(z) = \frac{\partial V(z)}{\partial z}f(z) + \frac{1}{m}W(z)\frac{\partial V(z)}{\partial z}Gz = -W(z) + \frac{1}{m}W(z)mV(z) = 0$ (we have used (14) for these equalities). On the other hand, w is orthogonal to z if and only if $w^\top z = \frac{\partial V(z)}{\partial z}z = 0$. Hence, we conclude that $z + \tau F(z) \neq 0$ for all $z \in \mathcal{S}_V$ and all $\tau \in \mathbb{R}_+^*$ if and only if for all $z \in \mathcal{S}_V$ such that $\frac{\partial V(z)}{\partial z}z = 0$ we have that $z^\top F(z) > 0$ (i.e. $F(z)$ and z have the same direction). 2. From the previous analysis it is clear that a sufficient condition to guarantee Assumption 1 is that $\frac{\partial V(z)}{\partial z}z \neq 0$ for all $z \in \mathcal{S}_V$, which is satisfied, for example, if the function $z \mapsto \frac{\partial V(z)}{\partial z}z$ is positive definite. Item (b) is proven as follows. If the set $\{z \in \mathbb{R}^n : V(z) \leq 1\}$ is convex, then homogeneity of V guarantees that the sets $\{z \in \mathbb{R}^n : V(z) \leq a\}$ are convex for all $a \in \mathbb{R}_+^*$. Thus, function V is quasi-convex (see, e.g. [6, Section 3.4.1]). Moreover, since V is positive definite, $z = 0$ is a global minimum. Hence, V is a pseudo-convex function (see, e.g. [11, Lemma 2.1]), therefore (by definition of pseudo-convexity), $\frac{\partial V(z)}{\partial z}z > 0$ for all $z \in \mathcal{S}_V$, see also [38, p. 40]. $\square$

Now, we can state the main result of this section.

Theorem 2. *Let* (1) *be* $\mathbf{r}$*−homogeneous of degree* μ *with a strict Lyapunov function* $V : \mathbb{R}^n \to \mathbb{R}$*, which is continuously differentiable and* $\mathbf{r}$*−homogeneous of degree* m*. Suppose that Assumption 1 holds. Consider the discrete-time approximation of* (1) *given by*

$$x_{k+1} = \psi(x_k) = \begin{cases} \Lambda^{\mathbf{r}}_{v_{k+1}^{\frac{1}{m}}} z_{k+1}, & x_k \neq 0, \\ 0, & x_k = 0, \end{cases} \qquad k \in \mathbb{Z}_+ \,, \tag{22}$$

where v_{k+1} *and* z_{k+1} *are given by* (20) *and* (21)*, respectively, with* $v_k = V(x_k)$*,* $z_k = \Lambda^{\mathbf{r}}_{V^{\frac{-1}{m}}(x_k)} x_k$*, and* $x_0 = x(0)$*. Then* V *is a Lyapunov function of* (22)*, and for all* $h \in \mathbb{R}_+^*$ *and all* $x(0) \in \mathbb{R}^n \setminus \{0\}$*,* $x_k \to 0$ *as* $k \to \infty$*. Moreover (with* α *as given in* (4)*):*

1. *if* $\mu = 0$*, then*
$$V(x_k) \leq V(x_0)\exp(-\alpha h k)\,, \quad \forall\, k \in \mathbb{Z}_+\,,$$
i.e. the origin of (22) *is exponentially stable;*

2. *if* $\mu > 0$*, then*
$$V(x_k) \leq \frac{V(x_0)}{\left(1 + \frac{\mu}{m}V^{\frac{\mu}{m}}(x_0)\alpha h k\right)^{\frac{m}{\mu}}}\,, \quad \forall\, k \in \mathbb{Z}_+\,,$$
i.e. the origin of (22) *is nearly fixed-time stable;*

3. *if* $\mu < 0$*, then* $V(x_k) \leq \bar{V}(x_k)$ *for all* $k \in \mathbb{Z}_+$*, where*
$$\bar{V}(x_k) = \begin{cases} \left(V^{\frac{-\mu}{m}}(x_0) - \frac{-\mu}{m}\alpha h k\right)^{\frac{m}{-\mu}}, & k < \frac{m}{-\mu\alpha h}V^{\frac{-\mu}{m}}(x_0), \\ 0, & k \geq \frac{m}{-\mu\alpha h}V^{\frac{-\mu}{m}}(x_0), \end{cases}$$

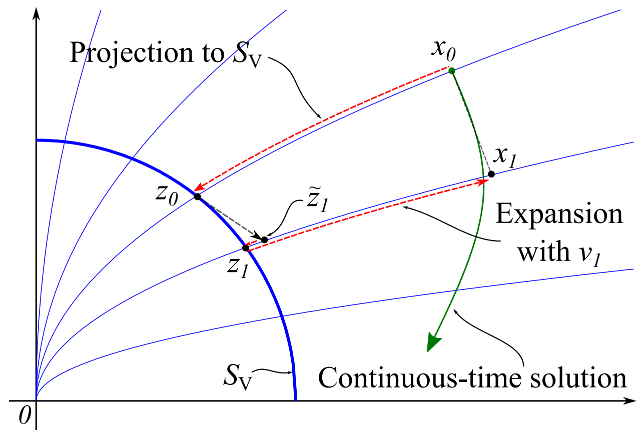

Figure 1: Graphical interpretation of one step of the discretisation scheme.

i.e. the origin of (22) *is finite-time stable.*

Proof. A one-step graphical interpretation of the discretisation scheme is shown in Fig. 1. Since $W(x) > 0$ for all $x \in \mathcal{S}_V$, and $z_k \in \mathcal{S}_V$, we see from (20) that v_k is strictly decreasing to zero, thus, from (22) it is straightforward that $x_k \to 0$ as $k \to \infty$.

Now, to obtain the convergence rates, we consider (20) whose solution is given by

$$\begin{aligned}
&v_k = v_0 \exp\big(-h\tilde{W}_k\big)\,, && \text{for } \mu = 0\,,\\
&v_k = \frac{v_0}{\Big(1 + \frac{\mu}{m} v_0^{\frac{\mu}{m}} h\tilde{W}_k\Big)^{\frac{m}{\mu}}}\,, && \text{for } \mu > 0\,,\\
&v_k = \begin{cases} \Big(v_0^{\frac{-\mu}{m}} - \frac{-\mu h}{m}\tilde{W}_k\Big)^{\frac{m}{-\mu}}, & \frac{-\mu h}{m}\tilde{W}_k < v_0^{\frac{-\mu}{m}},\\ 0, & \frac{-\mu h}{m}\tilde{W}_k \ge v_0^{\frac{-\mu}{m}}, \end{cases} && \text{for } \mu < 0\,,
\end{aligned}$$

where $\tilde{W}_k := \sum_{j=0}^{k-1} W(z_j)$. Now, since $z_k \in \mathcal{S}_V$ for all $k \in \mathbb{Z}_+$, we can use (4) to obtain

$$\begin{aligned}
&v_k \le v_0 \exp\Big(-hk\alpha\Big)\,, && \text{for } \mu = 0\,, && (23a)\\
&v_k \le \frac{v_0}{\Big(1 + \frac{\mu}{m} v_0^{\frac{\mu}{m}} hk\alpha\Big)^{\frac{m}{\mu}}}\,, && \text{for } \mu > 0\,, && (23b)\\
&v_k \le \begin{cases} \Big(v_0^{\frac{-\mu}{m}} - \frac{-\mu}{m}hk\alpha\Big)^{\frac{m}{-\mu}}, & k < \frac{m}{-\mu h\alpha}v_0^{\frac{-\mu}{m}},\\ 0, & k \ge \frac{m}{-\mu h\alpha}v_0^{\frac{-\mu}{m}}, \end{cases} && \text{for } \mu < 0\,. && (23c)
\end{aligned}$$

On the other hand, note from (22) that (by using homogeneity of V and the fact that $V(z_k) = 1$),

$$V(x_k) = V\big(\Lambda^{\mathbf{r}}_{v_k^{\frac{1}{m}}} z_k\big) = v_k V(z_k) = v_k\,. \tag{24}$$

Thus, $v_k = V(x_k)$ for all $k \in \mathbb{Z}_+$. By substituting this equality in (23) we obtain the three inequalities of the theorem. □

Remark 2. *Observe that Theorem 2 describes the main properties of the proposed method: 1) the discretisation is consistent in the sense that the stability and the convergence rate of the continuous-time solution is preserved; 2) the Lyapunov function from the continuous-time system is preserved, i.e., it is also a Lyapunov function for the discrete-time approximating system; 3) the discretisation scheme is explicit.*

Remark 3. *It is known that implicit discretisation methods have proven to be advantageous in contrast to standard explicit ones, e.g., for solving* stiff *problems [24]. However, a drawback of implicit methods is that the next step value depends on the solution to a (generally nonlinear) equation [9, p. 96], [23, p. 206]. This disadvantage is overcome in several special cases, where such an equation can be solved explicitly or by means of relatively simple numerical methods, e.g., linear systems, and systems with sliding mode controllers [13, 2]. Unfortunately, for the general nonlinear case, solving the implicit equations is not a simple task*[8]*. In contrast, no system of nonlinear equations must be solved for the explicit schemes. In this sense, explicit methods are generally easier to implement than implicit ones. Thus, explicitness of the proposed approach may constitute an advantage if we note that we have it in addition to the useful properties highlighted in Remark 2.*

Numerical convergence to the solutions

Consider (1) and its solution $x : [0, a] \to \mathbb{R}^n$ for some constant $a \in \mathbb{R}_+^*$. Assume that the discretisation step h is given by $h = a/N$, with $N \in \mathbb{Z}_+^*$, thus $h \to 0$ as $N \to \infty$.

A fundamental requirement for any discretisation method is that the step-function $t \mapsto \tilde{x}(t) := x_k$, $t \in [kh, (k+1)h)$ with $\{x_k\}_{k=0}^N$ generated by this method converges uniformly on $[0, a]$ to the solution x as the step size h tends to zero (equivalently, as N tends to infinity).

Thus, the standard technique to verify convergence is to confirm that the global truncation error[9] tends to zero as the step tends to zero. This task can be reduced to verify the existence of a function $\eta \in \mathcal{K}$ such that the local truncation error[10] $E(t+h) := x(t+h) - x_{k+1}$ satisfies (see, e.g. [50] and [23, pp. 37 and 159])

$$|E(t+h)| \leq h\eta(h)\,.$$

In this section we verify this local truncation error estimate to guarantee the convergence of the proposed discretisation method. Observe that in view of Theorem 1, we only have to prove that v_k and z_k do converge to v and z, respectively. Thus, we only need to study the local truncation errors of v_k and z_k.

[8]For example, consider the implicit Euler discretisation of (1) given by $x_{k+1} = x_k + hf(x_{k+1})$, which requires solving at each step the system of n nonlinear equations (in n variables x_{k+1}) $H(x_{k+1}) := x_{k+1} - hf(x_{k+1}) - x_k = 0$. Choosing an algorithm to solve such an equation is already an issue due to the large amount of available methods and their variations, see, e.g. [41]. Unfortunately, this is not the only disadvantage: most of the methods impose some smoothness restrictions on H (e.g. methods based on Jacobian Matrix), and usually require a *good initial approximation* for x_{k+1} to guarantee convergence [8, Chapter 10] (this is an important problem of optimization-based methods, which use Trust-Region algorithms [10]).

[9]The global truncation error is the accumulation of the errors made at each step in a given compact interval $[0, a]$, see, e.g. [33] or [23, p. 159].

[10]The local truncation error is the one-step error computed by assuming that $E(t) = 0$, i.e. $x(t) = x_k$.

Theorem 3. *Assume that the hypotheses of Theorem 2 hold with $v(t) = v_k$ and $z(t) = z_k$ for some $t \in \mathbb{R}_+$. Assume also that $\min\{V(x(t+h)), v_{k+1}\} \geq b$ for some $b \in \mathbb{R}_+^*$. Then, there exist functions $\eta_v, \eta_z \in \mathcal{K}$ such that in* (20), (21)*:*

$$|E^v(t+h)| := |v(t+h) - v_{k+1}| \leq h\,\eta_v(h)\,, \tag{25}$$
$$|E^z(t+h)| := |z(t+h) - z_{k+1}| \leq h\,\eta_z(h)\,. \tag{26}$$

For the proof of Theorem 3 we require the following lemma (the i-th element of the vector $\tilde{z}_k$ is denoted as $\tilde{z}_k^i$).

Lemma 4. *Consider* (21)*. Given $H \in \mathbb{R}_+^*$, under the assumptions of Theorem 3, there exist constants $\overline{\gamma}_i, \underline{\gamma}_i \in \mathbb{R}_+^*$ such that $\underline{\gamma}_i \leq |\tilde{z}_{k+1}^i| \leq \bar{\gamma}_i$ for all $h \leq H$ and all $i = 1, \ldots, n$.*

Proof. The existence of $\underline{\gamma}_i$ is guaranteed by Assumption 1. On the other hand, from (21) we can see that for every $i = 1, \ldots, n$,

$$|\tilde{z}_{k+1}^i| \leq \zeta_i + hv_k^{\frac{\mu}{m}}\left(\bar{f}_i + \tfrac{1}{m}\bar{\alpha}r_i\zeta_i\right),$$

where

$$\bar{f}_i = \sup_{z\in\mathcal{S}_V}|f_i(z)|\,, \quad \zeta_i = \sup_{z\in\mathcal{S}_V}|z_i|\,, \quad \bar{\alpha} = \sup_{z\in\mathcal{S}_V}|W(z)|\,.$$

Now, since v_k is decreasing, the hypotheses of the lemma ensure that

$$\begin{aligned}
\overline{\gamma}_i &= \zeta_i + H\left(\bar{f}_i + \tfrac{1}{m}\bar{\alpha}r_i\zeta_i\right), && \text{for } \mu = 0\,,\\
\overline{\gamma}_i &= \zeta_i + Hv_k^{\frac{\mu}{m}}\left(\bar{f}_i + \tfrac{1}{m}\bar{\alpha}r_i\zeta_i\right), && \text{for } \mu > 0\,,\\
\overline{\gamma}_i &= \zeta_i + Hb^{\frac{\mu}{m}}\left(\bar{f}_i + \tfrac{1}{m}\bar{\alpha}r_i\zeta_i\right), && \text{for } \mu < 0\,.
\end{aligned}$$

□

Proof of Theorem 3

First let us study E^v given by (25). From (19) and (20) we have that (denoting $E^v = E^v(t+h)$)

$$\begin{aligned}
|E^v| &\leq v(t)\left|\exp\left(-\hat{W}(t)\right) - \exp\left(-W(z_k)h\right)\right|, && \text{for } \mu = 0\,,\\
|E^v| &\leq v(t)\left|\left(1 + \tfrac{\mu}{m}v^{\frac{\mu}{m}}(t)\hat{W}(t)\right)^{-\frac{m}{\mu}} - \left(1 + \tfrac{\mu}{m}v^{\frac{\mu}{m}}(t)W(z_k)h\right)^{-\frac{m}{\mu}}\right|, && \text{for } \mu > 0\,,\\
|E^v| &\leq v(t)\left|\left(1 - \tfrac{-\mu}{m}v^{\frac{\mu}{m}}(t)\hat{W}(t)\right)^{\frac{m}{-\mu}} - \left(1 - \tfrac{-\mu}{m}v^{\frac{\mu}{m}}(t)W(z_k)h\right)^{\frac{m}{-\mu}}\right|, && \text{for } \mu < 0\,,
\end{aligned}$$

where $\hat{W}(t) := \int_t^{t+h} W(z(\tau))\,\mathrm{d}\tau$. Note that we have used the hypothesis $v(t) = v_k$. Let us rewrite these inequalities as follows

$$\begin{aligned}
|E^v| &\leq hv(t)\frac{\left|\exp\left(-\hat{W}(t)\right) - \exp\left(-W(z_k)h\right)\right|}{h}, && \text{for } \mu = 0\,,\\
|E^v| &\leq hv(t)\frac{\left|\left(1 + \tfrac{\mu}{m}v^{\frac{\mu}{m}}(t)\hat{W}(t)\right)^{-\frac{m}{\mu}} - \left(1 + \tfrac{\mu}{m}v^{\frac{\mu}{m}}(t)W(z_k)h\right)^{-\frac{m}{\mu}}\right|}{h}, && \text{for } \mu > 0\,,\\
|E^v| &\leq hv(t)\frac{\left|\left(1 - \tfrac{-\mu}{m}v^{\frac{\mu}{m}}(t)\hat{W}(t)\right)^{\frac{m}{-\mu}} - \left(1 - \tfrac{-\mu}{m}v^{\frac{\mu}{m}}(t)W(z_k)h\right)^{\frac{m}{-\mu}}\right|}{h}, && \text{for } \mu < 0\,.
\end{aligned}$$

Since W is continuous for all $z \in \mathbb{R}^n \setminus \{0\}$ and $z(t) = z_k$, it is clear (e.g. by using the L'Hôpital's rule) that

$$\lim_{h\to 0} \frac{\big|\exp\big(-\hat{W}(t)\big) - \exp\big(-hW(z_k)\big)\big|}{h} = 0\,,$$

$$\lim_{h\to 0} \frac{\Big|\big(1+\frac{\mu}{m}v^{\frac{\mu}{m}}(t)\hat{W}(t)\big)^{-\frac{m}{\mu}} - \big(1+\frac{\mu}{m}v^{\frac{\mu}{m}}(t)W(z_k)h\big)^{-\frac{m}{\mu}}\Big|}{h} = 0\,,$$

$$\lim_{h\to 0} \frac{\Big|\big(1-\frac{-\mu}{m}v^{\frac{\mu}{m}}(t)\hat{W}(t)\big)^{\frac{m}{-\mu}} - \big(1-\frac{-\mu}{m}v^{\frac{\mu}{m}}(t)W(z_k)h\big)^{\frac{m}{-\mu}}\Big|}{h} = 0\,.$$

Observe that $W(z)$ is positive and bounded for all $z \in \mathcal{S}_V$, therefore, (for each case of μ) there exists a function $\bar{\eta}_v \in \mathcal{K}$ (which does not depend on $z_k \in \mathcal{S}_V$) such that

$$\frac{\big|\exp\big(-\hat{W}(t)\big) - \exp\big(-hW(z_k)\big)\big|}{h} \le \bar{\eta}_v(h)\,, \text{ for } \mu = 0\,,$$

$$\frac{\Big|\big(1+\frac{\mu}{m}v^{\frac{\mu}{m}}(t)\hat{W}(t)\big)^{-\frac{m}{\mu}} - \big(1+\frac{\mu}{m}v^{\frac{\mu}{m}}(t)W(z_k)h\big)^{-\frac{m}{\mu}}\Big|}{h} \le \bar{\eta}_v(h)\,, \text{ for } \mu > 0\,,$$

$$\frac{\Big|\big(1-\frac{-\mu}{m}v^{\frac{\mu}{m}}(t)\hat{W}(t)\big)^{\frac{m}{-\mu}} - \big(1-\frac{-\mu}{m}v^{\frac{\mu}{m}}(t)W(z_k)h\big)^{\frac{m}{-\mu}}\Big|}{h} \le \bar{\eta}_v(h)\,, \text{ for } \mu < 0\,.$$

Thus, the result is obtained by taking $\eta_v(h) = v(t)\bar{\eta}_v(h)$.

Now, we study E^z given by (26). Observe from (21) that $E^z(t+h) = z(t+h) - \Lambda^{\mathbf{r}}_{V^{\frac{-1}{m}}(\tilde{z}_{k+1})}\tilde{z}_{k+1}$, and this equation can be rewritten as $E^z(t+h) = z(t+h) - \tilde{z}_{k+1} + \big(I - \Lambda^{\mathbf{r}}_{V^{\frac{-1}{m}}(\tilde{z}_{k+1})}\big)\tilde{z}_{k+1}$. Hence, for $i = 1, \dots, n$, we have that

$$E^z_i(t+h) = z_i(t+h) - \tilde{z}^i_{k+1} + \big(1 - V^{\frac{-r_i}{m}}(\tilde{z}_{k+1})\big)\tilde{z}^i_{k+1}\,. \tag{27}$$

Since $z(t+h) \in \mathcal{S}_V$, the term $1 - V^{\frac{-r_i}{m}}$ in (27) can be rewritten as follows

$$1 - V^{\frac{-r_i}{m}}(\tilde{z}_{k+1}) = \frac{1}{V^{\frac{r_i}{m}}(\tilde{z}_{k+1})}\big(V^{\frac{r_i}{m}}(\tilde{z}_{k+1}) - 1\big) = \frac{1}{V^{\frac{r_i}{m}}(\tilde{z}_{k+1})}\big(V^{\frac{r_i}{m}}(\tilde{z}_{k+1}) - V^{\frac{r_i}{m}}(z(t+h))\big)\,.$$

From Lemma 4, and for any $H \in \mathbb{R}^*_+$, we can assure that there exist constants $d_1, d_2 \in \mathbb{R}^*_+$ such that $d_1 \le V(\tilde{z}_{k+1}) \le d_2$ for all $h \le H$. Hence,[11]

$$\big|V^{\frac{r_i}{m}}(\tilde{z}_{k+1}) - V^{\frac{r_i}{m}}(z(t+h))\big| \le L_i\big|V(\tilde{z}_{k+1}) - V(z(t+h))\big| \le L_iL_v\big|\tilde{z}_{k+1} - z(t+h)\big|\,,$$

for some constants $L_i, L_v \in \mathbb{R}^*_+$. Thus, we obtain $\big|1 - V^{\frac{-r_i}{m}}(\tilde{z}_{k+1})\big| \le c_i\big|\tilde{z}_{k+1} - z(t+h)\big|$ with $c_i := d_1^{\frac{-r_i}{m}}L_iL_v$. Hence, we can bound (27) as follows

$$\begin{aligned} |E^z_i(t+h)| &\le |z_i(t+h) - \tilde{z}^i_{k+1}| + c_i\big|\tilde{z}_{k+1} - z(t+h)\big||\tilde{z}^i_{k+1}|\,,\\ &\le |\tilde{z}_{k+1} - z(t+h)| + c_i\big|\tilde{z}_{k+1} - z(t+h)\big||\tilde{z}^i_{k+1}|\,,\\ &\le (1 + c_i|\tilde{z}^i_{k+1}|)\big|\tilde{z}_{k+1} - z(t+h)\big|\,,\\ &\le \bar{c}_i\big|\tilde{z}_{k+1} - z(t+h)\big|\,, \quad \bar{c}_i := 1 + c_i\overline{\gamma}_i\,, \end{aligned} \tag{28}$$

[11] Since $[d_1, d_2] \subset \mathbb{R}$ is compact and $d_1 > 0$, the function $g : [d_1, d_2] \subset \mathbb{R}^*_+ \to \mathbb{R}$ given by $g(V) = V^{\frac{r_i}{m}}$ is Lipschitz continuous. Also, z_k and z belong to a compact subset of $\mathbb{R}^n$ on which V is Lipschitz continuous.

with $\overline{\gamma}_i$ as given in Lemma 4. To analyse the term $\left|\tilde{z}_{k+1} - z(t+h)\right|$ define $F(x) := f(x) + \frac{1}{m}W(x)Gx$. Thus, from (13) and (21) we have that $z(t+h) = z(t) + \int_t^{t+h} v^{\frac{\mu}{m}}(\tau)F(z(\tau))\,\mathrm{d}\tau$ and $\tilde{z}_{k+1} = z_k + hv_k^{\frac{\mu}{m}}F(z_k)$, respectively. By the Taylor's theorem, there exists a function $h \mapsto R(z(t), h)$ such that $z(t+h) = z(t) + v^{\frac{\mu}{m}}(t)F(z(t))h + R(z(t), h)$, and $\frac{1}{h}R(z(t), h) \to 0$ as $h \to 0$. Since $z_k = z(t)$ and $v_k = v(t)$,

$$\begin{aligned}\left|\tilde{z}_{k+1} - z(t+h)\right| &= \left|z_k + hv_k^{\frac{\mu}{m}}F(z_k) - z(t) - v^{\frac{\mu}{m}}(t)F(z(t))h - R(z(t), h)\right|,\\ &= |R(z(t), h)| = h\left|\frac{R(z(t), h)}{h}\right|, \quad \lim_{h\to 0}\left|\frac{R(z(t), h)}{h}\right| = 0\,. \end{aligned}\tag{29}$$

Therefore, from (28) and (29) we conclude that there exist $c \in \mathbb{R}_+^*$ and $\eta_z \in \mathcal{K}$ such that

$$|E^z(t+h)| \leq h\eta_z(h)\,, \quad \eta_z(h) \geq c\left|\frac{R(z(t), h)}{h}\right|.$$

□

5 Examples

In this section we illustrate the proposed discretisation method with three different systems. One of them being continuous and with the property of fixed-time to-a-ball convergence. The second and the third ones being discontinuous at the origin and with the property of finite-time stability.

5.1 Example 1

For this example we consider the following system

$$\dot{x}_1 = -k_1\lceil x_1\rfloor^{\frac{3}{2}} + x_2\,, \quad \dot{x}_2 = -k_2\lceil x_1\rfloor^2\,, \tag{30}$$

which is $\mathbf{r}-$homogeneous of degree $\mu = 1$ with $\mathbf{r} = [2, 3]^\top$. Consider the function $V : \mathbb{R}^2 \to \mathbb{R}$ given by

$$V(x) = \tfrac{2}{5}k_1|x_1|^{\frac{5}{2}} - x_1x_2 + \tfrac{3}{5}\alpha|x_2|^{\frac{5}{3}}\,.$$

This function[12] is $\mathbf{r}-$homogeneous of degree $m = 5$, and it can be proven that for any $k_1 \in \mathbb{R}_+^*$ there exist $\alpha, k_2 \in \mathbb{R}_+^*$ such that V is a Lyapunov function for (30). Then, the origin of (30) is nearly fixed-time stable. It is important to mention that the standard explicit Euler method is not suitable to discretise (30) since it produces unbounded trajectories for large initial conditions [37, 15].

In this case we obtain $W(x) = \left(k_1\lceil x_1\rfloor^{\frac{3}{2}} - x_2\right)^2 + k_2\left(\alpha\lceil x_1\rfloor^2\lceil x_2\rfloor^{\frac{2}{3}} - |x_1|^3\right)$. We consider the parameters: $k_1 = 2$, $k_2 = 1$, and $\alpha = 2$. Note that the function given by $\frac{\partial V(x)}{\partial x}x$ is positive definite. Therefore, Assumption 1 holds and we can use the discretisation method described

[12]This Lyapunov function was obtained by using the design method proposed in [46].

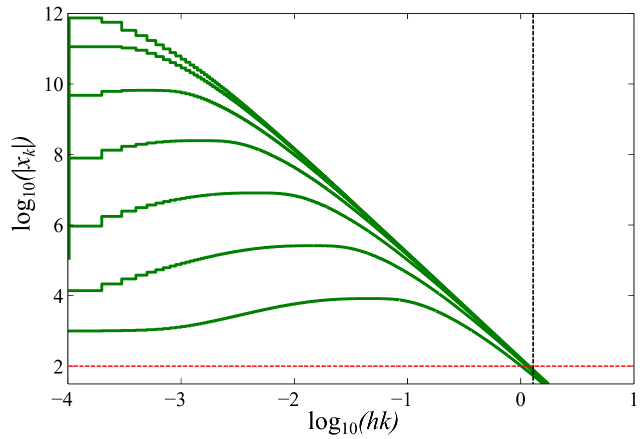

Figure 2: Norm of the states of the discrete-time approximation of (30) for different initial conditions.

in Theorem 2. Thus, the discrete-time approximation to the solution of (30) is given by $x_{k+1} = \text{diag}\Big(v_{k+1}^{\frac{2}{5}}, v_{k+1}^{\frac{3}{5}}\Big) z_{k+1}$, with $z_{k+1} = \text{diag}\Big(V^{-\frac{2}{5}}(\tilde{z}_{k+1}), V^{-\frac{3}{5}}(\tilde{z}_{k+1})\Big)\tilde{z}_{k+1}$,

$$v_{k+1} = \frac{v_k}{\Big(1 + \frac{h}{5} v_k^{\frac{1}{5}} W(z_k)\Big)^5}\,,$$

$$\tilde{z}_{k+1} = z_k + h v_k^{\frac{1}{5}} g(z_k)\,, \quad g(z_k) = \begin{bmatrix} \frac{2}{5} z_k^1 W(z_k) - k_1 \lceil z_k^1 \rfloor^{\frac{3}{2}} + z_k^2 \\ \frac{3}{5} z_k^2 W(z_k) - k_2 \lceil z_k^1 \rfloor^2 \end{bmatrix}\,,$$

where z_k^i denotes the i-th component of the vector $z_k = \Lambda^{\mathbf{r}}_{v_k^{-\frac{1}{m}}} x_k$, and $v_k = V(x_k)$.

We simulate the system 1.2 seconds by using 12000 steps, i.e. with a step length $h = 0.0001$, for the different initial conditions $x_2(0) = 0$ and $x_1(0) = 10^q$ with $q = 3, 4, \ldots, 9$. The norm of the system's states is shown in the logarithmic plot in Fig. 2, there we can appreciate the fixed-time convergence to the ball $\{x \in \mathbb{R}^2 : |x| \leq 100\}$.

5.2 Example 2

In this second example we consider a simple scalar system

$$\dot{x} = -\beta \text{sign}(x) + \delta\,, \quad \beta \in \mathbb{R}_+\,, \tag{31}$$

whose vector field is discontinuous at the origin. The disturbance δ is a Lebesgue-measurable function such that $\text{ess sup}_{t\in\mathbb{R}_+} |\delta(t)| \leq \bar{\delta}$. The origin of (31) is asymptotically stable and the trajectories converge to zero in a finite time for all $\beta > \bar{\delta}$. We use in this example three schemes to discretise (31). The first one is the explicit Euler discretisation given by

$$x_{k+1} = x_k - h\beta \text{sign}(x_k) + h\delta_k\,, \tag{32}$$

where $\delta_k = \delta(hk)$, and $h \in \mathbb{R}_+$ is the discretisation step. On the other hand, the implicit Euler discretisation is given by $x_{k+1} = x_k - h\beta \text{sign}(x_{k+1}) + h\delta_{k+1}$, whose explicit description is given by (see, e.g., [13, 2])

$$x_{k+1} = \begin{cases} x_k - h\beta \mathrm{sign}(x_k) + h\delta_{k+1}, & |x_k| > h(\beta - \delta_{k+1}\mathrm{sign}(x_k)), \\ 0, & |x_k| \leq h(\beta - \delta_{k+1}\mathrm{sign}(x_k)). \end{cases} \tag{33}$$

Now, observe that (31) is $\mathbf{r}-$homogeneous of degree $\mu = -1$ with $\mathbf{r} = 1$. Moreover, $V : \mathbb{R} \to \mathbb{R}$ given by $V(x) = x^2$ is a ($\mathbf{r}-$homogeneous of degree $m = 2$) Lyapunov function for (31). Since the function given by $\frac{\partial V(x)}{\partial x}x$ is positive definite, Assumption 1 holds. Thus, for the undisturbed case, i.e. $\delta(t) \equiv 0$, we can directly use the Lyapunov-based discretisation method described in Theorem 2. However, (31) is a non-autonomous equation in presence of the disturbance $\delta(t)$. Fortunately, the Lyapunov-based discretisation scheme is still useful for this case. Observe (for the disturbed case) that $\dot{V} = -W(x,t)$, where $W(x,t) := 2[\beta - \delta(t)\mathrm{sign}(x)]|x|$ is $\mathbf{r}-$homogeneous of degree 1 in the variable x. Thus, all the developments displayed in Section 3 still hold for this case. Now, for the results in Section 4, we only have to note that in (19) the function $\hat{W}$ is given by $\hat{W}(t) := \int_t^{t+h} W(z(\tau),\tau)\,\mathrm{d}\tau$. Hence, the forward-Euler approximation of $\hat{W}$ (used to obtain (20)) is given by $\hat{W}_k := 2h[\beta - \delta_k\mathrm{sign}(z_k)]|z_k|$. From Theorem 2 we have that $z_k = \Lambda^{\mathbf{r}}_{V^{\frac{-1}{m}}(x_k)} x_k = V^{-\frac{1}{2}}(x_k)x_k$, thus $z_k = x_k/|x_k| = \mathrm{sign}(x_k)$ and $|z_k| = 1$. It can also be verified from (21) that $\mathrm{sign}(z_{k+1}) = \mathrm{sign}(z_k)$. Thus, the discrete-time approximation to the solution of (31) is given by

$$x_{k+1} = \sqrt{v_{k+1}}\mathrm{sign}(x_k)\,, \quad v_{k+1} = \begin{cases} \left(\sqrt{v_k} - h[\beta - \delta_k\mathrm{sign}(x_k)]\right)^2, & v_k > h^2[\beta - \delta_k\mathrm{sign}(x_k)]^2, \\ 0, & v_k \leq h^2[\beta - \delta_k\mathrm{sign}(x_k)]^2. \end{cases} \tag{34}$$

with $v_0 = V(x(0))$.

For the simulations we consider the parameters $\beta = 3$, $h = 0.1$, and the initial condition $x(0) = 5$. Fig. 3 shows the simulation with the three schemes (32), (33), and (34), for the undisturbed case, i.e., $\delta(t) \equiv 0$. Note that, the explicit Euler discretisation exhibits *numerical chattering* in steady state, while the solutions with the implicit Euler and the Lyapunov-based methods converge exactly in finite-time to the origin. Fig. 4 shows the simulation with the three schemes (32), (33), and (34), for the case with $\delta(t) = 1/2 + 2\cos(10t)$. Note that, again, the explicit Euler discretisation exhibits a steady state oscillation produced by the effect of the disturbance and the numerical chattering. In contrast, the solutions with the implicit Euler and the Lyapunov-based methods converge exactly in finite-time to the origin despite the disturbance.

Remark 4. *From Theorem 2 we have that $v_k = V(x_k)$. Hence, we can rewrite* (34) *as follows (recall that $V(x) = x^2$)*

$$\begin{aligned} x_{k+1} &= \begin{cases} \left(\sqrt{V(x_k)} - h[\beta - \delta_k\mathrm{sign}(x_k)]\right)\mathrm{sign}(x_k), & V(x_k) > (h[\beta - \delta_k\mathrm{sign}(x_k)])^2, \\ 0, & V(x_k) \leq (h[\beta - \delta_k\mathrm{sign}(x_k)])^2, \end{cases} \\ &= \begin{cases} x_k - h\beta\mathrm{sign}(x_k) + h\delta_k, & |x_k| > h[\beta - \delta_k\mathrm{sign}(x_k)], \\ 0, & |x_k| \leq h[\beta - \delta_k\mathrm{sign}(x_k)]. \end{cases} \end{aligned} \tag{35}$$

Observe that for $\delta(t) \equiv 0$, (35) *coincides with* (33)*. Hence, for the undisturbed case of* (31)*, the implicit discretisation* (33) *is a particular case of the Lyapunov-based discretisation from*

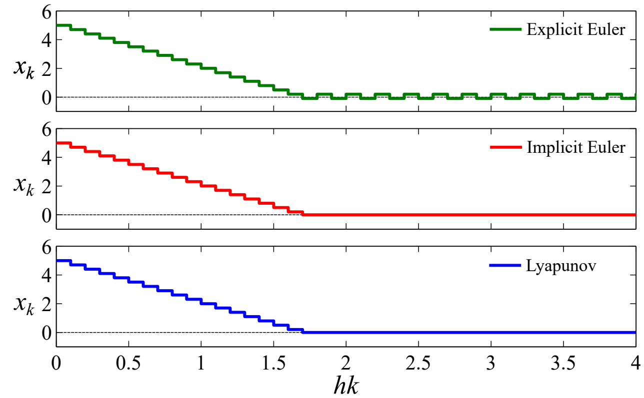

Figure 3: Discrete-time approximation of (31) with three different schemes for $\delta(t) \equiv 0$.

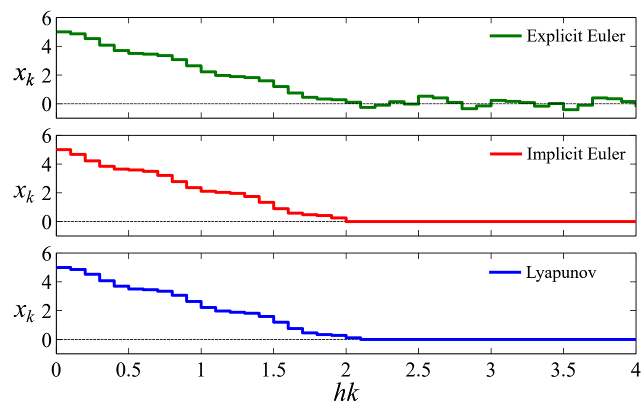

Figure 4: Discrete-time approximation of (31) with three different schemes for $\delta(t) = 1/2 + 2\cos(10t)$.

Theorem 2 by considering the Lyapunov function $V(x) = x^2$. Now, for the disturbed case, the right-hand side of (35) *only differs from* (33) *in depending on δ_k instead of δ_{k+1}. This difference produces an interesting consequence: as stated in [2],* (33) *does not coincide with the zero-order-hold discretisation method, in contrast,* (35) *possesses this feature.*

5.3 Example 3

In this third example we consider the controlled double integrator

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = u(x)\,, \quad u(x) = -k_1 \frac{x_1 + k_2\lceil x_2 \rfloor^2}{|x_1| + k_2|x_2|^2}\,, \tag{36}$$

where the controller u belongs to the class of high-order sliding-mode controllers known as quasi-continuous [36]. This controller is only discontinuous at the origin. The closed-loop system is $\mathbf{r}-$homogeneous of degree $\mu = -1$ with $\mathbf{r} = [2, 1]^\top$. We consider the function $V : \mathbb{R}^2 \to \mathbb{R}$ given by

$$V(x) = \tfrac{2}{3}\alpha|x_1|^{\frac{3}{2}} + x_1x_2 + \tfrac{1}{3}k_2|x_2|^3\,, \tag{37}$$

which is $\mathbf{r}-$homogeneous of degree $m = 3$. For this case we have that $W(x) = k_1\frac{(x_1+k_2\lceil x_2\rfloor^2)^2}{|x_1|+k_2|x_2|^2} - \alpha\lceil x_1\rfloor^{\frac{1}{2}}x_2 - x_2^2$. It can be proven that for any $k_2 \in \mathbb{R}^*_+$ there exist $\alpha, k_1 \in \mathbb{R}^*_+$ such that V is a Lyapunov function for (36). Then, the origin of (36) is asymptotically stable and the trajectories converge to zero in a finite time.

In this example we consider the parameters $k_1 = 2$, $k_2 = 2$, and $\alpha = 4$, which are such that the function given by $\frac{\partial V(x)}{\partial x}x$ is positive definite. Therefore, Assumption 1 holds and we can use the discretisation method described in Theorem 2. Thus, the discrete-time approximation to the solution of (36) is given by $x_{k+1} = \mathrm{diag}\Big(v_{k+1}^{\frac{2}{3}},\, v_{k+1}^{\frac{1}{3}}\Big)z_{k+1}$, with $z_{k+1} = \mathrm{diag}\Big(V^{-\frac{2}{3}}(\tilde{z}_{k+1}),\, V^{-\frac{1}{3}}(\tilde{z}_{k+1})\Big)\tilde{z}_{k+1}$,

$$v_{k+1} = \begin{cases} \Big(v_k^{\frac{1}{3}} - \frac{h}{3}W(z_k)\Big)^3, & hW(z_k) < 3v_k^{\frac{1}{3}}, \\ 0, & hW(z_k) \geq 3v_k^{\frac{1}{3}}, \end{cases}$$

$$\tilde{z}_{k+1} = \begin{cases} z_k + hv_k^{\frac{-1}{3}}g(z_k), & v_{k+1} > 0, \\ z_k, & v_{k+1} = 0, \end{cases} \qquad g(z_k) = \begin{bmatrix} \frac{2}{3}z_k^1W(z_k) + z_k^2 \\ \frac{1}{3}z_k^2W(z_k) + u(z_k) \end{bmatrix},$$

where z_k^i denotes the i-th component of the vector $z_k = \Lambda^{\mathbf{r}}_{v_k^{-\frac{1}{m}}}x_k$, and $v_k = V(x_k)$. We simulate 10 seconds with a step $h = 0.01$, and the initial conditions $x_1(0) = 0$, $x_2(0) = 3$. Fig. 5 shows the transient of the system's states converging to the origin. Nonetheless, in Fig. 6 we can see the exact finite-time convergence of the system's states to the origin. Observe that this exact convergence to zero is achieved by the discretisation method, which is explicit. This contrasts with the standard explicit Euler method, which produces a steady-state oscillation around the origin [15]. Now, the controller $u(x)$ is discontinuous at $x = 0$. In order to visualize the behaviour of u along the discrete-time solution x_k we define $u(x_k) = 0$ for $x_k = 0$. In Fig. 7 we can see the signal $u(x_k)$ which is bounded, and continuous except at the point the solutions reach the origin.

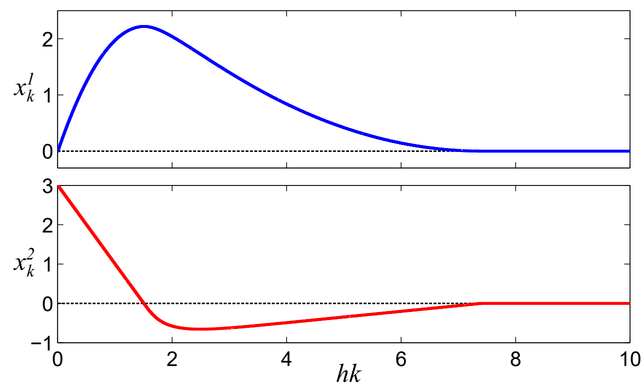

Figure 5: Discrete-time approximation of (36).

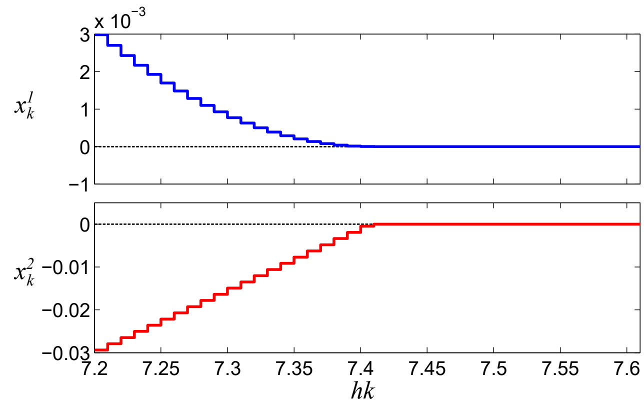

Figure 6: Close-up of the discrete-time approximation of (36).

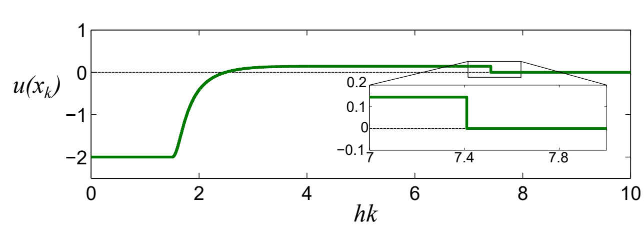

Figure 7: Control signal of the discrete-time approximation of (36).

6 Conclusion

In this paper we proposed a discretisation scheme for asymptotically stable homogeneous systems that preserves the stability and the convergence rate of the continuous-time system. The key ingredient of the method is the exploitation of the information provided by the Lyapunov function. It is important to mention that the proposed methodology does not restrict the discretisation of the projected dynamics to the explicit Euler method. Hence, a different technique could be used in such a process to obtain a different discretisation scheme, but preserving the main properties proven in this paper. One advantageous feature of the technique is that the discretisation and projection procedures are explicit, thus, no algebraic system has to be solved in each iteration.

Some interesting future developments include: the extension of the methodology to systems with discontinuities outside of the origin; study of different schemes obtained by modifying the discretisation technique for the projected dynamics; application of the proposed discretisation method to design sampled-data controllers.

Acknowledgements

This work was supported by the Agence Nationale de la Recherche project ANR-18-CE40-0008 “DIGITSLID.”

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