Floating Potential Boundary Condition in Smooth Domains in an Electroporation Context

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FLOATING POTENTIAL BOUNDARY CONDITION IN SMOOTH DOMAINS IN AN ELECTROPORATION CONTEXT

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Abstract. In electromagnetism, a conductor that is not connected to the ground is an equipotential whose value is implicitly determined by the constraint of the problem. It leads to a nonlocal constraints on the flux along the conductor interface, so-called floating potential problems. Unlike previous numerical study that tackle the floating potential problems with the help of advanced and complex numerical methods, we show how an appropriate use of Steklov-Poincaré operators enables to obtain the solution to this partial differential equations with a non local constraint as a linear (and well-designed) combination of $N+1$ Dirichlet problems, N being the number of conductors not connected to a ground potential. In the case of thin highly conductive inclusion, we perform an asymptotic analysis to approach the electroquasistatic potential at any order of accuracy. In particular, we show that the so-called floating potential approaches the electroquasistatic potential with a first order accuracy. This enables us to characterize the configurations for which floating potential approximation has to be used to accurately solve the electroquasistatic problem.

1. Introduction

The computation of the electroquasistatic electric field in high contrasted domains is a research field which is active for several decades in both electrical engineering and applied mathematics research areas [3, 1, 9, 4, 12]. The interest has increased a lot for the last decade with the use of pulse electric field for clinical ablation [7, 8, 5]. In particular, in the context of the insertion of multiple needles, the influence of the inactive electrods on the electric field distribution has to be precisely accounted to accurately determine the ablation region. The focus of this paper is to present an effective and rigorous way to compute the static electric field in the case of highly conductive thin inclusions.

The electroquasistatic theory states that the surface of a highly conductive conductor is an equipotential surface, whose value is determined implicitly by the constraints of the problem. This is the so-called floating potential problem which has been studied for several decades. In [1] Amann *et al.* have shown that the penalization method, which consists in imposing a high conductivity in the inclusion provides a less accurate electric potential than a well-designed numerical method for the floating potential problem. This result may seem strange, since the penalization is somehow the model of the real problem, while the floating potential is a perfect conductor approximation. Thus it is natural to wonder how the floating potential approaches the real electric potential, whether there is a relation between the size and the conductivity of the high conductive material which prevents the use of this approximation, and if it is possible to increase the accuracy with an asymptotic analysis. The aim of this paper is to address these questions for thin and highly conductive inclusions.

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1.1. **Preliminary numerical observations on concentric disks.** As preliminary, we investigate the observations of Amann *et al.* on a simplistic case, for which an explicit solution is available. We consider the case of a dielectric (low) conductive material Ω which is the annulus of radii $r_0 \in (0,1)$ and 1 and with conductivity equal to 1, surrounded by a high conductive sheet $\mathcal{O}_\varepsilon$ of thickness ε, and whose conductivity –after nondimensionalisation– is of order $1/\varepsilon^\ell$, where $\ell = 1$ or 2 and ε is a small parameter. The electroquasistatic potential u_ε satisfies the following elliptic problem

$$\frac{1}{r}\partial_r(r\partial_r u_\varepsilon) + \frac{1}{r^2}\partial_\theta^2 u_\varepsilon = 0 \quad \text{in} \quad (\{r_0 < r < 1\} \cup \{1 < r < 1+\varepsilon\}) \times \mathbb{R}/(2\pi\mathbb{Z}), \tag{1a}$$

with the following transmission conditions:

$$u_{\varepsilon|r=1^-} - u_{\varepsilon|r=1^+} = 0, \qquad \partial_r u_{\varepsilon|r=1^-} - \frac{1}{\varepsilon^\ell}\partial_r u_{\varepsilon|r=1^+} = 0, \tag{1b}$$

and the boundary conditions

$$\partial_r u_{\varepsilon|r=1+\varepsilon} = 0, \qquad u_{\varepsilon|r=r_0} = 1 + e^{\mathrm{i}\theta}. \tag{1c}$$

The corresponding floating potential problem consists in finding $(u, \alpha) \in H^1(\Omega) \times \mathbb{R}$ such that

$$\frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_\theta^2 u = 0 \quad \text{in} \quad \{r_0 < r < 1\} \times \mathbb{R}/(2\pi\mathbb{Z}), \tag{2a}$$

with the boundary conditions

$$u_{|r=1} = \alpha, \qquad \text{such that} \qquad \int_0^{2\pi} \partial_r u(1,\theta)\, d\theta = 0, \tag{2b}$$

$$u_{|r=r_0} = 1 + e^{\mathrm{i}\theta}, \tag{2c}$$

To prevent errors due to numerical computations, it is convenient to give the expression of the solution u^{exact} to Problem (1). In $\{r_0 < r < 1\} \times \mathbb{R}/(2\pi\mathbb{Z})$, u^{exact} reads:

$$u_\varepsilon^{exact}(r,\theta) = \frac{\varepsilon r_0 e^{i\theta}}{d(r_0,\varepsilon)}\left(\left(\varepsilon^{l+1} + 2\varepsilon^l + 2\varepsilon^{l-1} - \varepsilon - 2\right) r + \frac{(\varepsilon^{l+1} + 2\varepsilon^l + 2\varepsilon^{l-1} + \varepsilon + 2)}{r}\right) + 1,$$

and in $\{1 < r < 1+\varepsilon\} \times \mathbb{R}/(2\pi\mathbb{Z})$ it reads

$$u_\varepsilon^{exact}(r,\theta) = \frac{2\varepsilon^l r_0 e^{i\theta}}{d(r_0,\varepsilon)}\left(r + \frac{(\varepsilon+1)^2}{r}\right) + 1,$$

where $d(r_0,\varepsilon) = 2\varepsilon^l r_0^2(1 + (\varepsilon+1)^2) - \varepsilon(r_0^2 - 1)(\varepsilon^{l+1} + 2\varepsilon^l + 2\varepsilon^{l-1} + \varepsilon + 2)$.

This solution is then compared with the numerical resolution by standard second order finite difference scheme. This enables us to compare simultaneously how the solution to Problem (1) is approached by the solution to Problem (2), and how accurate is a standard second order numerical scheme for Problem (1). Numerical results are shown in Figure 1. Two main observations arise from these simplistic simulations.

First, for $\ell = 1$, the floating potential does not approach the solution to Problem (1), while it does with an order of accuracy in $O(\varepsilon)$ for $\ell = 2$, which means that floating potential cannot be used to approach the electric potential when the ratio R_{length} of the thickness of the conductor divided by the characteristic length of the dielectric is of the same order as the ratio R_{cond} of the conductivity of the dielectric divided by the conductivity of the high conductive sheet. Second, one can see that when ε becomes too small compared with the mesh grid, the numerical solution to the Problem (1) is not accurate, which provides an explanation to the statement by Amann *et al.* that the penalization method is less accurate than the floating potential.

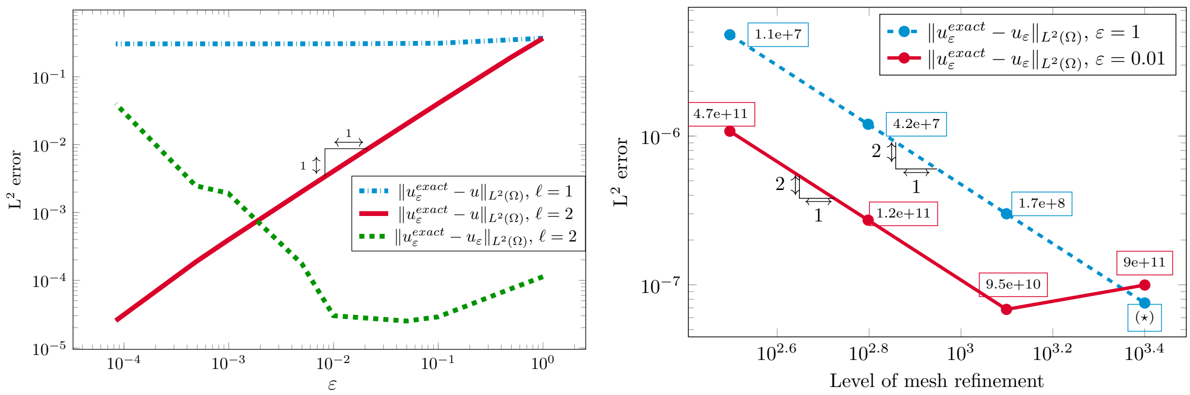

FIGURE 1. (Left): Comparison of the explicit solution u_ε^{exact} to Problem (1) with the floating potential problem for $\ell = 1$ (dashed and dotted blue line) and 2 (red line), and comparison of the explicit and the numerical solutions (1) (green dashed line) as ε tends to zero. One can see that for $\ell = 1$ the floating potential is not accurate: the conductive sheet is too thin to be an equipotential. For $\ell = 2$, the floating potential approaches the exact solution with an order of accuracy in $O(\varepsilon)$. Due a high *condition number* of the matrix, a direct numerical resolution of Problem (1) gives a worse approximation for smaller ε.
(Right): Numerical convergence with the steps $(d\theta, dr) \in \{(314, 200), (628, 400), (1256, 800), (2512, 1600)\}$ of the second order scheme to solve Problem (1) for $\varepsilon = 1$ (dashed blue line) and $\varepsilon = 0.01$ (red line). Condition number of the matrix for the discretization of Problem (1) grows considerably when ε approaches zero. With small discretization steps the instability increases and the numerical solution has lower accuracy.
($\star$) value not available due to not reasonable computational cost to compute it.

1.2. **Outline of the paper.** As shown on the previous simplistic example, the direct resolution of the electroquasistatic problem in a domain with highly conductive inclusion leads to *ill-conditioned* matrix and floating potentials are preferred to avoid the computational cost. The numerical resolution of such floating potential problems has been studied for several decades. One can cite for instance the paper by Dular *et al.* [3], where the authors proposed a finite element method, which consists in enriching the finite elements space with specific functions defined on the nodes of the interfaces Γ_k. Amann *et al.* proposed in [1] a boundary element method to tackle the problem using single boundary layer integral formulation of the solution. Note that recently, a hybrid Galerkin method has been proposed by Sala *et al.* for a similar problem in the context of ocular hemodynamic, the electric potential being replaced by the Darcy pressure [11].

The aim of the paper is twofold. On the one hand, after the proof of the well-posedness of the floating potential problem in the case of N multiple highly conductive inclusions, we propose a new numerical strategy to tackle the floating potential problem. Unlike the previous works cited above, our numerical strategy does not require any new specific numerical method. More precisely, it consists in characterising the solution to the floating potential problem as a linear combination of $N+1$ explicit Dirichlet problems thanks to the definition of well designed Steklov-Poincaré operators. On the second hand, we propose an asymptotic analysis of the electroquasistatic potential in the case of a highly conductive thin inclusion, in the asymptotic regime where the ratio of the conductivities R_{cond} is of order ε^2, while the ratio R_{length} is of order ε. In particular, we prove the convergence of the asymptotic approximation at any order as ε goes to 0.

2. Analysis and computation of the floating potential problem

Even though the use of well-designed numerical methods can be useful, they require deep changes in the computing software that prevent the use of standard softwares, which have been designed for Dirichlet, Neumann and/or Robin conditions in most cases. In the following, we show that the solution to the floating potential problem can be obtained as the linear combination of $N+1$ independent potentials with Dirichlet conditions. The parallelization of the independent problem implies that the floating potential is almost reduced to a Dirichlet problem, and the problem does not necessarily require the use of advanced numerical strategies.

Let us state precisely the problem. Let $\mathcal{O}$ be a domain of $\mathbb{R}^d$, $d=2,3$ and let $(\mathcal{O}_k)_{k=1}^N$ be N highly conductive inclusions embedded in $\mathcal{O}$. We denote by Γ_{out} the outer boundary of $\mathcal{O}$, and by Γ_k the boundary of $\mathcal{O}_k$ for $k=1,\cdots N$. Define $\Omega = \mathcal{O} \setminus \cup\overline{\mathcal{O}_k}$. Let $\sigma \in L^\infty(\Omega)$ be the conductivity map of Ω which satisfies for a given constant $a>0$

$$a \leq \|\sigma\|_{L^\infty(\Omega)} \leq 1/a.$$

Given $(g_k)_{k=1}^N \in \mathbb{R}$ and $f \in H^{-1}(\Omega)$, the floating potential problem[1] consists in finding the $N+1$-uple $(u, \alpha_1, \cdot, \alpha_N) \in H^1(\Omega)\times\mathbb{R}^N$ such that

$$-\nabla\cdot(\sigma\nabla u) = f \quad \text{in } \Omega, \qquad u|_{\Gamma_{\text{out}}} = 0, \tag{3a}$$

and on Γ_k, for $k=1,\cdots N$

$$u|_{\Gamma_k} = \alpha_k, \qquad \int_{\Gamma_k} \sigma\partial_n u\, ds = g_k. \tag{3b}$$

2.1. **Existence and Uniqueness of *floating potential* problem.** Even though the well-posedness of Problem (3) has been addressed by Amann *et al.* in [1] for one inclusion, we present a variant proof for N inclusions that will lead to our simple numerical strategy.

For $i=1,\cdots,N$, we consider the following Steklov-Poincaré operators defined as

$$\begin{aligned}
\Lambda_{\text{out}}^{(i)} : H^{-1}(\Omega) &\longrightarrow H^{-1/2}(\Gamma_i) \\
f \quad &\longmapsto \quad \sigma\partial_n v_{|\Gamma_i} \quad \text{s. t.} \quad \begin{cases} -\nabla\cdot(\sigma\nabla v) = f \quad \text{in } \Omega, \\ v_{|\Gamma_{\text{out}}} = 0, \quad v_{|\Gamma_\ell} = 0, \quad \text{for } \ell = 1,\cdots,N. \end{cases}
\end{aligned}$$

For $k=1,\cdots,N$, we define $\Lambda_k^{(i)}$ by

$$\begin{aligned}
\Lambda_k^{(i)} : H^{1/2}(\Gamma_k) &\longrightarrow H^{-1/2}(\Gamma_i) \\
\gamma \quad &\longmapsto \quad \sigma\partial_n v_{|\Gamma_i} \quad \text{s. t.} \quad \begin{cases} -\nabla\cdot(\sigma\nabla v) = 0 \quad \text{in } \Omega, \\ v_{|\Gamma_i} = \gamma, \\ v_{|\Gamma_{\text{out}}} = 0, \qquad v_{|\Gamma_\ell} = 0, \quad \text{for } \ell \neq i. \end{cases}
\end{aligned}$$

If it exists, the solution $(u, \alpha_1, \cdots, \alpha_N)$ to Problem (3) satisfies

$$\sigma\partial_n u_{|\Gamma_i} = \sum_{\ell=1}^N \alpha_\ell \Lambda_\ell^{(i)}(1) + \Lambda_{\text{out}}^{(i)}(f), \quad \text{for } i=1,\cdots,N,$$

and the nonlocal constraints (3b) read

$$g_i = \sum_{\ell=1}^N \alpha_\ell \int_{\Gamma_i} \Lambda_\ell^{(i)}(1) ds + \int_{\Gamma_i} \Lambda_{\text{out}}^{(i)}(f) ds, \qquad \text{for } i=1,\cdots,N. \tag{4}$$

Denoting by $\mathcal{M} = (\mathcal{M}_{ij})_{i,j=1,\cdots,N}$ the matrix defined as

$$\mathcal{M}_{ij} = \int_{\Gamma_i} \Lambda_j^i(1) ds, \tag{5a}$$

[1]Note that if the inclusion $\mathcal{O}_k$ is isolated, then g_k is nothing but 0.

and $\mathcal{B} = (\mathcal{B}_i)_{i=1,\cdots,N}$ the vector defined as

$$\mathcal{B}_i = g_i - \int_{\Gamma_i} \Lambda^i_{\text{out}}(f) ds. \tag{5b}$$

Then equality (4) reads

$$\mathcal{M} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} = \mathcal{B}, \tag{5c}$$

and the proof of the well-posedness of Problem (3) is reduced to proving the invertibility of $\mathcal{M}$.

Proposition 1. *Let Ω be a domain of $\mathbb{R}^d$, $d = 2, 3$. Let us endow the space $(L^2(\Omega))^d$, $d = 2, 3$ with the scalar product $< \cdot, \cdot >_{(L^2(\Omega))^d}$ defined by*

$$< F, G >^\sigma_{(L^2(\Omega))^d} = \int_\Omega \sigma F \cdot G \, dx, \qquad \forall (F, G) \in (L^2(\Omega))^d.$$

The matrix $\mathcal{M}$ defined by (5a) is a Gram matrix of the linearly independent vectors $\nabla v_1, \cdots, \nabla v_N$ of $(L^2(\Omega))^d$, where the functions $(v_\ell)_{\ell=1}^N$ are defined by

$$\begin{cases} -\nabla \cdot (\sigma \nabla v_\ell) = 0 \quad \textit{in } \Omega, \\ v_\ell|_{\Gamma_\ell} = 1, \\ v_\ell|_{\Gamma_{\text{out}}} = 0, \qquad v_\ell|_{\Gamma_k} = 0, \quad \textit{for } k \neq \ell. \end{cases} \tag{6}$$

Therefore $\mathcal{M}$ is invertible and there exists a unique $N+1$-uple $(u, \alpha_1, \cdots, \alpha_N) \in H^1(\Omega) \times \mathbb{R}^N$ solution to Problem (3).

Proof. Observe first that thanks to the Dirichlet boundary conditions on Γ_k for $k = 1, \cdots, N$, the vectors $(v_\ell)_{\ell=1}^N$ are linearly independent in $H^1(\Omega)$, hence the vectors $\nabla v_1, \cdots, \nabla v_N$ are linearly independent in $(L^2(\Omega))^d$.

By definition of $\mathcal{M}$ and by construction of v_i, one has

$$\mathcal{M}_{ij} = \int_{\Gamma_i} \sigma \partial_n v_j v_i dx = \int_\Omega \sigma \nabla v_j \cdot \nabla v_i \, dx.$$

Thus $\mathcal{M}$ is a Gram matrix of linearly independent vectors of $(L^2(\Omega))^d$, it is therefore invertible (see for instance [2]). □

2.2. **Numerical strategy to solve the floating potential problem.** Proposition 1 leads to a simple characterization of the solution to Problem (3), and thus a simple numerical strategy which is as follows.

• Compute v_ℓ given defined by (6) for $\ell = 1, \cdots, N$ and compute v_{out}, which is the solution in $H^1_0(\Omega)$ to

$$\begin{cases} -\nabla \cdot (\sigma \nabla v_{\text{out}}) = f \quad \text{in } \Omega, \\ v_{\text{out}|_{\partial\Omega}} = 0. \end{cases} \tag{7}$$

• Compute $(\mathcal{M}_{ij})$ and $(\mathcal{B}_{ij})$ given by (5a)–(5b) or equivalently

$$\mathcal{M}_{ij} = \int_\Omega \sigma \nabla v_i \cdot \nabla v_j \, dx, \qquad \mathcal{B}_i = g_i - \int_\Omega \sigma \nabla v_{\text{out}} \cdot \nabla v_i \, dx,$$

and deduce $(\alpha_1, \cdots, \alpha_N)$ by solving the linear system (5c).

• Then the solution u to the floating potential problem (3) is obtained by the following linear combination:

$$u = v_{\text{out}} + \sum_{\ell=1}^N \alpha_\ell v_\ell.$$

In other words, to compute u one just has to solve $N+1$ independent Dirichlet problems, which can be easily parallelized.

3. Asymptotic analysis and generalization of the floating potential problem for thin highly conductive sheets

3.1. **The conductivity problem.** In this section, we present the electroquasistatic problem in the case of one *thin high conductive* inclusion. We consider the asymptotic regime where the ratio between the dielectric/low conductive material R_{cond} is 2 order of magnitude greater than the ratio of the characteristic length R_{length} of the dielectric/low conductive material divided by the (small) thickness of sheet.

More precisely, we consider a smooth bounded domain Ω of $\mathbb{R}^d$, $d = 2$ or 3, which represents a conductive domain with a hole. We denote by Γ_{out} the external boundary of Ω, and by Γ the inner boundary corresponding the interface between Ω and the inner hole. The domain Ω is complemented with a thin highly conducting sheet coating the hole, and denoted by $\mathcal{O}_\varepsilon$, where ε is the ratio between the small thickness of the conductive sheet and the characteristic length of Ω. In addition we assume that the magnitude of the highly conductive sheet is of order $1/\varepsilon^2$. The domain $\mathcal{O}_\varepsilon$ may represent a inner passive electrode or a highly conductive thin inclusion as a surgical clip. We denote by Ω_ε the assembly $\Omega_\varepsilon = \Omega \cup \Gamma \cup \mathcal{O}_\varepsilon$, and Γ_ε is the interface between Ω_ε and the hole. Figure 2 provides a schematic of the geometrical framework.

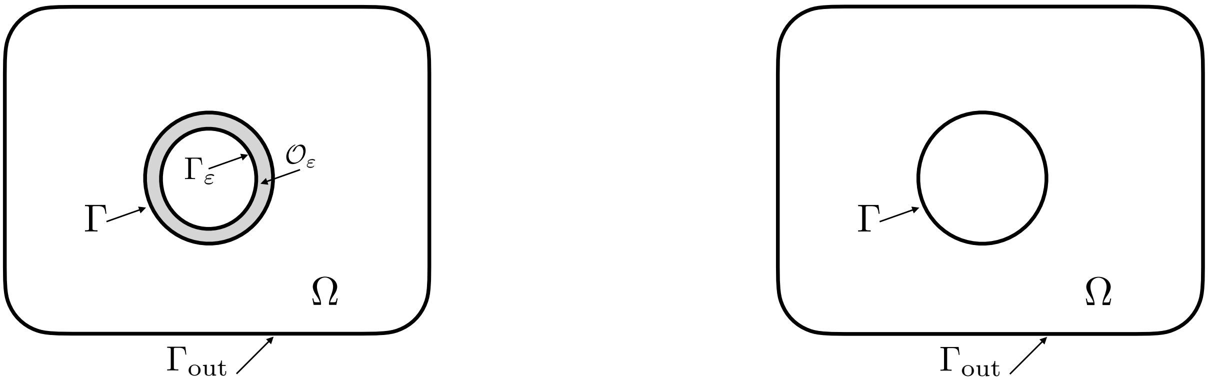

Figure 2. Schematics of the *toy model.* (Left): The domain with one thin highly conductive inclusion of thickness ε. The domain $\mathcal{O}_\varepsilon$ may represent a inner passive electrode or a highly conductive thin inclusion as a surgical clip. (Right): The domain Ω in the limit $\varepsilon = 0$.

The nondimensionalized conductivity[2] map σ_ε of the domain Ω_ε is given by

$$\sigma_\varepsilon(x) = \begin{cases} \sigma(x), & \text{if} \quad x \in \Omega, \\ \varepsilon^{-2}, & \text{if} \quad x \in \mathcal{O}_\varepsilon, \end{cases} \tag{8}$$

where σ is a strictly positive function in Ω.

The electroquasistatic potential u_ε in Ω satisfies the following elliptic problem

$$-\nabla \cdot (\sigma_\varepsilon \nabla u_\varepsilon) = \mathbb{1}_\Omega f \quad \text{in} \quad \Omega \cup \mathcal{O}_\varepsilon, \tag{9a}$$

with the transmission conditions on Γ:

$$u_{\varepsilon|\Gamma^+} - u_{\varepsilon|\Gamma^-} = 0, \qquad \sigma \partial_n u_{\varepsilon|\Gamma^+} - \frac{1}{\varepsilon^2}\partial_n u_{\varepsilon|\Gamma^-} = 0, \tag{9b}$$

and the boundary conditions

$$\partial_n u_{\varepsilon|\Gamma_\varepsilon} = 0, \qquad u_{\varepsilon|\Gamma_{\mathrm{out}}} = 0, \tag{9c}$$

where the source term $f \in H^{-1}(\Omega)$. The following a priori estimate holds.

[2]To simplify notation, we consider the non dimension conductivity map σ_ε, which is the conductivity map divided by the characteristic conductivity of the domain, which might the average of the conductivity on the low conductive domain.

Proposition 2 (A priori estimate)**.** *Let Ω and $\mathcal{O}_\varepsilon$ be smooth connected domains. Denote by $\Omega_\varepsilon = \Omega \cup \Gamma \cup \mathcal{O}_\varepsilon$. Let $f \in H^{-1}(\Omega)$. There exists a unique solution u_ε to Problem* (9) *in $H^1(\Omega_\varepsilon)$. Moreover there exists a constant $C > 0$ independent of ε such that*

$$\|u_\varepsilon\|_{H^1(\Omega)} + \frac{1}{\varepsilon}\|\nabla u_\varepsilon\|_{L^2(\mathcal{O}_\varepsilon)} \leq C\|f\|_{H^{-1}(\Omega)}.$$

Proof. The well-posedness of the elliptic problem (9) is standard by a straightforward application of the Lax-Migram lemma. The proof of the estimate is based on standard Poincaré estimate [6] in Ω since $u_\varepsilon|_\Omega \in H^1(\Omega)$ is such that $u_\varepsilon|_{\Gamma_{\text{out}}} = 0$. Indeed, multiplying by u_ε and integrating by parts leads to

$$\int_\Omega \sigma|\nabla u_\varepsilon|^2 dx + \frac{1}{\varepsilon^2}\int_{\mathcal{O}_\varepsilon} |\nabla u_\varepsilon|^2 dx \leq \|f\|_{H^{-1}(\Omega)}\|u_\varepsilon\|_{H^1(\Omega)} \leq C\|f\|_{H^{-1}(\Omega)}\|\nabla u_\varepsilon\|_{L^2(\Omega)},$$

where C is independent of ε, which ends the proof. □

The goal of this section is to provide the asymptotic expansion of u_ε as ε goes to 0.

3.2. **Local coordinates and Laplace operator.** To perform the asymptotic expansion it is natural to introduce the following change of variables which straightens up the thin inclusion. More precisely, let $\mathbf{x_T} = (\xi_1, \xi_2)$ be a system of local coordinates on $\Gamma = \{\Psi(\mathbf{x_T})\}$, where Ψ is a mapping of Γ. By abuse of notation, we denote by $\mathbf{x_T} \in \Gamma$ the point $\Psi(\mathbf{x_T}) \in \Gamma$. We define the following map Φ by

$$\Phi(\mathbf{x_T}, \xi_3) = \Psi(\mathbf{x_T}) + \xi_3\, \mathbf{n}(\mathbf{x_T}) \quad \forall(\mathbf{x_T}, \xi_3) \in \Gamma \times \mathbb{R},$$

where $\mathbf{n}$ is the outer normal vector of Γ. The layer $\mathcal{O}_\varepsilon$ is parameterized by

$$\mathcal{O}_\varepsilon = \{\Phi(\mathbf{x_T}, \xi_3) \mid (\mathbf{x_T}, \xi_3) \in \Gamma \times (0, \varepsilon)\}.$$

The Euclidean metric tensor $(g_{ij})_{i\,j=1,2,3}$, defined as

$$g_{ij} = \langle \partial_i\Phi, \partial_j\Phi\rangle$$

reads as follows [10]

$$\begin{aligned}
&g_{33} = 1, \quad g_{\alpha 3} = g_{3\alpha} = 0 \quad \forall \alpha \in \{1,2\},\\
&g_{\alpha\beta}(\mathbf{x_T}, \xi_3) = g^0_{\alpha\beta}(\mathbf{x_T}) + 2\,\xi_3\, b_{\alpha\beta}(\mathbf{x_T}, \xi_3) + {\xi_3}^2\, c_{\alpha\beta}(\mathbf{x_T}, \xi_3) \quad \forall \alpha, \beta \in \{1,2\}^2,
\end{aligned}$$

where

$$g^0_{\alpha\beta} = \langle \partial_\alpha\Psi, \partial_\beta\Psi\rangle,\ b_{\alpha\beta} = \langle \partial_\alpha\mathbf{n}, \partial_\beta\Psi\rangle,\ c_{\alpha\beta} = \langle \partial_\alpha\mathbf{n}, \partial_\beta\mathbf{n}\rangle.$$

The Laplace-Beltrami operator Δ_g in the system of local coordinates of $\mathcal{O}_\varepsilon$ reads then

$$\Delta_g = \frac{1}{\sqrt{g}}\sum_{i,j=1,2,3} \partial_i(\sqrt{g}g^{ij}\partial_j),$$

where $(g^{ij}) = (g_{ij})^{-1}$ and g the absolute value of the tensor metric determinant.
Define $\forall \ell \in \mathbb{N}$ the coefficients

$$\begin{aligned}
a^\ell_{ij} &= \partial_3^\ell\left(\frac{\partial_i(\sqrt{g}\,g^{ij})}{\sqrt{g}}\right)\Bigg|_{\xi_3=0}, \qquad \forall(i,j) \in \{1,2,3\}^2,\\
A^\ell_{\alpha\beta} &= \partial_3^\ell\left(g^{\alpha\,\beta}\right)\Big|_{\xi_3=0}, \qquad \forall(\alpha,\beta) \in \{1,2\}^2,
\end{aligned}$$

and let S^ℓ_Γ be the surface differential operator of order 2 on Γ defined as

$$S^\ell_\Gamma = \sum_{\alpha,\beta=1,2} a^l_{\alpha\beta}\partial_\beta + A^l_{\alpha\beta}\partial_\alpha\partial_\beta.$$

Remark 1. *As noticed in* [10]*, the operator S^0_Γ is nothing but the surface Laplace-Beltrami operator on Γ, so $S^0_\Gamma = \Delta_\Gamma$. Moreover a^0_{33} is the sum of the principal curvatures of Γ, in other words, denoting by $\mathcal{H}$ the mean curvature of Γ one has $a^0_{33} = 2\mathcal{H}$.*

The Laplace-Beltrami operator in $\mathcal{O}_\varepsilon$ can be rewritten as

$$\Delta_g = \partial_3^2 + \sum_{l\geq 0} \frac{\xi_3^l}{l!}(a_{33}^l \partial_3 + S_\Gamma^l) \quad \forall(\mathbf{x_T}, \xi_3) \in \Gamma \times (0, \varepsilon).$$

Performing the change of variable $\eta = \xi_3/\varepsilon$, we denote by $\Phi_\varepsilon(\mathbf{x_T}, \eta) = \Phi(\mathbf{x_T}, \varepsilon\eta)$, and we obtain

$$\Delta_g = \frac{1}{\varepsilon^2}\partial_\eta^2 + \frac{1}{\varepsilon} a_{33}^0 \, \partial_\eta + \sum_{l\geq 0} \varepsilon^l \frac{\eta^l}{l!} \left(\frac{\eta}{l+1} a_{33}^{l+1} \partial_\eta + S_\Gamma^l \right) \quad \forall(\mathbf{x_T}, \eta) \in \Gamma \times (0, 1).$$

3.3. **Formal expansion.** Denote by U_ε the electroquasistatic potential in local coordinates in $\mathcal{O}_\varepsilon$:

$$U_\varepsilon(\mathbf{x_T}, \eta) = u_\varepsilon \circ \Phi(\mathbf{x_T}, \varepsilon\eta), \qquad (\mathbf{x_T}, \eta) \in \Gamma \times (0, 1).$$

Thanks to this change of variables, Problem (9) reads as follows

$$\begin{cases} -\nabla \cdot (\sigma \nabla u_\varepsilon) = f \quad \text{in } \Omega, & (10a) \\ -\Delta_g U_\varepsilon = 0 \quad \text{on } \Gamma \times (0,1), & (10b) \\ u_{\varepsilon|_\Gamma} = U_{\varepsilon|_{\eta=0}} \circ \Psi^{-1}, & (10c) \\ \sigma\, \partial_n u_{\varepsilon|_\Gamma} = \varepsilon^{-3}\, \partial_\eta U_{\varepsilon|_{\eta=0}} \circ \Psi^{-1}, & (10d) \\ \partial_\eta U_{\varepsilon|_{\eta=1}} = 0, & (10e) \\ u_{\varepsilon|_{\Gamma_{\text{out}}}} = 0. & (10f) \end{cases}$$

We are now ready to derive formally the expansion. Set the following Ansatz:

$$u_\varepsilon(x) = \sum_{k\geq 0} \varepsilon^k\, u_k(x), \qquad \forall x \in \Omega, \tag{11a}$$

$$U_\varepsilon(\mathbf{x_T}, \eta) = \sum_{k\geq 0} \varepsilon^k\, \mathfrak{u}_k(\mathbf{x_T}, \eta), \qquad \forall(\mathbf{x_T}, \eta) \in \Gamma \times (0, 1). \tag{11b}$$

Injecting the formal series in Problem (10) and identifying the terms with the same power in ε lead to the following relations for any $p \geq 0$,

$$-\nabla \cdot (\sigma \nabla u_p) = \delta_p f \text{ in } \Omega, \tag{12a}$$

$$\partial_\eta^2 \mathfrak{u}_p = -a_{33}^0 \partial_\eta \mathfrak{u}_{p-1} - \sum_{l=0}^{p-2} \frac{\eta^l}{l!} \left(\frac{\eta}{l+1} a_{33}^{l+1} \partial_\eta \mathfrak{u}_{p-2-l} + S_\Gamma^l \mathfrak{u}_{p-2-l} \right) \text{ on } \Gamma \times (0,1), \tag{12b}$$

$$u_{p|_\Gamma} = \mathfrak{u}_{p|_{\eta=0}}, \tag{12c}$$

$$\sigma\, \partial_n u_{p-3|_\Gamma} = \partial_\eta \mathfrak{u}_{p|_{\eta=0}}, \tag{12d}$$

$$\partial_\eta \mathfrak{u}_{p|_{\eta=1}} = 0, \tag{12e}$$

$$u_{p|_{\Gamma_{\text{out}}}} = 0, \tag{12f}$$

where δ_p is the Kronecker symbol equal to 1 if $p = 0$ and 0 elsewhere, and with the convention u_p and $\mathfrak{u}_p$ are 0 if $p \leq 0$.

3.3.1. *Derivation of the 0th and 1rst order coefficients.* Using (12b) with $p = 0$ together with the boundary condition (12e), implies that $\partial_\eta \mathfrak{u}_0 = 0$ and thus $\mathfrak{u}_0 = \mathfrak{u}_0(\mathbf{x_T})$, and then similarly $\partial_\eta \mathfrak{u}_1 = 0$ and thus $\mathfrak{u}_1 = \mathfrak{u}_1(\mathbf{x_T})$. Then using (12b) for $p = 2$ implies that

$$\partial_\eta^2 \mathfrak{u}_2 = -\Delta_\Gamma \mathfrak{u}_0(\mathbf{x_T}).$$

The boundary conditions (12d)–(12e) imply thus that $\partial_\eta \mathfrak{u}_2 = 0$ and $-\Delta_\Gamma \mathfrak{u}_0 = 0$. Therefore we infer that $\mathfrak{u}_0$ is a constant denoted by α_0 and thus $u_{0|_\Gamma} = \alpha_0$. Then using (12b) for $p = 3$ implies that

$$\partial_\eta^2 \mathfrak{u}_3 = -\Delta_\Gamma \mathfrak{u}_1(\mathbf{x_T}), \tag{13}$$

since $\partial_\eta \mathfrak{u}_2 = 0$, and $S^1_\Gamma \mathfrak{u}_0 = 0$. We thus infer thanks to (12e) that

$$\int_\Gamma \partial_\eta \mathfrak{u}_3(\mathbf{x_T}, 1) d\mathbf{x_T} = -\int_\Gamma \Delta_\Gamma \mathfrak{u}_1 = 0,$$

from which we infer using (12d) the floating potential problem :

$$\begin{aligned}
&\text{Find } (u_0, \alpha_0) \in H^1(\Omega) \times \mathbb{R} \text{ such that} \\
&-\nabla \cdot (\sigma \nabla u_0) = f, \qquad \text{in } \Omega, \qquad u_{0|\Gamma_{\text{out}}} = 0, && (14a)\\
&u_{0|\Gamma} - \alpha_0 = 0, \qquad \text{such that} \qquad \int_\Gamma \sigma \partial_n u_0 \, ds = 0. && (14b)
\end{aligned}$$

Note also that the derivation process leads

$$\mathfrak{u}_0 = \alpha_0, \quad \partial_\eta \mathfrak{u}_1 = \partial_\eta \mathfrak{u}_2 = 0. \tag{15a}$$

To get the 1rst order coefficient, since $\partial_\eta \mathfrak{u}_1 = 0$, using (13) with (12d) implies that

$$\partial_\eta \mathfrak{u}_3(\mathbf{x_T}, \eta) = (1-\eta)\Delta_\Gamma \mathfrak{u}_1, \qquad \text{and} \qquad \Delta_\Gamma \mathfrak{u}_1 = \sigma \partial_n u_{0|\Gamma},$$

thus one can write $\mathfrak{u}_1 = g_1(\mathbf{x_T}) + \alpha_1$ where g_1 is uniquely determined by

$$\Delta_\Gamma g_1 = \sigma \partial_n u_{0|\Gamma}, \qquad \int_\Gamma g_1 ds = 0,$$

and thanks to (12c), $u_{1|\Gamma} - \alpha_1 = g_1 \circ \Psi^{-1}$. It remains to determine the constant α_1. Using (12b) with $p = 4$, one has

$$\begin{aligned}
\partial^2_\eta \mathfrak{u}_4 &= -a^0_{33} \partial_\eta \mathfrak{u}_3 - \sum_{l=0}^{2} \frac{\eta^l}{l!} \left(\frac{\eta}{l+1} a^{l+1}_{33} \partial_\eta \mathfrak{u}_{2-l} + S^l_\Gamma \mathfrak{u}_{2-l} \right) \\
&= -(1-\eta) a^0_{33} \Delta_\Gamma \mathfrak{u}_1 - \Delta_\Gamma \mathfrak{u}_2 - \eta S^1_\Gamma \mathfrak{u}_1,
\end{aligned}$$

and thanks to (12e) one infers

$$\partial_\eta \mathfrak{u}_4 = -\left(\eta - \eta^2/2 - 1/2\right) a^0_{33} \Delta_\Gamma \mathfrak{u}_1 - (\eta - 1)\Delta_\Gamma \mathfrak{u}_2 - (\eta^2/2 - 1/2) S^1_\Gamma \mathfrak{u}_1.$$

Note that since $\mathfrak{u}_2$ is not determined, the above equality does not define $\partial_\eta \mathfrak{u}_4$. However, integrating over Γ and using (12d), we obtain

$$\int_\Gamma \sigma \partial_n u_1 \, ds = \frac{1}{2} \int_\Gamma \left(a^0_{33} \Delta_\Gamma \mathfrak{u}_1 + S^1_\Gamma \mathfrak{u}_1 \right) ds = \frac{1}{2} \int_\Gamma \left(a^0_{33} \Delta_\Gamma g_1 + S^1_\Gamma g_1 \right) ds,$$

and then (u_1, α_1) is the solution to the following problem:

$$\begin{aligned}
&\text{Find } (u_1, \alpha_1) \in H^1(\Omega) \times \mathbb{R} \text{ such that :} \\
&-\nabla \cdot (\sigma \nabla u_1) = 0, \qquad \text{in } \Omega, \qquad u_{1|\Gamma_{\text{out}}} = 0, \\
&u_{1|\Gamma} - \alpha_1 = g_1 \circ \Psi^{-1}, \qquad \text{such that} \qquad \int_\Gamma \sigma \partial_n u_1 ds = h_1,
\end{aligned}$$

where g_1 and h_1 are given by

$$\begin{aligned}
&\Delta_\Gamma g_1 = \sigma \partial_n u_{0|\Gamma}, \qquad \int_\Gamma g_1 ds = 0, \\
&h_1 = \frac{1}{2} \int_\Gamma a^0_{33} \Delta_\Gamma g_1 + S^1_\Gamma g_1 ds.
\end{aligned}$$

Then one also has

$$\mathfrak{u}_1(\mathbf{x_T}) = g_1(\mathbf{x_T}) + \alpha_1, \qquad \partial_\eta \mathfrak{u}_2 = 0, \qquad \partial_\eta \mathfrak{u}_3 = (1-\eta)\Delta_\Gamma g_1.$$

3.3.2. *Derivation of the coefficients at any order k by induction.* Assume that there exists a smooth enough function g_k defined on Γ such that $\int_\Gamma g_k ds = 0$ and a constant h_k such that (u_k, α_k) is the solution to the following problem:

$$\begin{aligned}
&\text{Find } (u_k, \alpha_k) \in H^1(\Omega) \times \mathbb{R} \text{ such that}\\
&-\nabla \cdot (\sigma \nabla u_k) = \delta_k f, \qquad \text{in } \Omega, \qquad u_{k|_{\Gamma_{\text{out}}}} = 0,\\
&u_{k|_\Gamma} - \alpha_k = g_k \circ \Psi^{-1}, \qquad \text{such that} \qquad \int_\Gamma \sigma \partial_n u_k ds = h_k,
\end{aligned}$$

and assume that the following profile terms $\mathfrak{u}_\ell$ for $\ell = k, k+1, k+2$ read as

$$\begin{aligned}
&\mathfrak{u}_\ell(\mathbf{x_T}, \eta) = P_{\ell-1}(\mathbf{x_T}, \eta) + g_\ell(\mathbf{x_T}) + \alpha_\ell,\\
&\int_\Gamma \partial_\eta \mathfrak{u}_{k+3}(\mathbf{x_T}, 0) = h_k,
\end{aligned}$$

where $P_{\ell-1}$ is a given polynomial of order $\ell-1$ in η and vanishing in $\eta = 0$, for $\ell = k, k+1, k+2$, and g_ℓ and α_ℓ are unknown for $\ell = k+1, k+2$, with the constraint $\int_\Gamma g_\ell ds = 0$.

Then, using (12b)–(12e) with $p = k+3$, we infer that for any $r \in (0, 1)$

$$\begin{aligned}
\partial_\eta \mathfrak{u}_{k+3}(\mathbf{x_T}, r) - \int_r^1 \Delta_\Gamma \mathfrak{u}_{k+1} d\eta = \int_r^1 & a_{33}^0 \partial_\eta \mathfrak{u}_{k+2} + a_{33}^1 \eta \partial_\eta \mathfrak{u}_{k+1}\\
&+ \sum_{l=1}^{k+1} \frac{\eta^l}{l!} \left(\frac{\eta}{l+1} a_{33}^{l+1} \partial_\eta \mathfrak{u}_{k+1-l} + S_\Gamma^l \mathfrak{u}_{k+1-l} \right) d\eta.
\end{aligned}$$

We thus infer that $\partial_\eta \mathfrak{u}_{k+3}$ is polynomial of order $k+2$ or in other words $\mathfrak{u}_{k+3}$ reads as

$$\mathfrak{u}_{k+3}(\mathbf{x_T}, \eta) = P_{k+2}(\mathbf{x_T}, \eta) + g_{k+3}(\mathbf{x_T}) + \alpha_{k+3},$$

where P_{k+2} is explicitly given by the above equality vanishes in $\eta = 0$, and g_{k+3} is not determined but its mean value over Γ is 0 and α_{k+3} is a still undetermined constant. Using (12d) and the recurrence hypothesis, we also infer

$$\begin{aligned}
-\Delta_\Gamma g_{k+1} = -\sigma \partial_n u_k &+ \int_0^1 \eta \Delta_\Gamma P_k(\mathbf{x_T}, \eta) d\eta\\
&+ \int_0^1 a_{33}^0 \partial_\eta \mathfrak{u}_{k+2} + a_{33}^1 \eta \partial_\eta \mathfrak{u}_{k+1} \sum_{l=1}^{k+1} \frac{\eta^l}{l!} \left(\frac{\eta}{l+1} a_{33}^{l+1} \partial_\eta \mathfrak{u}_{k+1-l} + S_\Gamma^l \mathfrak{u}_{k+1-l} \right) ds,
\end{aligned}$$

which entirely determines g_{k+1} using the recurrence assumption since $\int_\Gamma g_{k+1} ds = 0$. It remains to determine the constant α_{k+1}. Using (12b)–(12e) with $p = k+4$ we infer that

$$\int_\Gamma \partial_\eta \mathfrak{u}_{k+4} ds = \int_\Gamma \int_\eta^1 a_{33}^0 \partial_\eta \mathfrak{u}_{k+3} + \sum_{l=0}^{k+2} \frac{r^l}{l!} \left(\frac{r}{l+1} a_{33}^{l+1} \partial_\eta \mathfrak{u}_{p-2-l} + S_\Gamma^l \mathfrak{u}_{p-2-l} \right) dr ds,$$

and (12d) leads to

$$\int_\Gamma \sigma \partial_n u_{k+1} ds = \int_\Gamma \int_0^1 a_{33}^0 \partial_\eta \mathfrak{u}_{k+3} + \sum_{l=0}^{k+2} \frac{r^l}{l!} \left(\frac{r}{l+1} a_{33}^{l+1} \partial_\eta \mathfrak{u}_{p-2-l} + S_\Gamma^l \mathfrak{u}_{p-2-l} \right) dr ds := h_{k+1}.$$

The condition (12c) implies then that (u_{k+1}, α_{k+1}) is the solution to the following problem:

$$\begin{aligned}
&\text{Find } (u_{k+1}, \alpha_{k+1}) \in H^1(\Omega) \times \mathbb{R} \text{ such that}\\
&-\nabla \cdot (\sigma \nabla u_{k+1}) = 0, \qquad \text{in } \Omega, \qquad u_{k+1|_{\Gamma_{\text{out}}}} = 0,\\
&u_{k+1|_\Gamma} - \alpha_{k+1} = g_{k+1} \circ \Psi^{-1}, \qquad \text{such that} \qquad \int_\Gamma \sigma \partial_n u_{k+1} ds = h_{k+1},
\end{aligned}$$

Remark 2. *It is worth noting that thanks to Proposition 1, the elementary problems are well-posed at any order.*

3.3.3. *Proof of the expansion.* Let us now prove the convergence of the expansion.

Theorem 1. *Let $N \geq 0$. Let $f \in \mathcal{C}^\infty(\Omega)$, such that the above inductive process to obtain the coefficients of the expansion* (11) *holds at any order. Let u_ε be the smooth solution to Problem* (9)*. Let $u_{\varepsilon,N}$ be the function defined by*

$$u_{\varepsilon,N} = \begin{cases} \sum_{k=0}^{N} \varepsilon^k u_k, & \text{in } \Omega, \\ \\ \sum_{k=0}^{N+2} \varepsilon^k \mathfrak{u}_k \circ \Phi_\varepsilon^{-1}, & \text{in } \mathcal{O}_\varepsilon, \end{cases}$$

where the functions $u_k, \mathfrak{u}_k$ are defined by the above inductive process.

Then there exists a constant C_N independent of ε such that

$$\|u_\varepsilon - u_{\varepsilon,N}\|_{H^1(\Omega)} + \frac{1}{\varepsilon}\|\nabla(u_\varepsilon - u_{\varepsilon,N})\|_{L^2(\mathcal{O}_\varepsilon)} \leq C_N \varepsilon^{N+1}.$$

Proof. Note that by hypothesis on f, $u_N \in C^\infty(\Omega)$ and $\mathfrak{u}_N \in C^\infty(\Gamma \times (0,1))$ (as well as their traces on Γ and derivatives) are uniformly bounded independently of ε.

Denote by $v_\varepsilon = u_\varepsilon - u_{\varepsilon,N}$. By construction of the expansion coefficients, v_ε satisfies the following problem:

$$\begin{cases} -\nabla \cdot (\sigma \nabla v_\varepsilon) = 0, \quad \text{in} \quad \Omega, & (16a) \\ -\Delta v_\varepsilon = \Delta u_{\varepsilon,N} \ (= O_{L^\infty(\Omega)}(\varepsilon^{N+1})), \quad \text{in} \quad \mathcal{O}_\varepsilon, & (16b) \\ v_\varepsilon|_{\Gamma^-} - v_\varepsilon|_{\Gamma^+} = 0, & (16c) \\ \sigma\partial_n v_\varepsilon|_{\Gamma^-} - \frac{1}{\varepsilon^2}\partial_n v_\varepsilon|_{\Gamma^+} = \sigma\partial_n u_{\varepsilon,N}|_{\Gamma^-} - \frac{1}{\varepsilon^2}\partial_n u_{\varepsilon,N}|_{\Gamma^+} \ (= O_{L^\infty(\Gamma)}(\varepsilon^N)), & (16d) \\ \partial_n v_\varepsilon|_{\Gamma_\varepsilon} = 0, & (16e) \\ v_\varepsilon|_{\Gamma_{\text{out}}} = 0. & (16f) \end{cases}$$

Using the fact that

$$\|O_{L^\infty(\Omega)}(\varepsilon^N)\|_{L^2(\mathcal{O}_\varepsilon)} \leq C\varepsilon^{N+1/2},$$

multiplying by v_ε and integrating lead to

$$\int_\Omega \sigma|\nabla v_\varepsilon|^2 dx + \frac{1}{\varepsilon^2}\int_{\mathcal{O}_\varepsilon} |\nabla v_\varepsilon|^2 dx \leq C\left(\varepsilon^{N+1/2}\|v_\varepsilon\|_{L^2(\mathcal{O}_\varepsilon)} + \varepsilon^{N+1}|v_\varepsilon|_{L^2(\Gamma)}\right).$$

Since the diameter of Ω_ε is bounded below by the diameter of Ω, uniform Poincaré estimate holds for any function in $H^1(\Omega_\varepsilon)$ vanishing on Γ_{out}. Thus thanks to Dirichlet trace estimate, there exists a constant C independent of ε such that

$$\begin{aligned} \int_\Omega \sigma|\nabla v_\varepsilon|^2 dx + \frac{1}{\varepsilon^2}\int_{\mathcal{O}_\varepsilon} |\nabla v_\varepsilon|^2 dx &\leq C\left(\varepsilon^{N+1/2}\|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)} + C\varepsilon^{N+1}|v_\varepsilon|_{L^2(\Gamma)}\right) \\ &\leq C\varepsilon^{N+1/2}\|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \end{aligned}$$

hence

$$\|u_\varepsilon - u_{\varepsilon,N}\|_{H^1(\Omega)} + \frac{1}{\varepsilon}\|\nabla(u_\varepsilon - u_{\varepsilon,N})\|_{L^2(\mathcal{O}_\varepsilon)} \leq C_N\varepsilon^{N+1/2}.$$

We similarly have

$$\|u_\varepsilon - u_{\varepsilon,N+1}\|_{H^1(\Omega)} + \frac{1}{\varepsilon}\|\nabla(u_\varepsilon - u_{\varepsilon,N+1})\|_{L^2(\mathcal{O}_\varepsilon)} \leq C_N\varepsilon^{N+3/2}.$$

Observing that $w_{\varepsilon,N}$ defined by $w_{\varepsilon,N} = u_{\varepsilon,N+1} - u_{\varepsilon,N}$, satisfies

$$\|w_{\varepsilon,N}\|_{H^1(\Omega)} + \frac{1}{\varepsilon}\|\nabla w_{\varepsilon,N}\|_{L^2(\mathcal{O}_\varepsilon)} \leq C_N\varepsilon^{N+1},$$

we infer the result. ∎

4. Conclusion

In this paper, we have proposed an asymptotic analysis to approach accurately the solution to the electroquasistatic potential in a smooth domain with a highly conductive inclusion. We have shown that the so-called floating potential approaches the electroquasistatic potential with a first order accuracy, and we have given the expansion at any order. For the sake of simplicity, we have only considered the case where the relative thickness of the inclusion is of order ε and the ratio of the conductivities (the conductivity of the conductive inclusion divided by the conductivity of the domain) is of order $1/\varepsilon^2$. It is worth noting that in the case of higher conductive thin inclusions – that is for ratios of the conductivities of order $1/\varepsilon^{2+s}$, with $s > 0$ – the floating potential provides an approximation of order $\varepsilon^{1+\lfloor s \rfloor}$, since the terms u_i for $i = 1, \cdots, \lfloor s \rfloor$ vanish. This observation, which easily comes from the formal derivation of Section 3.3 is left to the reader.

This paper also provides an efficient numerical method to compute accurately the floating potential problem in the case of N highly conductive inclusions, by replacing the PDE with nonlocal constraints on the total flux along the interfaces by the resolution of $N + 1$ Dirichlet problems uncoupled, the solution of the floating potential problem being obtained thanks to the inversion of a definite Gram matrix of size N. This efficient and rigorous approach has been used recently by the authors and colleagues in [12].

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