Conditions for some non stationary random walks in the quarter plane to be singular or of genus 0

Guy Fayolle, Roudol Iasnogorodski

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Guy Fayolle[*] *Roudol Iasnogorodski*[†]

Abstract

We analyze the *kernel* $K(x, y, t)$ of the basic functional equation associated with the tri-variate counting generating function (CGF) of walks in the quarter plane. In this short paper, taking $t \in]0, 1[$, we provide the conditions on the step set $\{p_{i,j}\}$ to decide whether the walks are *singular* or *regular*, as defined in [3, Section 2.3]. These conditions are independent of $t \in]0, 1[$ and given in terms of *step set configurations*. We also give the configurations for the kernel to be of genus 0, knowing that the genus is always $\leqslant 1$. All these conditions are very similar to the case $t = 1$ considered in [3]. Our results extend the work [2], which considers only very special situations, namely when $t \in]0, 1[$ is a transcendental number and the $p'_{i,j}s$ are rational.

1 Introduction and notation

Enumeration of planar lattice walks has become a classical topic in combinatorics. For a given set $\mathcal{S}$ of allowed jumps (or steps), it is a matter of counting the number of paths starting from some point and ending at some arbitrary point in a given time, and possibly restricted to some regions of the plane.

It will be convenient to denote by $\mathcal{S}$ the set of admissible *small steps*, included in the set of the eight nearest neighbors, so that

$$\mathcal{S} \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}.$$

Let $f(i, j, k)$ denote the probability for a walk in $\mathbb{Z}_+^2$ of reaching the point

[*]INRIA Paris-Saclay, SPECFUN Project-Team, guy.fayolle@inria.fr.
[†]Saint Petersburg State Chemical Pharmaceutical University, iasnogorodski@mail.ru

(i, j) after k steps, starting from $(0, 0)$. Then the corresponding CGF

$$F(x, y, t) = \sum_{i,j,k \geqslant 0} f(i, j, k) x^i y^j t^k \tag{1.1}$$

satisfies the functional equation (see [1] for details)

$$K(x, y, t)F(x, y, t) = K(x, 0, t)F(x, 0, t) + K(0, y, t)F(0, y, t) + l(x, y, t), \tag{1.2}$$

where

$$\begin{cases} K(x, y, t) = xy\big[tS(x, y) - 1\big], \\ S(s, y) = \displaystyle\sum_{(i,j)\in\mathcal{S}} p_{i,j} x^i y^j, \\ l(x, y, t) = -tp_{-1,-1}F(0, 0, t) - xy. \end{cases} \tag{1.3}$$

We assume $\sum_{(i,j)} p_{i,j} \in \mathcal{S} = 1$, which does not restrict the generality, so that $F(x, y, t)$ is sought to be convergent in the region $|x| \leq 1, |y| \leq 1, |t| \leq 1$. Usually, for fixed t, the algebraic curve corresponding to

$$\big\{(x, y) \in \mathbb{C}^2 : K(x, y, t) = 0\big\} \tag{1.4}$$

has genus 0 or 1. But this is is no more true when the kernel $K(x, y, t)$ may be factorized in $\mathbb{C}[x, y]$, in which case the walk is called *singular* according to the definition (see [3]).

Following the notation in [3], we define the triple $a(x)$, $b(x)$, $c(x)$, [resp. $\widetilde{a}(y), \widetilde{b}(y), \widetilde{c}(y)$] from (1.3) by

$$\begin{aligned} K(x, y, t) &= xy\Big(t\sum p_{ij}x^i y^j - 1\Big) && (1.5)\\ &\stackrel{\text{def}}{=} t[a(x)y^2 + b(x, t)y + c(x)] \stackrel{\text{def}}{=} t[\widetilde{a}(y)x^2 + \widetilde{b}(y, t)x + \widetilde{c}(y)], && (1.6) \end{aligned}$$

with

$$\begin{cases} a(x) = \displaystyle\sum_{i=0}^{2} p_{i-1,1} x^i, \\ b(x, t) = p_{-1,0} + (p_{0,0} - 1/t)x + p_{1,0}x^2, \\ c(x) = \displaystyle\sum_{i=0}^{2} p_{i-1,-1} x^i, \end{cases} \tag{1.7}$$

and similar equations for $\widetilde{a}(y), \widetilde{b}(y), \widetilde{c}(y)$.

Let us introduce the discriminant of (1.6) , always viewed as a polynomial in x

$$D(x, t) \stackrel{\text{def}}{=} b^2(x, t) - 4a(x)c(x). \tag{1.8}$$

Then it will be convenient to write

$$D(x,t) = \sum_{i=0}^{4} d_i(t)x^i, \tag{1.9}$$

where, by using (1.7),

$$\begin{cases} d_0(t) &= p_{-1,0}^2 - 4p_{-1,1}p_{-1,-1}, \\ d_1(t) &= 2[p_{-1,0}(p_{0,0} - 1/t) - 2(p_{-1,1}p_{0,-1} + p_{0,1}p_{-1,-1})], \\ d_2(t) &= (p_{0,0} - 1/t)^2 + 2p_{1,0}p_{-1,0} - 4(p_{1,1}p_{-1,-1} + p_{1,-1}p_{-1,1} + p_{0,1}p_{0,-1}), \\ d_3(t) &= 2p_{1,0}(p_{0,0} - 1/t) - 4(p_{1,1}p_{0,-1} + p_{0,1}p_{1,-1}), \\ d_4(t) &= p_{1,0}^2 - 4p_{1,1}p_{1,-1}, \end{cases} \tag{1.10}$$

noting that d_0 and d_4 are in fact independant of t. It is immediate to check that $d_2(t)$ is always strictly positive. Indeed we have the inequality

$$d_2(t) \geq (1 - 1/t)^2 + (p_{1,1} - p_{-1,-1})^2 + (p_{1,-1} - p_{-1,1})^2 + (p_{0,1} - p_{0,-1})^2.$$

Hence, the degree of $D(x,t)$ in x is always $\geqslant 2$.

We shall need the following simple notion of *positivity* for an arbitrary polynomial.

Definition 1.1. A polynomial $p(x)$ is said to be *positive (resp. negative)* to mean that its coefficients are all real positive (resp. negative) and not simultaneously 0.

2 The roots of the discriminant $\mathbf{D(x,t)}$

The main result of this section resides in he following theorem.

Theorem 2.1. *For all $t \in]0,1]$, $D(x,t)$ has 4 real roots, exactly two of them being inside the unit disk, and double roots can occur only at $x = 0$ or $x = \infty$.*

Proof. For all x satisfying $a(x)c(x) \geq 0$, we introduce

$$\phi_1(x,t) \stackrel{\text{def}}{=} b(x,t) + 2\sqrt{a(x)c(x)}, \quad \phi_2(x,t) \stackrel{\text{def}}{=} b(x,t) - 2\sqrt{a(x)c(x)}, \tag{2.1}$$

whence, according to (1.8),

$$D(x,t) = \phi_1(x,t)\phi_2(x,t).$$

Keeping in mind that $a(x)$ and $c(x)$ are *positive* (see Definition 1.1), they have no positive real roots. Our goal is to distribute the 4 roots of $D(x,t)$ between ϕ_1 and ϕ_2.

The roots of $\phi_1(x,t)$ and $\phi_2(x,t)$. The next inequalities are immediate

$$\phi_2(1,t) \leqslant \phi_1(1,t) \leqslant 1 - 1/t < 0, \quad \phi_1(0,t) \geqslant 0. \tag{2.2}$$

Let ξ_0 denote the largest real root of $a(x)c(x)$, when it exists. Then $\xi_0 \leqslant 0$ and $\phi_1(\xi_0,t) = \phi_2(\xi_0,t) = b(\xi_0,t) \geqslant 0$. Consequently, $\phi_1(x,t)$ and $\phi_2(x,t)$ each have a root on $]\xi_0, 1[$. On the other hand, if $\xi_0 \leqslant -1$, then $a(x)c(x) \geqslant 0$ for $x \in [-1,1]$ and

$$\phi_1(-1,t) \geqslant \phi_2(-1,t) > 0,$$

which, in tandem with (2.2), proves indeed that $\phi_2(x,t)$ *has always a root on* $]-1,1[$. This conclusion still holds when ξ_0 does not exist, since then $a(x)c(x)$ is positive and $\phi_1(x,t), \phi_2(x,t)$ are real $\forall x \in \mathbb{R}$.

So, when degree of $D(x,t) = 2$, there are two real roots inside the unit disk, harmoniously distributed between $\phi_1(x,t)$ and $\phi_2(x,t)$.

In addition, we note that, when degree of $D(x,t) \geqslant 3$,

$$\lim_{x\to+\infty} \phi_1(x,t) = +\infty, \tag{2.3}$$

in which case $\phi_1(x,t)$ has a root on $]1,\infty[$ and a root on $[0,1[$.

Degree of $\mathbf{D}(\mathbf{x},\mathbf{t}) = \mathbf{3}$, i.e., $\mathbf{d_4}(\mathbf{t}) = \mathbf{p}_{\mathbf{1,0}}^{\mathbf{2}} - \mathbf{4}\mathbf{p}_{\mathbf{1,1}}\mathbf{p}_{\mathbf{1,-1}} = \mathbf{0}$. Then two roots of $D(x,t)$ come from $\phi_1(x,t)$, and the third one from $\phi_2(x,t)$.

Degree of $\mathbf{D}(\mathbf{x},\mathbf{t}) = \mathbf{4}$, i.e. $\mathbf{d_4}(\mathbf{t}) \neq \mathbf{0}$. Letting $|x| \to \infty$, there are two possibilities.

(i) $d_4(t) > 0$. Then $\lim\limits_{x\to\infty} \phi_2(x,t) = +\infty$, and $\phi_2(x,t)$ has a positive real root on $]1,+\infty[$.

(ii) $d_4(t) < 0$. Let ξ_1 denote the smallest finite real root of $a(x)c(x)$, when it exists. Then

$$a(x)c(x \geqslant 0,\ \forall x \in]-\infty, \xi_1],\ \phi_2(\xi_1,t) = b(\xi_1,t) > 0,$$

and, since $\lim\limits_{x\to-\infty} \phi_2(x,t) = -\infty$, it follows that $\phi_2(x,t)$ has a negative real root on $]-\infty,\xi_1[$. If ξ_1 does not exist, then $\phi_2(x,t)$ has a root on $]-\infty,-1]$.

In conclusion, the distribution of the roots of $D(x,t)$ between $\phi_1(x,t)$ and $\phi_2(x,t)$ implies that a double root exists if, and only if,

$$\phi_1(x,t) = \phi_2(x,t) = 0 \Longleftrightarrow b(x,t) = a(x)c(x) = 0. \tag{2.4}$$

But the roots of $b(x,t)$ are real positive, while those of $a(x)$ and $c(x)$ have negative real parts. Hence (2.4) can only take place if $x = 0$ or $x = \infty$.

The proof of the theorem is terminated. ■

3 Classification of the singular random walks

Definition 3.1. A random walk is called *singular* (see [3]) if the associated polynomial $K(x, y, t)$ is either reducible or of degree 1 in at least one of the variables.

We establish a useful lemma, which is of an algebraic nature and gives conditions for the factorization of the kernel $K(x, y, t)$.

Lemma 3.2. *Let $\mathcal{A}[x]$ be the algebra of polynomials in x with coefficients in an arbitrary field $\mathcal{A}$ containing the rational numbers $\mathbb{Q}$. For given $a, b, c \in \mathcal{A}[x]$ satisfying*

$$b^2 - 4ac = kp^2, \tag{3.1}$$

where p is a polynomial and $k \in \mathcal{A}$, there exist $\alpha, \beta, f_1, f_2 \in \mathcal{B}[x]$, such that

$$ay^2 + by + c = (f_1 y - \alpha)(f_2 y - \beta), \tag{3.2}$$

where $\mathcal{B} \stackrel{\text{def}}{=} \mathcal{A} + \sqrt{k}$ denotes the field $\mathcal{A}$ to which is added the element $\sqrt{k}$). In addition, the relation (3.1) *is necessary to have the factorization* (3.2)*.*

Proof. Let Z_1, Z_2 be the roots of

$$Z^2 + bZ + ac = 0, \tag{3.3}$$

so that, with an obvious notation,

$$Z_{1,2} = -\frac{b}{2} \pm \frac{p}{2}\sqrt{k}.$$

In $\mathcal{B}[x]$, let f_1 be the greatest common divisor (g.c.d) of Z_1 and a, so that

$$Z_1 = \beta f_1 \quad \text{and } a = f_1 f_2,$$

where β and f_2 are relatively prime. Then, from $Z_1 Z_2 = ac$, we get $\beta Z_2 = cf_2$. Hence, the g.c.d. of β and f_2 being the unit element, f_2 divides Z_2 and we shall put $Z_2 = \alpha f_2$.

Setting for a while $Z = ay$, the announced factorization (3.2) follows directly from the chain of equalities

$$Z^2+bZ+ac = (Z-Z_1)(Z-Z_2) = (ay-\beta f_1)(ay-\alpha f_2) = a(f_2 y-\beta)(f_1 y-\alpha) \tag{3.4}$$

together with the identity

$$Z^2 + bZ + ac = a(ay^2 + by + c).$$

As for the necessity of (3.1) to have (3.2), we therefore assume

$$ay^2 + by + c = (f_1 y - \alpha)(f_2 y - \beta).$$

Letting $Z_1 \stackrel{\text{def}}{=} \alpha f_2, Z_2 \stackrel{\text{def}}{=} \beta f_1$, one sees that Z_1, Z_2 are the respective roots of (3.3), which both belong to $\in \mathcal{B}[x]$ and satisfy

$$(Z_1 - Z_2)^2 = (Z_1 + Z_2)^2 - 4Z_1 Z_2 = b^2 - 4ac. \tag{3.5}$$

On the other hand, as part of the hypothesis, we can set $Z_i \stackrel{\text{def}}{=} u_i + v_i\sqrt{k} \in \mathcal{B}[x]$, for $i = 1, 2$, where $u_i, v_i \in \mathcal{A}[x]$. Then, using the fact that a, b, c belong to $\mathcal{A}[x]$, together with the relations

$$\alpha f_2 + \beta f_1 = -b, \quad \text{and} \quad \alpha\beta f_1 f_2 = ac,$$

we get immediately

$$u_1 = u_2 = -b/2, \quad v_1 + v_2 = 0,$$

Hence

$$(Z_1 - Z_2)^2 = [2v_1\sqrt{k}]^2 = 4kv^2,$$

which, comparing with (3.5), yields exactly (3.1).

The proof of the lemma is terminated. ■

Remark 3.3. It is not difficult to see that in equation (3.1) there exists a version of p belonging to $\mathcal{A}[x]$.

Corollary 3.4. *Take $\mathcal{A} = \mathbb{R}$ in Lemma 3.2. Then, whenever $k > 0$ in equation (3.1), there exists a factorization (3.2) of $D(x,t)$ over $\mathbb{R}[x,y]$. On the other hand, if $k < 0$, then any factorization is over $\mathbb{C}[x,y]$.*

Proof. Assume $k > 0$ in Lemma 3.2. Then $\sqrt{k} \in \mathbb{R}$ and $\mathcal{B} = \mathcal{A} = \mathbb{R}$. ■

The classification of the family of random walks under study is given by the next theorem.

Theorem 3.5. *For $t \in]0,1]$ and $p_{i,j} \in [0,1], p_{0,0} \neq 1$, the random walk is singular if, and only if, one of the following conditions holds:*

(i) There exists $(i,j) \in \mathbb{Z}^2$, $|i| \leqslant 1$, $|j| \leqslant 1$, such that only p_{ij} and $p_{-i,-j}$ are different from 0 (see figure 3.1a and the three cases obtained by rotation);

(ii) There exists i, $|i| = 1$, such that for any j, $|j| \leqslant 1$, $p_{ij} = 0$ (see figure 3.1b,c);

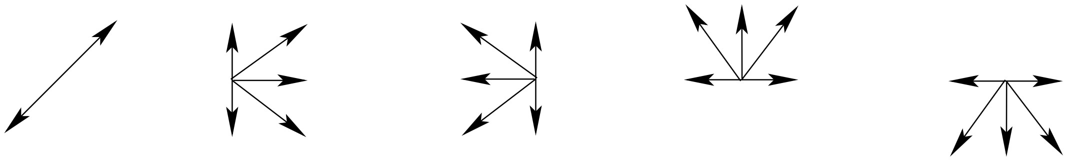


Fig. 3.1: Singular random walks.

(iii) There exists j, $|j| = 1$, such that for any i, $|i| \leqslant 1$, $p_{ij} = 0$ (see figure 3.1d,e).

Proof. Let us first eliminate three simple situations.

Case $\mathbf{a} \equiv \mathbf{0}$. Then $p_{1,1} = p_{0,1} = p_{-1,1} = 0$. The kernel is of degree 1 in y and this corresponds to the walks of type e in figure 3.1 and

$$K(x, y, t) = t(by + c).$$

Case $\mathbf{c} \equiv \mathbf{0}$. Then $p_{1,-1} = p_{0,-1} = p_{-1,-1} = 0$, giving the walks of type d in figure 3.1 and

$$K(x, y, t) = ty(ay + b).$$

By exchanging the variables x and y, and writing the kernel as a polynomial in x with coefficients $\widetilde{a}, \widetilde{b}, \widetilde{c}$ (see (1.5)), the cases $\widetilde{a} \equiv 0$ or $\widetilde{c} \equiv 0$ depict the walks of respective types c and b in figure (3.1).

From now on, we shall assume $ac \not\equiv 0$.

Case $\mathbf{K(x, y, t) = u(x, t)v(x, y, t)}$. Here u (resp. v) stands for a polynomial in x (resp. (x, y). It is immediate to see that the degree of $u(x, t)$ in x is 1. But this means that a, b, c have a common real root in x, which can occur only at $x = 0$. Indeed, by (1.7), it appears that $b(x, t)$ has always two positive real roots, while the roots of a and c cannot be positive. Thus the only possibility is $p_{-1,1} = p_{-1,0} = p_{-1,-1} = 0$, or equivalently $\widetilde{c} \equiv 0$, which corresponds to the walks of type b in figure (3.1). Hence,

$$\widetilde{c}(y) \equiv 0 \iff K(x, y, t) = tx[x\widetilde{a}(y) + \widetilde{b}(y, t)].$$

Similarly, by exchanging the roles of x and y, we obtain the walks of type d in figure (3.1). In particular,

$$c(x) \equiv 0 \iff K(x, y, t) = ty[ya(x) + b(x, t)].$$

Under Definition 3.1, all singular random walks satisfy (3.2) together with the system

$$\alpha f_2 + \beta f_1 = -b, \tag{3.6}$$
$$f_1 f_2 = a, \tag{3.7}$$
$$\alpha\beta = c, \tag{3.8}$$

where α, β, f_1, f_2 are polynomials of degree at most 2 in x, with real coefficients by Corollary 3.4. In the course of the proof, we shall omit the variables (x,t) for the sake of readability.

When a and c are positive (see Definition 1.1), (3.7) and (3.8) imply that (f_1, f_2) are both either positive or negative, and likwise for (α, β). This is true because a and c are of degree at most 2, and we shall choose (f_1, f_2) positive.

Recalling that $b \equiv b(x,t) = p_{-1,0} + (p_{0,0} - 1/t)x + p_{1,0}x^2$, the identity (3.6) leads to consider two different situations.

(i) $p_{1,0} + p_{-1,0} \neq 0$. Here it is impossible to satisfy (3.6) for $t \in]0,1]$, since the three coefficients of the polynomial b do not have the same sign.

(ii) $p_{1,0} = p_{-1,0} = 0$. Then (3.6) can hold only if (α, β) are positive and if $(\alpha f_2, \beta f_1)$ are proportional to x. Upon combining (3.7)-(3.8), we obtain

$$(\alpha f_2)(\beta f_1) = ac = Kx^2,$$

where K is a constant. Thus the coefficients of x^4, x^3, x, x^0 of the polynomial $a(x)c(x)$ must cancel out, yielding from (1.7) the respective equations

$$\begin{cases} p_{1,1}p_{1,-1} = 0, \\ p_{1,1}p_{0,-1} + p_{1,-1}p_{0,1} = 0, \\ p_{0,1}p_{-1,-1} + p_{-1,1}p_{0,-1} = 0, \\ p_{-1,1}p_{0,-1} = 0. \end{cases} \tag{3.9}$$

Now it is easy to check that the puzzle-like system (3.9) generates exactly the 4 following random walks:

$$\begin{cases} p_{1,-1} = p_{0,-1} = p_{-1,1} = p_{0,1} = 0, & \text{figure } (3.1,a); \\ p_{1,1} = p_{0,1} = p_{-1,-1} = p_{0,-1} = 0, & \text{figure } (3.1,a) \text{ rotated by } 90°; \\ p_{1,1} = p_{1,-1} = p_{-1,-1} = p_{-1,1} = 0, & \text{figure } (3.1,a) \text{ rotated by } 45°; \\ p_{1,1} = p_{0,1} = p_{-1,1} = p_{0,-1} = 0, & \text{sub-case of figure } (3.1,e); \end{cases} \tag{3.10}$$

By exchanging the roles of x and y, we also obtain the singular walk where only $p_{1,0}$ and $p_{-1,0}$ are different from zero.

The proof of the theorem is terminated. ■

Remark 3.6. When $p_{0,0} = 0$, one verifies easily that Theorem 3.5 holds in fact for all $t \in]0, +\infty[$.

4 Kernel of genus 0

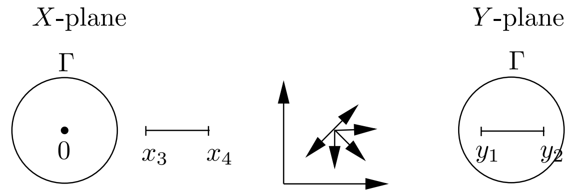
It is wellknown that, for all non-singular random walks, the Riemann surface $\mathbf{S}$ defined by the algebraic curve (1.4) has genus 0 or 1. Morever, the genus of $\mathbf{S}$ is 0 if, and only if, the discriminant $D(x, t)$ defined in (1.8) has a multiple zero in x, possibly infinite.

For the sake of completeness, we also introduce the branch points x_i (resp. y_i), $i = 1, \ldots, 4$, of the algebraic function $Y(x, t)$ satisfying $K(x, Y(x, t), t) = 0$ [resp. $X(y, t)$ satisfying $K(X(y, t), y, t) = 0$]. These branch points depend of course on t.

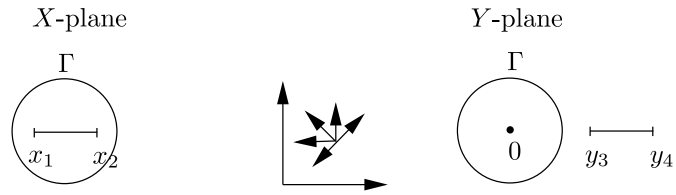
In the book [3, Section 2.3], the classification in the so-called stationary case $t = 1$ has been dealt with in great detail. The proof was heavily using the principle of *continuity of the roots with respect to the parameters* $p_{i,j}$, thus reducing the analysis to the case of the so-called *simple random walk*, i.e. when only $p_{1,0}, p_{0,1}, p_{-1,0}, p_{0,-1}$ are different from 0. Indeed, for $t \in]0, 1[$, exactly the same arguments hold. Nonetheless, we present hereafter a self-contained proof. The following theorem holds.

Theorem 4.1. *For all non singular random walks and for all $t \in]0, 1[$, the algebraic curve defined by $K(x, y, t) = 0$ has genus 0 if, and only if, $D(x, t)$ has a double root at $x = 0$ or $x = \infty$. That corresponds to the following pictures.*

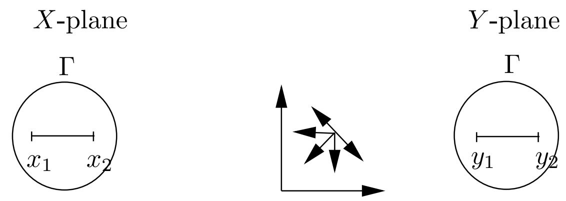
1. $\begin{cases} x_1 = x_2 = 0 \\ y_3 = y_4 = \infty. \end{cases}$



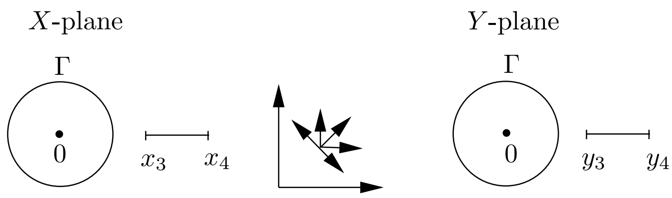
2. $\begin{cases} y_1 = y_2 = 0 \\ x_3 = x_4 = \infty. \end{cases}$



3. $\begin{cases} x_3 = x_4 = \infty \\ y_3 = y_4 = \infty. \end{cases}$



4. $\begin{cases} x_1 = x_2 = 0 \\ y_1 = y_2 = 0. \end{cases}$



In other words, multiple roots of $D(x,t)$ can occur only in either of the two following situations:

(a) $x = 0$, in which case $d_0(t) = d_1(t) = 0$;

(b) $x = \infty$, in which case $d_3(t) = d_4(t) = 0$.

Proof. Let us recall that, since $d_2(t) > 0$, the degree of $D(x,t)$ with respect to x is always $\geqslant 2$. From the outset, we shall solve the case where this degree is exactly 2.

Degree of $\mathbf{D(x,t) = 2}$. This means that $D(x,t$ has a double root at $x = \infty$, whence $d_3(t) = d_4(t) = 0$ and the genus of the kernel is 0 for all non singular random walks. Then, by using (1.10) and remarking that $d_3(t) \leqslant 0$, we get the system

$$p_{1,0} = p_{1,1}p_{0,-1} = p_{0,1}p_{1,-1} = p_{1,1}p_{1,-1} = 0. \tag{4.1}$$

It is now easy to check that the set of relations (4.1), without taking into account the singular random walk $p_{1,0} = p_{1,1}, = p_{1,-1} = 0$, is tantamount to the pictures *2.* and *3.*

Degree of $\mathbf{D(x,t) \geqslant 3}$. We have proved in Theorem 2.1 that a double root can only occur at $x = 0$. Then $d_0(t) = d_1(t) = 0$, whence by using (1.10)

$$p_{-1,0} = p_{-1,1}p_{0,-1} = p_{0,1}p_{-1,-1} = p_{-1,1}p_{-1,-1} = 0,$$

which corresponds to the pictures *1.* and *4.*

The proof of the theorem is terminated. ■

Remark 4.2. When $D(x,t)$ has a double root both at $x = 0$ and $x = \infty$, we know from Lemma 3.2 that a factorization of the form (3.2) holds, and the walk is singular. On the other hand, it is worth noting that, for all $t \in]0,1[$, the discriminant $D(x,t)$ can never have double root at $x = 1$. This case can occur for $t = 1$ and corresponds random walks having zero drift vectors (see [3, Section 2.3]).

Remark 4.3. The approach proposed in this paper applies verbatim to functional equations pertaining to the transient distribution of two-dimensional Markov processes, such as those encountered in queueing systems (see e.g.,[3, Section 8.4]).

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