Conditions for some non stationary random walks in the quarter plane to be singular or of genus 0

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January 10, 2021

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Abstract

We analyze the *kernel* $K(x, y, t)$ of the basic functional equation associated with the tri-variate counting generating function (CGF) of walks in the quarter plane. In this short paper, taking $t \in]0, 1[$, we provide the conditions on the jump probabilities $\{p_{i,j}$'s$\}$ to decide whether walks are *singular* or *regular*, as defined in [3, Section 2.3]. These conditions are independent of $t \in]0, 1[$ and given in terms of *step set configurations*. We also find the configurations for the kernel to be of genus 0, knowing that the genus is always $\leqslant 1$. All these conditions are very similar to that of the stationary case considered in [3]. Our results extend the work [2], which considers only the special situation where $t \in]0, 1[$ is a transcendental number over the field $\mathbb{Q}(p_{i,j})$. In addition, when $p_{0,0} = 0$, our classification holds for all $t \in]0, +\infty]$.

Keywords: Algebraic curve, functional equation, generating function, genus, quarter-plane, Riemann surface, singular random walk.

AMS 2000 Subject Classification: Primary 60G50; secondary 30F10, 30D05.

1 Historical introduction

Discrete two-dimensional random walks in domains with non-smooth boundaries are of interest to several groups in the mathematical community. In fact, these processes are encountered in pure probabilistic problems, as well as in applications involving queueing theory or enumerative combinatorics. Some four decades ago, original mathematical methods have been proposed to determine the invariant measure of such Markov processes, say $(M_t, N_t) \in \mathbb{Z}_+^2$ at time t, evolving in the positive quadrant, see the book [3]. When the jumps in the the interior of the quadrant are of size one, the bi-variate generating function $Q(x, y)$ of the probability distribution satisfies a functional equation of two complex variables, whose general solution,

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obtained via the so-called *kernel method*, amounts to solving a boundary value problem on a Riemann surface of genus 0 or 1, except for some degenerate walks, referred to as *singular*. The complete classification was obtained in [3, Section 2.3].

It turns out that the method can also be applied to analyze the transient behaviour of these walks, after switching to the Laplace transforms

$$F(x,y,s) = \int_0^\infty e^{-st} E[x^{M_t} y^{N_t}] dt, \quad \Re(s) \geq 0,$$

in which case $F(x,y,s)$ satisfies a functional equation, where s merely plays the role of a parameter.

In this article, we will consider the problem of counting walks in the quarter plane. The related counting generating function (CGF) is of a transient nature and satisfies the functional equation (2.2), in which the time parameter t is in a certain sense the analogous of s for the Laplace transforms. In this context we will provide the classification of the walks, which, as expected, turns out to be almost identical to that mentioned above.

2 Model and notation

Enumeration of planar lattice walks has become a classical topic in combinatorics. For given allowed jumps (or steps), it is a matter of counting the number of paths starting from some point and ending at some arbitrary point in a given time.

Here we assume the random walks evolve in the positive quarter plane, and the jumps have size 1. Hence, the following transitions can take place.

$$\begin{cases} (m,n) \to (m+i, n+j), \text{ with probability } p_{i,j}, |i|, |j| \leq 1, \forall (m,n) > (0,0); \\ (0,n) \to (i, n+j), \text{ with probability } p_{i,j}, 0 \leq i \leq 1, |j| \leq 1, \forall n > 0; \\ (m,0) \to (m+i, j), \quad \text{with probability } p_{i,j}, 0 \leq j \leq 1, |i| \leq 1, \forall m > 0; \\ (0,0) \to (i,j), \quad \text{with probability } p_{i,j}, 0 \leq i,j \leq 1. \end{cases}$$

Without restricting the generality, we suppose $\sum_{(i,j)\in\mathcal{S}} p_{i,j} = 1$.

It will be convenient to denote by $\mathcal{S}$ the set of the admissible jumps, which are included in the set of the eight nearest neighbors, so that

$$\mathcal{S} \subset \{-1,0,1\}^2 \setminus \{(0,0)\}.$$

Let $f(m,n,k)$ denote the probability for a walk in $\mathbb{Z}_+^2$ of reaching the point (m,n) after k steps, starting from $(0,0)$. Then the corresponding CGF

$$F(x,y,t) = \sum_{m,n,k \geqslant 0} f(m,n,k) x^m y^n t^k \tag{2.1}$$

satisfies the following functional equation (see [1] for the details)

$$K(x,y,t)F(x,y,t) = K(x,0,t)F(x,0,t) + K(0,y,t)F(0,y,t) + l(x,y,t), \tag{2.2}$$

where

$$\begin{cases} K(x,y,t) = xy\big[tS(x,y) - 1\big], \\ S(x,y) = \displaystyle\sum_{(i,j)\in\mathcal{S}} p_{i,j}x^i y^j, \\ l(x,y,t) = -tp_{-1,-1}F(0,0,t) - xy. \end{cases} \tag{2.3}$$

$F(x,y,t)$ is sought to be convergent in the region $|x| \le 1, |y| \le 1, |t| \le 1$. Usually, for fixed t, the algebraic curve corresponding to

$$\big\{(x,y) \in \mathbb{C}^2 : K(x,y,t) = 0\big\} \tag{2.4}$$

has genus 0 or 1. But this is is no more true when the kernel $K(x,y,t)$ may be factorized in $\mathbb{C}[x,y]$, in which case the walk is called *singular* according to the definition given in [3]).

Following the notation in [3], we define the triple $a(x)$, $b(x)$, $c(x)$, [resp. $\widetilde{a}(y), \widetilde{b}(y), \widetilde{c}(y)$] from (2.3) by

$$K(x,y,t) = xy\Big(t\sum_{(i,j)\in\mathcal{S}} p_{ij}x^i y^j - 1\Big) \tag{2.5}$$

$$\stackrel{\text{def}}{=} t[a(x)y^2 + b(x,t)y + c(x)] \stackrel{\text{def}}{=} t[\widetilde{a}(y)x^2 + \widetilde{b}(y,t)x + \widetilde{c}(y)], \tag{2.6}$$

with

$$\begin{cases} a(x) = \displaystyle\sum_{i=0}^{2} p_{i-1,1}x^i, \\ b(x,t) = p_{-1,0} + (p_{0,0} - 1/t)x + p_{1,0}x^2, \\ c(x) = \displaystyle\sum_{i=0}^{2} p_{i-1,-1}x^i, \end{cases} \tag{2.7}$$

and similar equations for $\widetilde{a}(y), \widetilde{b}(y), \widetilde{c}(y)$.

Let us introduce the discriminant of (2.6) , always viewed as a polynomial in x

$$D(x,t) \stackrel{\text{def}}{=} b^2(x,t) - 4a(x)c(x). \tag{2.8}$$

Then it will be convenient to write

$$D(x,t) = \sum_{i=0}^{4} d_i(t)x^i, \tag{2.9}$$

where, by using (2.7),

$$\begin{cases} d_0(t) &= p_{-1,0}^2 - 4p_{-1,1}p_{-1,-1}, \\ d_1(t) &= 2[p_{-1,0}(p_{0,0} - 1/t) - 2(p_{-1,1}p_{0,-1} + p_{0,1}p_{-1,-1})], \\ d_2(t) &= (p_{0,0} - 1/t)^2 + 2p_{1,0}p_{-1,0} - 4(p_{1,1}p_{-1,-1} + p_{1,-1}p_{-1,1} + p_{0,1}p_{0,-1}), \\ d_3(t) &= 2p_{1,0}(p_{0,0} - 1/t) - 4(p_{1,1}p_{0,-1} + p_{0,1}p_{1,-1}), \\ d_4(t) &= p_{1,0}^2 - 4p_{1,1}p_{1,-1}, \end{cases} \tag{2.10}$$

noting that d_0 and d_4 are in fact independant of t. It is immediate to check that $d_2(t)$ is always strictly positive. Indeed we have the inequality

$$d_2(t) \geq (1-1/t)^2 + (p_{1,1}-p_{-1,-1})^2 + (p_{1,-1}-p_{-1,1})^2 + (p_{0,1}-p_{0,-1})^2.$$

Hence, the degree of $D(x,t)$ in x is always $\geqslant 2$.

We shall need the following simple notion of *positivity* for an arbitrary polynomial.

Definition 2.1. A polynomial $p(x)$ is said to be *positive (resp. negative)* to mean that its coefficients are all real positive (resp. negative) and not simultaneously 0.

3 The roots of the discriminant $\mathbf{D}(\mathbf{x},\mathbf{t})$

The main result of this section resides in he following theorem.

Theorem 3.1. *For all $t \in]0,1]$, $D(x,t)$ has 4 real roots, exactly two of them being inside the unit disk, and double roots can occur only at $x=0$ or $x=\infty$.*

Proof. For all x satisfying $a(x)c(x) \geq 0$, we introduce

$$\phi_1(x,t) \stackrel{\text{def}}{=} b(x,t) + 2\sqrt{a(x)c(x)}, \quad \phi_2(x,t) \stackrel{\text{def}}{=} b(x,t) - 2\sqrt{a(x)c(x)}, \tag{3.1}$$

whence, according to (2.8),

$$D(x,t) = \phi_1(x,t)\phi_2(x,t).$$

Keeping in mind that $a(x)$ and $c(x)$ are *positive* (see Definition 2.1), they have no positive real roots. Our goal is to distribute the 4 roots of $D(x,t)$ between ϕ_1 and ϕ_2.

The roots of $\phi_1(x,t)$ and $\phi_2(x,t)$. The next inequalities are immediate

$$\phi_2(1,t) \leqslant \phi_1(1,t) \leqslant 1 - 1/t < 0, \quad \phi_1(0,t) \geqslant 0. \tag{3.2}$$

Let ξ_0 denote the largest real root of $a(x)c(x)$, when it exists. Then $\xi_0 \leqslant 0$ and $\phi_1(\xi_0,t) = \phi_2(\xi_0,t) = b(\xi_0,t) \geqslant 0$. Consequently, $\phi_1(x,t)$ and $\phi_2(x,t)$ each have a root on $]\xi_0,1[$. On the other hand, if $\xi_0 \leqslant -1$, then $a(x)c(x) \geqslant 0$ for $x \in [-1,1]$ and

$$\phi_1(-1,t) \geqslant \phi_2(-1,t) > 0,$$

which, in tandem with (3.2), proves indeed that $\phi_2(x,t)$ *has always a root on* $]-1,1[$. This conclusion still holds when ξ_0 does not exist, since then $a(x)c(x)$ is positive and $\phi_1(x,t), \phi_2(x,t)$ are real $\forall x \in \mathbb{R}$.

So, when degree of $D(x,t) = 2$, there are two real roots inside the unit disk, harmoniously distributed between $\phi_1(x,t)$ and $\phi_2(x,t)$.

In addition, we note that, when degree of $D(x,t) \geqslant 3$,

$$\lim_{x\to+\infty} \phi_1(x,t) = +\infty, \tag{3.3}$$

in which case $\phi_1(x,t)$ has a root on $]1,\infty[$ and a root on $[0,1[$.

Degree of $\mathbf{D(x,t) = 3}$, i.e., $\mathbf{d_4(t) = p_{1,0}^2 - 4p_{1,1}p_{1,-1} = 0}$. Then two roots of $D(x,t)$ come from $\phi_1(x,t)$, and the third one from $\phi_2(x,t)$.

Degree of $\mathbf{D(x,t) = 4}$, i.e. $\mathbf{d_4(t) \neq 0}$. Letting $|x| \to \infty$, there are two possibilities.

(i) $d_4(t) > 0$. Then $\lim_{x\to\infty} \phi_2(x,t) = +\infty$, and $\phi_2(x,t)$ has a positive real root on $]1,+\infty[$.

(ii) $d_4(t) < 0$. Let ξ_1 denote the smallest finite real root of $a(x)c(x)$, when it exists. Then

$$a(x)c(x \geqslant 0, \ \forall x \in]-\infty, \xi_1], \ \phi_2(\xi_1,t) = b(\xi_1,t) > 0,$$

and, since $\lim_{x\to-\infty} \phi_2(x,t) = -\infty$, it follows that $\phi_2(x,t)$ has a negative real root on $]-\infty, \xi_1[$. If ξ_1 does not exist, then $\phi_2(x,t)$ has a root on $]-\infty,-1]$.

In conclusion, the distribution of the roots of $D(x,t)$ between $\phi_1(x,t)$ and $\phi_2(x,t)$ implies that a double root exists if, and only if,

$$\phi_1(x,t) = \phi_2(x,t) = 0 \Longleftrightarrow b(x,t) = a(x)c(x) = 0. \tag{3.4}$$

But the roots of $b(x,t)$ are real positive, while those of $a(x)$ and $c(x)$ have negative real parts. Hence (3.4) can only take place if $x = 0$ or $x = \infty$.

The proof of the theorem is terminated. ■

4 Classification of the singular random walks

Definition 4.1. A random walk is called *singular* (see [3]) if the associated polynomial $K(x,y,t)$ is either reducible or of degree 1 in at least one of the variables.

We establish a useful lemma, which is of an algebraic nature and gives conditions for the factorization of the kernel $K(x,y,t)$.

Lemma 4.2. *Let $\mathcal{A}[x]$ be the algebra of polynomials in x with coefficients in an arbitrary field $\mathcal{A}$ containing the rational numbers $\mathbb{Q}$. For given $a,b,c \in \mathcal{A}[x]$ satisfying*

$$b^2 - 4ac = kp^2, \tag{4.1}$$

where p is a polynomial and $k \in \mathcal{A}$, there exist $\alpha, \beta, f_1, f_2 \in \mathcal{B}[x]$, such that

$$ay^2 + by + c = (f_1 y - \alpha)(f_2 y - \beta), \tag{4.2}$$

where $\mathcal{B} \stackrel{\text{def}}{=} \mathcal{A} + \sqrt{k}$ denotes the field $\mathcal{A}$ to which is added the element $\sqrt{k}$. In addition, the relation (4.1) *is necessary to have the factorization* (4.2).

Proof. Let Z_1, Z_2 be the roots of

$$Z^2 + bZ + ac = 0, \tag{4.3}$$

so that, with an obvious notation,

$$Z_{1,2} = -\frac{b}{2} \pm \frac{p}{2}\sqrt{k}.$$

In $\mathcal{B}[x]$, let f_1 be the greatest common divisor (g.c.d) of Z_1 and a, so that

$$Z_1 = \beta f_1 \quad \text{and } a = f_1 f_2,$$

where β and f_2 are relatively prime. Then, from $Z_1 Z_2 = ac$, we get $\beta Z_2 = c f_2$. Hence, the g.c.d. of β and f_2 being the unit element, f_2 divides Z_2 and we shall put $Z_2 = \alpha f_2$.

Setting for a while $Z = ay$, the announced factorization (4.2) follows directly from the chain of equalities

$$Z^2 + bZ + ac = (Z - Z_1)(Z - Z_2) = (ay - \beta f_1)(ay - \alpha f_2) = a(f_2 y - \beta)(f_1 y - \alpha) \quad (4.4)$$

together with the identity

$$Z^2 + bZ + ac = a(ay^2 + by + c).$$

As for the necessity of (4.1) to have (4.2), we therefore assume

$$ay^2 + by + c = (f_1 y - \alpha)(f_2 y - \beta).$$

Letting $Z_1 \stackrel{\text{def}}{=} \alpha f_2, Z_2 \stackrel{\text{def}}{=} \beta f_1$, one sees that Z_1, Z_2 are the respective roots of (4.3), which both belong to $\in \mathcal{B}[x]$ and satisfy

$$(Z_1 - Z_2)^2 = (Z_1 + Z_2)^2 - 4Z_1 Z_2 = b^2 - 4ac. \quad (4.5)$$

On the other hand, as part of the hypothesis, we can set $Z_i \stackrel{\text{def}}{=} u_i + v_i\sqrt{k} \in \mathcal{B}[x]$, for $i = 1, 2$, where $u_i, v_i \in \mathcal{A}[x]$. Then, using the fact that a, b, c belong to $\mathcal{A}[x]$, together with the relations

$$\alpha f_2 + \beta f_1 = -b, \quad \text{and} \quad \alpha\beta f_1 f_2 = ac,$$

we get immediately

$$u_1 = u_2 = -b/2, \quad v_1 + v_2 = 0,$$

Hence

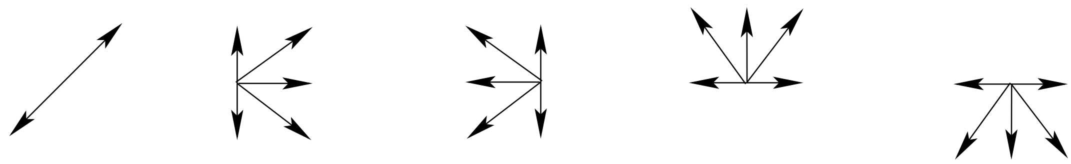
$$(Z_1 - Z_2)^2 = [2v_1\sqrt{k}]^2 = 4kv^2,$$

which, comparing with (4.5), yields exactly (4.1).

The proof of the lemma is terminated. ■

Remark 4.3. It is not difficult to see that in equation (4.1) there exists a version of p belonging to $\mathcal{A}[x]$.

Corollary 4.4. *Take $\mathcal{A} = \mathbb{R}$ in Lemma 4.2. Then, whenever $k > 0$ in equation* (4.1)*, there exists a factorization* (4.2) *of $D(x, t)$ over $\mathbb{R}[x, y]$. On the other hand, if $k < 0$, then any factorization is over $\mathbb{C}[x, y]$.*



a *b* *c* *d* *e*

Fig. 4.1: Singular random walks.

Proof. Assume $k > 0$ in Lemma 4.2. Then $\sqrt{k} \in \mathbb{R}$ and $\mathcal{B} = \mathcal{A} = \mathbb{R}$. ■

The classification of the family of random walks under study is given by the next theorem.

Theorem 4.5. *For $t \in]0,1]$ and $p_{i,j} \in [0,1], p_{0,0} \neq 1$, the random walk is singular if, and only if, one of the following conditions holds:*

(i) There exists $(i,j) \in \mathbb{Z}^2$, $|i| \leqslant 1$, $|j| \leqslant 1$, such that only $p_{i,j}$ and $p_{-i,-j}$ are different from 0 (see figure 4.1a and the three cases obtained by rotation);

(ii) There exists i, $|i| = 1$, such that for any j, $|j| \leqslant 1$, $p_{i,j} = 0$ (see figure 4.1b,c);

(iii) There exists j, $|j| = 1$, such that for any i, $|i| \leqslant 1$, $p_{i,j} = 0$ (see figure 4.1d,e).

Proof. Let us first eliminate three simple situations.

Case $\mathbf{a \equiv 0}$. Then $p_{1,1} = p_{0,1} = p_{-1,1} = 0$. The kernel is of degree 1 in y and this corresponds to the walks of type e in figure 4.1 and

$$K(x,y,t) = t(by + c).$$

Case $\mathbf{c \equiv 0}$. Then $p_{1,-1} = p_{0,-1} = p_{-1,-1} = 0$, giving the walks of type d in figure 4.1 and

$$K(x,y,t) = ty(ay + b).$$

By exchanging the variables x and y, and writing the kernel as a polynomial in x with coefficients $\widetilde{a}, \widetilde{b}, \widetilde{c}$ (see (2.5)), the cases $\widetilde{a} \equiv 0$ or $\widetilde{c} \equiv 0$ depict the walks of respective types c and b in figure (4.1).

From now on, we shall assume $ac \not\equiv 0$.

Case $\mathbf{K(x,y,t) = u(x,t)v(x,y,t)}$. Here u (resp. v) stands for a polynomial in x (resp. (x,y). It is immediate to see that the degree of $u(x,t)$ in x is 1. But this means that a, b, c have a common real root in x, which can occur only at $x = 0$. Indeed, by (2.7), it appears that $b(x,t)$ has always two positive real roots, while the roots of a and c cannot be positive. Thus the only possibility is $p_{-1,1} = p_{-1,0} = p_{-1,-1} = 0$, or equivalently $\widetilde{c} \equiv 0$, which corresponds to the walks of type b in figure (4.1). Hence,

$$\widetilde{c}(y) \equiv 0 \iff K(x,y,t) = tx[x\widetilde{a}(y) + \widetilde{b}(y,t)].$$

Similarly, by exchanging the roles of x and y, we obtain the walks of type d in figure (4.1). In particular,

$$c(x) \equiv 0 \iff K(x,y,t) = ty[ya(x) + b(x,t)].$$

Under Definition 4.1, all singular random walks satisfy (4.2) together with the system

$$\begin{aligned} \alpha f_2 + \beta f_1 &= -b, && (4.6)\\ f_1 f_2 &= a, && (4.7)\\ \alpha\beta &= c, && (4.8) \end{aligned}$$

where α, β, f_1, f_2 are polynomials of degree at most 2 in x, with real coefficients by Corollary 4.4. In the course of the proof, we shall omit the variables (x,t) for the sake of readability.

When a and c are positive (see Definition 2.1), (4.7) and (4.8) imply that (f_1, f_2) are both either positive or negative, and likwise for (α, β). This is true because a and c are of degree at most 2, and we shall choose (f_1, f_2) positive.

Recalling that $b \equiv b(x,t) = p_{-1,0} + (p_{0,0} - 1/t)x + p_{1,0}x^2$, the identity (4.6) leads to consider two different situations.

(i) $p_{1,0} + p_{-1,0} \neq 0$. Here it is impossible to satisfy (4.6) for $t \in]0,1]$, since the three coefficients of the polynomial b do not have the same sign.

(ii) $p_{1,0} = p_{-1,0} = 0$. Then (4.6) can hold only if (α, β) are positive and if $(\alpha f_2, \beta f_1)$ are proportional to x. Upon combining (4.7)-(4.8), we obtain

$$(\alpha f_2)(\beta f_1) = ac = Kx^2,$$

where K is a constant. Thus the coefficients of x^4, x^3, x, x^0 of the polynomial $a(x)c(x)$ must cancel out, yielding from (2.7) the respective equations

$$\begin{cases} p_{1,1}p_{1,-1} = 0,\\ p_{1,1}p_{0,-1} + p_{1,-1}p_{0,1} = 0,\\ p_{0,1}p_{-1,-1} + p_{-1,1}p_{0,-1} = 0,\\ p_{-1,1}p_{0,-1} = 0. \end{cases} \qquad (4.9)$$

Now it is easy to check that the puzzle-like system (4.9) generates exactly the 4 following random walks:

$$\begin{cases} p_{1,-1} = p_{0,-1} = p_{-1,1} = p_{0,1} = 0, & \text{figure } (4.1,a);\\ p_{1,1} = p_{0,1} = p_{-1,-1} = p_{0,-1} = 0, & \text{figure } (4.1,a) \text{ rotated by } 90°;\\ p_{1,1} = p_{1,-1} = p_{-1,-1} = p_{-1,1} = 0, & \text{figure } (4.1,a) \text{ rotated by } 45°;\\ p_{1,1} = p_{0,1} = p_{-1,1} = p_{0,-1} = 0, & \text{sub-case of figure } (4.1,e); \end{cases} \qquad (4.10)$$

By exchanging the roles of x and y, we also obtain the singular walk where only $p_{1,0}$ and $p_{-1,0}$ are different from zero.

The proof of the theorem is terminated. ∎

Remark 4.6. When $p_{0,0} = 0$, one verifies easily that Theorem 4.5 holds in fact for all $t \in]0, +\infty[$.

5 Kernel of genus 0

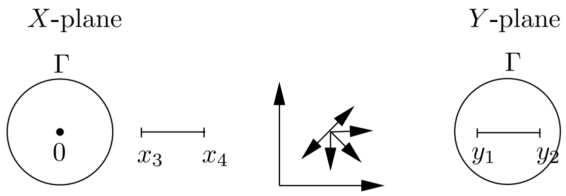
It is wellknown that, for all non-singular random walks, the Riemann surface $\mathbf{S}$ defined by the algebraic curve (2.4) has genus 0 or 1. Morever, the genus of $\mathbf{S}$ is 0 if, and only if, the discriminant $D(x,t)$ defined in (2.8) has a multiple zero in x, possibly infinite.

For the sake of completeness, we also introduce the branch points x_i (resp. y_i), $i = 1, \ldots, 4$, of the algebraic function $Y(x,t)$ satisfying $K(x, Y(x,t), t) = 0$ [resp. $X(y,t)$ satisfying $K(X(y,t), y, t) = 0$]. These branch points depend of course on t.

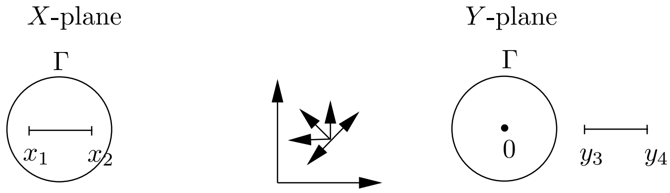
In the book [3, Section 2.3], the classification in the so-called stationary case $t = 1$ has been dealt with in great detail. The proof was heavily using the principle of *continuity of the roots with respect to the parameters* $p_{i,j}$, thus reducing the analysis to the case of the so-called *simple random walk*, i.e. when only $p_{1,0}, p_{0,1}, p_{-1,0}, p_{0,-1}$ are different from 0. Indeed, for $t \in]0, 1[$, exactly the same arguments hold. Nonetheless, we present hereafter a self-contained proof. The following theorem holds.

Theorem 5.1. *For all non singular random walks and for all $t \in]0,1[$, the algebraic curve defined by $K(x,y,t) = 0$ has genus 0 if, and only if, $D(x,t)$ has a double root at $x = 0$ or $x = \infty$. That corresponds to the following pictures.*

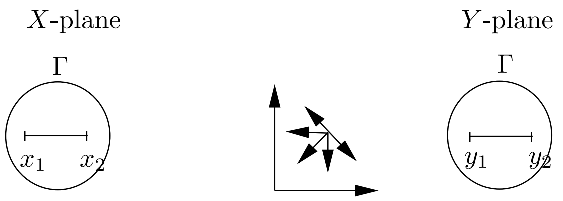
1. $\begin{cases} x_1 = x_2 = 0 \\ y_3 = y_4 = \infty. \end{cases}$



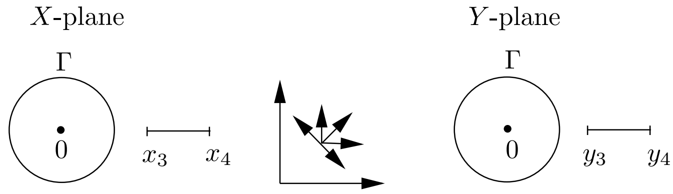
2. $\begin{cases} y_1 = y_2 = 0 \\ x_3 = x_4 = \infty. \end{cases}$



3. $\begin{cases} x_3 = x_4 = \infty \\ y_3 = y_4 = \infty. \end{cases}$



4. $\begin{cases} x_1 = x_2 = 0 \\ y_1 = y_2 = 0. \end{cases}$



In other words, multiple roots of $D(x,t)$ can occur only in either of the two following situations:

(a) $x = 0$, in which case $d_0(t) = d_1(t) = 0$;

(b) $x = \infty$, in which case $d_3(t) = d_4(t) = 0$.

Proof. Let us recall that, since $d_2(t) > 0$, the degree of $D(x,t)$ with respect to x is always $\geqslant 2$. From the outset, we shall solve the case where this degree is exactly 2.

Degree of $\mathbf{D(x,t) = 2}$. This means that $D(x,t$ has a double root at $x = \infty$, whence $d_3(t) = d_4(t) = 0$ and the genus of the kernel is 0 for all non singular random walks. Then, by using (2.10) and remarking that $d_3(t) \leqslant 0$, we get the system

$$p_{1,0} = p_{1,1}p_{0,-1} = p_{0,1}p_{1,-1} = p_{1,1}p_{1,-1} = 0. \tag{5.1}$$

It is now easy to check that the set of relations (5.1), without taking into account the singular random walk $p_{1,0} = p_{1,1}, = p_{1,-1} = 0$, is tantamount to the pictures *2.* and *3.*

Degree of $\mathbf{D(x,t) \geqslant 3}$. We have proved in Theorem 3.1 that a double root can only occur at $x = 0$. Then $d_0(t) = d_1(t) = 0$, whence by using (2.10)

$$p_{-1,0} = p_{-1,1}p_{0,-1} = p_{0,1}p_{-1,-1} = p_{-1,1}p_{-1,-1} = 0,$$

which corresponds to the pictures *1.* and *4.*

The proof of the theorem is terminated. ■

Remark 5.2. When $D(x,t)$ has a double root both at $x = 0$ and $x = \infty$, we know from Lemma 4.2 that a factorization of the form (4.2) holds, and the walk is singular. On the other hand, it is worth noting that, for all $t \in]0,1[$, the discriminant $D(x,t)$ can never have double root at $x = 1$. This case can occur for $t = 1$ and corresponds random walks having zero drift vectors (see [3, Section 2.3]).

Remark 5.3. The approach proposed in this paper applies verbatim to functional equations pertaining to the transient distribution of two-dimensional Markov processes, such as those encountered in queueing systems (see e.g.,[3, Section 8.4]).

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