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STATE-CONSTRAINED CONTROL-AFFINE PARABOLIC PROBLEMS II: SECOND ORDER SUFFICIENT OPTIMALITY CONDITIONS*

M. SOLEDAD ARONNA†, J. FRÉDÉRIC BONNANS‡, AND AXEL KRÖNER§

Abstract. In this paper we consider an optimal control problem governed by a semilinear heat equation with bilinear control-state terms and subject to control and state constraints. The state constraints are of integral type, the integral being with respect to the space variable. The control is multidimensional. The cost functional is of a tracking type and contains a linear term in the control variables. We derive second order sufficient conditions relying on the Goh transform.

Key words. Optimal control of partial differential equations, semilinear parabolic equations, state constraints, second order analysis, Goh transform, control-affine problems

AMS subject classifications. 49J20, 49K20, 35K58

1. Introduction.

This is the second part of two papers on necessary and sufficient optimality conditions for an optimal control problem governed by a semilinear heat equation containing bilinear terms coupling the control variables and the state, and subject to constraints on the control and state. While in the first part [5], first and second order necessary optimality conditions are shown, in this second part we derive second order sufficient optimality conditions. The control may have several components and enters the dynamics in a bilinear term and in an affine way in the cost. This does not allow to apply classical techniques of calculus of variations to derive second order sufficient optimality conditions. Therefore, we extend techniques that were recently established in the following articles, and that involve the Goh transform [12] in an essential way. Aronna, Bonnans, Dmitruk and Lotito [1] obtained second order necessary and sufficient conditions for bang-singular solutions of control-affine finite dimensional systems with control bounds, results that were extended in Aronna, Bonnans and Goh [2] when adding a state constraint of inequality type. An extension of the analysis in [1] to the infinite dimensional setting was done by Bonnans [6], for a problem concerning a semilinear heat equation subject to control bounds and without state constraints. For a quite general class of linear differential equations in Banach spaces with bilinear control-state couplings and subject to control bounds, Aronna, Bonnans and Kröner [3] provided second order conditions, that extended later to the complex Banach space setting [4].

There exists a series of publications on second order conditions for problems governed by control-affine ordinary differential equations, we refer to references in [5].

In the elliptic framework, regarding the case we investigate here, this is, when no quadratic control term is present in the cost (or what some authors call *vanishing Tikhonov term*), Casas in [7] proved second order sufficient conditions for bang-bang optimal controls of a semilinear equation, and for one containing a bilinear coupling of control and state in the recent joint work with D. and G. Wachsmuth [10].

Parabolic optimal control problems with state constraints are discussed in Rösch

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and Tröltzsch [16], who gave second order sufficient conditions for a linear equation with mixed control-state constraints. In the presence of pure-state constraints, Raymond and Tröltzsch [15], and Krumbiegel and Rehberg [13] obtained second order sufficient conditions for a semilinear equation, Casas, de Los Reyes, and Tröltzsch [8] and de Los Reyes, Merino, Rehberg and Tröltzsch [11] obtained sufficient second order conditions for semilinear equations, both in the elliptic and parabolic cases. The articles mentioned in this paragraph did not consider bilinear terms, and their sufficient conditions do not apply to the control-affine problems that we treat in the current work.

It is also worth mentioning the work [9] by Casas, Ryll and Tröltzsch that provided second order conditions for a semilinear FitzHugh-Nagumo system subject to control constraints in the case of vanishing Tikhonov term.

The contribution of this paper are second order sufficient optimality conditions for an optimal control problem for a semilinear parabolic equation with cubic nonlinearity, several controls coupled with the state variable through bilinear terms, pointwise control constraints and state constraints that are integral in space. The main challenge arises from the fact that both the dynamics and the cost function are affine with respect to the control, hence classical techniques are not applicable to derive second order sufficient conditions. We rely on the Goh transform [12] to derive sufficient optimality conditions for bang-singular solutions. In particular, the sufficient conditions are stated on a cone of directions larger than the one used for the necessary conditions.

The paper is organized as follows. In Section 2 the problem is stated and main assumptions are formulated. Section 3 is devoted to second order necessary conditions and Section 4 to second order sufficient conditions.

Notation. Let Ω be an open subset of $\mathbb{R}^n$, $n \leq 3$, with C^∞ boundary $\partial\Omega$. Given $p \in [1,\infty]$ and $k \in \mathbb{N}$, let $W^{k,p}(\Omega)$ be the Sobolev space of functions in $L^p(\Omega)$ with derivatives (here and after, derivatives w.r.t. $x \in \Omega$ or w.r.t. time are taken in the sense of distributions) in $L^p(\Omega)$ up to order k. Let $\mathcal{D}(\Omega)$ be the set of C^∞ functions with compact support in Ω. By $W_0^{k,p}(\Omega)$ we denote the closure of $\mathcal{D}(\Omega)$ with respect to the $W^{k,p}$-topology. Given a horizon $T > 0$, we write $Q := \Omega \times (0,T)$. $\|\cdot\|_p$ denotes the norm in $L^p(0,T)$, $L^p(\Omega)$ and $L^p(Q)$, indistinctly. When a function depends on both space and time, but the norm is computed only with respect of one of these variables, we specify both the space and domain. For example, if $y \in L^p(Q)$ and we fix $t \in (0,T)$, we write $\|y(\cdot,t)\|_{L^p(\Omega)}$. For the p-norm in $\mathbb{R}^m$, for $m \in \mathbb{N}$, we use $|\cdot|_p$. We set $H^k(\Omega) := W^{k,2}(\Omega)$ and $H_0^k(\Omega) := W_0^{k,2}(\Omega)$. By $W^{2,1,p}(Q)$ we mean the Sobolev space of $L^p(Q)$-functions whose second derivative in space and first derivative in time belong to $L^p(Q)$. We write $H^{2,1}(Q)$ for $W^{2,1,2}(Q)$ and, setting $\Sigma := \partial\Omega \times (0,T)$, we define the state space as

$$Y := \{y \in H^{2,1}(Q);\ y = 0 \text{ a.e. on } \Sigma\}. \tag{1.1}$$

If y is a function over Q, we use $\dot{y}$ to denote its time derivative in the sense of distributions. As usual we denote the spatial gradient and the Laplacian by ∇ and Δ. By $\operatorname{dist}(t,I) := \inf\{\|t-\bar{t}\|\ ;\ \bar{t} \in I\}$ for $I \subset \mathbb{R}$, we denote the distance of t to the set I.

2. Statement of the problem and main assumptions. In this section we introduce the optimal control problem and recall results on well-posedness of the state equation and existence of solutions of the optimal control problem from [5].

2.1. Setting. The *state equation* is given as

$$\begin{cases} \dot{y}(x,t) - \Delta y(x,t) + \gamma y^3(x,t) = f(x,t) + y(x,t)\displaystyle\sum_{i=0}^{m} u_i(t)b_i(x) \quad \text{in } Q, \\ y = 0 \;\text{ on } \Sigma, \quad y(\cdot,0) = y_0 \text{ in } \Omega, \end{cases} \tag{2.1}$$

with

$$y_0 \in H_0^1(\Omega), \quad f \in L^2(Q), \quad b \in W^{1,\infty}(\Omega)^{m+1}, \tag{2.2}$$

$\gamma \geq 0$, $u_0 \equiv 1$ is a constant, and $u := (u_1, \dots, u_m) \in L^2(0,T)^m$. Lemma A.1 below shows that for each control $u \in L^2(0,T)^m$, there is a unique associated solution $y \in Y$ of (2.1), called the *associated state*. Let $y[u]$ denote this solution. We consider control constraints of the form $u \in \mathcal{U}_{\rm ad}$, where

$$\mathcal{U}_{\rm ad} = \{u \in L^2(0,T)^m;\ \check{u}_i \leq u_i(t) \leq \hat{u}_i,\ i = 1, \dots, m\}, \tag{2.3}$$

for some constants $\check{u}_i < \hat{u}_i$, for $i = 1, \dots, m$. In addition, we have finitely many linear running state constraints of the form

$$g_j(y(\cdot,t)) := \int_\Omega c_j(x)y(x,t)\mathrm{d}x + d_j \leq 0, \quad \text{for } t \in [0,T], \;\; j = 1, \dots, q, \tag{2.4}$$

where $c_j \in H^2(\Omega) \cap H_0^1(\Omega)$ for $j = 1, \dots, q$, and $d \in \mathbb{R}^q$.

We call any $(u, y[u]) \in L^2(0,T)^m \times Y$ a *trajectory*, and if it additionally satisfies the control and state constraints, we say it is an *admissible trajectory.* The *cost function* is

$$\begin{aligned} J(u,y) :=& \tfrac{1}{2}\int_Q (y(x,t) - y_d(x))^2 \mathrm{d}x\mathrm{d}t \\ &+ \tfrac{1}{2}\int_\Omega (y(x,T) - y_{dT}(x))^2 \mathrm{d}x + \sum_{i=1}^m \alpha_i \int_0^T u_i(t)\mathrm{d}t, \end{aligned} \tag{2.5}$$

where

$$y_d \in L^2(Q), \quad y_{dT} \in H_0^1(\Omega), \tag{2.6}$$

and $\alpha \in \mathbb{R}^m$. We consider the optimal control problem

$$\operatorname*{Min}_{u \in \mathcal{U}_{\rm ad}} J(u, y[u]); \quad \text{subject to (2.4).} \tag{P}$$

For problem (P), assuming it in the sequel to be feasible, we consider two types of solution.

Definition 2.1. *We say that* $(\bar{u}, y[\bar{u}])$ *is an* L^2-local solution *(resp.,* L^∞-local solution*) if there exists* $\varepsilon > 0$ *such that* $(\bar{u}, y[\bar{u}])$ *is a minimum among the admissible trajectories* (u,y) *that satisfy* $\|u - \bar{u}\|_2 < \varepsilon$ *(resp.,* $\|u - \bar{u}\|_\infty < \varepsilon$*).*

The state equation is well-posed and has a solution in Y. Furthermore, the mapping $u \mapsto y$, $L^2(0,T) \to Y$ is of class C^∞. Since (P) has a bounded feasible set, it is easily checked that its set of solutions of (P) is non-empty. For details regarding these assertions see Appendix A.

2.2. First order optimality conditions. It is well-known that the dual of $C([0,T])$ is the set of (finite) Radon measures, and that the action of a finite Radon measure coincides with the Stieltjes integral associated with a bounded variation function $\mu \in BV(0,T)$. We may assume w.l.g. that $\mu(T)=0$, and we let $\mathrm{d}\mu$ denote the Radon measure associated to μ. Note that if $\mathrm{d}\mu$ belongs to the set $\mathcal{M}_+(0,T)$ of nonnegative finite Radon measures then we may take μ nondecreasing. Set

$$BV(0,T)_{0,+} := \big\{\mu \in BV(0,T);\ \mu(T)=0;\ \mathrm{d}\mu \geq 0\big\}. \tag{2.7}$$

Let $(\bar{u},\bar{y})$ be an admissible trajectory of problem (P). We say that $\mu \in BV(0,T)^q_{0,+}$ is *complementary to the state constraint* for $\bar{y}$ if

$$\int_0^T g_j(\bar{y}(\cdot,t))\mathrm{d}\mu_j(t) = \int_0^T \left(\int_\Omega c_j(x)\bar{y}(x,t)\mathrm{d}x + d_j\right)\mathrm{d}\mu_j(t) = 0, \quad j=1,\dots,q. \tag{2.8}$$

Let $(\beta,\mu) \in \mathbb{R}_+ \times BV(0,T)^q_{0,+}$. We say that $p \in L^\infty(0,T;H^1_0(\Omega))$ is the *costate associated* with $(\bar{u},\bar{y},\beta,\mu)$, or shortly with (β,μ), if (p,p_0) is solution of (B.6). As explained in appendix B.2, $p = p^1 - \sum_{j=1}^q c_j\mu_j$ for some $p^1 \in Y$. In particular $p(\cdot,0)$ and $p(\cdot,T)$ are well-defined and it can be checked that $p_0 = p(\cdot,0)$.

DEFINITION 2.2. *We say that the triple $(\beta,p,\mu) \in \mathbb{R}_+ \times L^\infty(0,T;H^1_0(\Omega)) \times BV(0,T)^q_{0,+}$ is a* generalized Lagrange multiplier *if it satisfies the following* first-order optimality conditions: *μ is complementary to the state constraint, p is the costate associated with (β,μ), the non-triviality condition $(\beta,\mathrm{d}\mu) \neq 0$ holds and, for $i=1$ to m, defining the* switching function *by*

$$\Psi_i^p(t) := \beta\alpha_i + \int_\Omega b_i(x)\bar{y}(x,t)p(x,t)\mathrm{d}x, \quad \textit{for } i=1,\dots,m, \tag{2.9}$$

one has $\Psi^p \in L^\infty(0,T)^m$ and

$$\sum_{i=1}^m \int_0^T \Psi_i^p(t)(u_i(t)-\bar{u}_i(t))\mathrm{d}t \geq 0, \quad \textit{for every } u \in \mathcal{U}_{\mathrm{ad}}. \tag{2.10}$$

We let $\Lambda(\bar{u},\bar{y})$ denote the set of generalized Lagrange multipliers associated with $(\bar{u},\bar{y})$. If $\beta=0$ we say that the corresponding multiplier is singular. *Finally, we write $\Lambda_1(\bar{u},\bar{y})$ for the set of pairs (p,μ) with $(1,p,\mu) \in \Lambda(\bar{u},\bar{y})$. When the nominal solution is fixed and there is no place for confusion, we just write Λ and Λ_1.*

We recall from [5, Lem. 6(i)] the following statement on first order conditions.

LEMMA 2.3. *If $(\bar{u},y[\bar{u}])$ is an L^2-local solution of (P), then the associated set Λ of multipliers is nonempty.*

Consider the *contact sets associated to the control bounds* defined, up to null measure sets, by $\check{I}_i := \{t \in [0,T];\ \bar{u}_i(t) = \check{u}_i\}$, $\hat{I}_i := \{t \in [0,T];\ \bar{u}_i(t) = \hat{u}_i\}$, $I_i := \check{I}_i \cup \hat{I}_i$. For $j=1,\dots,q$, the *contact set associated with the jth state constraint* is $I_j^C := \{t \in [0,T];\ g_j(\bar{y}(\cdot,t)) = 0\}$. Given $0 \leq a < b \leq T$, we say that (a,b) is a *maximal state constrained arc* for the jth state constraints, if I_j^C contains (a,b) but it contains no open interval strictly containing (a,b). We define in the same way a *maximal (lower or upper) control bound constraints arc* (having in mind that the latter are defined up to a null measure set).

We will assume the following *finite arc property:*

$$\left\{ \begin{array}{c} \text{the contact sets for the state and bound constraints are,} \\ \text{up to a finite set, the union of finitely many maximal arcs.} \end{array} \right. \tag{2.11}$$

In the sequel we identify $\bar{u}$ (defined up to a null measure set) with a function whose ith component is constant over each interval of time that is included, up to a zero-measure set, in either $\check{I}_i$ or $\hat{I}_i$. For almost all $t \in [0,T]$, the *set of active constraints at time* t is denoted by $(\check{B}(t), \hat{B}(t), C(t))$ where

$$\begin{cases} \check{B}(t) := \{1 \le i \le m;\ \bar{u}_i(t) = \check{u}_i\}, \\ \hat{B}(t) := \{1 \le i \le m;\ \bar{u}_i(t) = \hat{u}_i\}, \\ C(t) := \{1 \le j \le q;\ g_j(\bar{y}(\cdot,t)) = 0\}. \end{cases} \tag{2.12}$$

These sets are well-defined over open subsets of $(0,T)$ where the set of active constraints is constant, and by (2.11), there exist time points called *junction points* $0 =: \tau_0 < \cdots < \tau_r := T$, such that the intervals (τ_k, τ_{k+1}) are *maximal arcs with constant active constraints*, for $k = 0, \ldots, r-1$. We may sometimes call them shortly *maximal arcs.* For $m = 1$ we call junction points where a *BB junction* if we have active bound constraints on both neighbouring maximal arcs, a *CB junction* (resp. *BC junction*) if we have a state constrained arc and an active bound constrained arc.

DEFINITION 2.4. *For $k = 0, \ldots, r-1$, let $\check{B}_k, \hat{B}_k, C_k$ denote the set of indexes of active lower and upper bound constraints, and state constraints, on the maximal arc (τ_k, τ_{k+1}), and set $B_k := \check{B}_k \cup \hat{B}_k$.*

In the discussion that follows we fix k in $\{0, \ldots, r-1\}$, and consider a maximal arc (τ_k, τ_{k+1}), where the junction points. Set $\bar{B}_k := \{1, \ldots, m\} \setminus B_k$ and

$$M_{ij}(t) := \int_\Omega b_i(x) c_j(x) \bar{y}(x,t) \mathrm{d}x, \quad 1 \le i \le m,\ \ 1 \le j \le q. \tag{2.13}$$

Let $\bar{M}_k(t)$ (of size $|\bar{B}_k| \times |C_k|$) denote the submatrix of $M(t)$ having rows with index in $\bar{B}_k$ and columns with index in C_k. In the sequel we make the following assumption.

Hypothesis 2.5. We assume that $|C_k| \le |\bar{B}_k|$, for $k = 0, \ldots, r-1$, and that the following *(uniform) local controllability condition* holds:

$$\begin{cases} \text{there exists } \alpha > 0, \text{ such that } |\bar{M}_k(t)\lambda| \ge \alpha |\lambda|, \\ \text{for all } \lambda \in \mathbb{R}^{|C_k|}, \text{ a.e. on } (\tau_k, \tau_{k+1}), \text{ for } k = 0, \ldots, r-1. \end{cases} \tag{2.14}$$

3. Second order necessary conditions.

We start this section by recalling some results obtained in [5], the main one being the second order necessary condition of Theorem 3.5. We then introduce the *Goh transform* and apply it to the quadratic form and the critical cone, and then obtain necessary conditions on the transformed objects (see Theorem 3.12). We show later in Section 4 that these necessary conditions can be strengthened to get sufficient conditions for optimality (see Theorem 4.5).

Let us consider an admissible trajectory $(\bar{u}, \bar{y})$.

3.1. Assumptions and additional regularity.

For the remainder of the article we make the following set of assumptions.

Hypothesis 3.1. The following conditions hold:

1. the finite maximal arc property (2.11),
2. the problem is qualified (cf. also [5, Sec. 3.2.1])

$$\begin{cases} \text{there exists } \varepsilon > 0 \text{ and } u \in \mathcal{U}_{\mathrm{ad}} \text{ such that } v := u - \bar{u} \text{ satisfies} \\ g_j(\bar{y}(\cdot,t)) + g_j'(\bar{y}(\cdot,t)) z[v](\cdot,t) < -\varepsilon, \text{ for all } t \in [0,T], \text{ and } j = 1, \ldots, q. \end{cases} \tag{3.1}$$

3. the local (uniform) controllability condition (2.14) over each maximal arc (τ_k, τ_{k+1}),
4. the discontinuity of the derivative of the state constraints at corresponding junction points, i.e.,

$$\text{for some } c > 0\text{: } g_j(\bar{y}(\cdot,t)) \leq -c\,\mathrm{dist}(t, I_j^C), \text{ for all } t \in [0,T],\ j = 1,\ldots,q, \tag{3.2}$$

5. the uniform distance to control bounds whenever they are not active, i.e. there exists $\delta > 0$ such that,

$$\mathrm{dist}\left(\bar{u}_i(t), \{\check{u}_i, \hat{u}_i\}\right) \geq \delta, \quad \text{for a.a. } t \notin I_i, \text{ for all } i = 1,\ldots,m, \tag{3.3}$$

6. the following regularity for the data (we do not try to take the weakest hypotheses):

$$y_0, y_{dT} \in W_0^{1,\infty}(\Omega), \quad y_d, f \in L^\infty(Q), \quad b \in W^{2,\infty}(\Omega)^{m+1}, \tag{3.4}$$

7. the control $\bar{u}$ has left and right limits at the junction points $\tau_k \in (0,T)$, (this will allow to apply [5, Lem. 3.8]).

Remark 3.2. Hypotheses 3.1 4 and 5 are instrumental for constructing feasible perturbations of the nominal trajectory, used in the proof of Theorem 3.5 made in [5].

In view of point 3 above, we consider from now on $\beta = 1$ and thus we omit the component β of the multipliers.

THEOREM 3.3. *The following assertions hold.*

(i) For any $u \in L^\infty(0,T)^m$, the associated state $y[u]$ belongs to $C(\bar{Q})$. If u remains in a bounded subset of $L^\infty(0,T)^m$ then the corresponding states form a bounded set in $C(\bar{Q})$. In addition, if the sequence (u_ℓ) of admissible controls converges to $\bar{u}$ a.e. on $(0,T)$, then the associated sequence of states $(y_\ell := y[u_\ell])$ converges uniformly to $\bar{y}$ in $\bar{Q}$.

(ii) The set Λ_1 is nonempty and for every $(p, \mathrm{d}\mu) \in \Lambda_1$, one has that $\mu \in W^{1,\infty}(0,T)^q$ and p is essentially bounded in Q.

Proof. We refer to [5, Thm. 4.2]. Note that the non-emptiness of Λ_1 follows from (3.1). ∎

3.2. Second variation. For $(p,\mu) \in \Lambda_1$, set $\kappa(x,t) := 1 - 6\gamma\bar{y}(x,t)p(x,t)$, and consider the quadratic form

$$\mathcal{Q}[p](z,v) := \int_Q \left(\kappa z^2 + 2p\sum_{i=1}^m v_i b_i z\right)\mathrm{d}x\mathrm{d}t + \int_\Omega z(x,T)^2\mathrm{d}x. \tag{3.5}$$

Let (u,y) be a trajectory, and set

$$(\delta y, v) := (y - \bar{y}, u - \bar{u}). \tag{3.6}$$

Recall the definition of the operator A given in (B.1). Subtracting the state equation at $(\bar{u},\bar{y})$ from the one at (u,y), we get that

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\delta y + A\delta y = \displaystyle\sum_{i=1}^m v_i b_i y - 3\gamma\bar{y}(\delta y)^2 - \gamma(\delta y)^3 & \text{in } Q, \\ \delta y = 0 \quad \text{on } \Sigma, \quad \delta y(\cdot,0) = 0 & \text{in } \Omega. \end{cases} \tag{3.7}$$

Combining with the linearized state equation (B.2), we deduce that η given by $\eta := \delta y - z$, satisfies the equation

$$\begin{cases} \dot\eta - \Delta\eta = r\eta + \tilde r \quad \text{in } Q, \\ \eta = 0 \quad \text{on } \Sigma, \quad \eta(\cdot,0) = 0 \quad \text{in } \Omega \end{cases} \tag{3.8}$$

where r and $\tilde r$ are defined as

$$r := -3\gamma\bar y^2 + \sum_{i=0}^m \bar u_i b_i, \qquad \tilde r := \sum_{i=1}^m v_i b_i \delta y - 3\gamma\bar y(\delta y)^2 - \gamma(\delta y)^3. \tag{3.9}$$

See the definition of the Lagrangian function $\mathcal{L}$ given in equation (B.5) of the Appendix.

PROPOSITION 3.4. *Let $(p,\mu) \in \Lambda_1$, and let (u,y) be a trajectory. Then*

$$\begin{aligned} &\mathcal{L}[p,\mu](u,y,p) - \mathcal{L}[p,\mu](\bar u,\bar y,p) \\ &\qquad = \int_0^T \Psi^p(t)\cdot v(t)\mathrm{d}t + \tfrac12 \mathcal{Q}[p](\delta y, v) - \gamma\int_Q p(\delta y)^3 \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.10}$$

Here, we omit the dependence of the Lagrangian on (β, p_0) being equal to $(1, p(\cdot,0))$.

Proof. We refer to [5, Prop. 4.3]. ∎

3.3. Critical directions. Recall the definitions of $\check I_i, \hat I_i$ and I_j^C given in Section 2.2, and remember that we use $z[v]$ to denote the solution of the linearized state equation (B.2) associated to v.

We define the *cone of critical directions* at $\bar u$ in L^2, or in short *critical cone,* by

$$C := \left\{ \begin{array}{l} (z[v],v) \in Y \times L^2(0,T)^m; \\ v_i(t)\Psi_i^p(t) = 0 \text{ a.e. on } [0,T], \text{ for all } (p,\mu) \in \Lambda_1 \\ v_i(t) \ge 0 \text{ a.e. on } \check I_i,\ v_i(t) \le 0 \text{ a.e. on } \hat I_i, \text{ for } i = 1,\dots,m, \\ \displaystyle\int_\Omega c_j(x) z[v](x,t)\mathrm{d}x \le 0 \text{ on } I_j^C, \text{ for } j = 1,\dots,q \end{array} \right\}. \tag{3.11}$$

The *strict critical cone* is defined below, and it is obtained by imposing that the linearization of active constraints is zero,

$$C_{\mathrm{s}} := \left\{ \begin{array}{l} (z[v],v) \in Y \times L^2(0,T)^m;\ v_i(t) = 0 \text{ a.e. on } I_i, \text{ for } i = 1,\dots,m, \\ \displaystyle\int_\Omega c_j(x) z[v](x,t)\mathrm{d}x = 0 \text{ on } I_j^C, \text{ for } j = 1,\dots,q \end{array} \right\}. \tag{3.12}$$

Hence, clearly $C_{\mathrm{s}} \subseteq C$, and C_{s} is a closed subspace of $Y \times L^2(0,T)^m$.

3.4. Second order necessary condition. We recall from [5, Thm. 4.7].

THEOREM 3.5 (Second order necessary condition). *Let the admissible trajectory $(\bar u, \bar y)$ be an L^∞-local solution of (P). Then*

$$\max_{(p,\mu)\in\Lambda_1} \mathcal{Q}[p](z,v) \ge 0, \qquad \textit{for all } (z,v) \in C_{\mathrm{s}}. \tag{3.13}$$

3.5. Goh transform. Given a critical direction $(z[v], v)$, set

$$w(t) := \int_0^t v(s)\mathrm{d}s; \quad B(x,t) := \bar{y}(x,t)b(x); \quad \zeta(x,t) = z(x,t) - B(x,t)\cdot w(t). \tag{3.14}$$

Then ζ satisfies the initial and boundary conditions

$$\zeta(x,0) = 0 \text{ for } x \in \Omega, \quad \zeta(x,t) = 0 \text{ for } (x,t) \in \Sigma. \tag{3.15}$$

Remembering the definition (B.1) of the operator A, we obtain that

$$\dot{\zeta} + A\zeta = \left(\dot{z} + Az - \sum_{i=1}^m v_i B_i\right) - \sum_{i=1}^m w_i(AB_i + \dot{B}_i), \quad \zeta(\cdot,0) = 0, \quad \zeta(x,t) = 0 \text{ on } \Sigma. \tag{3.16}$$

In view of the linearized state equation (B.2), the term between the large parentheses in the latter equation vanishes. Since $\dot{B}_i = b_i\dot{\bar{y}}$ it follows that

$$\dot{\zeta}(x,t) + (A\zeta)(x,t) = B^1(x,t)\cdot w(t), \quad \zeta(\cdot,0) = 0, \quad \zeta(x,t) = 0 \text{ on } \Sigma, \tag{3.17}$$

where

$$B_i^1 := -fb_i + 2\nabla\bar{y}\cdot\nabla b_i + \bar{y}\Delta b_i - 2\gamma\bar{y}^3 b_i, \quad \text{for } i = 1,\ldots,m. \tag{3.18}$$

Equation (3.17) is well-posed since $b \in W^{2,\infty}(\Omega)$, and the solution ζ belongs to Y. We use $\zeta[w]$ to denote the solution of (3.17) corresponding to w.

3.6. Goh transform of the quadratic form. Recall that $(\bar{u}, \bar{y})$ is a feasible trajectory. Let $\bar{p} = p[\bar{u}]$ be the costate associated to $\bar{u}$, and set

$$W := Y \times L^2(0,T)^m \times \mathbb{R}^m. \tag{3.19}$$

Let $S(t)$ be the time dependent symmetric $m\times m-$matrix with generic term

$$S_{ij}(t) := \int_\Omega b_i(x)b_j(x)p(x,t)\bar{y}(x,t)\mathrm{d}x, \quad \text{for } 1 \le i,j \le m. \tag{3.20}$$

Set

$$\chi := \frac{\mathrm{d}}{\mathrm{d}t}(p\bar{y}) = pf + p\Delta\bar{y} - \bar{y}\Delta p + 2p\bar{y}^3 - \bar{y}(\bar{y} - y_d) - \bar{y}\sum_{j=1}^q c_j\dot{\mu}_j. \tag{3.21}$$

Observe that

$$\dot{S}_{ij}(t) = \int_\Omega b_i b_j \frac{\mathrm{d}}{\mathrm{d}t}(p\bar{y})\mathrm{d}x = \int_\Omega b_i b_j \chi \mathrm{d}x. \tag{3.22}$$

Since $\bar{y}$, p belong to $L^\infty(0,T,H_0^1(\Omega))$, and y_d, $\bar{y}^3$, $\dot{\mu}$ are essentially bounded, integrating by parts the terms in (3.21) involving the Laplacian operator and using (3.4), we obtain that $\dot{S}_{ij}$ is essentially bounded. So we can define the continuous quadratic form on W :

$$\widehat{\mathcal{Q}}[p,\mu](\zeta,w,h) := \int_0^T \hat{q}_I(t)\mathrm{d}t + \hat{q}_T, \tag{3.23}$$

where

$$(3.24)\quad \hat{q}_I := \int_\Omega \kappa \left(\zeta + \bar{y}\sum_{i=1}^m b_i w_i\right)^2 \mathrm{d}x - w^\top \dot{S} w$$
$$- 2\sum_{i=1}^m w_i \int_\Omega \Big[\zeta\big(- \Delta b_i p - 2\nabla b_i \cdot \nabla p + b_i(\bar{y} - y_d) + b_i \sum_{j=1}^q c_j \dot{\mu}_j\big) - pB^1 \cdot w\Big]\mathrm{d}x,$$

and

$$(3.25)\quad \hat{q}_T :=$$
$$\int_\Omega \left[\Big(\zeta(x,T) + \bar{y}(x,T)\sum_{i=1}^m h_i b_i(x)\Big)^2 + 2\sum_{i=1}^m h_i b_i(x) p(x,T)\zeta(x,T)\right]\mathrm{d}x + h^\top S(T)h.$$

LEMMA 3.6 (Transformed second variation). *For $v \in L^2(0,T)^m$, and $w \in AC([0,T])^m$ given by the Goh transform (3.14), and for all $(p,\mu) \in \Lambda_1$, we have*

$$(3.26)\qquad \mathcal{Q}[p](z[v],v) = \widehat{\mathcal{Q}}[p,\mu](\zeta[w],w,w(T)).$$

Proof. We first replace z by $\zeta + B\cdot w = \zeta + \bar{y}\sum_{i=1}^m w_i b_i$ in $\mathcal{Q}$, and define

$$(3.27)\quad \widetilde{\mathcal{Q}} := \int_Q \Big[\kappa(\zeta + \bar{y}\sum_{i=1}^m w_i b_i)^2 + 2p\sum_{i=1}^m v_i b_i(\zeta + \bar{y}\sum_{j=1}^m w_j b_j)\Big]\mathrm{d}x\mathrm{d}t$$
$$+ \int_\Omega \big(\zeta(T) + \bar{y}(T)\sum_{i=1}^m w_i(T) b_i\big)^2 \mathrm{d}x.$$

We aim at proving that $\widetilde{\mathcal{Q}}$ coincides with $\widehat{\mathcal{Q}}$. This will be done removing the dependence on v from the above expression. For this, we have to deal with the bilinear term in $\widetilde{\mathcal{Q}}$, namely with

$$(3.28)\qquad \widetilde{\mathcal{Q}}_b := \widetilde{\mathcal{Q}}_{b,1} + 2\sum_{i=1}^m \widetilde{\mathcal{Q}}_{b,2i},$$

where, omitting the dependence on the multipliers for the sake of simplicity of the presentation,

$$(3.29)\quad \widetilde{\mathcal{Q}}_{b,1} := 2\int_0^T v^\top S w \mathrm{d}t \quad \text{and} \quad \widetilde{\mathcal{Q}}_{b,2i} := \int_0^T v_i \int_\Omega b_i p \zeta \mathrm{d}x\mathrm{d}t, \text{ for } i = 1,\dots,m.$$

Concerning $\widetilde{\mathcal{Q}}_{b,1}$, since S is symmetric, we get, integrating by parts,

$$(3.30)\qquad \widetilde{\mathcal{Q}}_{b,1} = \big[w^\top S w\big]_0^T - \int_0^T w^\top \dot{S} w \mathrm{d}t.$$

Hence $\widetilde{\mathcal{Q}}_{b,1}$ is a function of w and $w(T)$. Concerning $\widetilde{\mathcal{Q}}_{b,2i}$ defined in (3.29), integrating by parts, we get

$$(3.31)\qquad \widetilde{\mathcal{Q}}_{2,bi} = w_i(T)\int_\Omega b_i p(x,T)\zeta(x,T)\mathrm{d}x - \int_0^T w_i \int_\Omega b_i \frac{\mathrm{d}}{\mathrm{d}t}(p\zeta)\mathrm{d}x\mathrm{d}t.$$

For the derivative inside the latter integral, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(p(x,t)\zeta(x,t)\big) = -\Delta p\zeta + p\Delta\zeta - \zeta\left((\bar{y}-y_d) + \sum_{j=1}^{q} c_j\dot{\mu}_j\right) + pB^1\cdot w. \tag{3.32}$$

By Green's Formula:

$$\int_Q w_i b_i\big(-\Delta p\zeta + p\Delta\zeta\big)\mathrm{d}x\mathrm{d}t = \int_Q w_i\big(\Delta b_i p + 2\nabla b_i\cdot\nabla p\big)\zeta\mathrm{d}x\mathrm{d}t. \tag{3.33}$$

Using (3.32) and (3.33) in the expression (3.31) yields

$$\begin{aligned}\widetilde{\mathcal{Q}}_{b,2i} = w_i(T)\int_\Omega b_i p(x,T)\zeta(x,T)\mathrm{d}x + \int_Q w_i\Big[\zeta\big(-\Delta b_i p - 2\nabla b_i\cdot\nabla p\\ + b_i(\bar{y}-y_d) + b_i\sum_{j=1}^{q}c_j\dot{\mu}_j\big) - pB^1\cdot w\Big]\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.34}$$

Hence, $\widetilde{\mathcal{Q}}_{b,2}$ is a function of $(\zeta, w, w(T))$. Finally, putting together (3.27), (3.28), (3.30) and (3.34) yields an expression for $\widetilde{\mathcal{Q}}$ that does not depend on v and coincides with $\widehat{\mathcal{Q}}$ (in view of its definition given in (3.23)-(3.25)). This concludes the proof. □

Remark 3.7. The matrix appearing as coefficient of the quadratic term w in $\widehat{\mathcal{Q}}$ (see (3.24)) is the symmetric $m\times m$ time dependent matrix $R(t)$ with entries

$$R_{ij} := \int_\Omega \Big(\kappa b_i b_j \bar{y}^2 - \dot{S}_{ij} + p(b_i B^1_j + b_j B^1_i)\Big)\,\mathrm{d}x, \quad \text{for } i,j = 1,\dots,m. \tag{3.35}$$

3.7. Goh transform of the critical cone. Here, we apply the Goh transform to the critical cone and obtain the cone PC in the new variables $(\zeta, w, w(T))$. We then define its closure PC_2, that will be used in the next section to prove second order sufficient conditions. In Proposition 3.11, we characterize PC_2 in the case of scalar control.

3.7.1. Primitives of strict critical directions. Define the set of primitives of strict critical directions as

$$PC := \left\{\begin{array}{l}(\zeta, w, w(T)) \in Y\times H^1(0,T)^m\times\mathbb{R}^m;\\ (\zeta, w) \text{ is given by (3.14) for some } (z,v)\in C_{\mathrm{s}}\end{array}\right\}, \tag{3.36}$$

which is obtained by applying the Goh transform (3.14) to C_{s}, and let

$$PC_2 := \text{closure of } PC \text{ in } Y\times L^2(0,T)^m\times\mathbb{R}^m. \tag{3.37}$$

Remember Definition 2.4 of the active constraints sets $\check{B}_k$, $\hat{B}_k$, $B_k = \check{B}_k\cup\hat{B}_k$, C_k.

LEMMA 3.8. *For any $(\zeta, w, h)\in PC$, it holds*

$$w_{B_k}(t) = \frac{1}{\tau_{k+1}-\tau_k}\int_{\tau_k}^{\tau_{k+1}} w_{B_k}(s)\mathrm{d}s, \quad \textit{for } k = 0,\dots,r-1. \tag{3.38}$$

Proof. Immediate from the constancy of w_{B_k} a.e. on each (τ_k, τ_{k+1}), for any $(\zeta, w, h)\in PC$. □

Take $(z,v) \in C_{\rm s}$, and let w and $\zeta[w]$ be given by the Goh transform (3.14). Let $k \in \{0,\dots,r-1\}$ and take an index $j \in C_k$. Then $0 = \int_\Omega c_j(x) z(x,t)\mathrm{d}x$ on (τ_k,τ_{k+1}). Therefore, letting $M_j(t)$ denote the jth column of the matrix $M(t)$ (defined in (2.13)), one has

$$M_j(t)\cdot w(t) = -\int_\Omega c_j(x)\zeta[w](x,t)\mathrm{d}x, \quad \text{on } (\tau_k,\tau_{k+1}), \text{ for } j \in C_k. \tag{3.39}$$

We can rewrite (3.38)-(3.39) in the form

$$\mathcal{A}^k(t)w(t) = \left(\mathcal{B}^k w\right)(t), \quad \text{on } (\tau_k,\tau_{k+1}), \tag{3.40}$$

where $\mathcal{A}^k(t)$ is an $m_k \times m$ matrix with $m_k := |B_k| + |C_k|$, and $\mathcal{B}^k\colon L^2(0,T)^m \to H^1(\tau_k,\tau_{k+1})^{m_k}$. We can actually consider $\mathcal{B} := (\mathcal{B}^1,\dots,\mathcal{B}^r)$ as a linear continuous mapping from $L^2(0,T)^m$ to $\Pi_{k=0}^{r-1} H^1(\tau_k,\tau_{k+1})^{m_k}$, and $\mathcal{A} := (\mathcal{A}^1,\dots,\mathcal{A}^r)$ as a linear continuous mapping from $L^2(0,T)^m$ into $\Pi_{k=0}^{r-1} L^2(\tau_k,\tau_{k+1})^{m_k}$. For each $t \in (\tau_k,\tau_{k+1})$, let us use $\mathcal{A}(t)$ to denote the matrix $\mathcal{A}^k(t)$. We have that, for a.e. $t \in (0,T)$, $\mathcal{A}(t)$ is of maximal rank, so that there exists a unique measurable $\lambda(t)$ (whose dimension is the rank of $\mathcal{A}(t)$ and depends on t) such that

$$w(t) = w_0(t) + \mathcal{A}(t)^\top \lambda(t), \quad \text{with } w_0(t) \in \operatorname{Ker}\mathcal{A}(t). \tag{3.41}$$

Observe that $\mathcal{A}(t)\mathcal{A}(t)^\top$ has a continuous time derivative and is uniformly invertible on $[0,T]$. So, $(\mathcal{A}(t)\mathcal{A}(t)^\top)^{-1}$ is linear continuous from H^1 into H^1 (with appropriate dimensions) over each arc, and $\mathcal{A}(t)\mathcal{A}(t)^\top\lambda(t) = (\mathcal{B}w)(t)$ a.e. We deduce that $t \mapsto (\lambda(t), w_0(t))$ belongs to H^1 over each arc (τ_k,τ_{k+1}). So, in the subspace $\operatorname{Ker}(\mathcal{A}-\mathcal{B})$, $w \mapsto \lambda(w)$ is linear continuous, considering the $L^2(0,T)^m$-topology in the departure set, and the $\Pi_{k=0}^{r-1} H^1(\tau_k,\tau_{k+1})^{m_k}$-topology in the arrival set. Since $(\mathcal{A}-\mathcal{B})$ is linear continuous over $L^2(0,T)^m$ we have that

$$w \in \operatorname{Ker}(\mathcal{A}-\mathcal{B}), \qquad \text{for all } (\zeta,w,h) \in PC_2. \tag{3.42}$$

While the inclusion induced by (3.42) could be strict, we see that for any $(\zeta,w,h) \in PC_2$, $\lambda(w)$ and $\mathcal{A}w$ are well-defined in the H^1 spaces, and the following initial-final conditions hold:

$$\begin{cases} \text{(i)} & w_i = 0 \ \text{ a.e. on } (0,\tau_1), \text{ for each } i \in B_0, \\ \text{(ii)} & w_i = h_i \ \text{ a.e. on } (\tau_{r-1},T), \text{ for each } i \in B_{r-1}, \\ \text{(iii)} & g_j'(\bar y(\cdot,T))[\zeta(\cdot,T) + B(\cdot,T)\cdot h] = 0 \text{ if } j \in C_{r-1}. \end{cases} \tag{3.43}$$

From the definitions of C_s (see (3.12)) and of PC_2, we can obtain additional continuity conditions at the *bang-bang* junction points:

$$\text{if } i \in B_{k-1}\cup B_k, \text{ then } w_i \text{ is continuous at } \tau_k, \quad \text{for all } (\zeta,w,h) \in PC_2. \tag{3.44}$$

Remark 3.9. Another example is when $m = 1$, the state constraint is active for $t < \tau$ and the control constraint is active for $t > \tau$, then w is continuous at time τ. This is similar to the ODE case studied in [2, Remark 5].

We have seen that over each arc (τ_k,τ_{k+1}), $\lambda^k := \lambda(w)$ is pointwise well-defined, and it possesses right limit at the entry point and left limit at the exit point, denoted by $\lambda(\tau_k^+)$ and $\lambda(\tau_{k+1}^-)$, respectively. Let $c_{k+1} \in \mathbb{R}^m$ be such that, for some ν^{k+i},

$$c_{k+1} = \mathcal{A}^{k+i}(\tau_{k+1})^\top \nu^{k+i}, \ \text{ for } i = 0,1, \tag{3.45}$$

meaning that c_{k+1} is a linear combination of the rows of $\mathcal{A}^{k+i}(\tau_{k+1})$ for both $i = 0,1$.

LEMMA 3.10. *Let $k = 0, \dots, r-1$, and let c_{k+1} satisfy (3.45). Then, the junction condition*

$$c_{k+1} \cdot \big(w(\tau_{k+1}^+) - w(\tau_{k+1}^-)\big) = 0, \tag{3.46}$$

holds for all $(\zeta, w, h) \in PC_2$.

Proof. Let (ζ, w, h) in PC, and set $c := c_{k+1}$ and $\tau := \tau_{k+1}$ in order to simplify the notation. Then

$$c \cdot w(\tau) = (\nu^k)^\top \mathcal{A}^k(\tau) w(\tau) = (\nu^k)^\top \mathcal{A}^k(\tau)(\mathcal{A}^k(\tau))^\top \lambda^k(\tau). \tag{3.47}$$

By the same relations for index $k+1$ we conclude that

$$(\nu^k)^\top \mathcal{A}^k(\tau)(\mathcal{A}^k(\tau))^\top \lambda^k(\tau) = (\nu^{k+1})^\top \mathcal{A}^{k+1}(\tau)(\mathcal{A}^{k+1}(\tau))^\top \lambda^{k+1}(\tau). \tag{3.48}$$

Now let $(\zeta, w, h) \in PC_2$. Passing to the limit in the above relation (3.48) written for $(\zeta[w_\ell], w_\ell, h_\ell) \in PC$, $w_\ell \to w$ in $L^2(0,T)^m$, $h_\ell \to h$ (which is possible since $\lambda(t)$ is uniformly Lipschitz over each arc), we get that (3.48) holds for any $(\zeta, w, h) \in PC_2$, from which the conclusion follows. ∎

By *junction conditions* at the junction time $\tau = \tau_k \in (0,T)$, we mean any relation of type (3.46). Set

$$PC_2' := \{(\zeta[w], w, h);\ w \in \mathrm{Ker}(\mathcal{A} - \mathcal{B}), (3.46) \text{ holds, for all } c \text{ satisfying } (3.45)\}. \tag{3.49}$$

We have proved that

$$PC_2 \subseteq PC_2'. \tag{3.50}$$

In the case of a scalar control ($m = 1$) we can show that these two sets coincide.

3.7.2. Scalar control case. The following holds:

PROPOSITION 3.11. *If the control is scalar, then*

$$PC_2 = \left\{ \begin{array}{l} (\zeta[w], w, h) \in Y \times L^2(0,T) \times \mathbb{R};\ \ w \in \mathrm{Ker}(\mathcal{A} - \mathcal{B}); \\ w \textit{ is continuous at BB, BC, CB junctions} \\ \lim_{t \downarrow 0} w(t) = 0 \textit{ if the first arc is not singular} \\ \lim_{t \uparrow T} w(t) = h \textit{ if the last arc is not singular} \end{array} \right\}. \tag{3.51}$$

For a proof we refer to [2, Prop. 4 and Thm. 3].

3.8. Necessary conditions after Goh transform. The following second order necessary condition in the new variables follows.

THEOREM 3.12 (Second order necessary condition). *If $(\bar u, \bar y)$ is an L^∞-local solution of problem (P), then*

$$\max_{(p,\mu) \in \Lambda_1} \widehat{\mathcal{Q}}[p,\mu](\zeta, w, h) \geq 0, \quad \textit{on } PC_2. \tag{3.52}$$

Proof. Let $(\zeta, w, h) \in PC_2$. Then there exists a sequence $(\zeta_\ell := \zeta[w_\ell], w_\ell, w_\ell(T))$ in PC with

$$(\zeta_\ell, w_\ell, w_\ell(T)) \to (\zeta, w, h), \quad \text{in } Y \times L^2(0,T) \times \mathbb{R}. \tag{3.53}$$

Let (z_ℓ, v_ℓ) denote, for each ℓ, the corresponding critical direction in $C_{\rm s}$. By Lemma 3.6 and Theorem 3.5, there exists $(p_\ell, \mu_\ell) \in \Lambda_1$ such that

$$0 \le \mathcal{Q}[p_\ell](z_\ell, v_\ell) = \widehat{\mathcal{Q}}[p_\ell, \mu_\ell](\zeta_\ell, w_\ell, h_\ell). \tag{3.54}$$

We have that $(\dot{\mu}_\ell)$ is bounded in $L^\infty(0,T)$ (this is an easy variant of [5, Cor. 3.12]). Extracting if necessary a subsequence, we may assume that $(\dot{\mu}_\ell)$ weak* converges in $L^\infty(0,T)$ to some $\mathrm{d}\mu$. Consequently, the corresponding solutions p_ℓ of (B.10) weakly converge to p in Y, p being the costate associated with μ, and $p_\ell(T)$ converges to $p(T)$ in $L^2(\Omega)$. In view of the definition of $\widehat{\mathcal{Q}}$ in (3.23), we get, by strong/weak convergence,

$$\lim_{\ell\to\infty} \widehat{\mathcal{Q}}[p_\ell, \mu_\ell](\zeta_\ell, w_\ell, h_\ell) = \lim_{\ell\to\infty} \widehat{\mathcal{Q}}[p_\ell, \mu_\ell](\zeta, w, h) = \widehat{\mathcal{Q}}[p, \mu](\zeta, w, h). \tag{3.55}$$ ◻

4. Second order sufficient conditions. In this section we derive second order sufficient optimality conditions.

Definition 4.1. *We say that an L^2-local solution $(\bar u, \bar y)$ satisfies the* weak quadratic growth condition *if there exist $\rho > 0$ and $\varepsilon > 0$ such that,*

$$F(u) - F(\bar u) \ge \rho(\|w\|_2^2 + |w(T)|^2), \tag{4.1}$$

where $(u, y[u])$ is an admissible trajectory, $\|u - \bar u\|_2 < \varepsilon$, $v := u - \bar u$, and $w(t) := \int_0^t v(s)\mathrm{d}s$.

Remark 4.2. Note that (4.1) is a quadratic growth condition in the L^2-norm of the perturbations $(w, w(T))$ obtained after Goh transform.

The main result of this part is given in Theorem 4.5 and establishes sufficient conditions for a trajectory to be a L^2-local solution with weak quadratic growth.

Throughout the section we assume Hypothesis 3.1. In particular, we have by Theorem 3.3 that the state and costate are essentially bounded.

Consider the condition

$$g_j'(\bar y(\cdot,T))(\bar\zeta(\cdot,T) + B(\cdot,T)\bar h) = 0, \ \text{ if } T \in I_j^C \text{ and } [\mu_j(T)] > 0, \text{ for } j = 1,\dots,q. \tag{4.2}$$

We define

$$PC_2^* := \left\{ \begin{matrix} (\zeta[w], w, h) \in Y \times L^2(0,T)^m \times \mathbb{R}^m;\ w_{B_k} \text{ is constant on each arc;} \\ (3.17), (3.39), (3.43)\text{(i)-(ii)}, (4.2) \text{ hold.} \end{matrix} \right\}. \tag{4.3}$$

Note that PC_2^* is a superset of PC_2.

Definition 4.3. *Let W be a Banach space. We say that a quadratic form $Q\colon W \to \mathbb{R}$ is a* Legendre form *if it is weakly lower semicontinuous, positively homogeneous of degree 2, i.e., $Q(tx) = t^2 Q(x)$ for all $x \in W$ and $t > 0$, and such that if $x_\ell \rightharpoonup x$ and $Q(x_\ell) \to Q(x)$, then $x_\ell \to x$.*

We assume, in the remainder of the article, the following strict complementarity conditions for the control and the state constraints.

Hypothesis 4.4. The following conditions hold:

$$\begin{cases} \text{(i)} \quad \text{for all } i = 1,\dots,m: \\ \quad \max\limits_{(p,\mu)\in\Lambda_1} \Psi_i^p(t) > 0 \text{ in the interior of } \check I_i, \text{ at } t=0 \text{ if } 0 \in \check I_i, \text{ at } t = T \text{ if } T \in \check I_i, \\ \quad \min\limits_{(p,\mu)\in\Lambda_1} \Psi_i^p(t) < 0 \text{ in the interior of } \hat I_i, \text{ at } t=0 \text{ if } 0 \in \hat I_i, \text{ at } t = T \text{ if } T \in \hat I_i, \\ \text{(ii)} \quad \text{there exists } (p,\mu) \in \Lambda_1 \text{ such that } \operatorname{supp} \mathrm{d}\mu_j = I_j^C, \text{ for all } j = 1,\dots,q. \end{cases} \tag{4.4}$$

THEOREM 4.5. *Let Hypotheses 3.1 and 4.4 hold. Then the following assertions hold.*

a) Assume that

(i) $(\bar{u}, \bar{y})$ is a feasible trajectory with nonempty associated set of multipliers Λ_1;

(ii) for each $(p, \mu) \in \Lambda_1$, $\widehat{\mathcal{Q}}[p,\mu](\cdot)$ is a Legendre form on the space $\{(\zeta[w], w, h) \in Y \times L^2(0,T)^m \times \mathbb{R}^m\}$; and

(iii) the uniform positivity *holds, i.e. there exists $\rho > 0$ such that*

$$\max_{(p,\mu)\in\Lambda_1} \widehat{\mathcal{Q}}[p,\mu](\zeta[w], w, h) \geq \rho(\|w\|_2^2 + |h|^2), \quad \text{for all } (w,h) \in PC_2^*. \tag{4.5}$$

Then $(\bar{u}, \bar{y})$ is a L^2-local solution satisfying the weak quadratic growth condition.

b) Conversely, for an admissible trajectory $(\bar{u}, y[\bar{u}])$ satisfying the growth condition (4.1), it holds

$$\max_{(p,\mu)\in\Lambda_1} \widehat{\mathcal{Q}}[p,\mu](\zeta[w], w, h) \geq \rho(\|w\|_2^2 + |h|^2), \quad \text{for all } (w,h) \in PC_2. \tag{4.6}$$

The remainder of this section is devoted to the proof of Theorem 4.5. We first state some technical results.

4.1. A refined expansion of the Lagrangian. Let $(\bar{u}, \bar{y})$ be an admissible trajectory. We start with a refinement of the expansion of the Lagrangian of Proposition 3.4.

LEMMA 4.6. *Let (u, y) be a trajectory, $(\delta y, v) := (u - \bar{u}, y - \bar{y})$, z be the solution of the linearized state equation (B.2), (w, ζ) given by the Goh transform (3.14) and $\eta := \delta y - z$. Then*

$$\begin{aligned}
&\text{(i) } \|z\|_{L^2(Q)} + \|z(\cdot,T)\|_{L^2(\Omega)} = O(\|w\|_2 + |w(T)|),\\
&\text{(ii.a) } \|\delta y\|_{L^2(Q)} + \|\delta y(\cdot,T)\|_{L^2(\Omega)} = O(\|w\|_2 + |w(T)|),\\
&\text{(ii.b) } \|\delta y\|_{L^\infty(0,T;H_0^1(\Omega))} = O(\|w\|_\infty),\\
&\text{(iii) } \|\eta\|_{L^\infty(0,T;L^2(\Omega))} + \|\eta(\cdot,T)\|_{L^2(\Omega)} = o(\|w\|_2 + |w(T)|).
\end{aligned} \tag{4.7}$$

Before doing the proof of Lemma 4.6, let us recall the following property:

PROPOSITION 4.7. *The equation*

$$\dot{\Phi} - \Delta\Phi + a\Phi = \hat{f}, \quad \Phi(x,0) = 0, \tag{4.8}$$

with $a \in L^\infty(Q)$, $\hat{f} \in L^1(0,T;L^2(\Omega))$, and homogeneous Dirichlet conditions on $\partial\Omega \times (0,T)$, has a unique solution Φ in $C([0,T];L^2(\Omega))$, that satisfies

$$\|\Phi\|_{C([0,T];L^2(\Omega))} \leq c\|\hat{f}\|_{L^1(0,T;L^2(\Omega))}. \tag{4.9}$$

Proof. This follows from the estimate for mild solutions in the semigroup theory, see e.g. [3, Theorem 2]. ∎

Proof of Lemma 4.6. (i) Since ζ is solution of (3.17), it satisfies (4.8) with

$$a := -3\gamma\bar{y}^2 + \sum_{i=0}^m \bar{u}_i b_i, \quad \hat{f} := \sum_{i=1}^m w_i B_i^1, \tag{4.10}$$

where B_i^1 is given in (3.18). One can see, in view of Hypothesis 3.1, that $\hat{f} \in L^1(0,T;L^2(\Omega))$ since the terms in brackets in (4.10) belong to $L^\infty(0,T;L^2(\Omega))$. Thus, from Proposition 4.7 we get that $\zeta \in C([0,T];L^2(\Omega))$ and

$$\|\zeta\|_{L^\infty(0,T;L^2(\Omega))} = O(\|\hat{f}\|_{L^1(0,T;L^2(\Omega))}) = O(\|w\|_1). \tag{4.11}$$

Thus, due to Goh transform (3.14) and Lemma A.1, we get that z belongs to $C([0,T];L^2(\Omega))$ and we obtain the estimate (i).

We next prove the estimate (ii) for δy. Set $\zeta_{\delta y} := \delta y - (w\cdot b)\bar{y}$. Then

$$\dot{\zeta}_{\delta y} - \Delta\zeta_{\delta y} + a_{\delta y}\zeta_{\delta y} = \hat{f}_{\delta y}, \tag{4.12}$$

with

$$\begin{aligned} a_{\delta y} &:= 3\gamma\bar{y}^2 + 3\gamma\bar{y}\zeta_{\delta y} + \gamma(\zeta_{\delta y})^2 - (\bar{u}\cdot b), \\ \hat{f}_{\delta y} &:= \sum_{i=1}^m w_i\left[\bar{y}\Delta b_i + \nabla b_i\cdot\nabla\bar{y} - b_i(2\gamma\bar{y}^3 + f)\right]. \end{aligned} \tag{4.13}$$

By Theorem 3.3, $\zeta_{\delta y}$ is in $L^\infty(Q)$, hence $a_{\delta y}$ is essentially bounded. Furthermore, in view of the regularity Hypothesis 3.1 and Lemma A.1, $\hat{f}_{\delta y} \in L^1(0,T;L^2(\Omega))$. We then get, using Proposition 4.7,

$$\|\zeta_{\delta y}\|_{L^\infty(0,T;H_0^1(\Omega))} \leq O(\|w\|_1). \tag{4.14}$$

From the latter equation and the definition of $\zeta_{\delta y}$ we deduce (ii.a). Since

$$\nabla(\delta y) = \nabla(\zeta_{\delta y}) + \sum_{i=1}^m w_i(\bar{y}\nabla b_i + b_i\nabla\bar{y}), \tag{4.15}$$

applying the $L^\infty(0,T;L^2(\Omega))$-norm to both sides, and using (4.14) and Lemma A.1 we get (ii.b).

The estimate in (iii) follows from the following consideration. To apply Proposition 4.7 to equation (3.8) we easily verify that r is in $L^\infty(Q)$ and $\tilde{r}$ in $L^1(0,T;L^2(\Omega))$. Consequently, we have

$$\begin{aligned} \|\eta\|_{C([0,T];L^2(\Omega))} &\leq c\left\|\sum_{i=1}^m v_i b_i\delta y - 3\gamma\bar{y}(\delta y)^2 - \gamma(\delta y)^3\right\|_{L^1(0,T;L^2(\Omega))} \\ &\leq \|v\|_2\|b\|_\infty\|\delta y\|_2 + 3\gamma\|\bar{y}\|_\infty\left\|(\delta y)^2\right\|_{L^1(0,T;L^2(\Omega))} + \gamma\left\|(\delta y)^3\right\|_{L^1(0,T;L^2(\Omega))}. \end{aligned} \tag{4.16}$$

Now, since $\|v\|_2 \to 0$ and $\|\delta y\|_\infty \to 0$ (by similar arguments to those of the proof of (i) in Theorem 3.3), we get (iii). ∎

PROPOSITION 4.8. *Let $(p, \mathrm{d}\mu) \in \Lambda_1$. Let $(u_\ell) \subset \mathcal{U}_{\rm ad}$ and let us write y_ℓ for the corresponding states. Set $v_\ell := u_\ell - \bar{u}$ and assume that $v_\ell \to 0$ a.e. Then,*

$$\begin{aligned} \mathcal{L}[p,\mu](\bar{u}+v_\ell, y_\ell) &= \mathcal{L}[p,\mu](\bar{u},\bar{y}) \\ &\quad + \int_0^T \Psi^p(t)\cdot v_\ell(t)\mathrm{d}t + \tfrac{1}{2}\widehat{\mathcal{Q}}[p,\mu](\zeta_\ell, w_\ell, w_\ell(T)) + o(\|w_\ell\|_2^2 + |w_\ell(T)|^2), \end{aligned} \tag{4.17}$$

where w_ℓ and ζ_ℓ are given by the Goh transform (3.14).

Proof. Since (v_ℓ) is bounded in $L^\infty(0,T)^m$ and converges a.e. to 0, it converges to zero in any $L^p(0,T)^m$. For simplicity of notation we omit the index ℓ for the remainder of the proof. Set $\delta y := y[\bar u + v] - \bar y$. By Proposition 3.4 it is enough to prove that

$$\left|\mathcal{Q}[p](\delta y, v) - \widehat{\mathcal{Q}}[p,\mu](w, w(T), \zeta)\right| = o(\|w\|_2^2 + |w(T)|^2), \tag{4.18}$$

$$\left|\int_Q p(\delta y)^3\right| = o(\|w\|_2^2 + |w(T)|^2). \tag{4.19}$$

We have, setting as before $\eta := \delta y - z$ where $z := z[v]$,

$$\begin{aligned} &\mathcal{Q}[p](\delta y, v) - \widehat{\mathcal{Q}}[p,\mu](\zeta, w, w(T)) = \mathcal{Q}[p](\delta y, v) - \mathcal{Q}[p](z, v) \\ &= 2\int_Q (v\cdot b)p\eta \mathrm{d}x\mathrm{d}t + \int_Q \kappa(\delta y + z)\eta \mathrm{d}x\mathrm{d}t + \int_\Omega (\delta y(x,T) + z(x,T))\eta(x,T)\mathrm{d}x, \end{aligned} \tag{4.20}$$

and therefore, since the state and costate are essentially bounded:

$$\begin{aligned} \left|\mathcal{Q}[p](\delta y, v) - \widehat{\mathcal{Q}}[p,\mu](\zeta, w, w(T))\right| &\le 2\left|\int_Q (v\cdot b)p\eta \mathrm{d}x\mathrm{d}t\right| + O(\|\delta y + z\|_2\|\eta\|_2) \\ &+ O(\|(\delta y + z)(\cdot, T)\|_{L^2(\Omega)}\|\eta(\cdot,T)\|_{L^2(\Omega)}). \end{aligned} \tag{4.21}$$

In view of lemma 4.6, the 'big O' terms in the r.h.s. are of the desired order and it remains to deal with the integral term. We have, integrating by parts in time,

$$\int_Q (v\cdot b)p\eta \mathrm{d}x\mathrm{d}t = \int_\Omega \big(w(T)\cdot b(x)\big)p(x,T)\eta(x,T)\mathrm{d}x - \int_Q (w\cdot b)\frac{\mathrm{d}}{\mathrm{d}t}(p\eta)\mathrm{d}x\mathrm{d}t. \tag{4.22}$$

For the first term in the r.h.s. of (4.22) we get, in view of (4.7)(ii),

$$\begin{aligned} &\left|\int_\Omega (w(T)\cdot b(x))p(x,T)\eta(x,T)\mathrm{d}x\right| \\ &\qquad = O(|w(T)|\|\eta(\cdot,T)\|_{L^2(\Omega)}) = o(\|w\|_2^2 + |w(T)|^2). \end{aligned} \tag{4.23}$$

And, for the second term in the r.h.s. of (4.22), since b is essentially bounded, and p and η satisfy (B.10) and (3.8), respectively, we have that,

$$\begin{gathered} \frac{\mathrm{d}}{\mathrm{d}t}(p\eta) = \varphi_0 + \varphi_1 + \varphi_2, \\ \varphi_0 := p\Delta\eta - \eta\Delta p;\ \varphi_1 := (v\cdot b)p\delta y;\ \varphi_2 := pe(\delta y)^2 - \eta\left(y - y_d + \sum_{j=1}^q c_j\dot\mu_j(t)\right). \end{gathered} \tag{4.24}$$

Contribution of φ_2. Since y, p and $\dot\mu$ are essentially bounded (see Theorem 3.3), we get

$$\left|\int_Q (w\cdot b)\varphi_2\right| = O\left(\|w(\delta y)^2 + w\eta\|_2\right) = o(\|w\|_2^2 + |w(T)|^2), \tag{4.25}$$

where the last equality follows from the estimates for δy and η obtained in Lemma 4.6. *Contribution of* φ_1. Integrating by parts in time, we can write the contribution of φ_1 as

$$\tfrac{1}{2}\int_Q \frac{\mathrm{d}}{\mathrm{d}t}(w\cdot b)^2 p\delta y = \tfrac{1}{2}\int_\Omega (w(T)\cdot b)^2 p(x,T)\delta y(x,T) - \tfrac{1}{2}\int_Q (w\cdot b)^2\frac{\mathrm{d}}{\mathrm{d}t}(p\delta y) \tag{4.26}$$

The contribution of the term at $t = T$ is of the desired order. Let us proceed with the estimate for the last term in the r.h.s. of (4.26). We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(p\delta y) &= (-\delta y\Delta p + p\Delta\delta y)\\ &\quad + \left(-(\bar y - y_d) - \sum_{j=1}^{q} c_j\dot\mu_j\right)\delta y + \left(\sum_{i=1}^{m} v_i b_i y - 3\gamma\bar y(\delta y)^2 - \gamma(\delta y)^3\right)p.\end{aligned} \tag{4.27}$$

For the contribution of first term in the r.h.s. of latter equation we get

$$\int_Q (w\cdot b)^2(-\delta y\Delta p + p\Delta\delta y) = \sum_{i,j=1}^{m}\int_0^T w_i w_j \int_\Omega \nabla(b_i b_j)\cdot(\delta y\nabla p - p\nabla\delta y). \tag{4.28}$$

Using [5, Lem. 2.2], since $\nabla(b_i b_j)$ is essentially bounded for every pair i, j, it is enough to prove that

$$\int_\Omega \nabla(b_i b_j)\cdot(\delta y\nabla p - p\nabla\delta y) \to 0 \tag{4.29}$$

uniformly in time. For this, in view of the estimate for $\|\delta y\|_{L^\infty(0,T;H_0^1(\Omega))}$ obtained in Lemma 4.6 item (ii.b), and since p is essentially bounded, it suffices to prove that p is in $L^\infty(0,T;H^1(\Omega))$ which follows from Corollary B.1.

Let us continue with the expression in (4.27). The terms containing δy go to 0 in $L^\infty(0,T;L^2(\Omega))$ and that is sufficient for our purpose. The only term that has to be estimated is

$$\begin{aligned}\int_Q (w\cdot b)^2(v\cdot b)yp &= \frac{1}{3}\int_Q \frac{\mathrm{d}}{\mathrm{d}t}(w\cdot b)^3 yp\\ &= \frac{1}{3}\int_\Omega (w(T)\cdot b)^3 y(\cdot,T)p(\cdot,T) - \frac{1}{3}\int_Q (w\cdot b)^3\frac{\mathrm{d}}{\mathrm{d}t}(yp).\end{aligned} \tag{4.30}$$

We consider the pair of state and costate equations with $g := y - y_d$ given as

$$\begin{aligned}\dot y - \Delta y + \gamma y^3 &= (u\cdot b)y + f; &\quad y(0) &= y_0;\\ -\dot p - \Delta p + \gamma y^2 p &= (u\cdot b)p + g + c\dot\mu; &\quad p(T) &= 0.\end{aligned} \tag{4.31}$$

and so for sufficiently smooth $\varphi\colon \Omega\times(0,T)\to\mathbb{R}$ we have

$$\begin{aligned}\int_Q \varphi\frac{\mathrm{d}}{\mathrm{d}t}(yp) &= \int_Q \varphi(\dot y p + y\dot p)\\ &= \int_Q \varphi\left[(\Delta y - \gamma y^3 + (u\cdot b)y + f)p + y(-\Delta p + \gamma y^2 p - (u\cdot b)p - g - c\dot\mu)\right]\\ &= \int_Q \varphi\left[fp - yg + c\dot\mu y\right] + \nabla\varphi\cdot(-p\nabla y + y\nabla p),\end{aligned} \tag{4.32}$$

and, consequently, we have for $\varphi = (w\cdot b)^3$,

$$\int_Q (w\cdot b)^3\frac{\mathrm{d}}{\mathrm{d}t}(yp) = \int_Q (w\cdot b)^3\left[fp - yg + c\dot\mu y\right] + \nabla(w\cdot b)^3\cdot(-p\nabla y + y\nabla p). \tag{4.33}$$

By Hypothesis 3.1, f and b are sufficiently smooth, $\dot\mu$ is essentially bounded, $y, p \in L^\infty(0,T;H_0^1(\Omega))$. We estimate

$$\left|\int_Q (w\cdot b)^3 \frac{\mathrm{d}}{\mathrm{d}t}(yp)\right| \le \|b\|_\infty^3 \|w\|_\infty \|w\|_2^2\, \|fp - yg + c\dot\mu y\|_{L^\infty(0,T;L^1(\Omega))}$$
$$+\, O(\|b\|_\infty^2 \|\nabla b\|_\infty)\|w\|_\infty \|w\|_2^2 \left(\|y\|_{L^\infty(0,T;H_0^1(\Omega))}\, \|p\|_{L^\infty(0,T;H_0^1(\Omega))}\right) = o(\|w\|^2).$$

Contribution of φ_0. Integrating by parts, we have that

$$\begin{aligned}\int_0^T w_i \int_\Omega b_i \varphi_0 &= \int_0^T w_i \int_\Omega b_i (p\Delta\eta - \eta\Delta p) = \int_0^T w_i \int_\Omega \nabla b_i \cdot (-p\nabla\eta + \eta\nabla p) \\ &= \int_0^T w_i \int_\Omega \Big(p\eta\Delta b_i + 2\eta\nabla p\cdot\nabla b_i\Big).\end{aligned} \tag{4.34}$$

Recalling that $b \in W^{2,\infty}(\Omega)$ (see (3.4)) and that p is essentially bounded (due to Theorem 3.3), we get for the first term in the r.h.s. of the latter display,

$$\left|\int_0^T w_i \int_\Omega p\eta\Delta b_i\right| \le \|\Delta b_i\|_\infty \|w_i\|_2 \|p\|_\infty \|\eta\|_{L^2(0,T;L^2(\Omega))}, \tag{4.35}$$

that is a small-o of $\|w\|_2^2$ in view of item (iii.a) of Lemma 4.6. For the second term in the r.h.s. of (4.34) we get

$$\left|\int_0^T w_i \int_\Omega \eta\nabla p\cdot\nabla b_i\right| \le \|\nabla b_i\|_\infty \|w_i\|_2 \|\eta\|_{L^2(0,T;L^2(\Omega))} \|\nabla p\|_{L^\infty(0,T;L^2(\Omega)^n)} \tag{4.36}$$

Since $p \in L^\infty(0,T;H^1(\Omega))$ as showed some lines above and in view of item (iii.a) of Lemma 4.6, we get that the r.h.s. of latter equation is a small-o of $\|w\|_2^2$, as desired.

Collecting the previous estimates, we get (4.18). Similarly, since $\delta y \to 0$ uniformly and the costate p is essentially bounded, with (4.7)(i) we get

$$\left|\int_Q pb(\delta y)^3 \mathrm{d}x\mathrm{d}t\right| = o\left(\|\delta y\|_2^2\right) = o\left(\|w\|_2^2 + |w(T)|^2\right). \tag{4.37}$$

The result follows. ∎

COROLLARY 4.9. *Let $u = \bar u + v$ be an admissible control. Then, setting $w(t) := \int_0^t v(s)\mathrm{d}s$, we have the reduced cost expansion*

$$F(u) = F(\bar u) + DF(\bar u)v + O(\|w\|_2^2 + |w(T)|^2). \tag{4.38}$$

PROPOSITION 4.10. *Let $(p, \mathrm{d}\mu) \in \Lambda_1$, and let $(z,v) \in Y \times L^2(0,T)^m$ satisfy the linearized state equation (B.2). Then,*

$$\int_0^T \Psi^p(t)\cdot v(t)\mathrm{d}t = DJ(\bar u,\bar y)(z,v) + \sum_{j=1}^q \int_0^T g_j'(\bar y(\cdot,t)z(\cdot,t)\mathrm{d}\mu_j(t), \tag{4.39}$$

where

$$DJ(\bar u,\bar y)(z,v) = \sum_{i=1}^m \int_0^T \alpha_i v_i \mathrm{d}t + \int_Q (\bar y - y_d) z \mathrm{d}x\mathrm{d}t + \int_\Omega (\bar y(T) - y_{dT}) z(T)\mathrm{d}x,$$

and it coincides with $DF(\bar u)v$.

Proof. It follows from (B.2), (B.8) and (2.9). ∎

4.2. Proof of Theorem 4.5. What remains to prove is similar to what has been done in Aronna, Bonnans and Goh [2, Theorem 5], except that here the control variable may be multidimensional and in [2] it is scalar.

We start by showing item a). If the conclusion does not hold, there must exist a sequence (u_ℓ, y_ℓ) of admissible trajectories, with u_ℓ distinct from $\bar{u}$, such that $v_\ell := u_\ell - \bar{u}$ converges to zero in $L^2(0,T)^m$, and

$$J(u_\ell, y_\ell) \leq J(\bar{u}, \bar{y}) + o(\Upsilon_\ell^2), \tag{4.40}$$

where (w_ℓ, ζ_ℓ) is obtained by Goh transform (3.14), $h_\ell := w_\ell(T)$ and

$$\Upsilon_\ell := \sqrt{\|w_\ell\|_2^2 + |w_\ell(T)|^2}.$$

Let $(p, \mathrm{d}\mu) \in \Lambda_1$. Adding $\int_0^T g(y_\ell)\mathrm{d}\mu \leq 0$ on both sides of (4.40) leads to

$$\mathcal{L}[p,\mu](u_\ell, y_\ell) \leq \mathcal{L}[p,\mu](\bar{u}, \bar{y}) + o(\Upsilon_\ell^2). \tag{4.41}$$

Set $(\bar{v}_\ell, \bar{w}_\ell, \bar{h}_\ell) := (v_\ell, w_\ell, h_\ell)/\Upsilon_\ell$. Then $(\bar{w}_\ell, \bar{h}_\ell)$ has unit norm in $L^2(0,T)^m \times \mathbb{R}^m$. Extracting if necessary a subsequence, we may assume that there exists $(\bar{w}, \bar{h})$ in $L^2(0,T)^m \times \mathbb{R}^m$ such that

$$\bar{w}_\ell \rightharpoonup \bar{w} \quad \text{and} \quad h_\ell \to \bar{h}, \tag{4.42}$$

where the first limit is given in the weak topology of $L^2(0,T)^m$. Set $\bar{\zeta} := \zeta[\bar{w}]$. The remainder of the proof is split in two parts:

Fact 1: The triple $(\bar{\zeta}, \bar{w}, \bar{h})$ belongs to PC_2^* (defined in (4.3)).

Fact 2: The inequality (4.40) contradicts the hypothesis of uniform positivity (4.5).

Proof of Fact 1. We divide this part in four steps: **(a)** $\bar{w}_i$ is constant on each maximal arc of I_i, for $i = 1, \dots, m$, **(b)** (3.43)(i),(ii) hold, **(c)** (3.39) holds, and **(d)** (4.2) holds.

(a) From Proposition 4.8, inequality (4.41), and (2.10) we have

$$-\widehat{\mathcal{Q}}[p, \mathrm{d}\mu](\zeta_\ell, w_\ell, h_\ell) + o(\Upsilon_\ell^2) \geq \sum_{i=1}^m \int_0^T \Psi_i^p(t) \cdot v_{\ell,i}(t)\mathrm{d}t \geq 0. \tag{4.43}$$

By the continuity of the quadratic form $\widehat{\mathcal{Q}}[p, \mathrm{d}\mu]$ over the space $L^2(0,T)^m \times \mathbb{R}^m$, we deduce that

$$0 \leq \int_0^T \Psi_i^p(t) v_{\ell,i}(t)\mathrm{d}t \leq O(\Upsilon_\ell^2), \quad \text{for all } i = 1, \dots, m. \tag{4.44}$$

Hence, since the integrand in previous inequality is nonnegative for all $\ell \in \mathbb{N}$, we have that

$$\lim_{\ell\to\infty} \int_0^T \Psi_i^p(t)\varphi(t)\bar{v}_{\ell,i}(t)\mathrm{d}t = 0 \tag{4.45}$$

for any nonnegative C^1 function $\varphi\colon [0,T] \to \mathbb{R}$. Let us consider, in particular, φ having its support $[c,d] \subset I_i$. Integrating by parts in (4.45) and using that $\bar{w}_\ell$ is a primitive of $\bar{v}_\ell$, we obtain

$$0 = \lim_{\ell\to\infty} \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}(\Psi_i^p\varphi)\bar{w}_{\ell,i}\mathrm{d}t = \int_c^d \frac{\mathrm{d}}{\mathrm{d}t}(\Psi_i^p(t)\varphi)\bar{w}_i\mathrm{d}t. \tag{4.46}$$

Over $[c,d]$, $\bar{v}_{\ell,i}$ has constant sign and, therefore, $\bar{w}_i$ is either nondecreasing or nonincreasing. Thus, we can integrate by parts in the latter equation to get

$$\int_c^d \Psi_i^p(t)\varphi(t)\mathrm{d}\bar{w}_i(t) = 0. \tag{4.47}$$

Take now any $t_0 \in (c,d)$. Assume, w.l.g. that $t_0 \in \check{I}_i$. By the strict complementary condition for the control constraint given in (4.4), there exists a multiplier such that the associated Ψ^p verifies $\Psi_i^p(t_0) > 0$. Hence, in view of the continuity of Ψ_i^p on I_i, there exists $\varepsilon > 0$ such that $\Psi_i^p > 0$ on $(t_0 - 2\varepsilon, t_0 + 2\varepsilon) \subset (c,d)$. Choose φ such that $\operatorname{supp}\varphi \subset (t_0-2\varepsilon, t_0+2\varepsilon)$, and $\Psi_i^p\varphi \equiv 1$ on $(t_0-\varepsilon, t_0+\varepsilon)$, then $\bar{w}_i(t_0+\varepsilon) - \bar{x}_i(t_0-\varepsilon) = 0$. Since $\mathrm{d}\bar{w}_i \geq 0$, we obtain $\mathrm{d}\bar{w}_i = 0$ a.e. on $\check{I}_i$. Since t_0 is an arbitrary point in the interior of I_i, we get

$$\mathrm{d}\bar{w}_i = 0 \quad \text{a.e. on } \check{I}_i. \tag{4.48}$$

This concludes step **(a)**.

(b) We now have to prove (3.43)(i),(ii). Assume now that $B_0 \neq \emptyset$ or, w.l.g., that $\check{B}_0 \neq \emptyset$, and let $i \in \check{B}_0$. By the previous step, $\bar{w}_i$ is equal to some constant θ a.e. over $(0,\tau_1)$. Let us show that $\theta = 0$. By the strict complementarity condition for the control constraint (4.4) there exist $t, \delta > 0$ and a multiplier such that the associated Ψ^p satisfies $\Psi_i^p > \delta$ on $[0,t] \subset [0,\tau_1)$. By considering in (4.45) a nonnegative Lipschitz continuous function $\varphi\colon [0,T] \to \mathbb{R}$ being equal to $1/\delta$ on $[0,t]$, with support included in $[0,\tau_1)$, and since $\bar{v}_{\ell,i} \geq 0$ a.e. on $[0,\tau_1]$, we obtain, for any $\tau \in [0,t]$,

$$\bar{w}_{\ell,i}(\tau) = \int_0^\tau \bar{v}_{\ell,i}(s)\mathrm{d}s \leq \int_0^t \Psi_i^p(s)\varphi(s)\bar{v}_{\ell,i}(s)\mathrm{d}s \to 0, \quad \text{when } \ell \to \infty. \tag{4.49}$$

Thus $\bar{w}_i = 0$ a.e. on $[0,t]$. Consequently, from (4.48) we get $\bar{w}_i = 0$ a.e. on $[0,\tau_1)$. The case when $i \in B_{r-1}$ follows by a similar argument. This yields item **(b)**.

(c) Let us prove (3.39). We have, since y_ℓ is admissible and g linear,

$$0 \geq g_j(y_\ell(\cdot,t)) - g_j(\bar{y}(\cdot,t)) = \int_\Omega c_j(x)(y_\ell - \bar{y})(x,t)\mathrm{d}x, \quad \text{on } [\tau_k, \tau_{k+1}], \tag{4.50}$$

whenever k, j are such that $k \in \{0,\dots,r-1\}$ and $j \in C_k$. Let z_ℓ denote the linearized state corresponding to v_ℓ, and let $\eta_\ell := y_\ell - \bar{y} - z_\ell$. By Lemma (4.6)(iii), we deduce that

$$\int_\Omega c_j(x)z_\ell(x,t)\mathrm{d}x \leq -\int c_j(x)\eta_\ell(x,t)\mathrm{d}x \leq o(\Upsilon_\ell), \quad \text{on } [\tau_k, \tau_{k+1}]. \tag{4.51}$$

Thus, by the Goh transform (3.14),

$$\int_\Omega c_j(x)(\bar{\zeta}_\ell(x,t) + B(x,t)\cdot\bar{w}_\ell(t))\mathrm{d}x \leq o(1), \quad \text{on } [\tau_k, \tau_{k+1}], \tag{4.52}$$

where $\bar{\zeta}_\ell$ is the solution of (3.17) corresponding to $\bar{w}_\ell$. Let φ be some time-dependent nonnegative continuous function with support included in I_j^C. From (4.52), we get that

$$\int_{\tau_k}^{\tau_{k+1}} \varphi \int_\Omega c_j(\bar{\zeta}_\ell + B\cdot\bar{w}_\ell)\mathrm{d}x\mathrm{d}t \leq o(1). \tag{4.53}$$

Taking the limit $\ell \to \infty$ yields

$$\int_{\tau_k}^{\tau_{k+1}} \varphi \int_\Omega c_j(\bar{\zeta} + B \cdot \bar{w}) \mathrm{d}x \mathrm{d}t \leq 0, \tag{4.54}$$

where $\bar{\zeta}$ is the solution of (3.17) associated to $\bar{w}$. Since (4.54) holds for any nonnegative φ, we get that

$$\int_\Omega c_j(\bar{\zeta}(x,t) + B(x,t) \cdot \bar{w}(t)) \mathrm{d}x \leq 0, \quad \text{a.e. on } [\tau_k, \tau_{k+1}]. \tag{4.55}$$

In particular, if $T \in I_j^C$, we get from (4.52) that

$$\int_\Omega c_j(\bar{\zeta}(x,T) + B(x,T) \cdot \bar{h}) \mathrm{d}x \leq 0. \tag{4.56}$$

We now have to prove the converse inequalities in (4.54) and (4.56).

By Proposition 4.10 and since u_ℓ is admissible, we have

$$\sum_{j=1}^q \int_0^T g_j'(\bar{y}(\cdot,t)) z(\cdot,t) \mathrm{d}\mu_j(t) + DJ(\bar{u},\bar{y})(z,v) = \int_0^T \Psi^p(t) \cdot v_\ell(t) \mathrm{d}t \geq 0. \tag{4.57}$$

By Proposition 4.9, we have $F(u_\ell) = F(\bar{u}) + DF(\bar{u})v_\ell + o(\Upsilon_\ell)$. This, together with (4.57), yield

$$0 \leq F(u_\ell) - F(\bar{u}) + o(\Upsilon_\ell) + \sum_{j=1}^q \int_0^T g_j'(\bar{y}(\cdot,t)) z(\cdot,t) \mathrm{d}\mu_j(t). \tag{4.58}$$

Using (4.40) in latter inequality implies that

$$-o(\Upsilon_\ell) \leq \sum_{j=1}^q \int_0^T g_j'(\bar{y}(\cdot,t)) z(\cdot,t) \mathrm{d}\mu_j(t), \tag{4.59}$$

thus

$$o(1) \leq \sum_{j=1}^q \int_0^T g_j'(\bar{y}(\cdot,t))(\bar{\zeta}_\ell(\cdot,t) + B(\cdot,t) \cdot \bar{w}_\ell(t)) \mathrm{d}\mu_j(t). \tag{4.60}$$

Since, for every $j = 1, \dots, q$, the measure $\mathrm{d}\mu_j$ has an essentially bounded density over $[0,T)$ (in view of Theorem 3.3), we have that

$$\begin{aligned} 0 &\leq \liminf_{\ell\to\infty} \sum_{j=1}^q \int_{[0,T]} g_j'(\bar{y}(\cdot,t))(\bar{\zeta}_\ell(\cdot,t) + B(\cdot,t) \cdot \bar{w}_\ell(t)) \mathrm{d}\mu_j \\ &= \lim_{\ell\to\infty} \sum_{j=1}^q \int_{[0,T)} g_j'(\bar{y}(\cdot,t))(\bar{\zeta}_\ell(\cdot,t) + B(\cdot,t) \cdot \bar{w}_\ell(t)) \mathrm{d}\mu_j. \end{aligned} \tag{4.61}$$

Using (4.55) and the strict complementarity for the state constraint (4.4)(ii), we get (3.39). This concludes the proof of item **(c)**.

(d) Let us now prove (4.2). Assume that $j \in \{1, \dots, q\}$ is such that $T \in I_j^C$. One inequality was already proved in (4.56). If we further have that $[\mu_j(T)] > 0$, condition (4.2) follows from (4.61).

We conclude that the limit direction $(\bar\zeta, \bar w, \bar h)$ belongs to PC_2^*.

Proof of Fact 2. From Proposition 4.8 we obtain

$$\begin{aligned} &\widehat{\mathcal{Q}}[p, \mathrm{d}\mu](\zeta_\ell, w_\ell, h_\ell) \\ &\quad = \mathcal{L}[p,\mu](u_\ell, y_\ell) - \mathcal{L}[p,\mu](\bar u, \bar y) - \int_0^T \Psi^p \cdot v_\ell \mathrm{d}t + o(\Upsilon_\ell^2) \le o(\Upsilon_\ell^2), \end{aligned} \tag{4.62}$$

where the last inequality follows from (4.41) and since $\Psi^p \cdot v_\ell \ge 0$ a.e. on $[0, T]$ in view of the first order condition (2.10). Hence,

$$\liminf_{\ell\to\infty} \widehat{\mathcal{Q}}[p,\mu](\bar\zeta_\ell, \bar y_\ell, \bar h_\ell) \le \limsup_{\ell\to\infty} \widehat{\mathcal{Q}}[p,\mu](\bar\zeta_\ell, \bar w_\ell, \bar h_\ell) \le 0. \tag{4.63}$$

Let us recall that, in view of the hypothesis (iii) of the current theorem, the mapping $\widehat{\mathcal{Q}}[p, \mathrm{d}\mu]$ is a Legendre form in the Hilbert space $\{(\zeta[w], w, h) \in Y \times L^2(0,T)^m \times \mathbb{R}^m\}$. Furthermore, for the critical direction $(\bar\zeta, \bar w, \bar h)$, due to the uniform positivity condition (4.5), there is a multiplier $(\bar p, \bar\mu) \in \Lambda_1$ such that

$$\rho(\|\bar w\|_2^2 + |\bar h|^2) \le \widehat{\mathcal{Q}}[\bar p, \bar\mu](\bar\zeta, \bar w, \bar h) = \liminf_{\ell\to\infty} \widehat{\mathcal{Q}}[\bar p, \bar\mu](\bar\zeta_\ell, \bar w_\ell, \bar h_\ell) \le 0, \tag{4.64}$$

where the equality holds since $\widehat{\mathcal{Q}}[\bar p, \bar\mu]$ is a Legendre form and the inequality is due to (4.62). From (4.64) we get $(\bar w, \bar h) = 0$ and $\lim_{k\to\infty} \widehat{\mathcal{Q}}[\bar p, \bar\mu](\bar\zeta_\ell, \bar w_\ell, \bar h_\ell) = 0$. Consequently, $(\bar w_\ell, \bar h_\ell)$ converges strongly to $(\bar w, \bar h) = 0$ which is a contradiction, since $(\bar w_\ell, \bar h_\ell)$ has unit norm in $L^2(0,T)^m \times \mathbb{R}^m$. We conclude that $(\bar u, \bar y)$ is an L^2-local solution satisfying the weak quadratic growth condition.

Conversely, assume that the weak quadratic growth condition (4.1) holds at $(\bar u, \bar y)$ for $\rho > 0$. Note that $(\bar u, \bar y, \bar w)$, with $\bar w(t) = \int_0^t \bar u(s)\mathrm{d}s$, is a L^2-local solution of the problem

$$\begin{aligned} &\min_{u\in\mathcal{U}_{\mathrm{ad}}} J(u, y[u]) - \rho\left(\int_0^T (w - \bar w)^2 \mathrm{d}t + |w(T) - \bar w(T)|^2\right), \\ &\quad \text{s.t. } \dot w = u,\ w(0) = 0,\ (2.4) \text{ holds,} \end{aligned} \tag{4.65}$$

Applying the second order necessary condition in Theorem 3.5 to this problem (4.65), followed by the Goh transform, yields the uniform positivity (4.6). For further details we refer to the corresponding statement for ordinary differential equations in [1, Theorem 5.5]. ☐

Appendix A. Well-posedness of state equation and existence of optimal controls. In this section we recall some statements from [5], for proofs we refer to the latter reference.

LEMMA A.1. *The state equation* (2.1) *has a unique solution* $y = y[u, y_0, f]$ *in* Y. *The mapping* $(y, y_0, f) \mapsto y[u, y_0, f]$ *is* C^∞ *from* $L^2(0,T)^m \times H_0^1(\Omega) \times L^2(Q)$ *to* Y, *and nondecreasing w.r.t.* y_0 *and* f. *In addition, there exist functions* C_i, $i = 1$ *to 2, not decreasing w.r.t. each component, such that*

$$\|y\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla y\|_2 \le C_1(\|y_0\|_2, \|f\|_2, \|u\|_2\|b\|_\infty), \tag{A.1}$$

$$\|y\|_Y \le C_2(\|y_0\|_{H_0^1(\Omega)}, \|f\|_2, \|u\|_2\|b\|_\infty). \tag{A.2}$$

Moreover, the state y also belongs to $C([0,T];H_0^1(\Omega))$, since Y is continuously embedded in that space [14, Theorem 3.1, p.23].

THEOREM A.2. *The mapping $u \mapsto y[u]$ is of class C^∞, from $L^2(0,T)^m$ to Y.*

THEOREM A.3. (i) *The function $u \mapsto J(u, y[u])$, from $L^2(0,T)^m$ to $\mathbb{R}$, is weakly sequentially l.s.c.* (ii) *The set of solutions of the optimal control problem (P) is weakly sequentially closed in $L^2(0,T)^m$.* (iii) *If (P) has a bounded minimizing sequence, the set of solutions of (P) is non empty. This is the case in particular if (P) is admissible and $\mathcal{U}_{\rm ad}$ is a bounded subset of $L^2(0,T)^m$.*

Appendix B. First order analysis. Here, we recall some properties from [5].

Throughout the section, $(\bar u, \bar y)$ is a trajectory of problem (P). We recall the hypotheses (2.2) and (2.6) on the data.

We fix a trajectory $(\bar u, \bar y = y[\bar u])$. Let A be linear continuous from $L^2(0,T;H^2(\Omega))$ to $L^2(Q)$ such that, for each $z \in L^2(0,T;H^2(\Omega))$ and $(x,t) \in Q$,

$$(Az)(x,t) := -\Delta z(x,t) + 3\gamma \bar y(x,t)^2 z(x,t) - \sum_{i=0}^m \bar u_i(t) b_i(x) z(x,t). \tag{B.1}$$

B.1. The linearized state equation. The *linearized state equation* at $(\bar u, \bar y)$ is given by

$$\dot z + Az = \sum_{i=1}^m v_i b_i \bar y \quad \text{in } Q; \quad z = 0 \text{ on } \Sigma, \quad z(\cdot,0) = 0 \text{ on } \Omega. \tag{B.2}$$

For $v \in L^2(0,T)^m$, equation (B.2) above possesses a unique solution $z[v] \in Y$ and the mapping $v \mapsto z[v]$ is linear from $L^2(0,T)^m$ to Y. Particularly, the following estimate holds:

$$\|z\|_{L^\infty(0,T;L^2(\Omega))} \le M_1 \sum_{i=1}^m \|b_i\|_\infty \|v_i\|_1, \tag{B.3}$$

where $M_1 := e^{\frac{T}{2} + \sum_{i=0}^m \|\bar u_i\|_1 \|b_i\|_\infty} \|\bar y\|_{L^\infty(0,T;L^2(\Omega))}$.

B.2. The costate equation. For $\mu \in BV(0,T)^q$ its distributional derivative $\mathrm{d}\mu$ is in the *space $\mathcal{M}(0,T)$ of finite Radon measures.* And, conversely, any element $\mathrm{d}\mu \in \mathcal{M}(0,T)$ can be identified with a function μ of bounded variation that vanishes at time T. Let us consider the set of positive finite Radon measures $\mathcal{M}_+(0,T)$ and identify it with the set

$$BV(0,T)^q_{0,+} := \{\mu \in BV(0,T)^q;\ \mu(T) = 0, \mathrm{d}\mu \ge 0\}. \tag{B.4}$$

The *generalized Lagrangian* of problem (P) is, choosing the multiplier of the state equation to be $(p, p_0) \in L^2(Q) \times H^{-1}(\Omega)$ and taking $\beta \in \mathbb{R}_+$, $\mathrm{d}\mu \in \mathcal{M}_+(0,T)$,

$$\begin{aligned} &\mathcal{L}[\beta, p, p_0, \mu](u,y) := \beta J(u,y) - \langle p_0, y(\cdot,0) - y_0\rangle_{H_0^1(\Omega)} \\ &+ \int_Q p\Big(\Delta y(x,t) - \gamma y^3(x,t) + f(x,t) + \sum_{i=0}^m u_i(t) b_i(x) y(x,t) - \dot y(x,t)\Big)\mathrm{d}x\mathrm{d}t \\ &+ \sum_{j=1}^q \int_0^T g_j(y(\cdot,t))\mathrm{d}\mu_j(t). \end{aligned} \tag{B.5}$$

The *costate equation* is the condition of stationarity of the Lagrangian $\mathcal{L}$ with respect to the state that is, for any $z \in Y$:

$$\text{(B.6)} \quad \int_Q p(\dot{z} + Az)\mathrm{d}x\mathrm{d}t + \langle p_0, z(\cdot,0)\rangle_{H_0^1(\Omega)} = \sum_{j=1}^{q} \int_0^T \int_\Omega c_j z \mathrm{d}x\mathrm{d}\mu_j(t)$$
$$+ \beta \int_Q (\bar{y} - y_d) z \mathrm{d}x\mathrm{d}t + \beta \int_\Omega (\bar{y}(x,T) - y_{dT}(x)) z(x,T)\mathrm{d}x.$$

To each $(\varphi, \psi) \in L^2(Q) \times H_0^1(\Omega)$, let us associate $z = z[\varphi, \psi] \in Y$, the unique solution of

$$\text{(B.7)} \qquad \dot{z} + Az = \varphi; \quad z(\cdot, 0) = \psi.$$

Since this mapping is onto, the costate equation (B.6) can be rewritten, for $z = z[\varphi, \psi]$ and arbitrary $(\varphi, \psi) \in L^2(Q) \times H_0^1(\Omega)$, as (see [5, Equation (3.7)])

$$\text{(B.8)} \quad \int_Q p\varphi \mathrm{d}x\mathrm{d}t + \langle p_0, \psi\rangle_{H_0^1(\Omega)} = \sum_{j=1}^{q} \int_0^T \int_\Omega c_j z \mathrm{d}x\mathrm{d}\mu_j(t),$$
$$+ \beta \int_Q (\bar{y} - y_d) z \mathrm{d}x\mathrm{d}t + \beta \int_\Omega (\bar{y}(x,T) - y_{dT}(x)) z(x,T)\mathrm{d}x.$$

Next consider the *alternative costates*

$$\text{(B.9)} \qquad p^1 := p + \sum_{j=1}^{q} c_j \mu_j; \quad p_0^1 := p_0 + \sum_{j=1}^{q} c_j \mu_j(0),$$

where $\mu \in BV(0,T)_{0,+}^q$ is the function of bounded variation associated with $\mathrm{d}\mu$. By [5, Cor. 3.3 and Lem. 5], $p^1 \in Y$ and $p^1(\cdot, 0) = p_0^1$. Therefore $p(\cdot, 0)$ makes sense as an element of $H_0^1(\Omega)$, and it follows that $p(\cdot, 0) = p^1(\cdot, 0) - \sum_{j=1}^{q} c_j \mu_j(0) = p_0$.

COROLLARY B.1. *If $\mu \in H^1(0,T)^q$ then $p \in Y$ and*

$$\text{(B.10)} \qquad -\dot{p} + Ap = \beta(\bar{y} - y_d) + \sum_{j=1}^{q} c_j \dot{\mu}_j.$$

B.3. An example. We recall an example from [5, Appendix B] satisfying Hypothesis 3.1. We will check that condition (a) of Theorem 4.5 is satisfied for this example.

We consider the following setting: Let $\Omega = (0, 1)$, and denote by $c_1(x) := \sqrt{2} \sin \pi x$ the first (normalized) eigenvector of the Laplace operator. We assume that $\gamma = 0$, the control is scalar ($m = 1$), $b_0 \equiv 0$ and $b_1 \equiv 1$ in Ω, and that $f \equiv 0$ in Q. Then the state equation with initial condition c_1 reads

$$\text{(B.11)} \quad \dot{y}(x,t) - \Delta y(x,t) = u(t)y(x,t); \quad (x,t) \in (0,1) \times (0,T), \quad y(x,0) = c_1(x), \quad x \in \Omega.$$

The state satisfies $y(x,t) = y_1(t)c_1(x)$, where y_1 is solution of

$$\text{(B.12)} \qquad \dot{y}_1(t) + \pi^2 y_1(t) = u(t)y_1(t); \quad t \in (0,T), \quad y_1(0) = y_{10} = 1.$$

We set $T = 3$ and consider the state constraint (2.4) with $q = 1$ and $d_1 := -2$, and the cost function (2.5) with $\alpha_1 = 0$. The state constraint reduces to

$$y_1(t) \le 2, \quad t \in [0,3]. \tag{B.13}$$

As target functions we take $y_{dT} := c_1$ and $y_d(x,t) := \hat{y}_d(t)c_1(x)$ with

$$\hat{y}_d(t) := \begin{cases} 1.5e^t & \text{for } t \in (0, \log 2), \\ 3 & \text{for } t \in (\log 2, 1), \\ 4 - t & \text{for } t \in (1,3). \end{cases} \tag{B.14}$$

We assume that the lower and upper bounds for the control are $\check{u} := -1$ and $\hat{u} := \pi^2 + 1$. The optimal control is given by

$$\bar{u}(t) := \begin{cases} \hat{u} & \text{for } t \in (0, \log 2), \\ \pi^2 & \text{for } t \in (\log 2, 2), \\ \pi^2 - 1/\hat{y}_d & \text{for } t \in (2,3). \end{cases} \tag{B.15}$$

and the optimal state by

$$\bar{y}_1(t) := \begin{cases} e^t & \text{for } t \in (0, \log 2), \\ 2 & \text{for } t \in (\log 2, 2), \\ 4 - t & \text{for } t \in (2,3). \end{cases} \tag{B.16}$$

The above control is feasible. The trajectory $(\bar{u}, \bar{y})$ is optimal. The costate equation is

$$-\dot{p} + Ap = c_1(\bar{y}_1 - \hat{y}_d) + c_1\dot{\mu}_1, \quad p(\cdot, T) = \bar{y}(T) - y_{dT} = 0. \tag{B.17}$$

Since $\bar{y}$ and y_d are colinear to c_1, it follows that $p(x,t) = p_1(t)c_1(x)$, and

$$-\dot{p}_1 + \pi^2 p_1 = \bar{u}p_1 + \bar{y}_1 - \hat{y}_d + \dot{\mu}_1; \quad p_1(3) = 0. \tag{B.18}$$

Over $(2,3)$, $\dot{\mu}_1 = 0$ (sate constraint not active) and $\bar{y}_1 = \hat{y}_d$, therefore p_1 and p identically vanish. Over $(\log 2, 2)$, $\bar{u}$ is out of bounds and therefore

$$0 = \int_\Omega p(x,t)\bar{y}(x,t) = p_1(t)\bar{y}_1(t)\int_\Omega c_1(x)^2 = 2p_1(t). \tag{B.19}$$

It follows that p_1 and p also vanish on $(\log 2, 2)$ and that

$$\dot{\mu}_1 = -(\bar{y}_1 - \hat{y}_d) > 0, \quad \text{a.a. } t \in (\log 2, 2). \tag{B.20}$$

Over $(0, \log 2)$, the control attains its upper bound, then

$$-\dot{p}_1 = p_1 - \tfrac{1}{2}e^t \tag{B.21}$$

with final condition $p_1(\log 2) = 0$, so that

$$p_1(t) = \frac{e^t}{4} - e^{-t}. \tag{B.22}$$

As expected, p_1 is negative.

LEMMA B.2. *The hypothesis (a) of Theorem 4.5 holds.*

Proof. (i) This has been obtained in part I [5]. Note that the multiplier is unique. (ii) We check the Legendre form condition. For this, we apply the Goh transformation to the example. For (v, z) solution of the linearized state equation we define

$$B := \bar{y}b = \bar{y}_1(t)c_1(x); \quad \xi := z - Bw = (z_1 - \bar{y}_1 w)c_1 \tag{B.23}$$

and we observe that $\xi = \xi_1 c_1$ is solution of

$$\dot{\xi} + A\xi = -(AB + \dot{B})w; \quad \xi(0) = 0; \tag{B.24}$$

where

$$AB + \dot{B} = (\pi^2 - \bar{u})B + \dot{\bar{y}}_1 c_1 = ((\pi^2 - \bar{u})\bar{y}_1 + \dot{\bar{y}}_1)c_1 \tag{B.25}$$

so that

$$\dot{\xi}_1 + (\pi^2 - \bar{u})\xi_1 = B^1 w, \; B^1 := (\pi^2 - \bar{u})\bar{y}_1 + \dot{\bar{y}}_1. \tag{B.26}$$

For checking the Legendre condition ((ii) of Theorem 4.5), we have to check the uniform positivity of the coefficient of w^2 in $\hat{Q}$. This trivially holds on the second and third arcs, since then p and therefore χ vanish, so that the coefficient of w^2 reduces to $\int_\Omega \kappa \bar{y}^2 = \bar{y}_1(t)^2 \geq 1$. We now detail the computation for the first arc. Replacing z by $\xi + Bw$ in the quadratic form $\mathcal{Q}[p](v, z)$ we have

$$\tilde{Q} = \int_Q ((\xi + Bw)^2 + pv(\xi + Bw))\mathrm{d}x\mathrm{d}t + \int_\Omega (\xi(\cdot, T) + B(\cdot, T)w(T))^2 \mathrm{d}x. \tag{B.27}$$

For the second term in the first integral we have

(B.28)

$$\begin{aligned}
\int_Q pv(\xi + Bw)\mathrm{d}x\mathrm{d}t &= \int_0^T \left(p_1 \xi \frac{\mathrm{d}}{\mathrm{d}t} w + \tfrac{1}{2} p_1 \bar{y}_1 \frac{\mathrm{d}}{\mathrm{d}t}(w^2) \right) \mathrm{d}t \\
&= -\int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}(p_1 \xi) w + \tfrac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t}(p_1 \bar{y}_1) w^2 \right) \mathrm{d}t + [\text{boundary-terms}]. \\
&= -\int_0^T \left(p_1 B^1 + \tfrac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t}(p_1 \bar{y}_1) \right) w^2 \mathrm{d}t + [\text{boundary-terms}].
\end{aligned}$$

Finally we obtain that over the first arc, the coefficient of w^2 in the integral term of $\tilde{Q}$ is $2 + e^{2t}/4$. It follows that $\widehat{\mathcal{Q}}[p](w, \xi[w])$ is a Legendre form.
(iii) We check the uniform positivity condition. Any $(w, h) \in PC_2^*$ is such that w vanishes on the two first arcs, and since the costate vanishes on the third arc we have that, using $\bar{y}(x, t) = (4 - t)c_1(x)$ on the third arc

(B.29)

$$\begin{aligned}
\widehat{\mathcal{Q}}[p, \mu](\xi, w, h) &= \int_2^3 \int_\Omega (\xi(x, t) + (4 - t)c_1(x)w(t))^2 \mathrm{d}x\mathrm{d}t + \int_\Omega (\xi(x, T) + hc_1(x))^2 \mathrm{d}x \\
&= \int_2^3 (\xi_1(t) + (4 - t)w(t))^2 \mathrm{d}t + (\xi_1(T) + h)^2.
\end{aligned}$$

This is a Legendre form over $L^2(2, 3)$, and so it is coercive iff it has positive values except at 0. If the value is zero then $w(t) = \xi_1(t)/(t - 4)$ so that ξ vanishes identically and therefore w also, and $h = 0$. The conclusion follows. ∎

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