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Abstract

An s-branching flow f in a network $\mathcal{N} = (D, u)$, such that u is the capacity function, is a flow that reaches every vertex in $V(D) \setminus \{s\}$ from s while loosing exactly one unit of flow in each vertex other than s. It is known that the hardness of the problem of finding k arc-disjoint s-branching flows in a network $\mathcal{N}$ is linked to the capacity u of the arcs in $\mathcal{N}$: for fixed c, the problem is solvable in polynomial time if every arc has capacity $n - c$ and, unless the Exponential Time Hypothesis (ETH) fails, there is no polynomial time algorithm for it for most other choices of the capacity function when every arc has the same capacity. The hardness of a few cases remains open. We further investigate a conjecture that aims to characterize networks admitting k arc-disjoint s-branching flows, generalizing a result that provides such characterization when all arcs have capacity $n-1$, based on Edmonds' branching theorem. We show that, in general, the conjecture is false. However, it holds for some special classes of digraphs, as branchings and spindles with parallel arcs.

1 Introduction

Let $D = (V, A)$ be a digraph. If e is an arc of D from a vertex v to a vertex w, we may refer to e as vw. The *in-degree* of a vertex $X \subseteq V(D)$, denoted by $d^-_D(v)$, is the number of arcs with tail outside of X and head inside of X. If $X = \{v\}$, we simply write $d^-_D(v)$ instead of $d^-_D(\{v\})$ (omitting the braces).

A *network* $\mathcal{N} = (D, u)$ is formed by a digraph $D = (V, A)$ with a *capacity function* $u : A(D) \to \mathbb{Z}_+$. If all arcs in D have capacity λ, we simply write $u \equiv \lambda$. A *flow* in $\mathcal{N}$ is a function $f : A(D) \to \mathbb{Z}_+$ such that $f(vw) \leq u(vw)$, $\forall vw \in A(D)$. For a vertex $v \in V(D)$, we define $f^+(v) = \sum_{vw \in A} f(vw)$ and $f^-(v) = \sum_{wv \in A} f(wv)$, that is, $f^+(v)$ and $f^-(v)$ is the amount of flow *leaving* and *entering* v, respectively. The *balance vector* of a flow f in a

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network $\mathcal{N} = (D, u)$ is the function $b_f : V(D) \to \mathbb{Z}$ associating each vertex $v \in V(D)$ to the value $f^+(v) - f^-(v)$. If $b_f(v) = \ell$, we may also say that v has *balance* ℓ. An (s,t)-*cut* on a network $\mathcal{N} = (D, c)$ is a set of arcs of the form $(S, \overline{S})$, where S, $\overline{S}$ form a partition of $V(D)$ such that $s \in S$ and $t \in \overline{S}$. The capacity of an (s,t)-cut is the sum of the capacities of arcs with tail in S and head in $\overline{S}$ and it is denoted by $u(S, \overline{S})$. For other concepts on graphs and digraphs, we follow standard terminology as in [3, 5].

Flows are widely studied as they allow, with a certain elegance and simplicity, modeling problems in different areas of study such as transportation, logistics and telecommunications. In the theoretical field, they are used to solve various problems in graphs and digraphs. A long list of results related to flows can be found in [1, 9]. In this work, we deal with one variation of the problem where, roughly speaking, k arc-disjoints flows, having in common the same source and relaxed conservation property have to be found. We show that an "Edmonds like" condition, proposed in [6], is not sufficient to guarantee the existence of such flows, even if that is the case for some classes of digraphs, as we are going to prove.

Since a flow in a network can be entirely described by its associated balance vector, the goal of a flow problem can be seen as finding a flow f whose balance vector respects a given set of properties. In the classical Maximum Flow problem, given a network $\mathcal{N} = (D, u)$ and a pair of vertices $s, t \in V(D)$, the objective is to find a flow f in $\mathcal{N}$ maximizing $b_f(s)$ such that $b_f(v) = 0$ for all $v \in V(D) \setminus \{s, t\}$ and $b_f(s) = -b_f(t)$. Such a flow f is known as an (s,t)-*flow* and, in this case, we say that f has *value* $b_f(s)$. This problem can be solved in polynomial time [10]. In the decision version of Maximum Flow, we are given an integer k and the goal is to decide if the network admits an (s,t)-flow of value at least k. By applying a simple reduction to this problem (see [3, Lemma 4.2.2]), we can solve in polynomial time any flow problem in which the aim is to find a flow g with $\sum_{v \in V(D)} b_g(v) = 0$.

The possibility of considering the existence of simultaneous flows in a network gives more power of modeling for this already so useful tool. In [8], a more general version of the flow problem was investigated. There, the goal was to find a collection of (s_i, t_i)-flows that sum a specific value r and such that the sum of all the flows in each arc respects its capacity. It was shown that this version of the problem is NP-hard.

Two flows f and g in a network $\mathcal{N} = (D, u)$ are arc-disjoint if $f(vw) \cdot g(vw) = 0$, $\forall vw \in A(D)$. In [2], the problem of finding arc-disjoint flows was introduced and studied. The authors considered many constraints showing that some generalize important problems, known to be hard, and that there are cases where polynomial-time algorithms are possible. Amongst other hardness results, they showed that the following are NP-complete: the problem of deciding if there are two arc-disjoint flows in a network where all arcs have capacity one, the problem of deciding if there are two arc-disjoint flows with the same balance vectors in a network where all arcs have capacity at most 2, and the problem of deciding if there are two arc-disjoint (s,t)-flows in a network where all arcs have capacity at most two.

We say that a digraph D is an s-*branching* if there is a directed path from s to every other vertex in D and the underlying graph of D is a tree. We also say that D has *root* s. There is an extensive literature concerning the study of branchings, given their relevance both from practical and theoretical point of view due to its numerous applications. A classical result of J. Edmonds [7] characterizes the digraphs containing k arc-disjoint s-branchings and a later proof of the same theorem done in [13] gives a polynomial-time algorithm to find such branchings if they exist. Another algorithm to find the branchings is given in [17].

Theorem 1.1 (Edmonds' branching theorem). *A digraph $D = (V, A)$ has k arc-disjoint s-branchings if and only if $d^-_D(X) \geq k$ for all $X \subseteq V(D) \setminus \{s\}$ with $X \neq \emptyset$.*

A stronger version of this result, that allows each branching to have its own set of roots, is proved in [15]. There are many applications for Edmonds' Theorem on arc-disjoint branchings: it can be used, for example, to prove Menger's Theorem [14], to characterize arc-connectivity [16], and to characterize branching cover [11]. The problem of finding disjoint branchings was recently studied in the setting of temporal graphs in [18].

An *s-branching flow* in a network $\mathcal{N} = (D, u)$ is a flow f such that $b_f(s) = n - 1$ and $b_f(v) = -1$ for every $v \in V(D) \setminus \{s\}$. In other words, f reaches all vertices of D and each vertex other than s *retains* one unit of flow. Finding one s-branching flow in a given network is easy: since $\sum_{v \in V(D)} b_f(v) = 0$, we can reduce this problem to the problem of finding one (s,t)-flow, as discussed above. We can also find k arc-disjoint s-branching flows in polynomial time when $u \equiv n-1$. In [2] it was shown that, in this case, $\mathcal{N}$ admits an s-branching flow if and only if D contains an s-branching. Thus, applying Theorem 1.1, the authors provided a characterization of networks admitting k arc-disjoint s-branching flows, which are exactly the networks constructed on digraphs containing k arc-disjoint s-branchings. They also provided a polynomial time algorithm that finds such flows if they exist. We now discuss how the tractability of this problem, in general, depends on the choice of the capacity function.

In [2], the authors showed that the problem of deciding if a network has k arc-disjoint s-branching flows is NP-complete if every arc has capacity at most two, and in [4] this result was extended to networks with capacity at most ℓ, for every fixed $\ell \geq 2$. For most choices of larger capacities, the problem remains hard. In [4] it was shown that, unless the *Exponential Time Hypothesis* [12] fails, there is no polynomial time algorithm for the problem of finding k arc-disjoint s-branching flows in a network $\mathcal{N} = (D, u)$ with $u \equiv \lambda$ for any choice of λ such that $n/2 \leq \lambda \leq n - (\log n)^{1+\varepsilon}$, $\varepsilon > 0$. In [6] the authors adapted this last proof to show that the same same holds if $(\log n)^{1+\varepsilon} \leq \lambda \leq n/2$. The last two results have an intersection point, which is when $u \equiv n/2$. The two constructions result in the same network in this case, and it is worth noticing that it gives a polynomial time reduction. So, for $u \equiv n/2$, the problem is NP-hard. On the positive side, in [4] it was also shown that the problem is solvable in polynomial time when $\lambda = n - c$ for fixed $c \geq 1$.

Let $\mathcal{N} = (D, u)$ be a network, $s \in V(D)$, and k, λ be non-negative integers. We say that D is *(k, λ, s)-sufficient* if $u \equiv \lambda$ and, for every $X \subseteq V(D) \setminus \{s\}$ with $X \neq \emptyset$, we have

$$d^-_D(X) \geq k \left\lceil \frac{|X|}{\lambda} \right\rceil. \tag{1}$$

In [6], the authors considered a natural extension of Theorem 1.1 to arc-disjoint s-branching flows and digraphs (k, λ, s)-sufficient. It is not hard to visualize the relationship between this property and the one in the statement of Theorem 1.1: if $\lambda = n - 1$, then every arc of D has enough capacity to send as many units of flow as needed to reach set $X \subseteq V(D)$. Thus, capacities are not an issue and Inequality 1 states that $d^-_D(X) \geq k$, as in Theorem 1.1, since $|X| \leq n - 1$. For other choices of λ, note that every s-branching flow on $\mathcal{N}$ must reach a set of vertices X with at least $|X|$ units of flow to cover it. Thus each s-branching flow in $\mathcal{N}$ uses at least $\lceil |X|/\lambda \rceil$ arcs to cover X. More formally, the following was proved.

Proposition 1.2. *[6] Let $\mathcal{N} = (D, u)$ be a network with $u \equiv \lambda$ and $s \in V(D)$. For all $1 \leq \lambda \leq n - 1$, If $\mathcal{N}$ admits k arc-disjoint s-branching flows, then D is (k, λ, s)-sufficient.*

In order to obtain a characterization similar to that one given in Theorem 1.1, Conjecture 1.3 was proposed.

Conjecture 1.3. *[6] Let $\mathcal{N} = (D, u)$ be a network with $u \equiv \lambda$ and $s \in V(D)$. For all $1 \leq \lambda \leq n - 1$, if D is (k, λ, s)-sufficient, then $\mathcal{N}$ admits k arc-disjoint s-branching flows.*

The authors showed that this conjecture is true when D is a *multi-path*, which is a directed path, with, eventually, parallel arcs. In this work, we further investigate Conjecture 1.3, showing that, although it is not true in general, it holds for some particular choices of k and λ and for some special classes of digraphs.

In Section 2 we show our positive results regarding Conjecture 1.3, namely, that it holds when $k = 1$ or $\lambda = 1$; when D is an s-branching where parallel arcs are allowed; and when D is a collection of internally disjoint multi-paths starting at s and all ending on the same vertex $t \in V(D)$. Finally, we show that a simple condition over $d^-_D(X)$, stronger than the one presented in the statement of Conjecture 1.3, guarantees the existence of k arc-disjoint s-branching flows in D. We remark that our proofs are constructive and yield polynomial time algorithms that find the flows or decide that they do not exist. In Section 3 we show that the conjecture is false in general, giving, for any choices of $k \geq 2$ and $\lambda \geq 2$, a network whose digraph satisfies the properties in the statement of Conjecture 1.3 and does not contain k arc-disjoint s-branching flows. In the same section, we also show that it is NP-complete to decide if a network $\mathcal{N} = (D, u)$ admits k arc-disjoint s-branching flows, even if D satisfies the conditions in the statement of Conjecture 1.3. Finally, we close the paper with some open questions (Section 4).

2 Special networks with k arc-disjoint branching flows

In this section, we show some positive occurrences of Conjecture 1.3. The results given here not only show that Conjecture 1.3 is valid for the corresponding cases, but they also imply in polynomial-time algorithms to find the desired flows in the networks $\mathcal{N} = (D, u)$ for which D is (k, λ, s)-sufficient.

We first consider the cases of k or λ with value 1, and for that we need the three following results.

Theorem 2.1 (*Max-flow Min-cut*)**.** *In any network $\mathcal{N} = (D, u)$ with source s and sink t, the value of a maximum flow is equal to the capacity of a minimum cut.*

Theorem 2.2 ([2])**.** *Let k be an integer and $\mathcal{N} = (D, u)$ be a network with $u \equiv 1$ and a prescribed balance vector b such that $b \not\equiv 0$. There exist k arc-disjoint flows in $\mathcal{N}$, all with balance vector b, if and only if $\mathcal{N}$ has a flow f with balance vector $b_f \equiv kb$. Hence one can decide the existence of these flows in polynomial time.*

Lemma 2.3 (**Adapted from [3]**)**.** *Given a network $\mathcal{N} = (D, u)$ and a prescribed balance vector b. Let $M = \sum_{v:b(v)>0}$ and let $\mathcal{N}' = (D', u')$ be a network defined as follows:*

- $V(D') = V(D) \cup \{s', t'\}$;
- $A(D') = A(D) \cup \{s'u : u \in V(D), b(u) > 0\} \cup \{vt' : v \in V(D), b(v) < 0\}$;
- $u'(a) = u(a), \forall a \in A(D)$, $u'(s'u) = b(u)$ *if* $b(u) > 0$ *and* $u'(vt') = -b(v)$ *if* $b(v) < 0$.

Then, $\mathcal{N}$ admits a flow f with balance b if and only if $\mathcal{N}'$ admits a (s', t')-flow f' with value M.

Theorem 2.4 (**Conjecture 1.3 for $\lambda = 1$ or $k = 1$**)**.** *Let D be a (k, λ, s)-sufficient digraph. Then, if $\lambda = 1$ or $k = 1$, the network $\mathcal{N} = (D, u)$, admits k arc-disjoint s-branching flows.*

Proof. We start by showing a result that is common to both cases. Let b_f be the following balance vector: $b_f(s) = k(n-1)$ and $b_f(v) = -k$, for all $v \in V(D) - s$, where $n = |V(D)|$. Let $\mathcal{N}'$ be the network obtained from $\mathcal{N}$ and b_f, as described in Lemma 2.3, and $(S, \overline{S})$ be a minimum (s', t')-cut in $\mathcal{N}'$. We show that $c(S, \overline{S}) = k(n-1)$ which, by the Theorem 2.1, implies that there is an (s', t')-flow with value $k(n-1)$ in $\mathcal{N}'$. Remark that, by construction of $\mathcal{N}'$, $u'(s's) = k(n-1)$, $u'(vt') = k$ and $u'(a) = \lambda$ for every $a \in A(D)$. It's easy to see that $c(S, \overline{S}) \le k(n-1)$ because $c(s', V(D') - s') = k(n-1)$. Then, we might assume that $s \in S$ and we define $X = \overline{S} \setminus \{t\}$. It follows that

$$c(S, \overline{S}) = c(S, X) + c(S, \{t'\}) = \lambda d_D^-(X) + k(n - |X| - 1). \tag{2}$$

Case $\lambda = 1$. Since D is $(k, 1, s)$-sufficient, $d_D^-(X) \ge k|X|$ and resuming (2), we have $c(S, \overline{S}) \ge k|X| + k(n - |X| - 1) = k(n-1)$. Thus, $c(S, \overline{S}) = k(n-1)$ and, by the Lemma 2.3, $\mathcal{N}$ admits a flow with balance b_f. The result follows from Theorem 2.2 which states that $\mathcal{N}$ admits k arc-disjoint s-branching flows if and only if it admits a single flow f with balance b_f.

Case $k = 1$. Here, D is $(1, \lambda, s)$-sufficient, which means that $d_D^-(X) \ge (|X|/\lambda)$, and so $\lambda d_D^-(X) \ge |X|$. Replacing this in (2), we obtain $c(S, \overline{S}) \ge |X| + n - |X| - 1 = n - 1$. Thus, $c(S, \overline{S}) = k(n-1)$. Similarly to the previous case, we conclude, by the Lemma 2.3, that $\mathcal{N}$ admits a flow f with balance b_f. As $k = 1$, we have that f is a branching flow and the result follows. □

We call a digraph $D = (V, A)$ a *multi-branching* if D is a out-branching when we ignore its parallel arcs (see example in Figure 1a).

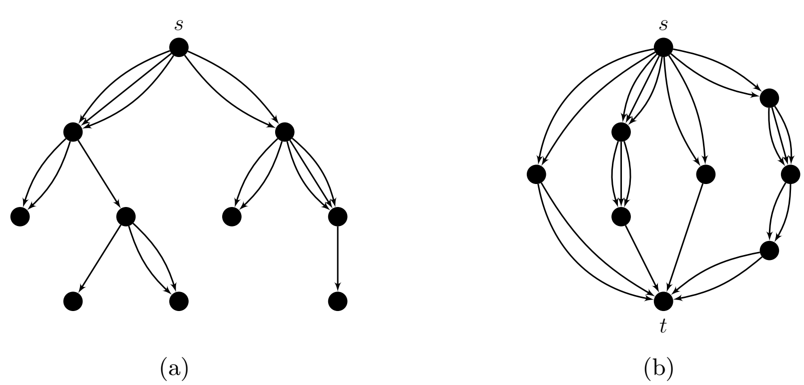

Figure 1: Examples of multi-branching (a) and multi-spindle (b)

Lemma 2.5. *Let D be a (k, λ, s)-sufficient digraph and let W be a nonempty subset of $V(D) \setminus \{s\}$ such that $N_D^+(V(D) \setminus W) = \{w\}$, $w \in W$. Then $D[W]$ is (k, λ, w)-sufficient.*

Proof. Since w is the unique vertex of W which has in-neighbours in $V(D) \setminus W$, then we have $d_{D[W]}^-(v) = d_D^-(v)$ for every $v \in W \setminus \{w\}$. Thus, for every $X \subseteq W \setminus \{w\}$ we have $d_{D[W]}^-(X) = d_D^-(X)$. □

Theorem 2.6 (**Conjecture 1.3 for multi-branchings**). *Let D be a (k, λ, s)-sufficient multi-branching. Then the network $\mathcal{N} = (D, u)$, with $u \equiv \lambda$, admits k arc-disjoint s-branching flows.*

Proof. We are going to use induction on the height h of the multi-branching with root s D to construct k arc-disjoint s-branching flows $f_1, f_2, \ldots, f_k$ in $\mathcal{N}$.

The base case is $h = 1$. In this case D is a star and, since it is (k, λ, s)-sufficient, there are at least k arcs from s to every other vertex and we can use one arc for each flow f_j, for $j \in [k]$.

Induction step: Suppose that the Conjecture 1.3 holds for multi-branchings of height $h < q$. Assume that D has height q and let $r_1, r_2, ..., r_p$ be the out-neighbours of s in D.

Let B_{r_i} be the subgraph of B_s that is a multi-branching with root r_i, for each $i \in [p]$. Observe that, by Lemma 2.5, B_{r_i} is (k, λ, r_i)-sufficient and has height $h_i \leq q - 1$, thus, by induction hypothesis, B_{r_i} has k arc-disjoint r_i-branching flows $f_1^i, f_2^i, \ldots, f_k^i$. We know that $d^-(r_i) \geq k \left\lceil \frac{|B_{r_i}|}{\lambda} \right\rceil$ and then we can use $\left\lceil \frac{|B_{r_i}|}{\lambda} \right\rceil$ of these arcs to send the proper amount of flow from s to r_i on each x_j to complete the k s-branching flows. □

Since a multi-path is also a multi-branching, Theorem 2.6 generalizes the result of [6] for multi-paths. We call a *multi-spindle* the class of digraphs $\mathcal{D}_\ell$, for $\ell \geq 1$, formed by a source vertex s, a sink vertex t, ℓ pairwise internally vertex-disjoint multi-paths $P_1, \ldots, P_\ell$ from s to t (see example in Figure 1b), each with $p_i \geq 1$ internal vertices, $1 \leq i \leq \ell$, respectively.

Theorem 2.7 (Conjecture 1.3 for multi-spindles). *If D is a (k, λ, s)-sufficient digraph in $\mathcal{D}_\ell$, then the network $\mathcal{N} = (D, u)$ with $u \equiv \lambda$ admits k arc-disjoint s-branching flows.*

Proof. Let D be a digraph in $\mathcal{D}_\ell$ that is (k, λ, s)-sufficient. By definition, D is composed by the multi-paths P_1, P_2, $\ldots$, P_ℓ and $V(P_i) \cap V(P_j) = \{s, t\}$ for every $1 \leq i < j \leq \ell$. We denote the p_i internal vertices of P_i by $v_{p_i}^i, v_{p_i-1}^i, \ldots, v_2^i, v_1^i$ in this order, for every $i \in [\ell]$. For every $j \in [p_i]$, we define $r_j^i \geq 0$ so that $d^-(v_j^i) = k\lceil j/\lambda \rceil + r_j^i$. Remark that $d^-(v_j^i) \geq k\lceil j/\lambda \rceil$, otherwise the set $\{v_j^i, v_{j-1}^i, \ldots, v_1^i\}$ would contradict the fact that D is (k, λ, s)-sufficient, since the only arcs entering in it are those arriving in v_j^i. Considering only the vertices with an index multiple of λ, that is, the vertices $v_{j\lambda}^i$ for every $1 \leq j \leq \lfloor p_i/\lambda \rfloor$, let

$$r(P_i) = \begin{cases} k, \text{ if } p_i \leq \lambda, \\ \min\{r_{j\lambda}^i \mid 1 \leq j \leq \lfloor p_i/\lambda \rfloor\}, \text{ otherwise.} \end{cases}$$

For all $i \in [\ell]$, we define e_i as the number of arcs from v_1^i to t and $m_i = \min\{r(P_i), e_i\}$. Let $k' = \sum_{i=1}^{\ell} m_i$.

Assume without loss of generality that $P_1, P_2, \ldots, P_q$ are the paths on which $r(P_i) \leq e_i$, for $i \in [\ell]$. Then $k' = \sum_{i=1}^{q} r(P_i) + \sum_{i=q+1}^{\ell} e_i$. Consider the set $X = \bigcup_{i=1}^{q} X_i \cup \{t\}$, where $X_i = \{v_{j_i\lambda}^i, v_{(j_i\lambda)-1}^i \ldots, v_1^i\}$ and $d^-(v_{j_i\lambda}^i) = kj_i + r(P_i)$. We have that

$$d_D^-(X) = \sum_{i=1}^{q} d^-(X_i) + \sum_{i=q+1}^{\ell} e_i = k\sum_{i=1}^{q} j_i + \sum_{i=1}^{q} r(P_i) + \sum_{i=q+1}^{\ell} e_i = k\sum_{i=1}^{q} j_i + k'. \tag{3}$$

Since D is (k, λ, s)-sufficient and $|X_i| = j_i\lambda$

$$d_D^-(X) \geq k\left\lceil \frac{|X|}{\lambda} \right\rceil = k\left\lceil \frac{1}{\lambda} + \frac{\lambda\sum_{i=1}^{q} j_i}{\lambda} \right\rceil = k\sum_{i=1}^{q} j_i + k. \tag{4}$$

Combining (3) and (4) we conclude that $k' \geq k$. To finally construct the k arc disjoint branching flows, we need the following claim.

Claim 2.8. *There are k arc-disjoint flows $x^i_1, \dots, x^i_k$ on the network $\mathcal{N}_i = (P_i, u)$, with $u \equiv \lambda$, such that $x^i_1, \dots x^i_{m_i}$ are branching flows in $\mathcal{N}_i$ and $x^i_{m_i+1}, \dots, x^i_k$ are branching flows in $\mathcal{N}_i - t$.*

Proof. We show how to construct the flows $x^i_1, x^i_2, \dots, x^i_k$. For every $j \in [p_i]$, the vertex v^i_j must receive $j+1$ units of flow on the first m_i flows and j units on remaining $k - m_i$ flows. Since $d^-_D(v^i_j) = d^-_{P_i}(v^i_j) \geq k\lceil j/\lambda \rceil$, we can use a distinct group of $\lceil j/\lambda \rceil$ arcs to send j units of flow on each one of the k flows and we only have to argue how to send the extra unit for the flows $x^i_1, \dots x^i_{m_i}$. If j is a multiple of λ, then there are at least another $r(P_i)$ extra arcs entering v^i_j that can be used to send the extra unit because $r(P_i) \leq m_i$ (note that, if $r(P_i) = 0$ then $m_i = 0$ and no extra unit is needed). Otherwise, j is not multiple of λ and when we send j units of flow through $\lceil j/\lambda \rceil$ arcs, there is an arc that will not be used in the maximum capacity and so we can use it to send the extra unit. Applying this method iteratively from $v^i_{p_i}$ to v^i_1, we arrive at v^i_1 with 2 units of flow on the flows $x^i_1, \dots x^i_{m_i}$ and we can send the extra unit of each flow to t because $m_i \geq e_i$. ♢

For each $i \in [\ell]$, we compute the flows $x^i_1, x^i_2, \dots, x^i_k$ as in the Claim 2.8. Observe that k' is the number of flows that reach t. If $k' = k$, then we can rename these k flows in such a way that each $x_j = \bigcup_{i=1}^{\ell} x^i_j$ is an s-branching flow on $\mathcal{N}$. Finally, if $k' > k$, we take a flow that reaches t and we modify it so that it doesn't reach t anymore, and we repeat this process until there are only k flows that reach t. □

It is worth to notice that the above proofs for multi-branchings and multi-spindles together with Proposition 1.2, besides giving a complete characterization of the digraphs in these classes having k arc-disjoint branching flows, they lead to polynomial-time algorithms to find such flows, once testing the (k, λ, s)-sufficiency for multi-branchings and multi-spindles can be done in polynomial time. The algorithms work for every value of λ, even those for which the problem of finding k arc-disjoint branching flows is known to be hard in general.

We end this section showing that, for particular choices of $d^-_(X)$ and λ, a simple stronger condition than the one presented in the statement of Conjecture 1.3 is sufficient to find k arc-disjoint s-branching flows in a given network.

Theorem 2.9. *Let $\mathcal{N} = (D, u)$ be a network with $u \equiv \lambda$, and p be an integer such that $p\lambda \geq n - 1$. If $d^-_D(X) \geq pk$ for all $X \subseteq V(D) \setminus \{s\}$ with $X \neq \emptyset$, then $\mathcal{N}$ admits k arc-disjoint s-branching flows.*

Proof. By Theorem 1.1, there are pk arc-disjoint s-branchings $B_1, \dots, B_{pk}$ in D. For $i \in [k]$, let D_i be the digraph formed by $B_{(i-1)p+1} \cup \cdots \cup B_{ip}$ and $\mathcal{N}_i = (D_i, c)$.

Now, for every non-empty $X \subseteq V(D) \setminus \{s\}$, we conclude that $d^-_{D_i}(X) \geq p$ since each B_i with $i \in [pk]$ must reach X at least once. Moreover, by our choice of p, we have that every $\mathcal{N}_i$ is $(1, \lambda, s)$-sufficient. Thus, by Theorem 2.4, we have that each $\mathcal{N}_i$ admits an s-branching flow x_i. Finally, as every arc of D appears in at most one digraph D_i, we conclude that the s-branching flows $x_1, \dots, x_k$ are pairwise arc-disjoint. The result follows since each of these flows can be promptly used to construct an s-branching flow in $\mathcal{N}$, by copying arcs and flow functions used by each flow x_j with $j \in [k]$. □

3 Counterexamples and hardness results

In this section, we first prove that Conjecture 1.3 is not always true, by showing how to construct a family of digraphs that contradicts it. Next, we show that it is NP-complete to decide if there are k s-branching flows in (k, λ, s)-sufficient digraphs.

Theorem 3.1. *For all $\lambda \geq 2$ and for all even $k \geq 2$, there exists a (k, λ, s)-sufficient digraph D such that the network $\mathcal{N} = (D, u)$, with $u \equiv \lambda$, does not admit k arc-disjoint s-branching flows.*

Proof. Given an even $k \geq 2$ and $\lambda \geq 2$, start the construction of $\mathcal{N}$ by adding the vertices s, a, b, c and d then add an arc from s to every other vertex along with the arcs ab, ac, bd, and cd. Subdivide the arc bd (cd) $\lambda - 2$ times and let B (C) be the union of $\{b\}$ ($\{c\}$) and the set of vertices obtained by the subdivisions of bd (cd) (Note that if $\lambda = 2$ nothing changes at this step, which means that $B = \{b\}$ and $C = \{c\}$), and then double the arcs between vertices of B (C). Let P be a path with λ vertices and denote by e and f, respectively, the first and last vertex of P. Triple the first $\lambda - 2$ arcs of P and double the last one (the one that enters f). After that, add P to D along with the arcs de, fa and two parallel arcs se. To complete the construction, replace each arc by $k/2$ copies of itself and set $u \equiv \lambda$. See Figure 2.

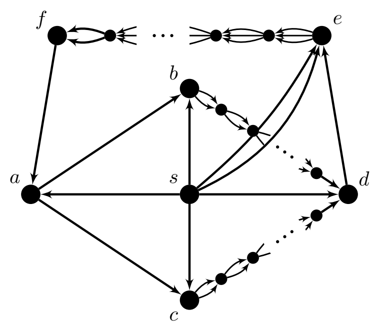

Figure 2: Counter-example for $k = 2$ and $\lambda \geq 2$

Here, we say that a set $X \subseteq V(D)$ is *satisfied* if $d^-_D(X) \geq k\lceil |X|/\lambda \rceil$.To prove that D is (k, λ, s)-sufficient, we show the cases where $k = 2$ because for a larger k we multiply the in-degree of each subset of $V(D) - s$ by $k/2$. Observe that, except for s, every vertex has in-degree at least 2 in D and that for every $X \subseteq V(D) - s$ each vertex from $\{a, b, c, d\}$ which belongs to X sums at least one unit to the its in-degree and the vertex e sums two. Consider a set $X \subseteq V(D) - s$. If $D[X]$ has a cycle, then $d^-_D(X) \geq 5$, once X necessarily contains a, d, P and either B or C (or both), but its also true that $|X| \geq 2\lambda + 1$, which means that should be at least one extra arc entering in X for it to be satisfied. To verify the existence of this extra arc assume without loss of generality that $B \subset X$ (the case where $C \subset X$ is symmetric) and take the longest (z, d)-path in $D[X]$ such that $z \in C$. If no such path exists, there is at least one arc going from C to d which was not counted before and if the path exists, then either $z = c$ or the two arcs yz enters X, where y is the in-neighbor of z which, by the choice of z, cannot be in X.

Now consider that $D[X]$ is acyclic. Note that each source in $D[X]$ contributes with at least two units in the in-degree of X. Since each weak component of $D[X]$ has at least one source, then every X with $|X| \leq \lambda$is satisfied. Similarly, the same also holds when $\lambda < |X| \leq 2\lambda$ if $D[X]$ has more then one weak component or only one component with many sources. Otherwise $D[X]$ has only one weak component W with a single source w, and we have 5 cases:

- $w \in B$. Observe that in this case $\{d, e\} \subset X$, once that $|X| \geq \lambda + 1$ and $|B| = \lambda - 1$, which means that $d^-_D(X) \geq 5$.
- $w \in C$. It's analogous to the previous one.

- $w = a$. In this case, there are at least two vertices from $\{b, c, d\}$ in X. That is, $d^-(X) \geq 4$.

- $w = d$. If d is the source, then $e \in X$, and thus $d^-_D(X) \geq 5$.

- $w \in V(P)$. Except for f, every vertex of P has in-degree three and since X must contains a, then $d^-_D(X) \geq 4$. When $w = f$, besides a, X must also contains at least one vertex from {b, c} which guaranties $d^-_D(X) \geq 4$.

When $2\lambda + 1 \leq |X| \leq 3\lambda$, the in-degree of X must be at least 6. We start with the case that $D[X]$ has just one weak component. If w is a source in $D[X]$, again we have 5 subcases:

- $w \in B$. In this case, since $|B \cup \{\text{d}\}| = \lambda$, either there is a vertex in C which is also a source in $D[X]$ or $a \in X$. In both situations, $d, e \in X$ because $|B \cup C| < 2\lambda$. Thus, $d^-_D(X) \geq 6$.

- $w \in C$. It's analogous to the previous one.

- $w = a$. Once $|\{a, d\} \cup B \cup C| = 2\lambda$, if a is a source, there is a path in $D[X]$ that starts in a, goes through B or C, d and ends in some vertex of P.

- $w = d$. When d is a source, for sure $\{a, b, c, d, e\} \subset X$.

- $w \in V(P)$. Here, in order to avoid cycles, we know that if $w = f$, necessarily a, b, c and d are also included in X and if $w \neq f$, at least a, b and c are included in X but, in this case, w already contributes with 3 for the in-degree of X. Thus $d^-_D(X) \geq 6$.

If $D[X]$ has more than 2 weak components we already know that $d^-_D(X) \geq 6$. If $D[X]$ has two weak components Y, Z, one of them, lets say Y, has at least $\lambda + 1$ vertices and hence, as in the previous cases, $d^-_D(Y) \geq 4$. Thus, $d^-(X) \geq 6$.

Notice that if there are k arc-disjoint s-branching flows $x_1, x_2, \ldots, x_k$ in $\mathcal{N}$, as $d^-_D(B) = d^-_D(C) = k$, each one of these k arcs has to be used by a different flow, and carry at least $\lambda - 1$ units of flow, that is, one unit for each vertex of B and C. Now observe that $k/2$ of the arcs which enter in B come from a and the same for the ones that enter C, and then a must receive λ units of flow on each of it's k incoming arcs in order to send the proper amount of flow to B and C. Assume without loss of generality that the $k/2$ copies of sa are going to be used by $x_1, x_2, \ldots, x_{k/2}$ and the $k/2$ copies of fa are left for $x_{k/2+1}, x_{k/2+2}, \ldots, x_k$. Since the in-degree of f is also equal to k, f can receive at most λ units of flow of each $x_1, x_2, \ldots, x_k$ and hence it can send at most $\lambda - 1$ units of any of these flows to a. Therefore, $\mathcal{N}$ does not admits $\mathcal{N}$ k arc-disjoint s-branching flows. □

So, by Theorem 3.1, there are networks $\mathcal{N} = (D, u)$, with D being (k, λ, s)-sufficient, which have k arc-disjoint s-branching flows and others that do not have such flows. We believe it is not always computationally easy to check the (k, λ, s)-sufficiency of a digraph, but, even if we know that D has such property, it is hard to decide if the desired flows exist.

Theorem 3.2. *Given a Network $\mathcal{N} = (D, u)$, with $u \equiv \lambda$ and a (k, λ, s)-sufficient digraph D, as inputs, it is* NP*-complete to decide whether $\mathcal{N}$ admits k arc-disjoint s-branching flows, for every fixed $k \geq 2$.*

Proof. We are going to reduce the PARTITION problem to our problem. The PARTITION problem consists in deciding whether a given set of natural numbers $S = \{a_1, a_2, \ldots, a_q\}$, on which $\sum_{i=1}^{q} a_i = 2\lambda$, admits $J \subset S$, such that $\sum_{a_j \in J} a_j = \lambda$. From S, we construct a network

$\mathcal{N} = (D, u)$ such that D is a (k, λ, s)-sufficient network that admits k arc-disjoint branching flows $x_1, x_2, \ldots, x_k$ if and only if S is a *yes* instance of the PARTITION problem. We can assume that $a_i \leq \lambda$, for $i \in [q]$, otherwise we have a *no*-instance.

We begin the construction by adding a vertex s, a multi-path $P_0 = (v_1^0, v_2^0, \ldots, v_\lambda^0)$ and, for every $i \in [q]$, multi-paths $P_i = (v_1^i, v_2^i, \ldots, v_{\lambda-a_i}^i)$, each one with k parallel arcs between consecutive vertices. Also, for every $i \in [q]$, we add one arc $v_{\lambda-a_i}^i v_1^0$ and k parallel arcs sv_1^i. We then add $k-2$ parallel arcs sv_1^0. See Figure 3. Finally, we set $u \equiv \lambda$ to conclude the construction.

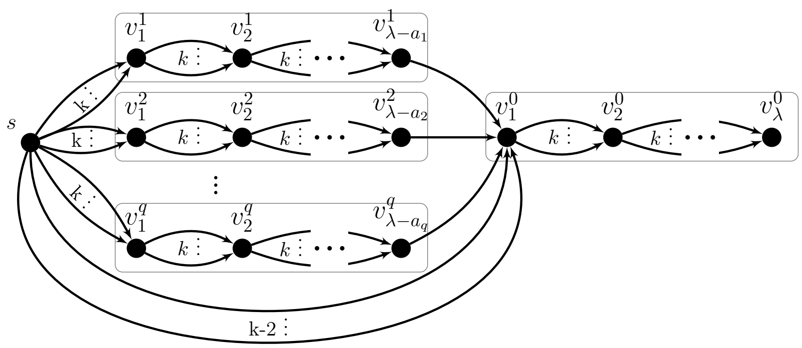

Figure 3: The network $\mathcal{N} = (D, c)$ constructed from an instance of the PARTITION problem

First, let us check that D is (k, λ, s)-sufficient. By contradiction, assume that there exists a set $X \subseteq V - s$ such that $d_D^-(X) < k\lceil |X|/\lambda \rceil$. Note that $X \neq V(D) - s$ since $|V - s| = (q-1)\lambda$ and $d^-(V - s) = kq + (k-2)$. Recall that the in-degree of every vertex of P_i is k and thus the in-degree of X is at least k. Therefore X cannot be entirely contained in a unique P_i, $i \in [q]$, once, in this case, $|X| \leq \lambda$ and then $\lceil |X|/\lambda \rceil = 1$. Similarly, X cannot be entirely contained in P_0, since beside the $k-2$ arcs from s, there are other q arcs arriving in v_1^0 and $q \geq 2$ (otherwise S is a *no*-instance of the PARTITION problem). Now, let $I = \{i : 1 \leq i \leq q, V(P_i) \cap X \neq \emptyset\}$. Observe that $d_D^-(X) \geq k|I|$. We are going now to consider the following cases:

(i) $V(P_0) \cap X = \emptyset$. Then $|X| \leq \lambda|I|$ and $\lceil |X|/\lambda \rceil \leq |I|$, so $d_D^-(X) \geq k|I| \geq k\lceil |X|/\lambda \rceil$, a contradiction to the choice of X.

(ii) $X \cap V(P_0) \neq \emptyset$. Then $|X| \leq \lambda|I| + \lambda = \lambda(|I| + 1)$. Take the small j such that $v_j^0 \in X$. We analyze two possibilities:

(ii.1) $j > 1$. Then there are k arcs $v_{j-1}^0 v_j^0$ that enter X, that is, $d^-(X) \geq k|I| + k \geq k\lceil |X|/\lambda \rceil$ since $|X| \leq \lambda(|I| + 1)$, a contradiction.

(ii.2) $j = 1$. Therefore $v_1^0 \in X$ and $d^-(X) \geq k|I| + k - 2$. Again, we consider some cases:

(ii.2.1) If there are $y, z \in [q]$ such that $v_{\lambda-a_y}^y, v_{\lambda-a_z}^z \notin X$, then $v_{\lambda-a_y}^y v_1^0, v_{\lambda-a_z}^z v_1^0$ contribute with two to the in-degree of X, and again we have $d^-(X) \geq k|I| + k$.

(ii.2.2) If there is only one $y \in [q]$ such that $v_{\lambda-a_y}^y \notin X$ while there is some other vertex of P_y in X, then $d^-(X) \geq kq+k-1$, which is a contradiction because $|X| < q\lambda$. Thus, $|I| = q-1$ and $d^-(X) \geq k(q-1) + k - 1 = kq - 1$. That is, $d^-(X) \geq kq - k$. As $|X| < \lambda(q-1)$, we have that $d^-(X) \geq k\lceil |X|\lambda \rceil$, a contradiction.

(ii.2.3) If all the last vertices are in X, then $d^-(X) \geq d^-(V-s)$, and we have another contradiction, since $|X| < |V(D)-s|$.

As we got a contradiction on each case, it follows that D is (k, λ, s)-sufficient.

Now we assume that there exists $J \subset S$, such that $\sum_{j\in J} a_j = \lambda$. In order to construct the arc-disjoint s-branching flows $x_1, x_2, \ldots, x_k$ in $\mathcal{N}$, we are going to specify only the amount of flow in the arcs that leave s on each flow. Note that this is sufficient because every vertex in $V(D)-s$ has a unique out-neighbour.

For every $j \in J$ and $i \in S-J$, we construct x_1 and x_2 in the following way: for two different copies of the arc sv_1^j, we set $x_1(sv_1^j) = \lambda$ and $x_2(sv_1^j) = \lambda - a_j$. Additionally, for two different copies of the arc sv_1^i, we set $x_1(sv_1^i) = \lambda - a_i$ and $x_2(sv_1^i) = \lambda$. We also set $x_1(sv_1^0) = x_2(sv_1^0) = 0$ for every copy of sv_1^0. Observe that $x_1(v^j_{\lambda-a_j}v_1^0) = a_j$ and $x_1(v^j_{\lambda-a_j}v_1^0) = 0$, that is, $\sum_{m\in[q]} x_1(v^m_{\lambda-a_m}v_1^0) = \lambda = |V(P_0)|$, and since the same holds for x_2, both x_1 and x_2 are s-branching flows.

For $2 < r \leq k$ and $\ell \in \{0\} \cup [q]$, we set $x_r(sv_1^\ell) = |V(P_\ell)|$, and we follow decreasing one unit of flow on each vertex of P_ℓ, always choosing different arcs for each flow, in order to make them arc-disjoint.

Finally, we assume that there exist k arc-disjoints s-branching flows $x_1, x_2, \ldots, x_k$ in $\mathcal{N}$. We first claim that $x_r(v^i_{\lambda-a_i}v_1^0) \leq a_i$, for all $r \in [k], i \in [q]$. Since $x_1, x_2, \ldots, x_k$ are arc-disjoints, there is only one flow x_t which is positive in $v^i_{\lambda-a_i}$. If $x_t(v^i_{\lambda-a_i}v_1^0) > a_i$, then, as $|V(P_i)| = \lambda - a_i$, v_1^i must receive more than λ units of flow in x_t, that is, x_t uses at least two copies of the arc sv_1^i. This is a contradiction because there are only k copies of sv_1^i and each flow must use exactly one copy. Thus, the claim follows, and $\sum_{r\in[k]}\sum_{i\in[q]} x_r(v^i_{\lambda-a_i}v_1^0) \leq 2\lambda$. In fact, this sum is equal to 2λ, because each x_r must reach v_1^0 with λ units of flow and there are only $k-2$ copies of sv_1^0. Let x_1 and x_2 be the flows that reach v_1^0 without using the copies of sv_1^0. Considering x_1, we define $J = \{a_j : x_1(v^j_{\lambda-a_j}v_1^0) = a_j\}$. Therefore, since $\sum_{j\in J} a_j = \sum_{i\in[q]} x_1(v^i_{\lambda-a_i}) = \lambda$, S is an *yes* instance of the PARTITION problem. □

Observe that, by the proof of Theorem 3.2, the digraphs constructed from negative instances of the NUMBER PARTITION problem are also counterexamples for Conjecture 1.3.

4 Concluding remarks

In this work, we studied the characterization of networks admitting k arc-disjoint s-branching flows. We showed that, in some cases (Theorems 2.6 and 2.7), an "Edmonds like" condition (see Equation 1) is enough to guarantee the existence of the k-arc disjoint s-branching flows, but, in other cases, a stronger statement will be needed (Section 3). This disproves Conjecture 1.3 in general, although the cases for which it is valid are still interesting because, among other reasons, they have been resulting in polynomial-time algorithms for them.

There are still many compelling cases of Conjecture 1.3 that worth to be analyzed. For instance, the case of networks with $\lambda = n - c$, for fixed $c \geq 1$, over (k, λ, s)-sufficient digraphs is more restricted than the one considered in Theorem 2.9. The result in [4] states that the problem of deciding whether such networks admit k arc-disjoint s-branching flows is XP with parameter c. It is not known if this problem is FPT with the same parameter, and we believe that Conjecture 1.3 holds for this case.

Another direction of this research would be to work in the characterization through a stronger condition, as cited before. Some interesting complexity questions are still open, such is a possible dichotomy in DAG's between the easy and hard cases. In other words, it would be very interesting to know if there a class of digraphs $\mathcal{H}$, all of which are DAGs, such that Conjecture 1.3 holds for every $D \in \mathcal{H}$ and not for every $D \notin \mathcal{H}$.

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