On an Integral Identity

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Abstract. We give three elementary proofs of a nice equality of definite integrals, recently proven by Ekhad, Zeilberger and Zudilin. The equality arises in the theory of bivariate hypergeometric functions, and has connections with irrationality proofs in number theory. We furthermore provide a generalization together with an equally elementary proof and discuss some consequences.

1. INTRODUCTION. The following infinite family of equalities between definite integrals was proven by S. B. Ekhad, D. Zeilberger, and W. Zudilin in [5], using the Almkvist–Zeilberger *creative telescoping algorithm* [2] for symbolic integration:

$$\int_0^1 \frac{x^n \left(1 - x\right)^n}{\left((x + a)\left(x + b\right)\right)^{n+1}} \, \mathrm{d}x = \int_0^1 \frac{x^n \left(1 - x\right)^n}{\left((a - b)\, x + (a + 1)\, b\right)^{n+1}} \, \mathrm{d}x, \qquad (1)$$

for any reals $a > b > 0$ and any nonnegative integer n.

As pointed out in [5], these integrals "are not taken from a pool of no-one-cares analytic creatures": they are related to rational approximations to some logarithmic values [1] and a trained eye could recognize in identity (1) a particular case of a known relation for Appell's bivariate hypergeometric function (see Section 5 below).

The proof provided in [5] is, without any doubt, elementary. It requires some clever (and at first sight, magic) auxiliary rational functions coming from the silicon guts of the first author of [5]. Of course, this is not objectable at all, and we are ourselves convinced that computer-assisted proofs are an increasingly important trend in mathematics. At the same time we think that, *pour l'honneur de l'esprit humain*, it is of some interest to offer a proof that a first-year undergraduate could not only follow but also create by hand.

In Sections 2–4 of this article we provide three elementary proofs of the identity (1). We also generalize the identity. In Section 5 we discuss some direct consequences, emphasizing the relation of the identities with known identities for hypergeometric functions. We also give a couple of combinatorical identities, and finally conclude that the Legendre polynomials are eigenfunctions of a certain differential operator.

2. FIRST PROOF: USING A RATIONAL CHANGE OF VARIABLES. Note that the natural range to assure the convergence of the integrals is $a, b > 0$ and $n \in \mathbb{R}_{>-1}$. Let us first demonstrate that an utterly simple change of variables proves the equality in this extended range and in fact an even more general equality:

Theorem 1. *If $a, b > 0$, $k, n \in \mathbb{R}$, and $s, \ell \in \mathbb{R}_{>-1}$, then*

$$\int_0^1 \frac{x^\ell (1 - x)^s}{(x + a)^{k+1} \, (x + b)^{n+1}} \, \mathrm{d}x = \frac{(b + 1)^{s-n}}{b^{n-\ell}} \int_0^1 \frac{x^s \, (1 - x)^\ell \, (x + b)^{n+k-\ell-s}}{\left((a - b)\, x + (a + 1)b\right)^{k+1}} \, \mathrm{d}x. \qquad (2)$$

Proof. Consider the rational change of variables

$$x = \frac{b(1 - u)}{b + u}, \quad \text{and consequently} \quad \mathrm{d}x = -\frac{b(1 + b)}{(b + u)^2} \, \mathrm{d}u.$$

Note that it is a promising change because it takes 0, 1, $b(a + 1)/(b - a)$, and ∞, which are the singularities of the second integrand, into 1, 0, $-a$, and $-b$, respectively,

which are singularities of the first integrand. The interval $[0, 1]$ is preserved. Thus, a direct calculation shows that the integral in the left-hand side is

$$
\begin{aligned}
I &= \int_0^1 \frac{\left(\frac{b(1-u)}{b+u}\right)^\ell \left(1 - \frac{b(1-u)}{b+u}\right)^s}{\left(\frac{b(1-u)}{b+u} + a\right)^{k+1}\left(\frac{b(1-u)}{b+u} + b\right)^{n+1}} \frac{b(1+b)}{(b+u)^2} \, du \\
&= \int_0^1 \frac{(1-u)^\ell u^s b^{\ell+1}(1+b)^{s+1}(u+b)^{k+n-\ell-s} \, du}{\left(b(1-u) + a(b+u)\right)^{k+1}\left(b(1-u) + b(b+u)\right)^{n+1}} \\
&= \frac{(b+1)^{s-n}}{b^{n-\ell}} \int_0^1 \frac{u^s(1-u)^\ell(b+u)^{k+n-\ell-s}}{\left((a-b)u + (a+1)b\right)^{k+1}} \, du,
\end{aligned}
$$

which proves the result. ∎

We will come back to (2) and draw some conclusions from this more general identity in Section 5.

3. SECOND PROOF: USING INDEFINITE INTEGRATION.

We still consider the extended convergence range $a, b > 0$, but now n is a nonnegative integer. The following proof is based on the generating functions [11] of the sequence of integrals when n varies.

Proof of (1) *for* $a, b > 0$ *and* $n \in \mathbb{Z}_{\geq 0}$. Let I_1 and I_2 be the generating functions of each side of (1), i.e., multiplying by t^n and summing from $n = 0$ to ∞. They clearly converge uniformly for small values of t and we have $I_j(t) = \int_0^1 dx/P_j(x, t)$ where

$$
\begin{aligned}
P_1(x, t) &= (x + a)(x + b) - tx(1 - x) \quad \text{and} \\
P_2(x, t) &= (a - b)x + (a + 1)b - tx(1 - x).
\end{aligned} \tag{3}
$$

If $P(x) = Ax^2 + Bx + C$ has no zero in $[0, 1]$, and $\Delta = B^2 - 4AC > 0$, then, using standard integration techniques, we obtain

$$
\int_0^1 \frac{dx}{P(x)} = \frac{1}{r} \log\left(\frac{B + 2C + r}{B + 2C - r}\right) \quad \text{with} \quad r = \sqrt{\Delta}.
$$

Note that for t small enough, P_1 and P_2 fulfill these conditions. A calculation shows that both polynomials have the same discriminant Δ and the same values of $B + 2C$. Hence $I_1(t) = I_2(t)$ in some interval containing the origin and then the integrals in (1), which are their Taylor coefficients, are equal. ∎

4. THIRD PROOF: CREATIVE TELESCOPING.

We cannot resist the temptation to offer a third proof, in the spirit of the one in [5], but based on a different kind of "creative telescoping." The starting point is the same as in Section 3, namely that the family of identities (1) is equivalent to the fact that the two integrals $I_1(t)$ and $I_2(t)$ between $x = 0$ and $x = 1$ of the rational functions $F_1 = 1/P_1$ and $F_2 = 1/P_2$, with P_j as in (3), are equal.

Creative telescoping [2], this time in the classical "differential-differential" setting, shows that F_1 and F_2 satisfy the equalities[1]

$$
(t - 2ab - a - b)F_j + \left(t^2 - 2t(2ab + a + b) + (a - b)^2\right)\frac{\partial F_j}{\partial t} + \frac{\partial}{\partial x}(F_j R_j) = 0,
$$

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[1]We have obtained these equalities with the help of the commands Zeilberger and creative_telescoping available in the Maple packages DEtools and Mgfun.

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where $R_1(t, x)$ and $R_2(t, x)$ are the rational functions

$$R_1(t, x) = \big((a + b + t + 2)x + 2ab + a + b - t\big)x$$

and

$$R_2(t, x) = \frac{\big((a - b)^2 - (2ab + a + b)t\big)x + b(a + 1)(a - b + t)}{t}.$$

Hence, by integration between $x = 0$ and $x = 1$, one obtains that both I_1 and I_2 are solutions of the differential equation

$$(t - 2ab - a - b)I(t) + \big(t^2 - 2t(2ab + a + b) + (a - b)^2\big)I'(t) + 2 = 0.$$

Therefore $I_1 = I_2$ by Cauchy's theorem, since $I_1 - I_2$ is the solution of a differential equation of first order with leading term nonvanishing at $t = 0$, and its evaluation at $t = 0$ is zero, as $I_1(0) = I_2(0) = \frac{1}{a-b} \ln\left(\frac{a(b+1)}{(a+1)b}\right)$.   □

5. SOME CONSEQUENCES.

Appell's identity. We will show that identity (2) from our Theorem 1 contains, as a particular case, a classical hypergeometric identity due to Appell. The relevant definitions needed to state this identity are the classical Gauss hypergeometric function $_2F_1(\alpha, \beta; \gamma; t)$ and the Appell bivariate hypergeometric function $F_1(\alpha; \beta, \beta'; \gamma; x, y)$, given respectively by

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{t^n}{n!} \quad \text{and} \quad \sum_{m,n \geq 0} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{x^m y^n}{m! n!},$$

where $(a)_n$ denotes the rising factorial $a(a + 1) \cdots (a + n - 1)$ for $n \in \mathbb{N}$ and it is assumed $|t|, |x|, |y| < 1$ to assure the convergence.

It is very classical that these hypergeometric functions admit the following integral representations, which hold as soon as $\beta, \beta' > 0$ and $\gamma := \beta + \beta' > \alpha > 0$:

$$_2F_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')} \int_0^1 \frac{t^{\beta-1}(1 - t)^{\beta'-1}}{(1 - tz)^{\alpha}} \, dt \tag{4}$$

and

$$F_1(\alpha; \beta, \beta'; \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 \frac{t^{\alpha-1}(1 - t)^{\gamma-\alpha-1}}{(1 - tx)^{\beta} (1 - ty)^{\beta'}} \, dt. \tag{5}$$

Equation (4) is due to Euler [**3**, Theorem 2.2.1], and (5) to Picard [**7**]; see also [**4**, equation (9)].

With these notations, we are able to state the following hypergeometric function identity, which appears on page 8 of Appell's classical memoir [**4**]:

Corollary 2 (Appell's identity). *If $\alpha, \beta, \beta' > 0$, $\beta + \beta' > \alpha$, $|x| < 1$, and $|y| < 1$ then*

$$_2F_1\left(\alpha, \beta; \beta + \beta'; \frac{y - x}{y - 1}\right) = (1 - y)^{\alpha} F_1(\alpha; \beta, \beta'; \beta + \beta'; x, y). \tag{6}$$

Equivalently, in terms of integrals:

$$\int_0^1 \frac{t^{\beta-1}(1-t)^{\beta'-1}}{((y-x)\,t+1-y)^\alpha}\,\mathrm{d}t = \frac{\Gamma(\beta)\Gamma(\beta')}{\Gamma(\alpha)\Gamma(\beta+\beta'-\alpha)}\int_0^1 \frac{t^{\alpha-1}(1-t)^{\beta+\beta'-\alpha-1}}{(1-tx)^\beta\,(1-ty)^{\beta'}}\,\mathrm{d}t.$$
(7)

Remark 3. Identity (1) is a particular case of (7), with $\alpha = \beta = \beta' = n+1$.

Proof of Corollary 2. If we set $n = \ell + s - k$ in (2), then the identity becomes

$$\int_0^1 \frac{x^\ell(1-x)^s}{(x+a)^{k+1}(x+b)^{\ell+s-k+1}}\,\mathrm{d}x$$
$$= \frac{(b+1)^{k-\ell}}{b^{s-k}}\int_0^1 \frac{x^s(1-x)^\ell}{((a-b)\,x+(a+1)b)^{k+1}}\,\mathrm{d}x.$$
(8)

Next we replace the integration variable x with $1-t$ in the first integral and with t in the second integral, then a with $-1/x$ and b with $-1/y$, to deduce the following equivalent form for $x, y < 0$:

$$\int_0^1 \frac{t^\ell(1-t)^s}{\left(1-\frac{y-x}{y-1}t\right)^{k+1}}\,\mathrm{d}t = (1-y)^{\ell+1}\int_0^1 \frac{t^\ell(1-t)^s}{(1-tx)^{k+1}(1-ty)^{\ell+s-k+1}}\,\mathrm{d}t.$$

This last integral can be analytically continued to the values of x and y as in the statement. Setting $\ell = \alpha - 1$, $k = \beta - 1$, and $s = \beta + \beta' - \alpha - 1$, using the integral representations (4) and (5) and the obvious symmetry $_2F_1(\alpha, \beta; \gamma; t) = {}_2F_1(\beta, \alpha; \gamma; t)$, we deduce (6). ∎

The close relationship between (8) and the univariate and bivariate hypergeometric functions is shown in the previous proof of Corollary 2. Taking into account that (7) is a formulation of this result not involving any hypergeometric function, it appears as a natural problem to provide a more direct proof of (7). We present an independent proof involving basic real and complex analysis.

Alternative proof of (7). We write $\gamma = \beta + \beta'$ as before. Changing $t \mapsto 1 - t$ in the first integral of (7) and multiplying the identity by $\Gamma(\alpha)\Gamma(\gamma - \alpha)$, we want to prove that the functions

$$G_1(x,y) = \Gamma(\alpha)\Gamma(\gamma-\alpha)\int_0^1 \frac{t^{\gamma-\beta-1}(1-t)^{\beta-1}}{(1-ty-(1-t)x)^\alpha}\,\mathrm{d}t$$

and

$$G_2(x,y) = \Gamma(\beta)\Gamma(\gamma-\beta)\int_0^1 \frac{t^{\alpha-1}(1-t)^{\gamma-\alpha-1}}{(1-tx)^\beta\,(1-ty)^{\gamma-\beta}}\,\mathrm{d}t$$

coincide. Both are analytic functions of x and y on $\mathrm{Re}(x), \mathrm{Re}(y) < 1/2$ (for instance by Morera's theorem applied separately in x and y). Then it is enough to prove

$$\frac{\partial^{n+m}G_1}{\partial x^n \partial y^m}(0,0) = \frac{\partial^{n+m}G_2}{\partial x^n \partial y^m}(0,0) \qquad \text{for every } m, n \in \mathbb{Z}_{\geq 0},$$

because in this case their Taylor coefficients coincide. Noting that the kth derivative of $(1-x)^{-\delta}$ is $\frac{\Gamma(\delta+k)}{\Gamma(\delta)}(1-x)^{-\delta-k}$, we see that this is the same as

$$\Gamma(\gamma-\alpha)\Gamma(\alpha+m+n)B(\gamma+m-\beta, \beta+n)$$
$$= \Gamma(\beta+n)\Gamma(\gamma+m-\beta)B(\alpha+m+n, \gamma-\alpha)$$

with $B(p, q) = \int_0^1 t^{p-1}(1 - t)^{q-1}\, \mathrm{d}t$. Using the well-known [**9**, Ex. 6.3.7] elementary evaluation $B(p, q) = \Gamma(p)\Gamma(q)/\Gamma(p + q)$, we complete the proof. $\blacksquare$

Combinatorial identities and Legendre polynomials. The equality (1) and its generalizations can be interpreted in combinatorial terms. If we set $a = b > 0$, $\ell = s = n$, and replace k by $k - 1$, then we get

$$\int_0^1 \frac{x^n(1 - x)^n}{(x + b)^{n+k+1}}\, \mathrm{d}x = \frac{1}{b^k(b + 1)^k} \int_0^1 \frac{x^n(1 - x)^n}{(x + b)^{n-k+1}}\, \mathrm{d}x. \tag{9}$$

Corollary 4. *The functions $f_n(x) = x^n(1 + x)^n$ with n a nonnegative integer satisfy*

$$f_k f_n^{(n+k)} = \frac{(n + k)!}{(n - k)!} f_n^{(n-k)} \qquad for\ 0 \leq k \leq n.$$

Proof. By Taylor expansion at $x = -b$, we have

$$x^n(1 - x)^n = (-1)^n f_n(-x) = \sum_{m=0}^{2n} c_m (x + b)^m \quad \text{with} \quad c_m = (-1)^{n+m} \frac{f_n^{(m)}(b)}{m!}.$$

If we substitute this in (9), a term with $\log(b + 1) - \log b$ appears in the left-hand side for $m = n + k$ (the other terms are rational functions of b) and in the integral of the right-hand side for $m = n - k$. Hence $c_{n+k} = c_{n-k}/\big(b(b + 1)\big)^k$. $\blacksquare$

By essentially comparing coefficients, one gets a triple binomial identity:

Corollary 5. *For each k, ℓ, n nonnegative integers with $\ell, k \leq n$,*

$$\sum_{m=0}^{k} \binom{k}{m}\binom{n}{m + \ell}\binom{2m + 2\ell}{k + n} = \binom{n}{\ell}\binom{2\ell}{n - k}.$$

Remark 6. This hypergeometric identity can be rewritten as

$$\binom{2\ell}{n + k} {}_3F_2\Big(-k, \ell - n, \ell + \frac{1}{2};\ \ell - \frac{k + n}{2} + 1, \frac{1 - n - k}{2} + \ell;\ 1\Big) = \binom{2\ell}{n - k},$$

so it is a particular case of the Pfaff–Saalschütz identity [**8**, §2.3.1] for the evaluation at 1 of well-poised ${}_3F_2$'s. Alternatively, it can be automatically obtained using Zeilberger's creative telescoping algorithm [**6**].

Proof. In Corollary 4 change the variable x to $(x - 1)/2$ to obtain, after clearing denominators,

$$(n - k)!(x^2 - 1)^k \frac{\mathrm{d}^{n+k}}{\mathrm{d}x^{n+k}}(x^2 - 1)^n = (n + k)! \frac{\mathrm{d}^{n-k}}{\mathrm{d}x^{n-k}}(x^2 - 1)^n. \tag{10}$$

We rewrite the left-hand side as

$$(n - k)! \sum_{\ell_1}(-1)^{k-\ell_1}\binom{k}{k - \ell_1} x^{2\ell_1} \cdot (n + k)! \sum_{\ell_2}(-1)^{n-\ell_2}\binom{n}{\ell_2}\binom{2\ell_2}{n + k} x^{2\ell_2 - n - k},$$

and the right-hand side as

$$(n + k)!(n - k)! \sum_{\ell}(-1)^{n-\ell}\binom{n}{\ell}\binom{2\ell}{n - k} x^{2\ell - n + k}.$$

Comparing coefficients and renaming $\ell_1 = k - m$, $\ell_2 = m + \ell$ yields the result. ∎

Remark 7. It turns out that (10) is a known identity expressing a symmetry of the associated Legendre polynomials. This can be obtained from Rodrigues' formula and from the general Legendre equation; see [**10**] for a simple elementary proof. The one obtained here (Corollary 4) is comparatively simple.

The definition of the Legendre polynomials ensures that they are eigenfunctions of the differential operator $\frac{d}{dx}(x^2 - 1)\frac{d}{dx}$. Although $\frac{d}{dx}$ and $(x^2 - 1)$ do not commute and apparently there is not a simple formula for the commutator of their powers, Legendre polynomials are also eigenfunctions of a simple operator composed by powers of these operators.

Corollary 8. *Let P_n be the nth Legendre polynomial. Then, for $0 \leq k \leq n$,*

$$L[P_n] = \frac{(n+k)!}{(n-k)!}P_n \qquad where \quad L = \frac{d^k}{dx^k}(x^2-1)^k\frac{d^k}{dx^k}.$$

Proof. Recalling that P_n is proportional to $\frac{d^n}{dx^n}(x^2-1)^n$ (Rodrigues' formula), the result follows from taking the kth derivative of (10). ∎

Actually, one could prove Corollary 4 from Corollary 8 by repeated integration, and noting that both sides in Corollary 4 are divisible by x^k.

REFERENCES

1. Alladi, K., Robinson, M. L. (1980). Legendre polynomials and irrationality. *J. Reine Angew. Math.* 318: 137–155. doi.org/10.1515/crll.1980.318.137
2. Almkvist, G., Zeilberger, D. (1990). The method of differentiating under the integral sign. *J. Symbolic Comput.* 10(6): 571–591. doi.org/10.1016/S0747-7171(08)80159-9
3. Andrews, G. E., Askey, R., Roy, R. (1999). *Special functions*. Cambridge: Cambridge Univ. Press.
4. Appell, P. (1925). Sur les fonctions hypergéométriques de plusieurs variables, les polynômes d'Hermite et autres fonctions sphériques dans l'hyperespace. *Mémorial des Sciences Mathématiques*. 3: 1–75.
5. Ekhad, S. B., Zeilberger, D., Zudilin, W. (2019). Two definite integrals that are definitely (and surprisingly!) equal. arxiv.org/abs/1911.01423
6. Petkovšek, M., Wilf, H. S., Zeilberger, D. (1996). $A = B$. Wellesley, MA: A K Peters, Ltd.
7. Picard. É. (1881). Sur une extension aux fonctions de deux variables du problème de Riemann relatif aux fonctions hypergéométriques. Annales scientifiques de l'École Normale Supérieure, Série 2. 10: 305–322.
8. Slater, L. J. (1966). *Generalized Hypergeometric Functions*. Cambridge: Cambridge Univ. Press.
9. Stein, E. M., Shakarchi, R. (2003). *Complex Analysis*. Princeton, NJ: Princeton Univ. Press.
10. Westra. D. B. Identities and properties for associated Legendre functions. Available at www.mat.univie.ac.at/~westra/associatedlegendrefunctions.pdf
11. Wilf, H. S. (2006). *generatingfunctionology*, 3rd ed. Wellesley, MA: A K Peters, Ltd.

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