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September 1, 2020

Abstract

We prove that the diagonal of any finite product of algebraic functions of the form

$$(1 - x_1 - \cdots - x_n)^R, \qquad R \in \mathbb{Q},$$

is a generalized hypergeometric function, and we provide explicit description of its parameters. The particular case $(1-x-y)^R/(1-x-y-z)$ corresponds to the main identity of Abdelaziz, Koutschan and Maillard in [1, §3.2]. Our result is useful in both directions: on the one hand it shows that Christol's conjecture holds true for a large class of hypergeometric functions, on the other hand it allows for a very explicit and general viewpoint on the diagonals of algebraic functions of the type above. Finally, in contrast to [1], our proof is completely elementary and does not require any algorithmic help.

1 Introduction

Let $\mathbb{K}$ be a field of characteristic zero and let $g \in \mathbb{K}[\![\mathbf{x}]\!]$ be a power series in $\mathbf{x} = (x_1, \ldots, x_n)$

$$g(\mathbf{x}) = \sum_{(i_1,\ldots,i_n)\in\mathbb{N}^n} g_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{K}[\![\mathbf{x}]\!].$$

The *diagonal* $\mathrm{Diag}(g)$ of $g(\mathbf{x})$ is the univariate power series given by

$$\mathrm{Diag}(g) \coloneqq \sum_{j\geq 0} g_{j,\ldots,j} t^j \in \mathbb{K}[\![t]\!].$$

A power series $h(\mathbf{x})$ in $\mathbb{K}[\![\mathbf{x}]\!]$ is called *algebraic* if there exists a non-zero polynomial $P(\mathbf{x}, T) \in \mathbb{K}[\mathbf{x}, T]$ such that $P(\mathbf{x}, h(\mathbf{x})) = 0$; otherwise, it is called *transcendental*.

If $g(\mathbf{x})$ is algebraic, then its diagonal $\mathrm{Diag}(g)$ is usually transcendental; however, by a classical result by Lipshitz [19], $\mathrm{Diag}(g)$ is D-finite, i.e., it satisfies a non-trivial linear differential equation with polynomial coefficients in $\mathbb{K}[t]$. Equivalently, the coefficients sequence $(g_{j,\ldots,j})_{j\geq 0}$ of $\mathrm{Diag}(g)$ is P-recursive, i.e., it satisfies a linear recurrence with polynomial coefficients (in the index j).

When a P-recursive sequence satisfies a recurrence of order 1, we say that it is *hypergeometric*. An important class of power series, whose coefficients sequence is hypergeometric by design, is that of generalized hypergeometric functions. Let $p, q \in \mathbb{N}$ and $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ be rational numbers such that $b_i + j \neq 0$ for any $i, j \in \mathbb{N}$. The *generalized hypergeometric function* ${}_pF_q$ with parameters $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ is the univariate power series in $\mathbb{K}[\![t]\!]$ defined by

$$ {}_pF_q([a_1, \ldots, a_p], [b_1, \ldots, b_q]; t) \coloneqq \sum_{j \geq 0} \frac{(a_1)_j \cdots (a_p)_j}{(b_1)_j \cdots (b_q)_j} \frac{t^j}{j!}, $$

where $(x)_j \coloneqq x(x+1)\cdots(x+j-1)$ is the rising factorial.

We are interested in this article by the following (dual) questions:

(i) what are the algebraic power series $g(\mathbf{x})$ whose diagonal $\mathrm{Diag}(g)$ is a generalized hypergeometric function ${}_pF_q$?[1]

(ii) what are the hypergeometric sequences $(a_j)_{j\geq 0}$ whose generating functions $\sum_{j\geq 0} a_j t^j$ can be written as diagonals of algebraic power series?

Already for $n \in \{1, 2\}$ these questions[2] are non-trivial. The classes of diagonals of bivariate rational power series and of algebraic power series coincide [21, 14]. Hence, questions (i) and (ii) contain as a sub-question the characterization of algebraic hypergeometric functions. This problem was only recently solved in a famous paper by Beukers and Heckman [3].

Another motivation for studying questions (i) and (ii) comes from the following conjecture, formulated in [10, 12]:

> **Christol's conjecture.** If a power series $f \in \mathbb{Q}[\![t]\!]$ is D-finite and globally bounded (i.e., it has non-zero radius of convergence in $\mathbb{C}$ and $\beta \cdot f(\alpha \cdot x) \in \mathbb{Z}[\![x]\!]$ for some $\alpha, \beta \in \mathbb{Z}$), then $f = \mathrm{Diag}(g)$ for some $n \in \mathbb{N}$ and some algebraic power series $g \in \mathbb{Q}[\![x_1, \ldots, x_n]\!]$.

Christol's conjecture is still largely open, even in the particular case when f is a generalized hypergeometric function. In this case, it has been proved [10, 12] in two extreme subcases: when all the bottom parameters b_i are integers (case of "minimal monodromy weight", in the terminology of [13]) and when they are all non-integers (case of "maximal monodromy weight"). In the first extremal case, the proof is based on the observation that

$$ {}_pF_q([a_1, \ldots, a_p], [1, \ldots, 1]; t) = (1-t)^{-a_1} \star \cdots \star (1-t)^{-a_p}, \tag{1} $$

where $\star$ denotes the Hadamard (term-wise) product, and on the fact that diagonals are closed under Hadamard product [11, Prop. 2.6]. In the second extremal case, it is based on the equivalence between being globally bounded and algebraic; this equivalence, proved by Christol [10, 12], is itself based on [3].

The other cases (of "intermediate monodromy weight") are widely open. A first explicit example of this kind, itself still open as of today, was given by Christol himself as soon as 1987 [10, §VII]:

> Is $f(t) = {}_3F_2\left(\left[\frac{1}{9}, \frac{4}{9}, \frac{5}{9}\right], \left[1, \frac{1}{3}\right]; t\right)$ the diagonal of an algebraic power series?

[1] Note that a necessary condition is that $q = p - 1$, since the radius of convergence must be finite and non-zero.

[2] From an algorithmic viewpoint, questions (i) and (ii) are very different in nature: while (i) is decidable (given an algebraic power series, one can decide if its diagonal is hypergeometric, for instance by combining the algorithms in [8] and [20]), the status of question (ii) is not known (does there exist an algorithm which takes as input a hypergeometric sequence and outputs an algebraic series whose diagonal is the generating function of the input sequence?).

Two decades later, Bostan et al. [5, 6] produced a large list of about 200 similar ${}_3F_2$ (globally bounded) functions, which are potential counter-examples to Christol's conjecture (in the sense that, like ${}_3F_2([1/9,4/9,5/9],[1,1/3];t)$, they are not easily reducible to the two known extreme cases, via closure properties of diagonals, e.g., with respect to Hadamard products). This year, Abdelaziz, Koutschan and Maillard [1, §3] managed to show that two members of that list, namely ${}_3F_2([1/9,4/9,7/9],[1,1/3];t)$ and ${}_3F_2([2/9,5/9,8/9],[1,2/3];t)$ are indeed diagonals. Precisely,

$$ {}_3F_2\left(\left[\frac{2}{9},\frac{5}{9},\frac{8}{9}\right],\left[1,\frac{2}{3}\right];27\,t\right)=\operatorname{Diag}\left(\frac{(1-x-y)^{1/3}}{1-x-y-z}\right) \tag{2} $$

and

$$ {}_3F_2\left(\left[\frac{1}{9},\frac{4}{9},\frac{7}{9}\right],\left[1,\frac{1}{3}\right];27\,t\right)=\operatorname{Diag}\left(\frac{(1-x-y)^{2/3}}{1-x-y-z}\right). \tag{3} $$

They moreover gave in [1, §3.2] the following extension of identities (2) and (3), to any $R\in\mathbb{Q}$:

$$ {}_3F_2\left(\left[\frac{1-R}{3},\frac{2-R}{3},\frac{3-R}{3}\right],[1,1-R];27\,t\right)=\operatorname{Diag}\left(\frac{(1-x-y)^{R}}{1-x-y-z}\right). \tag{4} $$

A common feature of the identities (2) and (3) (and their generalization (4)) is that the top parameters are in arithmetic progression, as opposed to Christol's initial example. However, they are the first known examples of generalized hypergeometric functions with intermediate monodromy weight, not reducible to the two known extreme cases, and which are provably diagonals.

Our first result extends identity (4) to a much larger class of (transcendental) generalized hypergeometric functions.

Theorem 1. *Let $R,S\in\mathbb{Q}$ and $n,N\in\mathbb{N}$ such that $S\neq 0$ and $0\leq n\leq N$. Set $s\coloneqq N-n$ and $Q\coloneqq S-R$. Then the generalized hypergeometric function*

$$ {}_{N+s}F_{N+s-1}\left(\left[\frac{Q}{N},\frac{Q+1}{N},\dots,\frac{Q+N-1}{N},\frac{S}{s},\dots,\frac{S+s-1}{s}\right],\left[\frac{Q}{s},\dots,\frac{Q+s-1}{s},1,\dots,1\right];N^N t\right) $$

is equal to the diagonal

$$ \operatorname{Diag}\left(\frac{(1-x_1-\cdots-x_n)^R}{(1-x_1-\cdots-x_N)^S}\right). $$

Note that identity (4) corresponds to the particular case $(n,N,S)=(2,3,1)$ of Theorem 1. The proof of (4) given in [1, §3.2] relies on an algorithmic technique called *creative telescoping* [17], which works in principle[3] on any diagonal of algebraic function, as long as the number $\max(n,N)$ of indeterminates is *fixed*. Our identity in Theorem 1 contains a number of indeterminates which is itself variable, hence it cannot be proved by creative telescoping in this generality. In §2 we offer instead a direct and elementary proof.

In §3 we will further generalize Theorem 1 in two distinct directions. The first extension (Theorem 3) shows that the diagonal of the product of an arbitrary number of arbitrary powers of linear forms of the type $1-x_1-\cdots-x_m$ is again a generalized hypergeometric function. The second

[3] Creative telescoping algorithms, such as the one in [8], compute a linear differential equation for $\operatorname{Diag}(g(\mathbf{x}))$. This equation is converted on a linear recurrence, whose hypergeometric solutions can be computed using Petkovšek's algorithm [20]. Note that the complexity (in time and space) of these algorithms increase with n,N,R and S.

extension (Theorem 4) shows that under a condition on the exponents the same stays true if the product is multiplied with another factor of the form $(1-x_1-\cdots-x_{m-2}-2\,x_{m-1})^b$. For instance, when restricted to $m=3$ variables, these results specialize respectively into the following identities:

Theorem 2. *For any $R, S, T \in \mathbb{Q}$, we have:*

$$\operatorname{Diag}\left((1-x)^R(1-x-y)^S(1-x-y-z)^T\right) = \tag{5}$$
$${}_6F_5\left(\left[\frac{-(R+S+T)}{3}, \frac{1-(R+S+T)}{3}, \frac{2-(R+S+T)}{3}, \frac{-(S+T)}{2}, \frac{1-(S+T)}{2}, -T\right];\right.$$
$$\left.\left[\frac{-(R+S+T)}{2}, \frac{1-(R+S+T)}{2}, -(S+T), 1, 1\right]; 27t\right)$$

and

$$\operatorname{Diag}\left((1-x)^R(1-x-2\,y)^S(1-x-y-z)^{-1}\right) = \tag{6}$$
$${}_4F_3\left(\left[\frac{1-(R+S)}{3}, \frac{2-(R+S)}{3}, \frac{3-(R+S)}{3}, \frac{1-S}{2}\right]; \left[\frac{1-(R+S)}{2}, \frac{2-(R+S)}{2}, 1\right]; 27t\right).$$

Note that (6) generalizes and explains the two following identities observed in [1, Eq. (30)–(31)]

$${}_3F_2\left(\left[\frac{1}{9}, \frac{4}{9}, \frac{7}{9}\right], \left[1, \frac{2}{3}\right]; 27\,t\right) = \operatorname{Diag}\left(\frac{(1-x-2\,y)^{2/3}}{1-x-y-z}\right) \tag{7}$$

and

$${}_3F_2\left(\left[\frac{2}{9}, \frac{5}{9}, \frac{8}{9}\right], \left[1, \frac{5}{6}\right]; 27\,t\right) = \operatorname{Diag}\left(\frac{(1-x-2\,y)^{1/3}}{1-x-y-z}\right). \tag{8}$$

Once again, our proofs of the (generalizations of) identities (5) and (6) are elementary, and do not rely on algorithmic tools.

One may wonder if other generalizations are possible, for instance if the coefficient 2 can be replaced by a different one in (6). The following example shows that this is not the case. Let

$$U(t) = \operatorname{Diag}\left(\frac{\sqrt[3]{1-ax}}{1-x-y}\right).$$

Then, the coefficients sequence $(u_j)_{j\geq 0}$ of $U(t)$ satisfies the second-order recurrence relation

$$\begin{aligned} 2\,a^2\,(6\,n+5)\,(3\,n+1)\,u_n \quad &+ \quad 9\,(a-1)\,(n+2)\,(n+1)\,u_{n+2} \\ &= \quad 3\,(n+1)\left(3\,\left(a^2+4\,a-4\right)n+2\,a^2+18\,a-18\right)u_{n+1}. \end{aligned}$$

When $a \in \{0,1,2\}$, the sequence $(u_j)_{j\geq 0}$ also satisfies a shorter recurrence, of order 1, as shown by our main results. In these cases, $U(t)$ is a hypergeometric diagonal. When $a \notin \{0,1,2\}$, the second-order recurrence is the minimal-order satisfied by $(u_j)_{j\geq 0}$, hence $U(t)$ is not a hypergeometric diagonal. This can be proved either using the explicit identity

$$U(t) = \sqrt[3]{\frac{a/2}{1-4\,t} + \frac{1-a/2}{(1-4\,t)^{\frac{3}{2}}}},$$

or by using the general approach in [9, §5].

An apparent weakness of our results is that they only provide examples with parameters in (unions of) arithmetic progressions. This is true, as long as identities are used alone. But symmetries may be broken by combining different identities and using for instance Hadamard products. As an illustration, by taking the Hadamard product in both sides of the following identities

$$ {}_3F_2\left(\left[\frac{Q}{3}, \frac{Q+1}{3}, \frac{Q+2}{3}\right], [1, Q]\,; t\right) = \mathrm{Diag}\left(\frac{(1-\frac{x_1}{3}-\frac{x_2}{3})^{1-Q}}{1-\frac{x_1}{3}-\frac{x_2}{3}-\frac{x_3}{3}}\right) $$

and

$$ {}_2F_1\left(\left[\frac{Q}{6}, \frac{Q+3}{6}\right], \left[\frac{Q}{3}\right]; t\right) = \mathrm{Diag}\left(\frac{(1-\frac{x_4}{2}-\frac{x_5}{2})^{1-Q/3}}{1-\frac{x_4}{2}-\frac{x_5}{2}}\right), $$

both particular cases of Theorem 1, one deduces that the *non-symmetric* hypergeometric function

$$ {}_4F_3\left(\left[\frac{Q}{6}, \frac{Q}{6}+\frac{1}{2}, \frac{Q}{3}+\frac{1}{3}, \frac{Q}{3}+\frac{2}{3}\right];\, [1, 1, Q]\,;\, t\right) $$

is equal to the following diagonal

$$ \mathrm{Diag}\left(\left(1-\frac{x_1}{3}-\frac{x_2}{3}\right)^{1-Q}\left(1-\frac{x_4}{2}\right)^{1-\frac{Q}{3}}\left(1-\frac{x_1}{3}-\frac{x_2}{3}-\frac{x_3}{3}\right)^{-1}\left(1-\frac{x_4}{2}-\frac{x_5}{2}\right)^{-1}\right). $$

Similarly, the *non-symmetric* hypergeometric function[4]

$$ {}_3F_2\left(\left[\frac{Q}{6}, \frac{Q}{6}+\frac{1}{2}, \frac{Q}{3}+\frac{1}{2}\right];\, \left[1, \frac{2\,Q}{3}\right];\, t\right) $$

is equal to the following diagonal

$$ \mathrm{Diag}\left(\left(1-\frac{x_1}{2}\right)^{1-\frac{Q}{3}}\left(1-\frac{x_3}{2}\right)^{1-\frac{2\,Q}{3}}\left(1-\frac{x_1}{2}-\frac{x_2}{2}\right)^{-1}\left(1-\frac{x_3}{2}-\frac{x_4}{2}\right)^{-1}\right). $$

A natural question is to prove (or, disprove) whether Christol's ${}_3F_2$ can be obtained in such a way.

As a final remark, one should not think that every generalized hypergeometric function which is a diagonal needs to have a representation like in our Theorems 1 or 2. For instance, the diagonal [7]

$$ \mathrm{Diag}\left(\frac{1}{1-(1+w)(x+y+z)}\right) $$

is equal to the generalized hypergeometric function

$$ {}_4F_3\left(\left[\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right];\, \left[1, 1, \frac{1}{2}\right];\, \frac{729}{4}\,t\right) = 1 + 18\,t + 1350\,t^2 + \cdots, $$

which is seemingly not of the form covered by any of our results.

[4]Amusingly, the above ${}_3F_2$ is not only asymmetric, but it also shares another similarity with Christol's example: the sum of two of the three top parameters is equal to the third one. This patters occurs in several other examples.

2 Proof of Theorem 1

Proof. By definition, for $k \in \mathbb{N}$, the coefficient of t^k in the hypergeometric function is given by

$$\frac{(Q/N)_k\,((Q+1)/N)_k \cdots ((Q+N-1)/N)_k (S/s)_k \cdots ((S+s-1)/s)_k}{(Q/s)_k \cdots ((Q+s-1)/s)_k (k!)^N} N^{Nk}.$$

Now we extract the coefficient of t^k from the diagonal. First note that

$$\operatorname{Diag}\left(\frac{(1-x_1-\cdots-x_n)^R}{(1-x_1-\cdots-x_N)^S}\right)(t) = \operatorname{Diag}\left((1+x_1+\cdots+x_n)^R(1+x_1+\cdots+x_N)^{-S}\right)((-1)^N t).$$

Recall the definition of the multinomial coefficient: for $m \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$ we set

$$\binom{m}{\alpha_1, \ldots, \alpha_n} \coloneqq \frac{m!}{\alpha_1! \cdots \alpha_n!},$$

if $|\alpha| \coloneqq \alpha_1 + \cdots + \alpha_n = m$. Otherwise the multinomial coefficient is defined to be zero. By the multinomial theorem we have:

$$\begin{aligned} g(\mathbf{x}) &\coloneqq (1+x_1+\cdots+x_n)^R(1+x_1+\cdots+x_N)^{-S} \\ &= \left(\sum_{\substack{i\geq 0,\\ \alpha\in\mathbb{N}^n}} \binom{R}{i}\binom{i}{\alpha_1,\ldots,\alpha_n} x_1^{\alpha_1}\cdots x_n^{\alpha_n}\right)\left(\sum_{\substack{j\geq 0,\\ \beta\in\mathbb{N}^N}} \binom{-S}{j}\binom{j}{\beta_1,\ldots,\beta_N} x_1^{\beta_1}\cdots x_N^{\beta_N}\right) \\ &= \sum_{\substack{i,j\geq 0,\\ \alpha\in\mathbb{N}^n,\beta\in\mathbb{N}^N}} \binom{R}{i}\binom{-S}{j}\binom{i}{\alpha_1,\ldots,\alpha_n}\binom{j}{\beta_1,\ldots,\beta_N} x_1^{\alpha_1+\beta_1}\cdots x_n^{\alpha_n+\beta_n} x_{n+1}^{\beta_{n+1}}\cdots x_N^{\beta_N}. \end{aligned}$$

Now for $k \in \mathbb{N}$ we can extract the coefficient, denoted $[\mathbf{x}^k]g(\mathbf{x})$, of $x_1^k \cdots x_N^k$ in $g(\mathbf{x})$:

$$\begin{aligned} &[\mathbf{x}^k](1+x_1+\cdots+x_n)^R(1+x_1+\cdots+x_N)^{-S} \\ &\quad= \sum_{\substack{\alpha,\gamma\in\mathbb{N}^n\\ \alpha_i+\gamma_i=k}} \binom{R}{|\alpha|}\binom{-S}{|\gamma|+(N-n)k}\binom{|\alpha|}{\alpha_1,\ldots,\alpha_n}\binom{|\gamma|+(N-n)k}{\gamma_1,\ldots,\gamma_n,k,\ldots,k} \\ &\quad= \sum_{\alpha\in\mathbb{N}^n} \binom{R}{|\alpha|}\binom{-S}{Nk-|\alpha|}\binom{|\alpha|}{\alpha_1,\ldots,\alpha_n}\binom{Nk-|\alpha|}{k-\alpha_1,\ldots,k-\alpha_n,k,\ldots,k} \\ &\quad= \sum_{\alpha\in\mathbb{N}^n} \binom{R}{|\alpha|}\binom{-S}{Nk-|\alpha|}\binom{|\alpha|}{\alpha_1,\ldots,\alpha_n}\binom{nk-|\alpha|}{k-\alpha_1,\ldots,k-\alpha_n}\binom{Nk-|\alpha|}{k,\ldots,k,nk-|\alpha|}. \end{aligned}$$

Recall the multinomial Chu-Vandermonde identity [28, Eq. (1.4)]:

$$\binom{m_1+m_2}{\gamma_1,\ldots,\gamma_n} = \sum_{\alpha\in\mathbb{N}^n}\binom{m_1}{\alpha_1,\ldots,\alpha_n}\binom{m_2}{\gamma_1-\alpha_1,\ldots,\gamma_n-\alpha_n}.$$

Using it twice we have

$$\begin{aligned}
&\sum_{\alpha\in\mathbb{N}^n}\binom{R}{|\alpha|}\binom{-S}{Nk-|\alpha|}\binom{|\alpha|}{\alpha_1,\dots,\alpha_n}\binom{nk-|\alpha|}{k-\alpha_1,\dots,k-\alpha_n}\binom{Nk-|\alpha|}{k,\dots,k,nk-|\alpha|}\\
&\quad=\sum_{j\geq 0}\binom{R}{j}\binom{-S}{Nk-j}\binom{Nk-j}{k,\dots,k,nk-j}\sum_{|\alpha|=j}\binom{j}{\alpha_1,\dots,\alpha_n}\binom{nk-j}{k-\alpha_1,\dots,k-\alpha_n}\\
&\quad=\sum_{j\geq 0}\binom{R}{j}\binom{-S}{Nk-j}\binom{Nk-j}{k,\dots,k,nk-j}\binom{nk}{k,\dots,k}\\
&\quad=\binom{(N-n)k}{k,\dots,k}\binom{nk}{k,\dots,k}\sum_{j\geq 0}\binom{R}{j}\binom{-S}{Nk-j}\binom{Nk-j}{nk-j}\\
&\quad=\binom{(N-n)k}{k,\dots,k}\binom{nk}{k,\dots,k}\binom{-S}{(N-n)k}\sum_{j\geq 0}\binom{R}{j}\binom{-S-(N-n)k}{nk-j}\\
&\quad=\binom{(N-n)k}{k,\dots,k}\binom{nk}{k,\dots,k}\binom{-S}{(N-n)k}\binom{R-S-(N-n)k}{nk}.
\end{aligned}$$

Henceforth, in order to prove Theorem 1, it suffices to ensure that

$$\begin{aligned}
&(-1)^{Nk}\binom{(N-n)k}{k,\dots,k}\binom{nk}{k,\dots,k}\binom{-S}{(N-n)k}\binom{R-S-(N-n)k}{nk}\\
&\qquad\stackrel{!}{=}\frac{(Q/N)_k\,((Q+1)/N)_k\cdots((Q+N-1)/N)_k(S/s)_k\cdots((S+s-1)/s)_k}{(Q/s)_k\cdots((Q+s-1)/s)_k(k!)^N}N^{Nk}.
\end{aligned}$$

However this is easily verified by canceling obvious factors and then using the fact that

$$(a/b)_k((a+1)/b)_k\cdots((a+b-1)/b)_k\cdot b^{bk}=(a)_{bk}. \qquad \square$$

3 General case

This section contains several parts: first we introduce in §3.1 and §3.2 some notation and state the two general Theorems 3 and 4. Then we explain them in §3.3 by means of four examples, showing that both Theorem 1 and Theorem 2 are special cases. Further, we continue in §3.4 with several lemmas and their proofs. Finally, the general theorems are proven in §3.5 and §3.6.

3.1 First Statement

Let $N\in\mathbb{N}\setminus\{0\}$ and $b_1,\dots,b_N\in\mathbb{Q}$ with $b_N\neq 0$. We want to prove that the diagonal of

$$R(x_1,\dots,x_N):=(1+x_1)^{b_1}(1+x_1+x_2)^{b_2}\cdots(1+x_1+\cdots+x_N)^{b_N} \tag{9}$$

can be expressed as a hypergeometric function. For each $k=1,\dots,N$ we define the tuple

$$u^k:=\left(\frac{B(k)}{N-k+1},\frac{B(k)+1}{N-k+1},\dots,\frac{B(k)+N-k}{N-k+1}\right),$$

where $B(k) := -(b_k + \cdots + b_N)$. For $k = 1, \dots, N-1$ we set

$$v^k := \left(\frac{B(k)}{N-k}, \frac{B(k)+1}{N-k}, \dots, \frac{B(k)+N-k-1}{N-k} \right).$$

Moreover set $v^N := (1, 1, \dots, 1)$ with exactly $N-1$ ones. It follows by construction that the lengths of the tuples

$$u := (u^1, \dots, u^N) \quad \text{and}$$
$$v := (v^1, \dots, v^N)$$

are given by $M := N + \cdots + 2 + 1 = N(N+1)/2$ and $M-1$ respectively. We have the following generalization of Theorem 1:

Theorem 3. *It holds that*

$$\operatorname{Diag}(R(x_1, \dots, x_N)) = {}_M F_{M-1}(u; v; (-N)^N t).$$

3.2 Second Statement

Let $N \in \mathbb{N} \setminus \{0\}$ and $b_1, \dots, b_N \in \mathbb{Q}$ with $b_N \neq 0$. Assume that $b_N \neq 0$ and $b_{N-1} + b_N = -1$. We will prove that, for any $b \in \mathbb{Q}$, we can express

$$(1 + x_1 + \cdots + x_{N-2} + 2\, x_{N-1})^b \cdot R(x_1, \dots, x_N)$$

as a hypergeometric function as well. Again, let $B(k) := -(b_k + \cdots + b_N)$. For each $k = 1, \dots, N-2$ we define the tuple

$$u^k := \left(\frac{B(k)-b}{N-k+1}, \frac{B(k)-b+1}{N-k+1}, \dots, \frac{B(k)-b+N-k}{N-k+1} \right)$$

and set $u^{N-1} := -(b_{N-1} + b_N + b)/2 = (1-b)/2$ and $u^N := -b_N$. Moreover, for $k = 1, \dots, N-2$ we set

$$v^k := \left(\frac{B(k)-b}{N-k}, \frac{B(k)-b+1}{N-k}, \dots, \frac{B(k)-b+N-k-1}{N-k} \right),$$

and $v^{N-1} := (1, 1, \dots, 1)$ with exactly $N-1$ ones. It follows by construction that the lengths of the tuples

$$u := (u^1, \dots, u^N) \quad \text{and}$$
$$v := (v^1, \dots, v^{N-1})$$

are given by $M - 1 = N + \cdots + 4 + 3 + 1 + 1 = N(N+1)/2 - 1$ and $M-2$ respectively.

Theorem 4. *It holds that*

$$\operatorname{Diag}((1 + x_1 + \cdots + x_{N-2} + 2\, x_{N-1})^b \cdot R(x_1, \dots, x_N)) = {}_{M-1} F_{M-2}(u; v; (-N)^N t).$$

3.3 Examples

Let us list some examples of the general theorems and draw the connection to previous statements.

1. First we emphasize that Theorem 1 follows promptly from the more general Theorem 3 by letting all $b_j = 0$ except $b_n = R$ and $b_N = -S$. Clearly, the change $\mathbf{x} \mapsto -\mathbf{x}$ in the algebraic function is reflected by the change $t \mapsto (-1)^N t$ in its diagonal.

2. Letting $N = 3$ in Theorem 3 we obtain immediately the first part of Theorem 2

$$\operatorname{Diag}\left((1+x)^R(1+x+y)^S(1+x+y+z)^T\right) = {}_6F_5\left(\left[\frac{-(R+S+T)}{3}, \frac{-(R+S+T)+1}{3}, \frac{-(R+S+T)+2}{3}, \frac{-(S+T)}{2}, \frac{-(S+T)+1}{2}, -T\right]; \left[\frac{-(R+S+T)}{2}, \frac{-(R+S+T)+1}{2}, -(S+T), 1, 1\right]; -27t\right).$$

3. If moreover $T = -1$ in the previous example, we achieve a cancellation of the last parameter and are left with

$$\operatorname{Diag}\left(\frac{(1+x)^R(1+x+y)^S}{1+x+y+z}\right) = {}_5F_4\left(\left[\frac{1-(R+S)}{3}, \frac{2-(R+S)}{3}, \frac{3-(R+S)}{3}, \frac{1-S}{2}, \frac{2-S}{2}\right]; \left[\frac{1-(R+S)}{2}, \frac{2-(R+S)}{2}, 1-S, 1\right]; -27t\right).$$

4. Comparing with the similar situation of Theorem 4 in the case $N = 3$ and $b_{N-1} = -1-b_N = 0$, we see that a family of ${}_4F_3$ functions remains and covers the second statement of Theorem 2:

$$\operatorname{Diag}\left((1+x+2\,y)^T \cdot (1+x)^R(1+x+y+z)^{-1}\right) = {}_4F_3\left(\left[\frac{1-(R+T)}{3}, \frac{2-(R+T)}{3}, \frac{3-(R+T)}{3}, \frac{1-T}{2}\right]; \left[\frac{1-(R+T)}{2}, \frac{2-(R+T)}{2}, 1\right]; -27t\right).$$

3.4 Lemmas and Proofs

In this section we will state and prove necessary lemmas for the proofs of Theorems 3 and 4.

Lemma 1. *Let N be a positive integer and $b_1, \ldots, b_N \in \mathbb{Q}$ such that $b_N \neq 0$. It holds that*

$$[x_1^{k_1} \cdots x_N^{k_N}](1+x_1)^{b_1}(1+x_1+x_2)^{b_2} \cdots (1+x_1+\cdots+x_N)^{b_N} = \binom{b_N}{k_N}\binom{b_{N-1}+b_N-k_N}{k_{N-1}} \cdots \binom{b_1+\cdots+b_N-k_N\cdots-k_2}{k_1}.$$

This result contains the core identity of the present paper, since it enables the connection between the algebraic functions $R(\mathbf{x})$ of the form (9) and hypergeometric sequences. It can be proven in two ways: a direct approach generalizes the proof of Theorem 1 and works by multiplying the left-hand side out using the multinomial theorem, picking the needed coefficient and reducing the sum using the Chu-Vandermonde identity several times. This procedure is rather tedious and not instructive, therefore we present a combinatorially inspired proof.

Proof. We have

$$[x_1^{k_1}\cdots x_N^{k_N}](1+x_1)^{b_1}\cdots(1+x_1+\cdots+x_N)^{b_N}$$
$$= [x_1^{k_1}\cdots x_N^{k_N}](1+x_1)^{b_1}\cdots(1+x_1+\cdots+x_{N-1})^{b_{N-1}}\cdot x_N^{b_N}\left(1+\frac{1+x_1+\cdots+x_N}{x_N}\right)^{b_N}$$
$$= [x_1^{k_1}\cdots x_N^{k_N}](1+x_1)^{b_1}\cdots(1+x_1+\cdots+x_{N-1})^{b_{N-1}}\cdot x_N^{b_N}\sum_{j\geq 0}\binom{b_N}{j}\left(\frac{1+x_1+\cdots+x_{N-1}}{x_N}\right)^j.$$

Because $(1+x_1)^{b_1}\cdots(1+x_1+\cdots+x_{N-1})^{b_{N-1}}$ does not depend on x_N, we must have $k_N = b_N - j$. This reduces the sum to one term, namely $j = b_N - k_N$, and we obtain

$$[x_1^{k_1}\cdots x_N^{k_N}](1+x_1)^{b_1}\cdots(1+x_1+\cdots+x_N)^{b_N}$$
$$= \binom{b_N}{k_N}\cdot[x_1^{k_1}\cdots x_{N-1}^{k_{N-1}}](1+x_1)^{b_1}\cdots(1+x_1+\cdots+x_{N-1})^{b_{N-1}}(1+x_1+\cdots+x_{N-1})^{b_N-k_N}$$
$$= \binom{b_N}{k_N}\cdot[x_1^{k_1}\cdots x_{N-1}^{k_{N-1}}](1+x_1)^{b_1}\cdots(1+x_1+\cdots+x_{N-2})^{b_{N-2}}(1+x_1+\cdots+x_{N-1})^{b_{N-1}+b_N-k_N}.$$

Now the claim follows by iteration. □

Note that Lemma 1 shares some similarities with Straub's [25, Theorem 3.1], which provides explicit expressions of rational power series of the form

$$\Big((1+x_1+\cdots+x_{\lambda_1})(1+x_{\lambda_1+1}+\cdots+x_{\lambda_1+\lambda_2})\cdots(1+x_{\lambda_1+\cdots+\lambda_{\ell-1}}+\cdots+x_N)-\alpha\cdot x_1x_2\cdots x_N\Big)^{-1}.$$

In Lemma 1, we allow products of linear forms with arbitrary exponents but no term $\alpha\cdot x_1x_2\cdots x_N$, while in [25, Theorem 3.1] the linear forms have disjoint variables and appear at exponent 1. Setting $\alpha = 0$ in Straub's formula also yields a product of binomial coefficients.

It is legitimate to wonder whether there is a common generalization of Lemma 1 and Thm. 3.1 in [25]. For instance, one may ask for which values of α is the diagonal

$$\operatorname{Diag}\left(\left(\sqrt{1-x}\,(1-y)-\alpha xy\right)^{-1}\right) = 1+(\alpha+1/2)\,t+\left(\alpha^2+2\,\alpha+3/8\right)t^2+\cdots$$

hypergeometric? For a general α, the minimal recurrence satisfied by the coefficients of the diagonal is of order 4, for $\alpha = \pm i/2$ it is of order 3, and it seems that the only rational value of α for which there exists a shorter recurrence is $\alpha = 0$, in which case the diagonal is hypergeometric.

Now we want to verify a similar statement for the situation as in Theorem 4, so the case where we deal with the coefficient sequence of

$$(1+x_1+\cdots+x_{N-2}+2\,x_{N-1})^b\cdot R(x_1,\ldots,x_N).$$

We lay the grounds for a lemma similar to Lemma 1, by starting with a rather surprising identity.

Lemma 2. *Let $k\in\mathbb{N}$ and $b\in\mathbb{Q}$ arbitrary. It holds that*

$$[x^k]\frac{(1+2x)^b}{(1+x)^{k+1}} = 4^k\binom{(b-1)/2}{k}$$

Proof. First notice that for arbitrary a, b we can compute

$$[x^k](1+2x)^b(1+x)^a = [x^k]\left(\sum_{i\geq 0} 2^i \binom{b}{i} x^i\right)\left(\sum_{j\geq 0}\binom{a}{j}x^j\right) = \sum_{j=0}^{k} 2^j \binom{b}{j}\binom{a}{k-j}.$$

So we set $a = -(k+1)$ and obtain

$$[x^k]\frac{(1+2x)^b}{(1+x)^{k+1}} = \sum_{j=0}^{k} 2^j\binom{b}{j}\binom{-k-1}{k-j} = 2^k\sum_{j=0}^{k}(-1)^j 2^{-j}\binom{b}{k-j}\binom{k+j}{k}.$$

It remains to prove the following identity[5]

$$\sum_{j=0}^{k}(-2)^{-j}\binom{b}{k-j}\binom{k+j}{k} = 2^k\binom{(b-1)/2}{k}. \tag{10}$$

To do this, we note that

$$\sum_{j=0}^{k}(-1)^j\binom{b}{k-j}\binom{k+j}{k}u^j = \binom{b}{k}{}_2F_1([-k,k+1];[b+1-k];u),$$

and

$${}_2F_1([-k,k+1];[b+1-k];1/2) = \frac{\Gamma((b+1-k)/2)\Gamma((b+2-k)/2)}{\Gamma((b+1-2k)/2)\Gamma((b+2)/2)} = 2^k\frac{\binom{(b-1)/2}{k}}{\binom{b}{k}},$$

by Kummer's identity [18, Eq. 3, p. 134]. □

The proof above explains the special role of the coefficient $a = 2$ mentioned in the introduction: it is one of the few values, along with 1 and -1, for which there exists a closed form expression for the evaluation of a ${}_2F_1$ hypergeometric function of u at $u = 1/a$.

Now we can step forward and prove the essential lemma for Theorem 4. Note that contrary to Lemma 1 the following statement is purely about diagonal coefficients and not for general exponents anymore. Except for the missing factor $\binom{b_{N-1}+b_N-k}{k}$ and the new two factors 4^k and $\binom{(b-1)/2}{k}$ the statement is completely analogous.

Lemma 3. *Let N be a positive integer and $b_1, \ldots, b_N \in \mathbb{Q}$ such that $b_{N-1} + b_N = -1$. For any $b \in \mathbb{Q}$ the coefficient of $x_1^k \cdots x_N^k$ in*

$$(1 + x_1 + \cdots + x_{N-2} + 2\, x_{N-1})^b \cdot (1+x_1)^{b_1}\cdots(1+x_1+\cdots+x_N)^{b_N}$$

is given by

$$4^k\binom{(b-1)/2}{k}\binom{b_N}{k}\cdot\binom{b_{N-2}+b_{N-1}+b_N+b-2k}{k}\cdots\binom{b_1+\cdots+b_N+b-(N-1)k}{k}.$$

[5] Note that identity (10) could alternatively be proven by using Zeilberger's creative telescoping algorithm [27], or derived from identity (3.42) in [15, p. 27] by setting $2n - x = b$, multiplying with 2^k and reverting the summation.

Proof. By the same argument as in the proof of Lemma 1, the left-hand side is equal to $\binom{b_N}{k}$ multiplied with the coefficient of $x_1^k \cdots x_{N-1}^k$ in

$$(1 + x_1 + \cdots + x_{N-2} + 2\, x_{N-1})^b \cdot \prod_{j=1}^{N-2} \left(1 + \sum_{i=1}^{j} x_i\right)^{b_j} \cdot (1 + x_1 + \cdots + x_{N-2} + x_{N-1})^{b_{N-1}+b_N-k}.$$

Because the product in the middle does not depend on x_{N-1} and since we assumed $b_{N-1}+b_N = -1$, we can first compute

$$[x_{N-1}^k](1 + 2\, x_{N-1})^b(1 + x_{N-1})^{-1-k} = 4^k \binom{(b-1)/2}{k},$$

by Lemma 2. Therefore we are left with

$$4^k \binom{(b-1)/2}{k}\binom{b_N}{k} \cdot [x_1^k \cdots x_{N-2}^k] \prod_{j=1}^{N-3} \left(1 + \sum_{i=1}^{j} x_i\right)^{b_j} \cdot (1 + x_1 + \cdots + x_{N-2})^{b_{N-2}+b_{N-1}+b_N+b-2k},$$

which is easily computed using Lemma 1. □

Note that the requirement $b_{N-1} + b_N = -1$ comes from the $+1$ in the denominator of the left-hand side in Lemma 2. Since this identity is itself surprising and does not allow for obvious generalizations, the condition on the relationship of b_{N-1} and b_N is necessary.

3.5 Proof of Theorem 3

For the proof of Theorem 3 we will only use Lemma 1 and algebraic manipulations similar to the proof of Bober's Lemma 4.1 in [4].

By Lemma 1 we obtain the coefficient of t^n for any $n \in \mathbb{N}$ on the left-hand side:

$$[t^n]\mathrm{Diag}((1 + x_1)^{b_1} \cdots (1 + x_1 + \cdots + x_N)^{b_N}) = \binom{b_N}{n} \cdots \binom{b_1 + \cdots + b_N - (N-1)n}{n}.$$

For the right-hand side we use the fact that for all a, b and non-negative integers n it holds

$$(a/b)_n((a + 1)/b)_n \cdots ((a + b - 1)/b)_n \cdot b^{bn} = (a)_{bn}.$$

Then

$$U_k := \prod_{i=1}^{N-k+1} (u_i^k)_n = \frac{(-b_k - \cdots - b_N)_{(N-k+1)n}}{(N-k+1)^{(N-k+1)n}},$$

for all $k = 1, \ldots, N$. Similarly,

$$V_k := \prod_{i=1}^{N-k} (v_i^k)_n = \frac{(-b_k - \cdots - b_N)_{(N-k)n}}{(N-k)^{(N-k)n}},$$

for all $k = 1, \dots, N-1$. Clearly $V_N \coloneqq \prod_{i=1}^{N-1} (v_i^N)_n = (n!)^{N-1}$. We deduce that

$$[t^n]_M F_{M-1}(u; v; (-N)^N t) = \frac{(-N)^{nN}}{n!} \prod_{i=1}^{N} \frac{U_i}{V_i} = \frac{(-1)^{nN}}{(n!)^N} \prod_{i=1}^{N} \frac{(-b_i - \cdots - b_N)_{(N-i+1)n}}{(-b_i - \cdots - b_N)_{(N-i)n}}$$

$$= \frac{(-1)^{nN}}{(n!)^N} \prod_{i=1}^{N} (-b_i - \cdots - b_N + (N-i)n)_n.$$

The claim of Theorem 3 follows from the fact that

$$(-1)^n \frac{(-b_k - \cdots - b_N + (N-k)n)_n}{n!} = (-1)^n \binom{-b_k - \cdots - b_N + (N-k+1)n - 1}{n}$$

$$= \binom{b_k + \cdots + b_N - (N-k)n}{n}.$$

□

3.6 Proof of Theorem 4

The proof of Theorem 4 is very similar: we will use Lemma 3 and the same reasoning as before. The only difference lies in the fact that because the hypergeometric function has one parameter less, we need to redefine U_{N-1}, V_{N-1} and V_N. Recall that the denominator of U_k was given by $(N-k+1)^{(N-k+1)n}$ and it cancelled with the denominator of V_{k-1}. In the present case U_{N-1} will have no denominator and therefore 2^{2n} from V_{N-2} survives. This fits with the 4^k in the statement of Lemma 3 and is another indicator for the importance and essence of the constant $a = 2$.

Using Lemma 3 we obtain the coefficient of t^n for any $n \in \mathbb{N}$ on the left-hand side:

$$[t^n]\mathrm{Diag}((1 + x_1 + \cdots + 2\, x_{N-1} + x_N)^b R(x_1, \dots, x_N))$$
$$= 4^n \binom{(b-1)/2}{n} \binom{b_N}{n} \cdot \binom{b_{N-2} + b_{N-1} + b_N + b - 2n}{n} \cdots \binom{b_1 + \cdots + b_N + b - (N-1)n}{n}.$$

By the same reasoning as before, we have for all $k = 1, \dots, N-2, N$

$$U_k \coloneqq \prod_{i=1}^{N-k+1} (u_i^k)_n = \frac{(-b_k - \cdots - b_N - b)_{(N-k+1)n}}{(N-k+1)^{(N-k+1)n}},$$

and similarly

$$V_k \coloneqq \prod_{i=1}^{N-k} (v_i^k)_n = \frac{(-b_k - \cdots - b_N - b)_{(N-k)n}}{(N-k)^{(N-k)n}},$$

for $k = 1, \dots, N-2$. Clearly $V_{N-1} \coloneqq \prod_{i=1}^{N-1} (v_i^{N-1})_n = (n!)^{N-1}$ and we set $V_N \coloneqq 1$. Moreover, this time we have

$$U_{N-1} \coloneqq (u^{N-1})_n = ((1-b)/2)_n.$$

Altogether, we find

$$
\begin{aligned}
&[t^n]_M F_{M-1}(u; v; (-N)^N t) = \frac{(-N)^{nN}}{n!} \prod_{i=1}^{N} \frac{U_i}{V_i} \\
=& \frac{(-1)^{nN} 2^{2n} ((1-b)/2)_n}{(n!)^N} \prod_{i=1}^{N-2} \frac{(-b_i - \cdots - b_N - b)_{(N-i+1)n}}{(-b_i - \cdots - b_N - b)_{(N-i)n}} \cdot \frac{(-b_N - b)_n}{1} \\
=& 4^n \frac{(-1)^n ((1-b)/2)_n}{n!} \cdot \prod_{i=1}^{N-2} \frac{(-1)^n (-b_i - \cdots - b_N + (N-i)n)_n}{n!} \cdot \frac{(-1)^n (-b_N - b)_n}{n!}.
\end{aligned}
$$

Using the same final observation as before we conclude the proof. □

4 Algebraicity and Hadamard grade

4.1 Algebraic cases

We address here the following question: given $b_1, \ldots, b_N \in \mathbb{Q}$, $b_N \neq 0$, when is the diagonal

$$\mathrm{Diag}(R(\mathbf{x})) = \mathrm{Diag}((1+x_1)^{b_1} \cdots (1 + x_1 + \cdots + x_N)^{b_N})$$

an algebraic function?

Corollary 1. *$\mathrm{Diag}(R(\mathbf{x}))$ is algebraic if and only if $N = 2$ and $b_2 \in \mathbb{Z}$, or $N = 1$.*

In the proof below we will use several times the following useful fact [13, Thm. 33]: if a generalized hypergeometric function is algebraic, then its monodromy weight is zero, that is the number of integer bottom parameters is at most equal to the number of integer top parameters.

Proof. By Theorem 3 it is sufficient to study the algebraicity of the generalized hypergeometric function $H(t)$ defined by

$${}_{N(N+1)/2}F_{N(N+1)/2-1}\left([u^1, \ldots, u^{N-1}, -b_N]; [v^1, \ldots, v^{N-1}, 1, 1, \ldots, 1]; t\right),$$

where u^k and v^k, $k = 1, \ldots, N-1$ are defined like in §3.1:

$$u^k := \left(\frac{b}{\ell+1}, \frac{b+1}{\ell+1}, \ldots, \frac{b+\ell}{\ell+1}\right)$$

and

$$v^k := \left(\frac{b}{\ell}, \frac{b+1}{\ell}, \ldots, \frac{b+\ell-1}{\ell}\right),$$

for $b = -(b_k + b_2 + \cdots + b_N)$ and $\ell = N - k$.

By definition, $N-1$ of the bottom parameters are ones. We claim that each tuple u^k contains at most one integer and if it does contain one, then v^k does as well. From the definition of u^k it follows that if some $u_i^k \in \mathbb{Z}$ then $b \in \mathbb{Z}$ and $b \equiv -i + 1 \mod \ell + 1$. This shows that for any $k = 1, \ldots, N-1$ at most one $u_i^k \in \mathbb{Z}$. Because of the definition of v^k we see that if $b \in \mathbb{Z}$ and $b \equiv i \mod \ell$ for some $i \in \{1, \ldots, \ell\}$, then $v_i^k \in \mathbb{Z}$. This proves the claim.

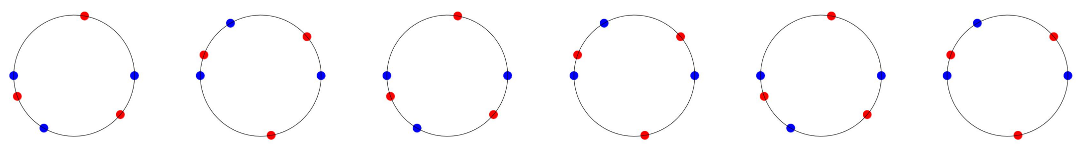

Figure 1: A pictorial proof of the algebraicity of ${}_3F_2\left(\left[\frac{1-R}{3}, \frac{2-R}{3}, \frac{3-R}{3}\right], [\frac{1}{2}, 1-R]; t\right)$ for $R \in \{\frac{1}{3}, \frac{2}{3}\}$. There are $\varphi(18) = 6$ conditions to check, which lead to two distinct interlacing configurations.

Henceforth in order to introduce new integer parameters on the top, while not creating equally many on the bottom, it is only possible to choose $-b_N$ integer. Therefore in order to archive monodromy weight zero – a necessary condition for algebraicity of $H(t)$ – we need to have $N-1 \leq 1$. From the same argument it follows that in the case $N-1=1$, we need to have $-b_N \in \mathbb{Z}$.

Obviously for $N = 1$ the diagonal is algebraic, so it remains to prove that, conversely, when $-b_2 =: S$ is an integer and $b_1 =: R \in \mathbb{Q}$ arbitrary, then the diagonal in

$$\mathrm{Diag}\left(\frac{(1-x_1)^R}{(1-x_1-x_2)^S}\right) = {}_3F_2\left(\left[\frac{S-R}{2}, \frac{S-R+1}{2}, S\right], [1, S-R]\,; 4\,t\right). \tag{11}$$

is an algebraic function. If R is an integer too, this follows from by [21, 14]. In the general case, one can rewrite the ${}_3F_2$ in (11) as the Hadamard product

$${}_3F_2\left(\left[\frac{S-R}{2}, \frac{S-R+1}{2}, S\right], [1, S-R]\,; t\right) = {}_2F_1\left(\left[\frac{S-R}{2}, \frac{S-R+1}{2}\right], [S-R]\,; t\right) \star (1-t)^{-S}. \tag{12}$$

The ${}_2F_1$ is algebraic as it corresponds to Case I in Schwarz's table [23]. Since S is an integer, $(1-t)^{-S}$ is a rational function. We conclude by applying Jungen's theorem [16, Thm. 8]: the Hadamard product of an algebraic and a rational function is algebraic, see also [24, Prop. 6.1.11]. □

4.2 Hadamard grade

Recall that the *Hadamard grade* [2] of a power series $S(t)$ is the least positive integer $h = h(S)$ such that $S(t)$ can be written as the Hadamard product of h algebraic power series, or ∞ if no such writing exists. Since algebraic power series are diagonals [14, §3], and diagonals are closed under Hadamard product [11, Prop. 2.6], any power series with finite Hadamard grade is a diagonal [2, Thm. 7]. Conversely, it is not clear whether diagonals always have finite Hadamard grade[6].

A natural question in relation with Corollary 1 is the following: given $b_1, \ldots, b_N \in \mathbb{Q}$, determine the Hadamard grade of $\mathrm{Diag}((1+x_1)^{b_1} \cdots (1+x_1+\cdots+x_N)^{b_N})$, or at least decide if it is finite or not. For instance, by Theorem 1, when $S = 1$, $R = 1/2$, the diagonal

$$\mathrm{Diag}\left(\frac{(1-x_1-x_2)^R}{(1-x_1-x_2-x_3)^S}\right)$$

[6]There exist diagonals of any prescribed finite grade [22, Cor. 1 and 2], assuming the Rohrlich–Lang conjecture [26, Conj. 22]. If moreover Christol's conjecture is true, then there also exist diagonals of infinite grade [22, Prop. 1].

is a transcendental ${}_2F_1$, which can be written as the Hadamard product of two algebraic functions

$$ {}_2F_1\left(\left[\frac{1}{6},\frac{5}{6}\right],\left[\frac{1}{2}\right];t\right)\star(1-t)^{-1/2}, $$

(the ${}_2F_1$ being algebraic by Schwarz's classification [23]) hence its Hadamard grade is 2.

Similarly, the diagonals from (2) and (3) have Hadamard grade 2 due to the identities

$$ {}_3F_2\left(\left[\frac{2}{9},\frac{5}{9},\frac{8}{9}\right],\left[1,\frac{2}{3}\right];t\right)={}_3F_2\left(\left[\frac{2}{9},\frac{5}{9},\frac{8}{9}\right],\left[\frac{1}{2},\frac{2}{3}\right];t\right)\star(1-t)^{-1/2} $$

and

$$ {}_3F_2\left(\left[\frac{1}{9},\frac{4}{9},\frac{7}{9}\right],\left[1,\frac{1}{3}\right];t\right)={}_3F_2\left(\left[\frac{1}{9},\frac{4}{9},\frac{7}{9}\right],\left[\frac{1}{2},\frac{1}{3}\right];t\right)\star(1-t)^{-1/2} $$

and to the fact that the two ${}_3F_2$'s on the right-hand side are algebraic by the interlacing criterion [3, Thm. 4.8]; see Figure 1 for a pictorial proof, where red points correspond to top parameters, and blue points to bottom parameters (and the additional parameter 1). More generally, the diagonal from (4) has Hadamard grade 2 due to the identity

$$ {}_3F_2\left(\left[\frac{1-R}{3},\frac{2-R}{3},\frac{3-R}{3}\right],[1,1-R];t\right)={}_3F_2\left(\left[\frac{1-R}{3},\frac{2-R}{3},\frac{3-R}{3}\right],\left[\frac{1}{2},1-R\right];t\right)\star(1-t)^{-\frac{1}{2}}, $$

since the ${}_3F_2$ on the right-hand side is an algebraic function for any $R\in\mathbb{Q}$ (with Fig. 1 replaced by a similar one, containing only interlacing blue right triangles and red equilateral triangles).

This observation provides an alternative (and probably the shortest) proof that the hypergeometric functions in (2), (3) and (4) are diagonals of algebraic functions.

The same observation also quickly solves two more cases amongst the 16 cases in the list [5, p. 58], namely those

$$ {}_3F_2\left(\left[\frac{N_1}{9},\frac{N_2}{9},\frac{N_3}{9}\right],\left[1,\frac{M_1}{3}\right];t\right) $$

for which $(N_1,N_2,N_3;M_1)$ is $(1,4,7;2)$ or $(2,5,8;1)$.

Furthermore, using the interlacing criterion it is easy to see that the hypergeometric function ${}_3F_2([1/9,4/9,7/9];[a,b];t)$ is algebraic if (a,b) or (b,a) occurs in the set

$$ \{(3/4,1/4),(2/3,1/3),(2/3,1/6),(1/2,1/3),(1/2,1/6)\}. $$

Similarly, ${}_3F_2([2/9,5/9,8/9];[a,b];t)$ is algebraic if (a,b) or (b,a) is part of

$$ \{(5/6,1/2),(5/6,1/3),(3/4,1/4),(2/3,1/2),(2/3,1/3)\}. $$

Similarly, both ${}_3F_2\left(\left[\frac{1}{4},\frac{3}{8},\frac{7}{8}\right],\left[\frac{2}{3},\frac{1}{3}\right]\right)$ and ${}_3F_2\left(\left[\frac{1}{8},\frac{3}{4},\frac{5}{8}\right],\left[\frac{2}{3},\frac{1}{3}\right]\right)$ are algebraic.

The previous analysis proves the following corollary.

Corollary 2. *The hypergeometric function*

$$ {}_3F_2\left([A,B,C],[1,D];t\right) $$

has Hadamard grade 2 (hence is a diagonal) for $(A, B, C; D)$ *in the following set*

$$\Big\{(1/4, 3/8, 7/8; 1/3),\ 1/4, 3/8, 7/8; 2/3),\ (1/8, 5/8, 3/4; 1/3),\ (1/8, 5/8, 3/4; 2/3),$$
$$(1/9, 4/9, 7/9; 1/2),\ (1/9, 4/9, 7/9; 1/3),\ (1/9, 4/9, 7/9; 1/4),\ (1/9, 4/9, 7/9; 1/6),$$
$$(1/9, 4/9, 7/9; 2/3),\ (1/9, 4/9, 7/9; 3/4),\ (2/9, 5/9, 8/9; 1/2),\ (2/9, 5/9, 8/9; 1/3),$$
$$(2/9, 5/9, 8/9; 1/4),\ (2/9, 5/9, 8/9; 2/3),\ (2/9, 5/9, 8/9; 3/4),\ (2/9, 5/9, 8/9; 5/6)\Big\}.$$

Note that the authors of [5, 6] produced in 2011 a list of 116 potential counter-examples to Christol's conjecture; they displayed a sublist of 18 cases in the preprint [5, Appendix F], of which they selected 3 cases that were published in [6, §5.2]. As of today, to our knowledge, the 3 cases in [6] are still unsolved[7], while 2 of the 18 cases in [5] have been solved in [1] (in red, above) and 2 others in the current paper (in orange, above). From the list of 116 cases, only 2 were previously solved, in [1]. Corollary 2 solves 14 cases more, raising the number of solved cases to 16 (out of 116).

Acknowledgements. We are grateful to Jakob Steininger, who provided a combinatorial idea which led us to the proof of Lemma 1, and to Herwig Hauser for his interest and engaging questions which led to Section 4. Alin Bostan was supported in part by DeRerumNatura ANR-19-CE40-0018 and Sergey Yurkevich was fully supported by the Austrian Science Fund FWF, P-313338.

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