Propagation of chaos for stochastic particle systems with singular mean-field interaction of $L^q - -L^p$ type

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Abstract

In this work, generalizing the techniques introduced by Jabir-Talay-Tomasevic [3], we prove the well-posedness and propagation of chaos for a stochastic particle system in mean-field interaction under the assumption that the interacting kernel belongs to a suitable $L^q_t - L^p_x$ space. Contrarty to the large deviation principle approach by [2], the main ingredient of the proof here are the partial Girsanov transformations introduced in [3].

1 Introduction

In this work, we prove the propagation of chaos of the particle system

$$\begin{cases} dX^{i,N}_t = \frac{1}{N}\sum_{j=1}^N b(t, X^{i,N}_t, X^{j,N}_t)\ dt + \sqrt{2} dW^i_t, \\ X^{i,N}_0 \text{ i.i.d. and independent of } W := (W^i, 1 \le i \le N), \end{cases} \tag{1.1}$$

towards the non-linear stochastic differential equation

$$\begin{cases} dX_t = \int (b(t, X_t, y)\rho_t(y)\ dy\ dt + \sqrt{2} dW_t, \quad t > 0, \\ \rho_s(y)dy := \mathcal{L}(X_s), \quad X_0 \sim \rho_0(x)dx, \end{cases} \tag{1.2}$$

under the assumption that

($\mathbf{H}^b$) $|b(t, x, y)| \le h_t(x - y)$ for some $h \in L^q_{loc}(\mathbb{R}_+; L^p(\mathbb{R}^d))$, where $p, q \in (2, \infty)$ satisfy $\frac{d}{p} + \frac{2}{q} < 1$.

In (1.1) and (1.2) W^i's and W are independent standard d dimensional Brownian motions defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$

The propagation of chaos of (1.1) towards (1.2) has been very recently established in the literature using large deviation principle theory [2]. In this work, we propose a completely different approach relying on the techniques introduced in [3] (where the present author is one of the co-authors).

The problem treated in [3] is the case of probabilistic interpretation of the parabolic-parabolic Keller-Segel model in one-dimensional setting. The particles interaction is both non-Markovian and singular and as such does not enter in the framework of [2]. However, the propagation of chaos was established and it seemed important to adapt the techniques from [3] to the setting in (1.1) and make them available for other authors, rather than treating the problems with singular interactions in a case by case studies.

Despite the one-dimensional setting, the problem treated in [3] presents a notable difficulty due to the unusual interaction between the particles. Namely, instead of $b(t, X^{i,N}_t, X^{j,N}_t)$ in (2.1), one has $\int_0^t K_{t-s}(X^{i,N}_t - X^{j,N}_s)ds$ where K is the singular kernel $K_t(x) = \frac{-x}{\sqrt{2\pi}t^{3/2}} e^{-\frac{x^2}{2t}}$.
Inspired by [5], the authors in [3] first prove the weak well-posedness of the particle system by means of a Girsanov transformation exploting the integrability properties (in time and space) of the interaction kernel. However, in order to get propagation of chaos, the above transformation is not helpful as the estimate on the exponential moment of the drift (appearing in the Novikov condition) explodes as $N \to \infty$. This difficulty is circumvented by an original idea to perform on the particle system the so-called *partial Girsanov transformations*. These partial transformations fix a finite number of particles $k < N$ and transform them to independent Brownian motions while they also take off their dependence of the rest $N - k$ particles. Their advantage is that the estimates on the respective exponential martingales do not explode with N.
When one proves by hand the propagation of chaos, these transformations naturally appear as the particle

system is exchangeable. Hence, as it was noted in Sznitman [8, p. 180], what one needs to control are functionals of finite number of particles (usually at most 4).

In this paper, we adapt the above techniques to the setting in (1.1). The fact the interaction is Markovian simplifies lightly the computations. However, the multidimensional setting and the general assumptions on b, complicate the computations.

We finish this section by noting that in [6, Theorem 1.1] strong well-posedness of (1.2) is established. Also, a Gaussian estimate on the one dimensional time marginal densities is proved. In this work, as we show the convergence in law of the empirical measure of (1.1) towards the law of the solution to (1.2), we will only work with the martingale problem related to (1.2) that we formulate precisely in the next section.

2 Main Results

Let $t > 0$. As the interaction kernel $b(t,\cdot,\cdot)$ has only integrability properties w.r.t. the second or third variable, it may be unbounded or not well defined in certain points. Let us denote by $\mathcal{N}_b(t)$ the following set:

$$\mathcal{N}_b(t) = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : \lim_{(x',y') \to (x,y)} |b(t,x',y')| = \infty \quad \text{or} \quad \lim_{(x',y') \to (x,y)} b(t,x',y') \text{ does not exist}\}.$$

However, as $|b(t,x,y)| \leq h_t(x-y)$ and $h_t \in L^p(\mathbb{R}^d)$, the set $\mathcal{N}_b(t)$ is of Lebesgue's measure zero in $\mathbb{R}^d \times \mathbb{R}^d$.

Thus, in order to ensure that all the termes appearing in it are well defined, the particle system reads

$$\begin{cases} dX_t^{i,N} = \frac{1}{N}\sum_{j=1,j\neq i}^N b(t, X_t^{i,N}, X_t^{j,N}) \mathbb{1}_{\{(X_t^{i,N},X_t^{j,N}) \notin \mathcal{N}_b(t)\}}\, dt + \sqrt{2} dW_t^i, \\ X_0^{i,N} \text{ i.i.d. and independent of } W := (W^i, 1 \leq i \leq N). \end{cases} \tag{2.1}$$

Here W^i's are N independent standard d-dimensional Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Notice here that we assume that a particle does not interact with itself.

In practical examples, h_t explodes (or it is not well defined) only in one point (zero) and thus, $b(t,\cdot,\cdot)$ may explode (or not be well defined) on the line $x = y$. In order to keep the result as general as possible, we will rather work with the sets $(\mathcal{N}_b(t))_{t\geq 0}$. However, the reader should have in mind that this is just a technicality and that, in practice $\mathbb{1}_{\{(X_t^{i,N},X_t^{j,N}) \notin \mathcal{N}_b(t)\}}$ becomes $\mathbb{1}_{\{X_t^{i,N} \neq X_t^{j,N}\}}$ (as in [3]).

It is reasonable to assume that $b(t,\cdot,\cdot)$ is continuos outside of $\mathcal{N}_b(t)$ as in practice one deals with interaction kernels in the form of convolutions that are well defined and continuos almost everywhere (like $\pm\frac{x}{|x|^r}$).

Our first main result is weak well-posedness of (2.1).

Theorem 2.1. *Assume* $(\mathbf{H}^b)$. *Given* $0 < T < \infty$ *and* $N \in \mathbb{N}$, *there exists a weak solution* $(\Omega, \mathcal{F}, (\mathcal{F}_t; 0 \leq t \leq T), \mathbb{Q}^N, W, X^N)$ *to the* N*-interacting particle system* (2.1) *that satisfies, for any* $1 \leq i \leq N$,

$$\mathbb{Q}^N\left(\int_0^T \left(\frac{1}{N}\sum_{j=1,j\neq i}^N b(t, X_t^{i,N}, X_t^{j,N}) \mathbb{1}_{\{(X_t^{i,N},X_t^{j,N}) \notin \mathcal{N}_b(t)\}}\right)^2 dt < \infty\right) = 1. \tag{2.2}$$

In view of Karatzas and Shreve [4, Chapter 5, Proposition 3.10], one has the following uniqueness result:

Corollary 2.2. *Weak uniqueness holds in the class of weak solutions satisfying* (2.2).

Before we state the propagation of chaos result, we formulate the martingale problem associated to (1.2).

Definition 2.3. $\mathbb{Q} \in \mathcal{P}(C[0,T];\mathbb{R}^d)$ *is a solution to (MP) if:*

(i) $\mathbb{Q}_0 = \mu_0$;

(ii) For any $t \in (0,T]$ *and any* $r > 1$, *the one dimensional time marginal* $\mathbb{Q}_t$ *of* $\mathbb{Q}$ *has a density* ρ_t *w.r.t. Lebesgue measure on* $\mathbb{R}^d$ *which belongs to* $L^r(\mathbb{R}^d)$ *and satisfies*

$$\exists C_T, \quad \forall\, 0 < t \leq T, \quad \|\rho_t\|_{L^r(\mathbb{R}^d)} \leq \frac{C_T}{t^{\frac{d}{2}(1-\frac{1}{r})}};$$

(iii) Denoting by $(x(t);\, t \leq T)$ the canonical process of $C([0,T];\mathbb{R}^d)$, we have: For any $f \in C_b^2(\mathbb{R}^d)$, the process defined by

$$M_t := f(x(t)) - f(x(0)) - \int_0^t \left(\nabla f(x(s)) \cdot \Big(\int b(s,x(s),y)\rho_s(y)dy \Big) + \triangle f(x(s)) \right) ds$$

is a $\mathbb{Q}$-martingale w.r.t. the canonical filtration.

Remark 2.4. *Under the assumption (H_b) and that the measure μ_0 has a finite β-order moment for some $\beta > 2$, the martingale problem (MP) admits a unique solution according to [6, Thm. 1.1]. In fact, [6, Thm. 1.1] gives strong well-posedness of (1.2) and proves the Gaussian estimates punctually on the marginal densities. In the martingale formulation it is enough to impose such estimates in L^r-norms as along with (H_b) this will ensure that all the terms in the definition of the process (M) are well defined.*

It is classical that a suitable notion of weak solution to (1.2) is equivalent to the notion of solution to (MP) (see e.g. [4]).

We are ready to state our second main result, the propagation of chaos of (2.1).

Theorem 2.5. *In addition to $(\mathbf{H}^b)$, assume that for any $t > 0$, $b(t,\cdot,\cdot)$ is continuous outside of the set $\mathcal{N}_b(t)$. Assume that the $X_0^{i,N}$'s are i.i.d. and that the initial distribution of $X_0^{1,N}$ is the measure μ_0 that for some $\beta > 2$ has finite β-order moment . Then, the empirical measure $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}}$ of (2.1) converges in the weak sense, when $N \to \infty$, to the unique weak solution of (1.2).*

3 Proof of Theorem 2.1

We start from a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t;\, 0 \leq t \leq T), \mathbb{W})$ on which d - dimensional Brownian motions $(W^1, \dots, W^N)$ and the random variables $X_0^{i,N}$(see (2.1)) are defined. Set $\bar{X}_t^{i,N} := X_0^{i,N} + W_t^i$ $(t \leq T)$ and $\bar{X} := (\bar{X}^{i,N}, 1 \leq i \leq N)$. Denote the drift terms in (2.1) by $b_t^{i,N}(x)$, $x \in C([0,T];\mathbb{R}^d)^N$, and the vector of all the drifts as $B_t^N(x) = (b_t^{1,N}(x), \dots, b_t^{N,N}(x))$. For a fixed $N \in \mathbb{N}$, consider

$$Z_T^N := \exp\left\{ \int_0^T B_t^N(\bar{X}) \cdot dW_t - \frac{1}{2}\int_0^T \left|B_t^N(\bar{X})\right|^2 dt \right\}.$$

To prove Theorem 2.1, it suffices to prove the following Novikov condition holds true (see e.g. [4, Chapter 3, Proposition 5.13]): For any $T > 0$, $N \geq 1$, $\kappa > 0$, there exists $C(T,N,\kappa)$ such that

$$\mathbb{E}_{\mathbb{W}}\left(\exp\left\{ \kappa \int_0^T |B_t^N(\bar{X})|^2 dt \right\} \right) \leq C(T,N,\kappa). \tag{3.1}$$

Drop the index N for simplicity. Using the definition of (B_t^N) and Jensen's inequality one has

$$\mathbb{E}_{\mathbb{W}}\left[\exp\left\{ \kappa \int_0^T \left|B_t^N(\bar{X})\right|^2 dt \right\} \right] \leq \mathbb{E}_{\mathbb{W}}\left[\exp\left\{ \frac{1}{N}\sum_{i=1}^N \frac{1}{N} \sum_{j=1, j\neq i}^N \int_0^T \kappa N |b(t, \bar{X}_t{}^i, \bar{X}_t{}^j)|^2 dt \right\} \right],$$

from which we deduce

$$\mathbb{E}_{\mathbb{W}}\left[\exp\left\{ \kappa \int_0^T \left|B_t^N(\bar{X})\right|^2 dt \right\} \right] \leq \frac{1}{N}\sum_{i=1}^N \frac{1}{N} \sum_{j=1, j\neq i}^N \mathbb{E}_{\mathbb{W}}\left[\exp\left\{ \kappa N \int_0^T |b(t, \bar{X}_t{}^i, \bar{X}_t{}^j)|^2 dt \right\} \right].$$

Assume for a moment that for $i, j \leq N$ such that $j \neq i$ one has

$$\mathbb{E}_{\mathbb{W}}\left[\exp\left\{ \kappa N \int_0^T |b(t, \bar{X}_t{}^i, \bar{X}_t{}^j)|^2 dt \right\} \right] \leq C(T,N). \tag{3.2}$$

Then (3.1) is satisfied and the proof is finished.

The rest of the proof will be devoted to establishing (3.2). Actually, we will prove the following more general statement:

Proposition 3.1. *Let $T > 0$. Suppose the hypothesis (H_b). Let $w := (w_t)$ be a $(\mathcal{G}_t)$-Brownian motion with an arbitrary initial distribution μ_0 on some probability space equipped with a probability measure $\mathbb{P}$ and a filtration $(\mathcal{G}_t)$. Suppose that the filtered probability space is rich enough to support a continuous process Y independent of (w_t). For any $\alpha > 0$, one has*

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left\{\alpha\int_0^T |b(t,w_t,Y_t)|^2 dt\right\}\right] \leq C(T,\alpha),$$

where $C(T,\alpha)$ depends only on T and α, but does neither depend on the law $\mathcal{L}(Y)$ nor of μ_0.

To prove Proposition 3.1, we need some preparation in form of two auxiliary Lemmas.

First, for $(t,x) \in [0,T] \times \mathbb{R}^d$, denote by $g_t(x) := \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}}$. Note that for any $t > 0$ and any $p \geq 1$, one has

$$\|g_t\|_p = \frac{C_p}{t^{\frac{d}{2}(1-\frac{1}{p})}}. \tag{3.3}$$

Lemma 3.2. *Suppose the hypothesis (H_b). Let $w := (w_t)$ be a $(\mathcal{G}_t)$-Brownian motion with an arbitrary initial distribution μ_0 on some probability space equipped with a probability measure $\mathbb{P}$ and a filtration $(\mathcal{G}_t)$. There exists a universal real number $C_0 > 0$ such that*

$$\forall x \in C([0,T];\mathbb{R}), \quad \forall 0 \leq t_1 \leq t_2 \leq T, \quad \int_{t_1}^{t_2} \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{t_1}} |b(t,w_t,x_t)|^2 dt \leq C_0(T)(t_2-t_1)^{\frac{q-2}{q}-\frac{d}{p}}.$$

Proof. Using the assumption (H_b), one has

$$I := \int_{t_1}^{t_2} \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{t_1}} |b(t,w_t,x_t)|^2 dt \leq \int_{t_1}^{t_2}\int h_t^2(y+w_{t_1}-x_t) g_{t-t_1}(y) dy\ dt$$

Applying Hölder inequality in space with $\frac{p}{2} > 1$ and afterwards in time with $\frac{q}{2} > 1$, one has

$$I \leq \int_{t_1}^{t_2} \|h_t\|^2_{L^p(\mathbb{R}^d)} \|g_{t-t_1}\|_{L^{\frac{p}{p-2}}(\mathbb{R}^d)} \leq \|h\|_{L^q((0,T);L^p(\mathbb{R}^d))} \left(\int_{t_1}^{t_2} \left(\|g_{t-t_1}\|_{L^{\frac{p}{p-2}}(\mathbb{R}^d)}\right)^{\frac{q}{q-2}} dt\right)^{\frac{q-2}{q}}.$$

According to (3.3), we have

$$I \leq C_0(T)\left(\int_{t_1}^{t_2} \frac{1}{(t-t_1)^{\frac{d}{p}\frac{q}{q-2}}} dt\right)^{\frac{q-2}{q}}.$$

For the last integral to be finite, one needs to have $\frac{d}{p} < \frac{q-2}{q} = 1 - \frac{2}{q}$. This is exactly the constraint in the hypothesis (H_b). Thus, one obtains

$$I \leq C_0(T) t^{\frac{q-2}{q}-\frac{d}{p}}.$$

□

Lemma 3.3. *Same assumptions as in Lemma 3.2. Let $C_0(T)$ be as in Lemma 3.2. For any $\kappa > 0$, there exists $C(T,\kappa)$ independent of μ_0 such that, for any $0 \leq T_1 \leq T_2 \leq T$ satisfying $T_2 - T_1 < (C_0(T)\kappa)^{-\frac{1}{\frac{q-2}{q}-\frac{d}{p}}}$,*

$$\forall x \in C([0,T];\mathbb{R}), \quad \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}}\left[\exp\left\{\kappa\int_{T_1}^{T_2} |b(t,w_t,x_t)|^2 dt\right\}\right] \leq C(T,\kappa).$$

Proof. We adapt the proof of Khasminskii's lemma in Simon [7]. Admit for a while we have shown that there exists a constant $C(\kappa,T)$ such that for any $M \in \mathbb{N}$

$$\sum_{k=1}^{M} \frac{\kappa^k}{k!} \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}} \left(\int_{T_1}^{T_2} |b(t,w_t,x_t)|^2\ dt\right)^k \leq C(T,\kappa), \tag{3.4}$$

provided that $T_2 - T_1 < (C_0(T)\kappa)^{-\frac{1}{\frac{q-2}{q}-\frac{d}{p}}}$. The desired result then follows from Fatou's lemma.

We now prove (3.4).

By the tower property of conditional expectation,

$$\mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}}\left[\left(\int_{T_1}^{T_2}|b(t,w_t,x_t)|^2dt\right)^k\right]=k!\int_{T_1}^{T_2}\int_{t_1}^{T_2}\int_{t_2}^{T_2}\cdots\int_{t_{k-2}}^{T_2}\int_{t_{k-1}}^{T_2}\mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}}\Big[|b(t_1,w_{t_1},x_{t_1})|^2|b(t_2,w_{t_2},x_{t_2})|^2$$
$$\times\cdots\times|b(t_{k-1},w_{t_{k-1}},x_{t_{k-1}})|^2\left(\mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{t_{k-1}}}|b(t_k,w_{t_k},x_{t_k})|^2\right)\Big]\,dt_k\,dt_{k-1}\cdots dt_2\,dt_1.$$

In view of Lemma 3.2,

$$\int_{t_{k-1}}^{T_2}\mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{t_{k-1}}}|b(t_k,w_{t_k},x_{t_k})|^2\,dt_k\le C_0(T)(T_2-t_{k-1})^{\frac{q-2}{q}-\frac{d}{p}}\le C_0(T)(T_2-T_1)^{\frac{q-2}{q}-\frac{d}{p}}.$$

Therefore, by Fubini's theorem,

$$\mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}}\left[\left(\int_{T_1}^{T_2}|b(t,w_t,x_t)|^2dt\right)^k\right]\le k!C_0(T)(T_2-T_1)^{\frac{q-2}{q}-\frac{d}{p}}\int_{T_1}^{T_2}\int_{t_1}^{T_2}\int_{t_2}^{T_2}\cdots\int_{t_{k-2}}^{T_2}\mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}}\Big[|b(t_1,w_{t_1},x_{t_1})|^2|b(t_2,w_{t_2},x_{t_2})|^2$$
$$\times\cdots\times|b(t_{k-1},w_{t_{k-1}},x_{t_{k-1}})|^2\Big]dt_{k-1}\cdots dt_2\,dt_1.$$

Now we repeatedly condition with respect to $\mathcal{G}_{t_{k-i}}$ $(i\ge 2)$ and combine Lemma 3.2 with Fubini's theorem. It comes:

$$\mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}}\left(\int_{T_1}^{T_2}|b(t,w_t,x_t)|^2dt\right)^k\le k!(C_0(T)(T_2-T_1)^{\frac{q-2}{q}-\frac{d}{p}})^{k-1}\int_{T_1}^{T_2}\mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}}|b(t_1,w_{t_1},x_{t_1})|^2dt_1\le k!(C_0(T)(T_2-T_1)^{\frac{q-2}{q}-\frac{d}{p}})^k.$$

Thus, (3.4) is satisfied provided that $T_2-T_1<(C_0(T)\kappa)^{-\frac{1}{\frac{q-2}{q}-\frac{d}{p}}}$. □

Finally, the proof of Proposition 3.1 is identical to the proof of [3, Prop. 3.3] where F is replaced by b^2 and the previous lemma is used instead of [3, Lemma 3.2].

4 Propagation of chaos

4.1 Girsanov transform for $1\le r<N$ particles

For any integer $1\le r<N$, proceeding as in the proof of Theorem 2.1 one gets the existence of a weak solution on $[0,T]$ to

$$\begin{cases} d\widehat{X}_t^{l,N}=dW_t^l, \quad 1\le l\le r,\\ d\widehat{X}_t^{i,N}=\left\{\frac{1}{N}\sum_{j=r+1}^N b(t,\widehat{X}_t^{i,N},\widehat{X}_t^{j,N})\right\}dt+dW_t^i, \quad r+1\le i\le N,\\ \widehat{X}_0^{i,N}\text{ i.i.d. and independent of }(W):=(W^i,1\le i\le N).\end{cases}\tag{4.1}$$

Below we set $\hat{X}:=(\hat{X}^{i,N},1\le i\le N)$ and we denote by $\mathbb{Q}^{r,N}$ the probability measure under which $\hat{X}$ is well defined. Notice that $(\widehat{X}^{l,N},1\le l\le r)$ is independent of $(\widehat{X}^{i,N},r+1\le i\le N)$. We now study the exponential local martingale associated to the change of drift between (2.1) and (4.1). For $x\in C([0,T];\mathbb{R}^d)^N$ set

$$\beta_t^{(r)}(x):=\Big(b_t^{1,N}(x),\dots,b_t^{r,N}(x),\frac{1}{N}\sum_{i=1}^r b(t,x_t^{r+1},x_t^i),\dots,\frac{1}{N}\sum_{i=1}^r b(t,x_t^N,x_t^i)\Big).$$

In the sequel we will need uniform w.r.t N bounds for moments of

$$Z_T^{(r)}:=\exp\left\{-\int_0^T\beta_t^{(r)}(\widehat{X})\cdot dW_t-\frac{1}{2}\int_0^T|\beta_t^{(r)}(\widehat{X})|^2dt\right\}.\tag{4.2}$$

Proposition 4.1. *For any $T>0$, $\gamma>0$ and $r\geq 1$ there exists $N_0\geq r$ and $C(T,\gamma,r)$ s.t.*

$$\forall N\geq N_0,\quad \mathbb{E}_{\mathbb{Q}^{r,N}}\exp\left\{\gamma\int_0^T|\beta_t^{(r)}(\widehat{X})|^2dt\right\}\leq C(T,\gamma,r).$$

Proof. For $x\in C([0,T];\mathbb{R}^d)^N$, one has

$$|\beta_t^{(r)}(x)|^2=\sum_{i=1}^r\left(\frac{1}{N}\sum_{j=1}^N b(t,x_t^i,x_t^j)\right)^2+\frac{1}{N^2}\sum_{j=1}^{N-r}\left(\sum_{i=1}^r b(t,x_t^{r+j},x_t^i)\right)^2.$$

By Jensen's inequality,

$$|\beta_t^{(r)}|^2\leq\frac{1}{N}\sum_{i=1}^r\sum_{j=1}^N|b(t,x_t^i,x_t^j)|^2+\frac{r}{N^2}\sum_{j=1}^{N-r}\sum_{i=1}^r|b(t,x_t^{r+j},x_t^i)|^2.$$

For simplicity we below write $\mathbb{E}$ (respectively, $\hat{X}^i$) instead of $\mathbb{E}_{\mathbb{Q}^{r,N}}$ (respectively, $\hat{X}^{i,N}$). Observe that

$$\begin{aligned}
&\mathbb{E}\exp\left\{\gamma\int_0^T|\beta_t^{(r)}(\widehat{X})|^2dt\right\}\\
&\leq\Big(\mathbb{E}\exp\Big\{\sum_{i=1}^r\frac{2\gamma}{N}\sum_{j=1}^N\int_0^T|b(t,\widehat{X}_t^i,\widehat{X}_t^j)|^2dt\Big\}\Big)^{1/2}\Big(\mathbb{E}\exp\Big\{\frac{2\gamma r}{N^2}\sum_{j=1}^{N-r}\sum_{i=1}^r\int_0^T|b(t,\widehat{X}_t^{r+j},\widehat{X}_t^i)|^2dt\Big\}\Big)^{1/2}\\
&\leq\Big(\prod_{i=1}^r\frac{1}{N}\sum_{j=1}^N\mathbb{E}\exp\Big\{2\gamma r\int_0^T|b(t,\widehat{X}_t^i,\widehat{X}_t^j)|^2dt\Big\}\Big)^{\frac{1}{2r}}\Big(\prod_{j=1}^{N-r}\frac{1}{r}\sum_{i=1}^r\mathbb{E}\exp\Big\{\frac{2\gamma r^2}{N}\int_0^T|b(t,\widehat{X}_t^{r+j},\widehat{X}_t^i)dt\Big\}\Big)^{\frac{1}{2(N-r)}}.
\end{aligned}$$

In view of Proposition 3.1, the proof is finished. ☐

4.2 Tightness

We start with showing the tightness of $\{\mu^N\}$ and of an auxiliary empirical measure which is needed in the sequel.

Lemma 4.2. *Let $\mathbb{Q}^N$ be as above. The sequence $\{\mu^N\}$ is tight under $\mathbb{Q}^N$. In addition, let $\nu^N:=\frac{1}{N^4}\sum_{i,j,k,l=1}^N\delta_{X_\cdot^{i,N},X_\cdot^{j,N},X_\cdot^{k,N},X_\cdot^{l,N}}$. The sequence $\{\nu^N\}$ is tight under $\mathbb{Q}^N$.*

Proof. The tightness of $\{\mu^N\}$, respectively $\{\nu^N\}$, results from the tightness of the intensity measure $\{\mathbb{E}_{\mathbb{Q}^N}\mu^N(\cdot)\}$, respectively$\{\mathbb{E}_{\mathbb{Q}^N}\nu^N(\cdot)\}$: See Sznitman [8, Prop. 2.2-ii]. By symmetry, in both cases it suffices to check the tightness of $\{\text{Law}(X^{1,N})\}$. We aim to prove

$$\exists C>0,\forall N\geq N_0,\quad \mathbb{E}_{\mathbb{Q}^N}[|X_t^{1,N}-X_s^{1,N}|^4]\leq C_T|t-s|^2,\quad 0\leq s,t\leq T,\tag{4.3}$$

where N_0 is as in Proposition 4.1. Let $Z_T^{(1)}$ be as in (4.2). One has

$$\mathbb{E}_{\mathbb{Q}^N}[|X_t^{1,N}-X_s^{1,N}|^4]=\mathbb{E}_{\mathbb{Q}^{1,N}}[(Z_T^{(1)})^{-1}|\widehat{X}_t^{1,N}-\widehat{X}_s^{1,N}|^4].$$

As $\widehat{X}^{1,N}$ is a one dimensional Brownian motion under $\mathbb{Q}^{1,N}$,

$$\mathbb{E}_{\mathbb{Q}^N}[|X_t^{1,N}-X_s^{1,N}|^4]\leq(\mathbb{E}_{\mathbb{Q}^{1,N}}[(Z_T^{(1)})^{-2}])^{1/2}(\mathbb{E}_{\mathbb{Q}^{1,N}}[|\widehat{X}_t^{1,N}-\widehat{X}_s^{1,N}|^8])^{1/2}\leq(\mathbb{E}_{\mathbb{Q}^{1,N}}[(Z_T^{(1)})^{-2}])^{1/2}C|t-s|^2.$$

Observe that, for a Brownian motion $(W^\sharp)$ under $\mathbb{Q}^{1,N}$,

$$\mathbb{E}_{\mathbb{Q}^{1,N}}[(Z_T^{(1)})^{-2}]=\mathbb{E}_{\mathbb{Q}^{1,N}}\exp\left\{2\int_0^T\beta_t^{(1)}(\hat{X})\cdot dW_t^\sharp-\int_0^T|\beta_t^{(1)}(\hat{X})|^2dt\right\}.$$

Adding and subtracting $3\int_0^T|\beta_t^{(1)}|^2dt$ and applying again the Cauchy-Schwarz inequality,

$$\mathbb{E}_{\mathbb{Q}^{1,N}}[(Z_T^{(1)})^{-2}]\leq\left(\mathbb{E}_{\mathbb{Q}^{1,N}}\exp\left\{6\int_0^T|\beta_t^{(1)}(\hat{X})|^2dt\right\}\right)^{1/2}.$$

Applying Proposition 4.1 with $k=1$ and $\gamma=6$, we obtain the desired result. ☐

4.3 Convergence

To prove Theorem 2.5 we have to show that any limit point of $\{\text{Law}(\mu^N)\}$ is $\delta_{\mathbb{Q}}$, where $\mathbb{Q}$ is the unique solution to (MP). Since the particles interact through an unbounded possibly singular function, we adapt the arguments in Bossy and Talay [1, Thm. 3.2].

Let $\phi \in C_b(\mathbb{R}^{pd})$, $f \in C_b^2(\mathbb{R}^d)$, $0 < t_1 < \cdots < t_p \leq s < t \leq T$ and $m \in \mathcal{P}(C[0,T];\mathbb{R}^d)$. Set

$$G(m) := \int_{(C[0,T];\mathbb{R}^d)^2} \phi(x^1_{t_1},\dots,x^1_{t_p})\Big(f(x^1_t) - f(x^1_s) - \frac{1}{2}\int_s^t \triangle f(x^1_u)du - \int_s^t \nabla f(x^1_u)\cdot b(u,x^1_u,x^2_u)du\Big)dm(x^1)\otimes dm(x^2).$$

We start with showing that

$$\lim_{N\to\infty} \mathbb{E}[\big(G(\mu^N)\big)^2] = 0. \tag{4.4}$$

Observe that

$$G(\mu^N) = \frac{1}{N}\sum_{i=1}^N \phi(X^{i,N}_{t_1},\dots,X^{i,N}_{t_p})\Big(f(X^{i,N}_t) - f(X^{i,N}_s) - \frac{1}{2}\int_s^t \triangle f(X^{i,N}_u)du - \frac{1}{N}\sum_{j=1}^N \int_s^t \nabla f(X^{i,N}_u)\cdot b(u,X^{i,N}_u,X^{j,N}_u)\,du\Big).$$

Apply Itô's formula to $\frac{1}{N}\sum_{i=1}^N (f(X^{i,N}_t) - f(X^{i,N}_s))$. It comes:

$$\mathbb{E}[\big(G(\mu^N)\big)^2] \leq \frac{C}{N^2}\mathbb{E}\Big(\sum_{i=1}^N \int_s^t f'(X^{i,N}_u)dW^i_u\Big)^2 \leq \frac{C}{N}.$$

Thus, (4.4) holds true.

Suppose for a while we have proven the following lemma:

Lemma 4.3. *Let $\Pi^\infty \in \mathcal{P}(\mathcal{P}(C([0,T];\mathbb{R}^d)^4))$ be a limit point of $\{law(\nu^N)\}$. Then*

$$\lim_{N\to\infty} \mathbb{E}[\big(G(\mu^N)\big)^2] = \int_{\mathcal{P}(C([0,T];\mathbb{R}^d)^4)} \Big\{\int_{C([0,T];\mathbb{R}^d)^4} \Big[f(x^1_t) - f(x^1_s) - \frac{1}{2}\int_s^t \triangle f(x^1_u)du - \int_s^t \nabla f(x^1_u)\cdot b(u,x^1_u,x^2_u)du\Big] \times \phi(x^1_{t_1},\dots,x^1_{t_p})d\nu(x^1\dots,,x^4)\Big\}^2 d\Pi^\infty(\nu), \tag{4.5}$$

and

i) Any $\nu \in \mathcal{P}(C([0,T];\mathbb{R}^d)^4)$ belonging to the support of Π^∞ is a product measure: $\nu = \nu^1\otimes\nu^1\otimes\nu^1\otimes\nu^1$.

ii) For any $t \in (0,T]$, the time marginal ν^1_t of ν^1 has a density ρ^1_t which satisfies for any $r > 1$

$$\exists C_T,\ \forall 0 < t \leq T,\quad \|\rho^1_t\|_{L^r(\mathbb{R}^d)} \leq \frac{C_T}{t^{\frac{d}{2}(1-\frac{1}{r})}}.$$

Then, combining (4.4) with the above result, we get

$$\int_{C([0,T];\mathbb{R}^d)} \phi(x^1_{t_1},\dots,x^1_{t_p})\Big[f(x^1_t) - f(x^1_s) - \frac{1}{2}\int_s^t \triangle f(x_u)du - \int_s^t \nabla f(x^1_u)\cdot b(u,x^1_u - y)\rho^1_u(y)dydu\Big]d\nu^1(x^1) = 0.$$

We deduce that ν^1 solves (MP) and thus that $\nu^1 = \mathbb{Q}$. As by definition Π^∞ is a limit point of $\text{Law}(\nu^N)$, it follows that any limit point of $\text{Law}(\mu^N)$ is $\delta_{\mathbb{Q}}$, which ends the proof.

4.3.1 Proof of Lemma 4.3

Proof of (4.5): Step 1. Notice that

$$\mathbb{E}[\big(G(\mu^N)\big)^2] = \frac{1}{N^2}\mathbb{E}\sum_{i,k=1}^{N}\Phi_2(X^{i,N},X^{k,N}) + \frac{1}{N^3}\mathbb{E}\sum_{i,k,l=1}^{N}\Phi_3(X^{i,N},X^{k,N},X^{l,N})$$
$$+\frac{1}{N^3}\mathbb{E}\sum_{i,j,k=1}^{N}\Phi_3(X^{k,N},X^{i,N},X^{j,N}) + \frac{1}{N^4}\mathbb{E}\sum_{i,j,k,l=1}^{N}\Phi_4(X^{i,N},X^{j,N},X^{k,N},X^{l,N}), \tag{4.6}$$

where

$$\Phi_2(X^{i,N},X^{k,N}) := \phi(X^{i,N}_{t_1},\dots,X^{i,N}_{t_p})\ \phi(X^{k,N}_{t_1},\dots,X^{k,N}_{t_p})$$
$$\times\Big(f(X^{i,N}_t)-f(X^{i,N}_s)-\frac{1}{2}\int_s^t \triangle f(X^{i,N}_u)du\Big)\Big(f(X^{k,N}_t)-f(X^{k,N}_s)-\frac{1}{2}\int_s^t \triangle f(X^{k,N}_u)du\Big),$$
$$\Phi_3(X^{i,N},X^{k,N},X^{l,N}) := -\phi(X^{i,N}_{t_1},\dots,X^{i,N}_{t_p})\ \phi(X^{k,N}_{t_1},\dots,X^{k,N}_{t_p})$$
$$\times\Big(f(X^{i,N}_t)-f(X^{i,N}_s)-\frac{1}{2}\int_s^t \triangle f(X^{i,N}_{u_1})du_1\Big)\int_s^t \nabla f(X^{k,N}_u)\cdot b(u,X^{k,N}_u,X^{l,N}_u)\mathbb{1}_{\{(X^{k,N}_u,X^{l,N}_u)\notin\mathcal{N}_b(u)\}}\,du,$$
$$\Phi_4(X^{i,N},X^{j,N},X^{k,N},X^{l,N}) := \phi(X^{i,N}_{t_1},\dots,X^{i,N}_{t_p})\phi(X^{k,N}_{t_1},\dots,X^{k,N}_{t_p})$$
$$\times\int_s^t\int_s^t \nabla f(X^{i,N}_{u_1})\cdot b(u_1,X^{i,N}_{u_1},X^{j,N}_{u_1})\mathbb{1}_{\{(X^{i,N}_{u_1},X^{j,N}_{u_1})\notin\mathcal{N}_b(u_1)\}}$$
$$\times\nabla f(X^{k,N}_{u_2})\cdot b(u_2,X^{k,N}_{u_2},X^{l,N}_{u_2})\mathbb{1}_{\{(X^{i,N}_{u_2},X^{j,N}_{u_2})\notin\mathcal{N}_b(u_2)\}}\,du_1\,du_2.$$

Let C_N be the last term in the r.h.s. of (4.6). In Steps 2-4 below we prove that C_N converges as $N\to\infty$ and we identify its limit. We have: Define the function F on $\mathbb{R}^{(2p+4)d}$ as

$$F(x^1,\dots,x^{2p+4}) := \phi(x^5,\dots,x^{p+4})\ \phi(x^{p+5},\dots,x^{2p+4})\ \nabla f(x^1)\cdot b(u_1,x^1,x^2)\nabla f(x^3)\cdot b(u_2,x^3,x^4)$$
$$\times\mathbb{1}_{\{(x^1,x^2)\notin\mathcal{N}_b(u_1)\}}\mathbb{1}_{\{(x^3,x^4)\notin\mathcal{N}_b(u_2)\}}. \tag{4.7}$$

We set $C_N=\int_s^t\int_s^t A_N\ du_1\ du_2$ with

$$A_N := \frac{1}{N^4}\sum_{i,j,k,l=1}^{N}\mathbb{E}(F(X^{i,N}_{u_1},X^{j,N}_{u_1},X^{k,N}_{u_2},X^{l,N}_{u_2},X^{i,N}_{t_1},\dots,X^{i,N}_{t_p},X^{k,N}_{t_1},\dots,X^{k,N}_{t_p})).$$

We now aim to show that A_N converges pointwise (Step 2), that $|A_N|$ is bounded from above by an integrable function w.r.t. $d\theta_1\ d\theta_2\ du_1\ du_2$ (Step 3), and finally to identify the limit of C_N (Step 4).

Proof of (4.5): Step 2. Fix $u_1,u_2\in[s,t]$. Define τ^N as

$$\tau^N := \frac{1}{N^4}\sum_{i,j,k,l=1}^{N}\delta_{X^{i,N}_{u_1},X^{j,N}_{u_1},X^{k,N}_{u_2},X^{l,N}_{u_2},X^{i,N}_{t_1},\dots,X^{i,N}_{t_p},X^{k,N}_{t_1},\dots,X^{k,N}_{t_p}}.$$

Define the measure $\mathbb{Q}^N_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}$ on $(\mathbb{R}^{2p+4})^d$ as $\mathbb{Q}^N_{u_1,u_2,t_1,\dots,t_p}(A)=\mathbb{E}(\tau^N(A))$. The convergence of $\{\text{law}(\nu^N)\}$ implies the weak convergence of $\mathbb{Q}^N_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}$ to the measure on $(\mathbb{R}^{2p+6})^d$ defined by

$$\mathbb{Q}_{u_1,u_2,t_1,\dots,t_p}(A) := \int_{\mathcal{P}(C([0,T];\mathbb{R}^d)^4)}\int_{C([0,T];\mathbb{R}^d)^4}\mathbb{1}_A(x^1_{u_1},x^2_{u_1},x^3_{u_2},x^4_{u_2},x^1_{t_1},\dots,$$
$$x^1_{t_p},x^3_{t_1},\dots,x^3_{t_p})d\nu(x^1,x^2,x^3,x^4)d\Pi^\infty(\nu).$$

Let us show that this probability measure has an L^2-density w.r.t. the Lebesgue measure on $(\mathbb{R}^{2p+4})^{\times d}$ (L^2 could be replaced with any L^r). Let $h\in C_c(\mathbb{R}^{(2p+4)d})$. By weak convergence,

$$|<\mathbb{Q}_{u_1,u_2,t_1,\dots,t_p},h>|$$
$$=\left|\lim_{N\to\infty}\frac{1}{N^4}\sum_{i,j,k,l=1}^{N}\mathbb{E}h(X^{i,N}_{u_1},X^{j,N}_{u_1},X^{k,N}_{u_2},X^{l,N}_{u_2},X^{i,N}_{t_1},\dots,X^{i,N}_{t_p},X^{k,N}_{t_1},\dots,X^{k,N}_{t_p})\right|.$$

When, in the preceding sum, at least two indices are equal, we bound the expectation by $\|h\|_\infty$. When $i \neq j \neq k \neq l$, we apply Girsanov's transform in Section 4.1 with four particles and Proposition 4.1. This procedure leads to

$$\left|< \mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}, h >\right| \leq \lim_{N\to\infty} \Big(\|h\|_\infty \frac{C}{N} + \frac{C_T}{N^4} \sum_{i\neq j\neq k\neq l} \Big(\mathbb{E} h^2(\hat{X}^{i,N}_{u_1}, \hat{X}^{j,N}_{u_1}, \hat{X}^{k,N}_{u_2}, \hat{X}^{l,N}_{u_2}, \hat{X}^{i,N}_{t_1}, \dots, \hat{X}^{i,N}_{t_p}, \hat{X}^{k,N}_{t_1}, \dots, \hat{X}^{k,N}_{t_p}) \Big)^{1/2} \Big).$$

All the processes $\hat{X}^{i,N}, \dots, \hat{X}^{l,N}$ being independent Brownian motions we deduce that

$$\left|< \mathbb{Q}_{u_1,u_2,t_1,\dots,t_p}, h >\right| \leq C_{u_1,u_2,\theta_1,\theta_2,t_1,\dots,t_p} \|h\|_{L^2(\mathbb{R}^{2p+6})}.$$

It follows from Riesz's representation theorem that $\mathbb{Q}_{u_1,u_2,t_1,\dots,t_p}$ has a density w.r.t. Lebesgue's measure in $L^2(\mathbb{R}^{(2p+4)d})$. Therefore, the functional F is continuous $\mathbb{Q}_{u_1,u_2,t_1,\dots,t_p}$ - a.e. Since for any fixed $u_1, u_2 \in [s,t]$ F is also bounded $\mathbb{Q}_{u_1,u_2,t_1,\dots,t_p}$ - a.e. we have

$$\lim_{N\to\infty} A_N = < \mathbb{Q}_{u_1,u_2,t_1,\dots,t_p}, F > .$$

Proof of (4.5): Step 3. In view of the definition (4.7) of F we may restrict ourselves to the case $i \neq j$ and $k \neq l$. Use the Girsanov transforms from Section 4.1 with $r_{i,j,k,l} \in \{2,3,4\}$ according to the respective cases $(i = k, j = l)$, $(i = k, j \neq l)$, $(i \neq k, j \neq l)$, etc. Below we write r instead of $r_{i,j,k,l}$. By exchangeability it comes:

$$A_N = \Big| \frac{1}{N^4} \sum_{i\neq j, k\neq l} \mathbb{E}_{\mathbb{Q}^{r,N}}(Z_T^{(r)} F(\cdots)) \Big| \leq \frac{1}{N^4} \sum_{i\neq j, k\neq l} \Big(\mathbb{E}_{\mathbb{Q}^{r,N}}(Z_T^{(r)})^2 \Big)^{1/2} \Big(\mathbb{E}_{\mathbb{Q}^{r,N}}(F^2(\cdots)) \Big)^{1/2}.$$

By Proposition 4.1, $\mathbb{E}_{\mathbb{Q}^{r,N}}(Z_T^{(r)})^2$ can be bounded uniformly w.r.t. N. As the functions f and ϕ are bounded we deduce

$$\sqrt{\mathbb{E}_{\mathbb{Q}^{r,N}}(F^2(\cdots))} \leq C \left(\mathbb{E}_{\mathbb{Q}^{r,N}}(h^2_{u_1}(W^i_{u_1} - W^j_{u_1}) h^2_{u_2}(W^k_{u_2} - W^l_{u_2})) \right)^{1/2},$$

for $i \neq j$, $k \neq l$ and $r \equiv r_{i,j,k,l}$. We consider the three cases:

- $i \neq k,\ j \neq l$: As all 4 Brownian motions are independent, one can separate this into a product of expectations and using the same computations as in Lemma 3.2, one has

$$\left(\mathbb{E}_{\mathbb{Q}^{r,N}}(h^2_{u_1}(W^i_{u_1} - W^j_{u_1}) h^2_{u_2}(W^k_{u_2} - W^l_{u_2})) \right)^{1/2} \leq \sqrt{\|h_{u_1}\|^2_{L^p(\mathbb{R}^d)} \|g_{u_1}\|_{L^{\frac{p}{p-2}}(\mathbb{R}^d)} \|h_{u_2}\|^2_{L^p(\mathbb{R}^d)} \|g_{u_2}\|_{L^{\frac{p}{p-2}}(\mathbb{R}^d)}}$$

- $i = k,\ j = l$: As we ony have two independent Brownian motions, we condition by the smaller time index and by one of the two independent Brownian motions. It comes

$$\left(\mathbb{E}_{\mathbb{Q}^{r,N}}(h^2_{u_1}(W^i_{u_1} - W^j_{u_1}) h^2_{u_2}(W^i_{u_2} - W^j_{u_2})) \right)^{1/2} \leq \mathbb{1}_{\{u_1<u_2\}} \Big(\|h_{u_1}\|^2_{L^p(\mathbb{R}^d)} \|g_{u_1}\|_{L^{\frac{p}{p-2}}(\mathbb{R}^d)}$$
$$\times \|h_{u_2}\|^2_{L^p(\mathbb{R}^d)} \|g_{u_2-u_1}\|_{L^{\frac{p}{p-2}}(\mathbb{R}^d)} \Big)^{\frac{1}{2}} + \mathbb{1}_{\{u_2<u_1\}} \Big(\|h_{u_1}\|^2_{L^p(\mathbb{R}^d)} \|g_{u_1-u_2}\|_{L^{\frac{p}{p-2}}(\mathbb{R}^d)} \times \|h_{u_2}\|^2_{L^p(\mathbb{R}^d)} \|g_{u_2}\|_{L^{\frac{p}{p-2}}(\mathbb{R}^d)} \Big)^{\frac{1}{2}}$$

- $i = k,\ j \neq l$: Same bound as above is obtained by conditioning by the smaller time index and W^j and W^l.

In any of the above cases, in view of the assumption $(\mathbf{H}^b)$, the bounds are integrable in $L^1((0,T)^2)$. We thus have obtained:

$$A_N \leq C H(u_1, u_2),$$

where H belongs to $L^1((0,T)^2)$.

Proof of (4.5)**: Step 4.** Steps 2 and 3 allow us to conclude that

$$\lim_{N\to\infty} C_N = \int_s^t \int_s^t < \mathbb{Q}_{u_1,u_2,t_1,\dots,t_p}, F > \; du_1 du_2.$$

By definition of $\mathbb{Q}_{u_1,u_2,t_1,\dots,t_p}$ and F we thus have obtained that

$$\begin{aligned}\lim_{N\to\infty} C_N = \int_{P(C([0,T];\mathbb{R})^4)} \int_s^t \int_s^t \int_{C([0,T];\mathbb{R}^d)^4} & \phi(x^1_{t_1},\dots,x^1_{t_p})\phi(x^3_{t_1},\dots,x^3_{t_p}) \\ & \times \nabla f(x^1_{u_1})\cdot b(u_1,x^1_{u_1},x^2_{u_1})\nabla f(x^3_{u_2})\cdot b(u_2,x^3_{u_2},x^4_{u_2})\mathbb{1}_{\{x^1_{u_1}\neq x^2_{u_1}\}}\mathbb{1}_{\{x^3_{u_2}\neq x^4_{u_2}\}} \\ & d\nu(x^1,x^2,x^3,x^4)\; d\theta_1\; d\theta_2\; du_1\; du_2\; d\Pi^\infty(\nu).\end{aligned}$$

A similar procedure is applied to the three other terms in the r.h.s. of (4.6). Together with the preceding, we obtain (4.5)

Proof of i) and ii). Now, we prove the claims i) and ii) of Lemma 4.3.

i) For any measure $\nu \in \mathcal{P}(C([0,T];\mathbb{R})^4)$, denote its first marginal by ν^1. One easily gets Π^∞ a.e., $\nu = \nu^1\otimes\nu^1\otimes\nu^1\otimes\nu^1$ (see [1, Lemma 3.3]).

ii) Take $\varphi \in C_c(\mathbb{R}^d)$ and fix $r > 1$. Let $\alpha \in (1, r')$ where r' is the conjugate of r. Using similar arguments as in the above Step 1, for any $0 < t \le T$ one has $\Pi^\infty(d\nu)$ a.e.,

$$\begin{aligned}< \nu^1_t, \varphi > &= \lim_{N\to\infty} \mathbb{E}_{\mathbb{Q}^N} < \mu^N_t, h > = \lim_{N\to\infty} \mathbb{E}_{\mathbb{Q}^N}(\varphi(X^{1,N}_t)) = \lim_{N\to\infty} \mathbb{E}_{\mathbb{Q}^{1,N}}(Z^{(1)}_T \varphi(W^{1,N}_t)) \\ &\le C\left(\mathbb{E}_{\mathbb{Q}^{1,N}}(Z^{(1)}_T)^{\alpha'}\right)^{\frac{1}{\alpha'}} \left(\mathbb{E}_{\mathbb{Q}^{1,N}}(\varphi(X^{1,N}_t))^\alpha\right)^{\frac{1}{\alpha}} \le C\|\varphi\|_{L^{r'}(\mathbb{R}^d)}\|g_t\|^{\frac{1}{\alpha}}_{L^{(r'/\alpha)'}} \le C\|\varphi\|_{L^{r'}(\mathbb{R}^d)}\frac{1}{t^{\frac{d}{2}\frac{1}{r'}}}\end{aligned}$$

Thus, one has

$$< \nu^1_t, \varphi > \le C\|\varphi\|_{L^{r'}(\mathbb{R}^d)}\frac{1}{t^{\frac{d}{2}(1-\frac{1}{r})}}.$$

Apply the Riesz representation theorem to conclude the proof.

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