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# Decomposition of $L^2$ -vector fields on Lipschitz surfaces: characterization via null-spaces of the scalar potential

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## Abstract

For  $\partial\Omega$  the boundary of a bounded and connected strongly Lipschitz domain in  $\mathbb{R}^d$  with  $d \geq 3$ , we prove that any field  $f \in L^2(\partial\Omega; \mathbb{R}^d)$  decomposes, in an unique way, as the sum of three silent vector fields—fields whose magnetic potential vanishes in one or both components of  $\mathbb{R}^d \setminus \partial\Omega$ . Moreover, this decomposition is orthogonal if and only if  $\partial\Omega$  is a sphere. We also show that any  $f$  in  $L^2(\partial\Omega; \mathbb{R}^d)$  is uniquely the sum of two silent fields and a Hardy function, in which case the sum is orthogonal regardless of  $\partial\Omega$ ; we express the corresponding orthogonal projections in terms of layer potentials. When  $\partial\Omega$  is a sphere, both decompositions coincide and match what has been called the Hardy-Hodge decomposition in the literature.

## 1 Introduction

Orthogonal direct sum decompositions provide an important tool in harmonic analysis and applied sciences. Orthogonal direct sums are closely related to the method of orthogonal projections, which was introduced into potential theory by Weyl [1]. Further developed by Vishik [2, 3] and Gårding [4], and combined with the results by Lax and Milgram [5] this method quickly developed and serves now as a powerful approach to elliptic and parabolic boundary value problems (see for example [6] or [7]). On a sphere, there are two sums that are particularly interesting for magnetic inverse problems: one is the decomposition of fields into

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contributions that do not create a magnetic field inside or outside of the sphere; the other is the decomposition of fields into contributions that can be harmonically continued inside and outside the sphere. In this paper, we show how these sums generalize to hold on Lipschitz surfaces.

The first decomposition appears in the following setting: Consider a ball whose very thin boundary layer is magnetized. The magnetization of that layer defines a vector field supported near the boundary of the ball. Idealizing, we can say that this vector field is defined on a sphere. Such a magnetization will generate a magnetic field inside and outside the sphere. By measuring the surrounding magnetic field we thus can “see” this magnetization. However, it is possible that a magnetization will be invisible—meaning that its magnetic field will vanish at least inside or outside of the sphere. Inverse problems in magneto-statics are non-unique precisely because invisible magnetizations exist, and for a generic shape it is important to understand when a magnetization can be invisible. In this paper we answer this question for a fairly general class of Lipschitz surfaces.

For a sphere, it is known that every sufficiently regular vector field (say, of  $L^2$ -class) is a sum of three invisible magnetizations: loosely said, one that is invisible inside; one that is invisible outside; and one that is invisible everywhere. To make this statement more precise let  $\Omega$  be an open region with the boundary  $\partial\Omega$  and consider the following spaces:

- $D(\partial\Omega)$  the space of square integrable fields that are invisible everywhere;
- $I(\partial\Omega)$  the space, orthogonal to  $D(\partial\Omega)$ , of fields invisible only inside the region  $\Omega$ , and the zero field;
- $O(\partial\Omega)$  the space, orthogonal to  $D(\partial\Omega)$ , of fields invisible only outside the region  $\Omega$ , and the zero field.

If  $\partial\Omega$  is a sphere,  $I(\partial\Omega)$  and  $O(\partial\Omega)$  are mutually orthogonal and the space of  $\mathbb{R}^3$ -valued square integrable fields splits into an orthogonal direct sum

$$L^2(\partial\Omega; \mathbb{R}^3) = I(\partial\Omega) + O(\partial\Omega) + D(\partial\Omega). \quad (1)$$

Additionally, the space  $I(\partial\Omega)$  defines precisely those magnetizations that one can “see” from the outside. Because of this, decomposition (1) becomes an important tool to solve magnetic inverse problems on a sphere. For example, in medical imaging it is used to process EEG/MEG measurements of the human head models [8]; in scanning magnetic microscopy it appears in the study of planetary rock samples [9]; it has been used to separate magnetic fields measured on satellite orbits with respect to their sources [10–13]; and to invert magnetic fields for spherical source currents [14] as well as for the lithospheric magnetization [15–18].

The second important decomposition on the sphere is the so called Hardy-Hodge decomposition

$$L^2(\partial\Omega; \mathbb{R}^d) = H_+^2(\partial\Omega) + H_-^2(\partial\Omega) + D_f(\partial\Omega); \quad (2)$$

see [15, 19, 20] for the Hardy-Hodge decomposition on a plane or a sphere in various smoothness classes. The space  $D_f(\partial\Omega)$  is the space of divergence-free fields. The spaces  $H_+^2(\partial\Omega)$  and  $H_-^2(\partial\Omega)$  in (2) are the Hardy-spaces, initially introduced on half spaces in [21], studied over  $C^1$ -hypersurfaces in [22, 23], and on Lipschitz surfaces in [24, 25]. In a nutshell, these spaces define vector fields that have a harmonic extension inside or outside of the sphere. Because harmonic functions have much stronger properties than merely square integrable functions, the Hardy-Hodge decomposition is a powerful tool in mathematical analysis. In fact it features the Calderon-Zygmund operator, which gave rise to the elliptic regularity theory [26, 27]. Eventually, on the sphere it holds that  $I(\partial\Omega) = H_+^2(\partial\Omega)$  and  $O(\partial\Omega) = H_-^2(\partial\Omega)$  which again makes (2) pertinent to various inverse problems.

If  $\partial\Omega$  is not a sphere, the decomposition (1) seems new. Also the Hardy-Hodge decomposition seems unknown if  $\partial\Omega$  is Lipschitz (see [28] for recent results). Consequently, also the relation between the spaces  $I(\partial\Omega)$ ,  $O(\partial\Omega)$  and  $H_+^2(\partial\Omega)$ ,  $H_-^2(\partial\Omega)$  is missing. In this paper, we address these questions for Lipschitz surfaces and vector fields of  $L^2$ -class. We believe that the answer will allow one to extend existing analytical techniques known on the sphere to Lipschitz surfaces. Also it could improve the accuracy of computations in cases where the sphere would only be an approximation of the actual underlying geometry.

The main result of this paper is to prove the following statement that we have separated into several theorems for clarity:

**Main result.** *Let  $\Omega$  be a bounded and connected Lipschitz domain in  $\mathbb{R}^d$  ( $d \geq 3$ ) with a connected boundary  $\partial\Omega$ . The space of square integrable  $\mathbb{R}^d$ -valued vector fields on  $\partial\Omega$  decomposes into the following orthogonal direct sums*

$$L^2(\partial\Omega; \mathbb{R}^d) = I(\partial\Omega) \oplus H_-^2(\partial\Omega) \oplus D(\partial\Omega) \quad (\text{Theorem 3.8})$$

$$L^2(\partial\Omega; \mathbb{R}^d) = H_+^2(\partial\Omega) \oplus O(\partial\Omega) \oplus D(\partial\Omega) \quad (\text{Theorem 3.20});$$

and into the following topological direct sums

$$L^2(\partial\Omega; \mathbb{R}^d) = H_+^2(\partial\Omega) + H_-^2(\partial\Omega) + D_f(\partial\Omega) \quad (\text{Theorem 4.1})$$

$$L^2(\partial\Omega; \mathbb{R}^d) = I(\partial\Omega) + O(\partial\Omega) + D(\partial\Omega) \quad (\text{Theorem 4.3}).$$

*These topological direct sums are orthogonal if and only if  $\partial\Omega$  is a sphere (Theorem 5.1 and Corollary 5.2); and in this case  $H_+^2(\partial\Omega) = I(\partial\Omega)$  and  $H_-^2(\partial\Omega) = O(\partial\Omega)$  (Corollary 5.3). Moreover, regardless of  $\partial\Omega$  it always holds that  $D_f(\partial\Omega) = D(\partial\Omega)$ .*

In Section 2 we set up notational conventions and discuss layer potentials. The latter are well-studied in the literature (for example, [24, 25, 29]) and lie at the core of all arguments in this paper. Although the main physical interest attaches to surfaces in  $\mathbb{R}^3$ , it would be artificial to restrict to dimension 3 and for that reason we present the material in  $\mathbb{R}^d$  for  $d \geq 3$ .

## 2 Preliminaries and Notation

### 2.1 Conventions

In this paper we reserve the symbol  $\Omega$  to denote a bounded and connected strongly Lipschitz domain in the  $d$ -dimensional Euclidean space  $\mathbb{R}^d$  ( $d \geq 3$ ). Throughout, we assign the symbol  $\partial\Omega$  to denote the boundary of  $\Omega$ , which is a connected and closed hypersurface, locally given as the graph of a Lipschitz function (see appendix A for more details). We call  $\Omega$  the inside of  $\partial\Omega$  and  $\Omega^o = \mathbb{R}^d \setminus \bar{\Omega}$  the outside. The surface measure on  $\partial\Omega$  will be denoted by  $\sigma$ ; it is the restriction to  $\partial\Omega$  of the  $(d-1)$ -Hausdorff measure; since  $\partial\Omega$  is compact,  $\sigma$  is finite. Statements made almost everywhere (a.e.) on  $\partial\Omega$  are always understood with respect to  $\sigma$ .

On  $\mathbb{R}^d$ , we denote the Euclidean scalar product by  $\langle x, y \rangle_{\mathbb{R}^d}$  and the Euclidean norm by  $|x| = \sqrt{\langle x, x \rangle_{\mathbb{R}^d}}$ . If  $f$  is a differentiable function defined on an open region in  $\mathbb{R}^d$ , then  $\nabla f$  denotes the Euclidean gradient of  $f$ . We denote the Euclidean gradient by  $\Delta$  that reads in coordinates  $\Delta f = \sum_{i=1}^d \partial^2 f / \partial x_i^2$ .

The space  $L^2(\partial\Omega)$  ( $= L^2(\partial\Omega, \sigma)$ ) comprises scalar-valued functions on  $\partial\Omega$  that are square integrable with respect to  $\sigma$ . As usual, we identify functions that coincide a.e. on  $\partial\Omega$ . The subspace  $L^2_{\text{zm}}(\partial\Omega) \subset L^2(\partial\Omega)$  consists of functions with zero mean; note that  $L^2(\partial\Omega)$ -functions indeed have a well-defined mean on  $\partial\Omega$ , since  $\sigma$  is finite. We call an  $\mathbb{R}^d$ -valued function on  $\partial\Omega$  a field, and we denote the space of square integrable fields by  $L^2(\partial\Omega; \mathbb{R}^d)$  ( $= L^2(\partial\Omega; \mathbb{R}^d, \sigma)$ ). If  $f$  and  $g$  are two fields in  $L^2(\partial\Omega; \mathbb{R}^d)$ , their scalar product is  $\langle f, g \rangle = \int_{\partial\Omega} \langle f(q), g(q) \rangle_{\mathbb{R}^d} d\sigma(q)$  and the norm of  $f$  in  $L^2(\partial\Omega; \mathbb{R}^d)$  is  $\|f\| = \sqrt{\langle f, f \rangle}$ . Clearly,  $L^2(\partial\Omega; \mathbb{R}^d)$  is a Hilbert space.

The tangent space to  $\partial\Omega$  at  $x$  is denoted by  $T_x(\partial\Omega)$ ; it is well-defined a.e. and so is the outer unit normal field  $\eta$  (see appendix A for more details). We identify  $T_x(\partial\Omega)$  with a  $(d-1)$ -dimensional hyperplane in  $\mathbb{R}^d$ . If  $f$  is a field in  $L^2(\partial\Omega; \mathbb{R}^d)$  and for a.e.  $x$  the vector  $f(x)$  lies in  $T_x(\partial\Omega)$ , we call  $f$  a tangent field. We denote the space of all tangent fields by  $T(\partial\Omega)$ ; it is a closed subspace of  $L^2(\partial\Omega; \mathbb{R}^d)$ . We will often split a field  $f \in L^2(\partial\Omega; \mathbb{R}^d)$  into a normal part  $f_\eta \in L^2(\partial\Omega)$  and the tangent part  $f_T \in T(\partial\Omega)$ , and write  $f = \eta f_\eta + f_T$ .

A Lipschitz function  $f : \partial\Omega \rightarrow \mathbb{R}$  is differentiable a.e. on  $\partial\Omega$  and therefore it has a well-defined tangential gradient  $\nabla_T f(x) \in T_x(\partial\Omega)$  at almost every  $x$ . We define the Sobolev space  $W^{1,2}(\partial\Omega)$  on  $\partial\Omega$  to be the completion of Lipschitz functions for the norm  $(\|f\|^2 + \|\nabla_T f\|^2)^{1/2}$ . If  $f$  is in  $W^{1,2}(\partial\Omega)$ , *a fortiori*  $f$  is in  $L^2(\partial\Omega)$  and  $\nabla_T f$  is a tangent field in  $T(\partial\Omega)$  (see appendix A for details).

If  $A$  is a bounded operator then  $\mathcal{N}[A]$  denotes its null space and  $\mathcal{R}[A]$  its range. For  $N$  a subspace of a Hilbert space, we let  $N^\perp$  designate its orthogonal complement. When saying that  $A$  is invertible, we always mean that  $A$  has a bounded inverse.

**Regular family of cones.** We will use a regular family of cones. Even though this family does not explicitly appear in what follows, it is fundamental to the very definition of boundary values on  $\partial\Omega$  and the limiting behavior of potentials that we invoke repeatedly. The existence of a regular family of cones is folklore, but it is hard to locate a proof, see [28].

More precisely, for  $\theta \in (0, \pi/2)$  and  $y, z \in \mathbb{R}^d$  with  $|z| = 1$ , we put  $C_{\theta, z}(y)$  for the open, right circular, positive cone with vertex at  $y$ , axis directed by  $z$ , and aperture angle  $2\theta$ ; the cone being truncated to some fixed suitable length. We do not make the length explicit in the notation, for it plays no role provided that it is small enough (how small depends on  $\Omega$ ). To each point of  $\partial\Omega$  we can attach two “natural” cones with fixed aperture such that: (i) their direction is that of the graph or opposite to it; (ii) their length is small enough that one of them lies in  $\Omega$  and the other in  $\Omega^o$ .

Let  $\mathbb{S}^{d-1}$  denote the  $(d-1)$ -dimensional unit sphere and  $Z: \partial\Omega \rightarrow \mathbb{S}^{d-1}$  be a continuous function with the following property: for some  $\theta_1 < \theta < \theta_2$  independent of  $q$ , we require that the cone  $C_{\theta, \pm Z(q)}(q)$ , truncated to suitable length independent of  $q$ , contains a natural cone of aperture  $2\theta_1$  and is contained in another natural cone of aperture  $2\theta_2$ . We may assume, replacing  $Z$  by  $-Z$  if necessary, that  $C_{\theta, Z(q)}(q) \subset \Omega$  and  $C_{\theta, -Z(q)}(q) \subset \Omega^o$ .

A regular family of cones for  $\Omega$  (resp.  $\Omega^o$ ) is a map that associates to every  $q \in \partial\Omega$  a cone  $C_{\theta, Z(q)}(q) \subset \Omega$  (resp.  $C_{\theta, -Z(q)}(q) \subset \Omega^o$ ) with  $Z$  as above, compare the definition in [24]. Hereafter, we fix such a family once and for all, and we write  $\Gamma_i(q)$  for the inner cone at  $q$  in this family,  $\Gamma_o(q)$  for the outer cone at  $q$ , and  $\Gamma(q) = \Gamma_i(q) \cup \Gamma_o(q)$  for the double cone.

Associated to the regular family of cones is a nontangential maximal function defined as follows. If  $g_i: \Omega \rightarrow \mathbb{R}^d$  (resp.  $g_o: \Omega^o \rightarrow \mathbb{R}^d$ ) is a function defined on  $\Omega$  (resp.  $\Omega^o$ ), we denote the nontangential maximal functions of  $g$  at  $p \in \partial\Omega$  as,

$$\begin{aligned} g_i^M(p) &= \sup \{|g(x)| : x \in \Gamma_i(p)\}, \\ \text{(resp. } g_o^M(p) &= \sup \{|g(x)| : x \in \Gamma_o(p)\} \text{)}. \end{aligned} \tag{3}$$

When  $g$  is defined on  $\Omega \cup \Omega^o$ , we set

$$g^M(p) = \sup \{|g(x)| : x \in \Gamma(p)\}. \tag{5}$$

A function  $h$  on an open set  $O \subset \mathbb{R}^d$  is harmonic if it satisfies  $\Delta h = 0$ . As soon as  $\Delta h$  exists in the distributional sense,  $h$  is infinitely differentiable. It follows from [30, Sec. 5, thm.] and [31, thm. 1] that every harmonic function on  $\Omega$  (resp.  $\Omega^o$ ) whose maximal function is in  $L^2(\partial\Omega)$  has a nontangential limit a.e. on  $\partial\Omega$ , and that limit function is in  $L^2(\partial\Omega)$ .

## 2.2 Potentials

In this section, we summarize known results about the single and the double layer potentials. Most of the statements and references or proofs for them can be found

in [24]. In the following we denote the surface area of a  $d$ -dimensional sphere by  $\omega_d$ .

**Single layer potential.** The single layer potential of  $f \in L^2(\partial\Omega)$  is

$$\mathcal{S}f(x) \doteq \frac{-1}{\omega_d(d-2)} \int_{\partial\Omega} \frac{1}{|x-q|^{d-2}} f(q) d\sigma(q) \quad (x \in \mathbb{R}^d \setminus \partial\Omega). \quad (6)$$

It is harmonic on  $\mathbb{R}^d \setminus \partial\Omega$ . The result below follows for instance from the combination of [32, thm. 1] and [33, ch. 15, thm. 1], see also [24, thm. 08.D and lem. 1.3].

**Lemma 2.1.** *For a.e.  $p \in \partial\Omega$ , the limit,*

$$\lim_{\substack{x \rightarrow p \\ x \in \Gamma(p)}} \mathcal{S}f(x) = \text{p.v.} \frac{-1}{\omega_d(d-2)} \int_{\partial\Omega} \frac{1}{|x-q|^{d-2}} f(q) d\sigma(q) \doteq Sf(p), \quad (7)$$

*exists, and the limit function is in  $W^{1,2}(\partial\Omega)$ . Moreover,  $\|(Sf)^M\| \leq C\|f\|$  for some constant  $C = C(\partial\Omega)$ .*

The operator  $S: L^2(\partial\Omega) \rightarrow W^{1,2}(\partial\Omega)$ , defined by the (weakly) singular integral in (7), is a bounded linear operator [24, lem. 1.8]; when no confusion is possible, we also call it the single layer potential.

The following property of  $S$  will be important in what follows.

**Theorem 2.2** ([24, thm 3.3]). *The operator  $S: L^2(\partial\Omega) \rightarrow W^{1,2}(\partial\Omega)$  is invertible.*

*Remark 2.3.* The previous results hold for a more general range of exponents, but for the purpose of this paper statements in  $L^2$  will suffice.

**The gradient of  $\mathcal{S}f$ .** The Euclidean gradient of the single layer potential is,

$$\nabla \mathcal{S}f(x) \doteq \frac{1}{\omega_d} \int_{\partial\Omega} \frac{x-q}{|x-q|^d} f(q) d\sigma(q) \quad (x \in \mathbb{R}^d \setminus \partial\Omega). \quad (8)$$

Each of the vector components of  $\nabla \mathcal{S}f$  is a harmonic function on  $\mathbb{R}^d \setminus \partial\Omega$  and it holds that  $\|(\nabla \mathcal{S}f)^M\| < C\|f\|$  for some constant  $C = C(\Omega)$ . This last fact follows from [33, ch. 15, thm. 1] (see also [24, lem. 1.3]). Thus,  $\nabla \mathcal{S}f$  has nontangential limits a.e. on  $\partial\Omega$  from each side, that define two fields in  $T(\partial\Omega)$ . We shall describe their tangential and normal components separately. We begin with the tangential component, which is the same from either side (compare [24, thm 1.6.]):

**Lemma 2.4.** *For every tangent field  $\tau \in T(\partial\Omega)$  the limit*

$$\begin{aligned} \lim_{\substack{x \rightarrow p \\ x \in \Gamma(p)}} \langle \tau(p), \nabla \mathcal{S}f(x) \rangle_{\mathbb{R}^d} &= \text{p.v.} \frac{1}{\omega_d} \int_{\partial\Omega} \frac{\langle \tau(p), p-q \rangle_{\mathbb{R}^d}}{|p-q|^d} f(q) d\sigma(q) \\ &= \langle \tau(p), \nabla_{\text{T}} Sf(p) \rangle_{\mathbb{R}^d} \end{aligned} \quad (9)$$

*exists at a.e.  $p \in \partial\Omega$ , and defines the tangent field  $\nabla_{\text{T}} Sf \in T(\partial\Omega)$ . Moreover,  $\|\nabla_{\text{T}} Sf\| \leq C\|f\|$ .*

*Proof.* See Lemma A.1 in Appendix A. ■

We can define the operator  $\nabla_{\mathbb{T}}S: L^2(\partial\Omega) \rightarrow T(\partial\Omega)$  such that  $(\nabla_{\mathbb{T}}S)f = \nabla_{\mathbb{T}}(Sf)$ . By the above lemma,  $\nabla_{\mathbb{T}}S$  is bounded and linear.

**The scalar potential.** The scalar potential of a field  $f \in L^2(\partial\Omega; \mathbb{R}^d)$  is

$$\mathcal{P}f(x) \doteq \frac{1}{\omega_d} \int_{\partial\Omega} \frac{\langle q-x, f(q) \rangle_{\mathbb{R}^d}}{|q-x|^d} d\sigma(q) \quad (x \in \mathbb{R}^d \setminus \partial\Omega). \quad (10)$$

It is harmonic on  $\mathbb{R}^d \setminus \partial\Omega$ .

By definition, the scalar potential is a constant multiple of the single layer potential of the divergence of  $f$ , and it behaves much like the Euclidean gradient of a single layer potential except that it acts on fields rather than functions. Note that the integral kernel of  $\mathcal{P}$  differs from the integral kernel of  $\nabla\mathcal{S}$  by a minus sign. As the next lemma shows, it is convenient to define the scalar potential in this way.

**Lemma 2.5.** *For every tangent field  $\tau \in T(\partial\Omega)$ , the limit*

$$\lim_{\substack{x \rightarrow p \\ x \in \Gamma(p)}} \mathcal{P}\tau(x) = \text{p.v.} \frac{1}{\omega_d} \int_{\partial\Omega} \frac{\langle q-p, \tau(q) \rangle_{\mathbb{R}^d}}{|p-q|^d} d\sigma(q) \doteq (\nabla_{\mathbb{T}}S)^* \tau(p), \quad (11)$$

*exists at a.e.  $p \in \partial\Omega$ , and the limit function is in  $L^2(\partial\Omega)$ .*

The operator  $(\nabla_{\mathbb{T}}S)^*: T(\partial\Omega) \rightarrow L^2(\partial\Omega)$ , defined by (11), is linear and bounded, and one can see from (9) that  $(\nabla_{\mathbb{T}}S)^*$  is indeed the  $L^2$ -adjoint of  $\nabla_{\mathbb{T}}S$ .

The limits in (9) and (11) are independent of the components of the cone  $\Gamma(p)$ . That is, the tangent part of  $\nabla\mathcal{S}$  transitions continuously through the boundary  $\partial\Omega$ . Contrary to this, the normal component of  $\nabla\mathcal{S}$  has a jump across the boundary. This behavior will be described by the double layer potential.

*Remark 2.6.* The operator  $(\nabla_{\mathbb{T}}S)^*$  is closely connected with the divergence operator: for  $f \in T(\partial\Omega)$ , its divergence  $\text{div}_{\mathbb{T}}f$  is a continuous functional on  $W^{1,2}(\partial\Omega)$  acting on  $h$  by the rule  $\langle \text{div}_{\mathbb{T}}f, h \rangle = -\langle f, \nabla_{\mathbb{T}}h \rangle$ . Now, for  $g \in L^2(\partial\Omega)$ , we get that

$$\langle (\nabla_{\mathbb{T}}S)^* f, g \rangle = \langle f, \nabla_{\mathbb{T}}Sg \rangle = -\langle \text{div}_{\mathbb{T}}f, Sg \rangle.$$

If we identify the dual space of  $L^2(\partial\Omega)$  with  $L^2(\partial\Omega)$  itself and the dual space of  $W^{1,2}(\partial\Omega)$  with the Sobolev space  $W^{-1,2}(\partial\Omega)$  of negative exponent<sup>1</sup>, it holds that  $(\nabla_{\mathbb{T}}S)^* = S^* \text{div}_{\mathbb{T}}$  where  $S^*: W^{-1,2}(\partial\Omega) \rightarrow L^2(\partial\Omega)$ . In particular, since  $S$  is invertible by Theorem 2.2, so is  $S^*$  and we conclude that  $\mathcal{N}[(\nabla_{\mathbb{T}}S)^*]$  consists exactly of divergence-free vector fields in  $T(\partial\Omega)$ .

<sup>1</sup>This goes as in the Euclidean case, see [34, sec. 3.13]



**Double layer potential.** Let  $\eta$  denote the outward unit normal vector on  $\partial\Omega$ . For  $f_\eta \in L^2(\partial\Omega)$ , the double layer potential of  $f_\eta$  is

$$\mathcal{K}f_\eta(x) = \frac{1}{\omega_d} \int_{\partial\Omega} \frac{\langle q-x, \eta(q) \rangle_{\mathbb{R}^d}}{|q-x|^d} f_\eta(q) d\sigma(q) = \mathcal{P}(\eta f_\eta)(x), \quad (x \in \mathbb{R}^d \setminus \partial\Omega)$$

and the (boundary) double layer potential is

$$Kf_\eta(p) = \text{p.v.} \frac{1}{\omega_d} \int_{\partial\Omega} \frac{\langle q-p, \eta(q) \rangle_{\mathbb{R}^d}}{|q-p|^d} f_\eta(q) d\sigma(q), \quad (12)$$

defined for a.e.  $p$  on  $\partial\Omega$ . The operator  $K: L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)$  is linear and bounded; we denote its  $L^2$ -adjoint by  $K^*$ . These operators qualify the boundary behavior of  $\mathcal{K}$  by the following Lemma [24, thms. 1.10 and 1.11].

**Lemma 2.7.** *For  $f \in L^2(\partial\Omega)$  and for a.e.  $p \in \partial\Omega$  it holds that*

$$\lim_{\substack{x \rightarrow p \\ x \in \Gamma_i(p)}} \mathcal{K}f(x) = \left( \frac{1}{2} + K \right) f(p), \quad (13)$$

$$\lim_{\substack{x \rightarrow p \\ x \in \Gamma_o(p)}} \mathcal{K}f(x) = - \left( \frac{1}{2} - K \right) f(p), \quad (14)$$

and

$$\lim_{\substack{x \rightarrow p \\ x \in \Gamma_i(p)}} \langle \eta(p), \nabla \mathcal{S}f(x) \rangle_{\mathbb{R}^d} = - \left( \frac{1}{2} - K^* \right) f(p), \quad (15)$$

$$\lim_{\substack{x \rightarrow p \\ x \in \Gamma_o(p)}} \langle \eta(p), \nabla \mathcal{S}f(x) \rangle_{\mathbb{R}^d} = \left( \frac{1}{2} + K^* \right) f(p). \quad (16)$$

The next two results will be much used in what follows.

**Lemma 2.8** ([24, thm. 3.3]). *The operators*

$$\left( \frac{1}{2} + K \right) : L^2(\partial\Omega) \rightarrow L^2(\partial\Omega) \quad \text{and} \quad \left( \frac{1}{2} - K^* \right) : L^2_{\text{ZM}}(\partial\Omega) \rightarrow L^2_{\text{ZM}}(\partial\Omega),$$

are invertible.

**Lemma 2.9.** *There exists a unique function  $\nu_o \in L^2_{\text{ZM}}(\partial\Omega)$ , possibly zero, such that  $S(1 - \nu_o)$  is a non-zero constant with*

$$\left( \frac{1}{2} - K^* \right) (1 - \nu_o) = 0, \quad \text{and} \quad \nabla_{\text{T}} S(1 - \nu_o) = 0. \quad (17)$$

Further, there exists a constant  $C > 0$  that depends on  $\Omega$ , such that for every  $f \in L^2(\partial\Omega)$  it holds

$$\left\| \left( \frac{1}{2} - K^* \right) f \right\| \leq C \|\nabla_{\text{T}} S f\|. \quad (18)$$

The first statement of Lemma 2.9 was shown in the proof of [24, thm. 3.3. (ii)]. The second was established in the proof of [24, thm. 2.1]. Note that the equalities  $\nabla_{\text{T}}S1 = 0 = \left(\frac{1}{2} - K^*\right) 1$  hold if and only if  $\nu_o = 0$ . This fact will be used at several places below.

*Remark 2.10.* Since  $\left(\frac{1}{2} - K^*\right)$  is invertible on  $L^2_{\text{ZM}}(\partial\Omega)$  and maps  $L^2(\partial\Omega)$  into  $L^2_{\text{ZM}}(\partial\Omega)$  [24, thm. 3.3], its null space on  $L^2(\partial\Omega)$  is 1-dimensional and consists exactly of multiples of  $1 - \nu_o$ , by (17). In another connection, the null space of  $\nabla_{\text{T}}S$  comprises those  $f$  for which  $Sf$  is constant, and since  $S$  is injective by Theorem 2.2 it follows from (17) that the null space of  $\nabla_{\text{T}}S$  coincides with the null space of  $\left(\frac{1}{2} - K^*\right)$ . In fact, the first identity in (17) means that  $\mathcal{A}(1 - \nu_o)$  is the solution to a harmonic Neumann problem in  $\Omega$  with nontangential maximal function of its gradient in  $L^2(\partial\Omega)$  and zero boundary data, while the second identity says that  $\mathcal{A}(1 - \nu_o)$  is the solution to a harmonic Dirichlet problem in  $\Omega$  with nontangential maximal function in  $L^2(\partial\Omega)$  and constant boundary data. In both cases, this means that  $\mathcal{A}(1 - \nu_o)$  is constant on  $\Omega$ .

*Remark 2.11.* Equation (17) entails that the Newtonian equilibrium measure  $\mu$  of  $\bar{\Omega}$ —a well studied object in potential theory [35]—is given by  $d\mu = (1/\sigma(\partial\Omega))(1 - \nu_o) d\sigma$ . Indeed,  $\mu$  is known to be a probability measure supported on  $\partial\Omega$ , and it is absolutely continuous with  $L^2$ -density with respect to  $\sigma$ , by fundamental results on harmonic measure for Lipschitz domains [31, cor. to thm. 3] and standard relations between harmonic and equilibrium measures [36, ch. IV, sec. 5, §20]. Thus,  $d\mu = kd\sigma$  for some  $k \in L^2(\partial\Omega)$ , and since  $\Omega$  is non thin at each point of  $\partial\Omega$  (for example, by [36, thm. 5.4 and thm. 5.10], a characteristic property of  $\mu$  is that its potential is constant on  $\Omega \cup \partial\Omega$  since it is constant approximately everywhere [36, ch. II, sec. 1, §3]. This amounts to require that  $Sk$  is constant on  $\partial\Omega$ , that is  $\nabla_{\text{T}}Sk = 0$ , and by Remark 2.10 this is equivalent to require that  $\left(\frac{1}{2} - K^*\right)k = 0$ . Altogether, we conclude that  $1 - \nu_o$ , which has mass  $\sigma(\partial\Omega)$ , is equal to  $\sigma(\partial\Omega)k$ , as announced. We also obtain the extra-information that  $\nu_o \leq 1$  a.e. on  $\partial\Omega$ , because  $\mu$  is positive. Gruber’s conjecture asserts that  $\nu_o$  is zero if and only if  $\partial\Omega$  is a sphere; and this is known to hold when  $\Omega$  is convex [37, thm. 4.12]. Even though a discussion of Gruber’s conjecture is beyond the scope of the present paper, our analysis will stress a link between the equilibrium measure and the properties of projectors onto Hardy-spaces.

### 3 Orthogonal decomposition of fields

In this section, we present two orthogonal decompositions of the space  $L^2(\partial\Omega; \mathbb{R}^d)$  based on the null space of the scalar potential. Hereafter, we will denote the space of harmonic functions in a region  $N \subset \mathbb{R}^d$  by  $\mathcal{H}(N)$ .

We begin with a definition of inner and outer scalar potentials.

**Definition 3.1.** Let  $g|_\Omega$  denote the restriction of the function  $g$  to the region  $\Omega$ , and recall from (10) the scalar potential  $\mathcal{P}$ . We define the inner scalar potential as

$$\mathcal{P}_i: L^2(\partial\Omega; \mathbb{R}^d) \rightarrow \mathcal{H}(\Omega) \quad (19)$$

$$f \mapsto (\mathcal{P}f)|_\Omega \quad (20)$$

and the outer scalar potential as

$$\mathcal{P}_o: L^2(\partial\Omega; \mathbb{R}^d) \rightarrow \mathcal{H}(\Omega^o) \quad (21)$$

$$f \mapsto (\mathcal{P}f)|_{\Omega^o}. \quad (22)$$

Both operators are linear.

The operators  $\mathcal{P}_i$  and  $\mathcal{P}_o$  have non-trivial null spaces, leading to orthogonal direct sum decompositions of  $L^2(\partial\Omega; \mathbb{R}^d)$  as  $\mathcal{N}[\mathcal{P}_i] \oplus \mathcal{N}[\mathcal{P}_i]^\perp$  and  $\mathcal{N}[\mathcal{P}_o] \oplus \mathcal{N}[\mathcal{P}_o]^\perp$ .

When  $\partial\Omega$  is a sphere, the spaces  $\mathcal{N}[\mathcal{P}_i]$  and  $\mathcal{N}[\mathcal{P}_o]$  relate to nontangential limits of harmonic gradients—the Hardy spaces. Below, we recall the definition of Hardy spaces.

**Hardy spaces.** The inner Hardy space,  $H_+^2(\Omega)$ , is the space of gradients of harmonic functions in  $\Omega$  whose nontangential maximal function is in  $L^2(\partial\Omega)$ :

$$H_+^2(\Omega) = \{ \nabla g : g \in \mathcal{H}(\Omega), \|(\nabla g)_i^M\| < \infty \}. \quad (23)$$

The outer Hardy space  $H_-^2(\Omega)$  is defined similarly on the outside of  $\partial\Omega$ :

$$H_-^2(\Omega) = \left\{ \nabla g : g \in \mathcal{H}(\Omega^o), \|(\nabla g)_o^M\| < \infty, \lim_{x \rightarrow \infty} |g(x)| = 0 \right\}. \quad (24)$$

For each  $\nabla g \in H_+^2(\Omega)$  there is a square integrable boundary field  $f \in L^2(\partial\Omega; \mathbb{R}^d)$ , such that  $\nabla g$  converges nontangentially from inside to  $f$ , a.e. on  $\partial\Omega$  (see for example [29, thm. 7.9]). The space of boundary fields obtained in this way is denoted by  $H_+^2(\partial\Omega)$ . The same limiting procedure for  $H_-^2(\Omega)$ -functions leads to the space  $H_-^2(\partial\Omega)$ . We still call the space  $H_+^2(\partial\Omega)$  (or  $H_-^2(\partial\Omega)$ ) the inner (or outer) Hardy space, with no fear of confusion because the nontangential limit and the  $L^2$ -boundedness of the nontangential maximal function characterize harmonic functions.

If  $\partial\Omega$  is a sphere then  $H_+^2(\partial\Omega)$  and  $H_-^2(\partial\Omega)$  are orthogonal; and denoting by  $D(\partial\Omega) \subset T(\partial\Omega)$  the space of divergence-free tangent fields on  $\partial\Omega$ , we have that  $\mathcal{N}[\mathcal{P}_i] = H_+^2(\partial\Omega) \oplus D(\partial\Omega)$  and  $\mathcal{N}[\mathcal{P}_i]^\perp = H_-^2(\partial\Omega)$ . This is the Hardy-Hodge decomposition on a sphere—an orthogonal direct sum decomposition of  $L^2(\partial\Omega; \mathbb{R}^d)$  as  $L^2(\partial\Omega; \mathbb{R}^d) = H_+^2(\partial\Omega) \oplus H_-^2(\partial\Omega) \oplus D(\partial\Omega)$ . Moreover, if we use the null space of  $\mathcal{P}_i$  instead of the null space of  $\mathcal{P}_o$ , the role of  $H_+^2(\partial\Omega)$  and  $H_-^2(\partial\Omega)$  get swapped.

On Lipschitz domains, the Hardy-Hodge decomposition still exists, but the inner and outer Hardy-spaces are no longer orthogonal to each other when  $\partial\Omega$  is not a sphere (see Corollary 5.2).

### 3.1 Inner decomposition

Below we derive an orthogonal direct sum for  $L^2(\partial\Omega; \mathbb{R}^d)$ , based on the null space of the inner scalar potential. For  $f$  a field, recall its normal component  $f_\eta$  and its tangent component  $f_\tau$  such that  $f = \eta f_\eta + f_\tau$ .

**Definition 3.2.** Define the operator,  $B_i : L^2(\partial\Omega; \mathbb{R}^d) \rightarrow L^2(\partial\Omega)$ , as,

$$B_i f \doteq \left( \frac{1}{2} + K \right) f_\eta + (\nabla_\tau S)^* f_\tau; \quad (25)$$

it is linear and bounded. We call  $B_i$  the inner boundary operator.

**Theorem 3.3.** *The inner scalar potential of  $f \in L^2(\partial\Omega; \mathbb{R}^d)$  can be expressed as*

$$\mathcal{P}_i f(x) = \mathcal{K} \left( \frac{1}{2} + K \right)^{-1} B_i f(x) \quad (x \in \Omega). \quad (26)$$

Moreover, the null space of  $\mathcal{P}_i$  is given by

$$\mathcal{N}[\mathcal{P}_i] = \mathcal{N}[B_i] = \left\{ f \in L^2(\partial\Omega; \mathbb{R}^d) : \left( \frac{1}{2} + K \right) f_\eta = -(\nabla_\tau S)^* f_\tau \right\}. \quad (27)$$

*Proof.* Let  $f$  be in  $L^2(\partial\Omega; \mathbb{R}^d)$ . For  $x \in \Omega$ , we get that

$$\mathcal{P}_i f(x) = \mathcal{P}_i(\eta f_\eta)(x) + \mathcal{P}_i f_\tau(x) = \mathcal{K} f_\eta(x) + \mathcal{P}_i f_\tau(x). \quad (28)$$

Taking the nontangential limit on  $\partial\Omega$  from  $\Omega$  on both sides of the above equation and using (11) and (13) yields

$$\lim_{x \rightarrow q} \mathcal{P}_i f(x) = \left( \frac{1}{2} + K \right) f_\eta(q) + (\nabla_\tau S)^* f_\tau(q) = B_i f(q). \quad (29)$$

By Lemma 2.8 the operator  $\left( \frac{1}{2} + K \right)^{-1}$  is well defined. Hence,  $\mathcal{P}_i f - \mathcal{K} \left( \frac{1}{2} + K \right)^{-1} B_i f$  is harmonic in  $\Omega$ , and converges nontangentially to zero a.e. on  $\partial\Omega$ , and has  $L^2$ -bounded nontangential maximal function. By uniqueness of the Dirichlet problem (see for example [24, cor. 3.2]) that function is zero; this proves (26). It follows that  $\mathcal{N}[\mathcal{P}_i] = \mathcal{N}[B_i]$  and the second equality in (27) is immediate from the definition of  $B_i$ . ■

**Corollary 3.4.** *For  $f \in \mathcal{N}[\mathcal{P}_i]$ , the normal component of  $f$  is uniquely determined by the tangent part of  $f$  through the relation,*

$$f_\eta = \left( \frac{1}{2} + K \right)^{-1} (\nabla_\tau S)^* f_\tau. \quad (30)$$

Theorem 3.3 shows that the operator  $B_i$  determines the null space for  $\mathcal{P}_i$ . The following lemma shows that the adjoint  $B_i^*$  of  $B_i$  determines the orthogonal complement of the null space of  $\mathcal{P}_i$ .

**Lemma 3.5.** *The operator  $B_i^*: L^2(\partial\Omega) \rightarrow L^2(\partial\Omega; \mathbb{R}^d)$  can be written as*

$$B_i^* g = \eta \left( \frac{1}{2} + K^* \right) g + \nabla_{\mathbb{T}} S g. \quad (31)$$

*The range of  $B_i^*$  satisfies  $\mathcal{R}[B_i^*] = H_-^2(\partial\Omega) = \mathcal{N}[P_i]^\perp$ .*

*Proof.* Let  $f = \eta f_\eta + f_{\mathbb{T}}$  be in  $L^2(\partial\Omega; \mathbb{R}^d)$ . For every  $g \in L^2(\partial\Omega)$ , we get that

$$\langle B_i f, g \rangle = \left\langle \left( \frac{1}{2} + K \right) f_\eta + (\nabla_{\mathbb{T}} S)^* f_{\mathbb{T}}, g \right\rangle \quad (32)$$

$$= \left\langle f_\eta, \left( \frac{1}{2} + K^* \right) g \right\rangle + \left\langle f_{\mathbb{T}}, \nabla_{\mathbb{T}} S g \right\rangle = \langle f, B_i^* g \rangle. \quad (33)$$

Hence, using (16) we have that

$$B_i^* g(p) = \eta \left( \frac{1}{2} + K^* \right) g(p) + \nabla_{\mathbb{T}} S g(p) = \lim_{x \rightarrow p} \nabla \mathcal{S} g(x) \quad (34)$$

when  $x$  approaches  $p$  within  $\Gamma_o$ . Since  $\mathcal{S}g$  is harmonic in  $\Omega^o$  and has  $L^2$ -bounded maximal function, the right hand side of (34) defines a function in  $H_-^2(\partial\Omega)$ .

To see that  $B_i^*$  is onto, consider  $f \in H_-^2(\partial\Omega)$ . By the well-posedness of the Neumann problem, there exists a unique function  $\varphi$  that is harmonic in  $\Omega^o$ , vanishes at infinity, and whose normal derivative equals  $f_\eta$  a.e. on  $\partial\Omega$ , while  $\|(\nabla\varphi)^M\|$  is finite. On the other hand, since  $(\frac{1}{2} + K^*)$  is invertible because its adjoint is by Lemma 2.8, we get from(16) that  $\varphi = \mathcal{S}(\frac{1}{2} + K^*)^{-1} f_\eta$ , and so the function  $g = (\frac{1}{2} + K^*)^{-1} f_\eta \in L^2(\partial\Omega)$  is such that

$$f(p) = \lim_{x \rightarrow p} \nabla \varphi(p) = \lim_{x \rightarrow p} \nabla \mathcal{S} g(x)$$

where the nontangential limits are taken from  $\Omega^o$ . Consequently,  $B_i^*$  is onto and  $\mathcal{R}[B_i^*] = H_-^2(\partial\Omega)$ .

For the last statement, observe by general properties of operators on Hilbert space that  $\mathcal{N}[B_i] = \mathcal{R}[B_i^*]^\perp = H_-^2(\partial\Omega)^\perp$ . The conclusion now follows since  $H_-^2(\partial\Omega)$  is closed.  $\blacksquare$

**Corollary 3.6.** *The space  $L^2(\partial\Omega; \mathbb{R}^d)$  splits into an orthogonal sum as*

$$L^2(\partial\Omega; \mathbb{R}^d) = H_-^2(\partial\Omega) \oplus \mathcal{N}[P_i]. \quad (35)$$

## 3.2 Orthogonal projections for the inner decomposition

For the rest of the paper, we make the following definitions.

**Definition 3.7.** We define a space of tangent fields with zero mean as

$$T_{\text{ZM}}(\partial\Omega) = \{f \in T(\partial\Omega) : (\nabla_{\text{T}}S)^* f \in L_{\text{ZM}}^2(\partial\Omega)\},$$

and the space of tangent gradient fields as

$$G(\partial\Omega) = \{\nabla_{\text{T}}\varphi : \varphi \in W^{1,2}(\partial\Omega)\},$$

as well as the space of tangent divergence-free vector fields (see Remark 2.6):

$$D(\partial\Omega) = \{f \in T(\partial\Omega) : (\nabla_{\text{T}}S)^* f = 0\}.$$

Since  $S$  is onto  $W^{1,2}(\partial\Omega)$ , we can equivalently write the space of gradient fields as  $G(\partial\Omega) = \{\nabla_{\text{T}}Sf : f \in L^2(\partial\Omega)\}$ . Then, it is immediate that  $D(\partial\Omega)$  is the orthogonal complement of  $G(\partial\Omega)$ ; this fact is also apparent from the definition of the divergence. The spaces  $T_{\text{ZM}}(\partial\Omega)$  and  $D(\partial\Omega)$  are closed, since  $(\nabla_{\text{T}}S)^*$  is continuous, and therefore there exist orthogonal projections  $P_{\text{ZM}}: T(\partial\Omega) \rightarrow T_{\text{ZM}}(\partial\Omega)$  and  $P_{\text{D}}: T(\partial\Omega) \rightarrow D(\partial\Omega)$ .

The goal of this section is to prove the following theorem.

**Theorem 3.8.** *The operator*

$$P_{\text{--}} = B_i^* (B_i B_i^*)^{-1} B_i \tag{36}$$

*defines the orthogonal projection from  $L^2(\partial\Omega; \mathbb{R}^d)$  onto  $H_{\text{--}}^2(\partial\Omega)$ .*

*Let  $P_{\text{D}}$  be as in definition 3.7. Then,*

$$P_i \doteq \mathbf{1} - P_{\text{D}} - P_{\text{--}} \tag{37}$$

*is the orthogonal projection from  $L^2(\partial\Omega; \mathbb{R}^d)$  onto  $I(\partial\Omega) = \mathcal{N}[\mathcal{P}_i] \cap \mathcal{N}[\mathcal{P}_o]^\perp$ ; and the space  $L^2(\partial\Omega; \mathbb{R}^d)$  splits into an orthogonal direct sum as*

$$L^2(\partial\Omega; \mathbb{R}^d) = I(\partial\Omega) \oplus H_{\text{--}}^2(\partial\Omega) \oplus D(\partial\Omega). \tag{38}$$

We will proceed in several steps. First, we derive an explicit expression for the projection  $P_{\text{D}}$  that leads to an explicit formula, in terms of potentials, for the Helmholtz-Hodge-decomposition of tangent fields on Lipschitz surfaces. Next, we show that the operator  $B_i B_i^*$  is invertible, thereby implying that  $P_{\text{--}}$  introduced in the theorem is a well defined orthogonal projection. We conclude the proof by showing that the ranges of  $P_{\text{--}}$  and  $P_i$  are as stated.

In what follows, we let  $R_{\text{ZM}}$  denote the canonical projection of  $L^2(\partial\Omega)$  onto  $L_{\text{ZM}}^2(\partial\Omega)$ .

**Lemma 3.9.** *The operator  $R_{\text{ZM}} (\nabla_{\text{T}}S)^* \nabla_{\text{T}}S: L_{\text{ZM}}^2(\partial\Omega) \rightarrow L_{\text{ZM}}^2(\partial\Omega)$  is self-adjoint and invertible.*

*Proof.* The operator  $R_{\text{ZM}} (\nabla_{\text{T}} S)^* \nabla_{\text{T}} S$  is self-adjoint, since for  $f, g \in L_{\text{ZM}}^2(\partial\Omega)$  we have:

$$\begin{aligned} \langle R_{\text{ZM}} (\nabla_{\text{T}} S)^* \nabla_{\text{T}} S f, g \rangle &= \langle (\nabla_{\text{T}} S)^* \nabla_{\text{T}} S f, g \rangle \\ &= \langle f, (\nabla_{\text{T}} S)^* \nabla_{\text{T}} S g \rangle = \langle f, R_{\text{ZM}} (\nabla_{\text{T}} S)^* \nabla_{\text{T}} S g \rangle. \end{aligned} \quad (39)$$

By the Riesz lemma it defines a symmetric and continuous bilinear form, say  $M$ , such that for  $f, g \in L_{\text{ZM}}^2(\partial\Omega)$

$$M(f, g) \doteq \langle R_{\text{ZM}} (\nabla_{\text{T}} S)^* \nabla_{\text{T}} S f, g \rangle. \quad (40)$$

From (18) and Lemma 2.8, it follows that  $M$  is coercive, and by the Lax-Milgram theorem  $R_{\text{ZM}} (\nabla_{\text{T}} S)^* \nabla_{\text{T}} S$  has bounded inverse.  $\blacksquare$

**Lemma 3.10.** *Let  $P_{\text{ZM}}$  and  $P_{\text{D}}$  be the projections from definition 3.7. Then,*

$$P_{\text{D}} = P_{\text{ZM}} \left( \mathbf{1} - \nabla_{\text{T}} S (R_{\text{ZM}} (\nabla_{\text{T}} S)^* \nabla_{\text{T}} S)^{-1} (\nabla_{\text{T}} S)^* \right) P_{\text{ZM}}. \quad (41)$$

*Proof.* Let  $\nu_{\circ}$  be as in Lemma 2.9. The projection onto  $T_{\text{ZM}}(\partial\Omega)$  can be written as

$$P_{\text{ZM}} f = (\mathbf{1} - P_{\text{S}}) f, \quad \text{with} \quad P_{\text{S}} f = \begin{cases} \langle f, \nabla_{\text{T}} S \mathbf{1} \rangle \frac{\nabla_{\text{T}} S \mathbf{1}}{\|\nabla_{\text{T}} S \mathbf{1}\|^2} & \text{if } \nu_{\circ} \neq 0, \\ 0 & \text{if } \nu_{\circ} = 0. \end{cases} \quad (42)$$

To see this, observe that  $P_{\text{ZM}}$  and  $P_{\text{S}}$  as defined above are complementary projections. Then, for  $f \in T(\partial\Omega)$ , we have that

$$\int_{\partial\Omega} (\nabla_{\text{T}} S)^* P_{\text{ZM}} f = \int_{\partial\Omega} (\nabla_{\text{T}} S)^* (\mathbf{1} - P_{\text{S}}) f \, d\sigma(q) = \langle f, \nabla_{\text{T}} S \mathbf{1} \rangle - \langle f, \nabla_{\text{T}} S \mathbf{1} \rangle = 0,$$

and thus  $P_{\text{ZM}} f$  is in  $T_{\text{ZM}}(\partial\Omega)$ . Conversely, if  $f$  is in  $T_{\text{ZM}}(\partial\Omega)$  then  $P_{\text{S}} f = 0$  and  $P_{\text{ZM}} f = f$ . Hence,  $P_{\text{ZM}} T(\partial\Omega) = T_{\text{ZM}}(\partial\Omega)$ .

Next, we write the operator  $P_{\text{D}}$  from (41) as

$$P_{\text{D}} = P_{\text{ZM}} (\mathbf{1} - A) P_{\text{ZM}} \quad \text{with} \quad A = \nabla_{\text{T}} S (R_{\text{ZM}} (\nabla_{\text{T}} S)^* \nabla_{\text{T}} S)^{-1} (\nabla_{\text{T}} S)^*, \quad (43)$$

and we show that it is the orthogonal projection onto  $D(\partial\Omega)$ .

The operator  $P_{\text{ZM}} A P_{\text{ZM}}$  is well defined, because  $P_{\text{ZM}}$  projects  $T(\partial\Omega)$  onto  $T_{\text{ZM}}(\partial\Omega)$  and  $(\nabla_{\text{T}} S)^*$  maps the latter into  $L_{\text{ZM}}^2(\partial\Omega)$ , where the operator  $(R_{\text{ZM}} (\nabla_{\text{T}} S)^* \nabla_{\text{T}} S)^{-1}$  is well defined by Lemma 3.9. Also, the operator  $A$  preserves the space  $T_{\text{ZM}}(\partial\Omega)$ ; that is, for  $f \in T_{\text{ZM}}(\partial\Omega)$ , we have  $Af \in T_{\text{ZM}}(\partial\Omega)$ . Indeed, observe that if  $f$  is in  $T_{\text{ZM}}(\partial\Omega)$ , then by (17) and Lemma 3.9 we have:

$$\begin{aligned} \langle (\nabla_{\text{T}} S)^* A f, \mathbf{1} \rangle &= \langle A f, \nabla_{\text{T}} S \nu_{\circ} \rangle = \langle (R_0 (\nabla_{\text{T}} S)^* \nabla_{\text{T}} S)^{-1} (\nabla_{\text{T}} S)^* f, (\nabla_{\text{T}} S)^* \nabla_{\text{T}} S \nu_{\circ} \rangle \\ &= \langle (R_0 (\nabla_{\text{T}} S)^* \nabla_{\text{T}} S)^{-1} (\nabla_{\text{T}} S)^* f, R_0 (\nabla_{\text{T}} S)^* \nabla_{\text{T}} S \nu_{\circ} \rangle = \langle f, \nabla_{\text{T}} S \nu_{\circ} \rangle = \langle f, \nabla_{\text{T}} S \mathbf{1} \rangle = 0. \end{aligned}$$

Consequently,  $Af$  is in  $T_{\text{ZM}}(\partial\Omega)$ . By a similar calculation,  $AA P_{\text{ZM}} = A P_{\text{ZM}}$  and thus  $A P_{\text{ZM}} A P_{\text{ZM}} = A A P_{\text{ZM}} = A P_{\text{ZM}}$ .

From the above we see that  $P_{\text{D}}$  is idempotent, since

$$P_{\text{D}} P_{\text{D}} = P_{\text{ZM}} (\mathbb{1} - A) P_{\text{ZM}} (\mathbb{1} - A) P_{\text{ZM}} = P_{\text{ZM}} (\mathbb{1} - A) P_{\text{ZM}} = P_{\text{D}}. \quad (44)$$

Also,  $P_{\text{D}}$  is self-adjoint, because  $\mathbb{1}$ ,  $P_{\text{ZM}}$ , and  $A$  are self-adjoint. Thus,  $P_{\text{D}}$  is an orthogonal projection.

We claim that the range of  $P_{\text{D}}$  is  $D(\partial\Omega)$ . To support this claim, it is enough to show that  $f \in D(\partial\Omega)$  holds if and only if  $P_{\text{D}}f = f$ . Assume that  $P_{\text{D}}f = f$  holds. Since the range of  $P_{\text{D}}$  is in the range of  $P_{\text{ZM}}$ , it follows that  $P_{\text{ZM}}f = f \in T_{\text{ZM}}(\partial\Omega)$ , and since  $A$  preserves  $T_{\text{ZM}}(\partial\Omega)$  we get from (43) that

$$Af = P_{\text{ZM}} A P_{\text{ZM}} f = P_{\text{ZM}} f - P_{\text{D}} f = f - f = 0. \quad (45)$$

This implies  $(\nabla_{\text{T}} S)^* f = 0$ , as the remaining operators in the definition of  $A$  are injective on functions with zero mean, by Lemma 2.9. Therefore,  $f$  is in  $D(\partial\Omega)$ .

Conversely, take  $f \in D(\partial\Omega)$  and write,  $f = c \nabla_{\text{T}} S \mathbb{1} + P_{\text{ZM}} f (= P_{\text{S}} f + P_{\text{ZM}} f)$  for some constant  $c \in \mathbb{R}$ . Then,

$$0 = \langle (\nabla_{\text{T}} S)^* f, \mathbb{1} \rangle = c \langle \nabla_{\text{T}} S \mathbb{1}, \nabla_{\text{T}} S \mathbb{1} \rangle + \langle P_{\text{ZM}} f, \nabla_{\text{T}} S \mathbb{1} \rangle = c \|\nabla_{\text{T}} S \mathbb{1}\|^2. \quad (46)$$

Hence, either  $\nabla_{\text{T}} S \mathbb{1} = 0$  or  $\nabla_{\text{T}} S \mathbb{1} \neq 0$ , and in the latter case  $c = 0$ ; thus,  $P_{\text{ZM}} f = f$  in all cases, so that  $f$  is in  $T_{\text{ZM}}(\partial\Omega)$  and hence,

$$A P_{\text{ZM}} f = \nabla_{\text{T}} S (R_{\text{ZM}} (\nabla_{\text{T}} S)^* \nabla_{\text{T}} S)^{-1} (\nabla_{\text{T}} S)^* f = 0,$$

ensuing that  $P_{\text{D}} f = P_{\text{ZM}} f + P_{\text{ZM}} A P_{\text{ZM}} f = f$ . This shows that the range of  $P_{\text{D}}$  is  $D(\partial\Omega)$  and completes the proof.  $\blacksquare$

Since divergence-free fields are orthogonal to gradients, it follows that

$$P_{\text{G}} \doteq \mathbb{1} - P_{\text{D}} = P_{\text{S}} + P_{\text{ZM}} A P_{\text{ZM}}, \quad (47)$$

is the orthogonal projection onto  $G(\partial\Omega)$ , so we can write the Helmholtz-Hodge decomposition explicitly as

$$T(\partial\Omega) = P_{\text{G}} T(\partial\Omega) + P_{\text{D}} T(\partial\Omega) = G(\partial\Omega) + D(\partial\Omega). \quad (48)$$

*Remark 3.11.* The existence of the Helmholtz-Hodge decomposition is trivial, for it reduces to the fact that a Hilbert space decomposes as the sum of a closed subspace and its orthogonal complement. However, the point in (48) is the explicit expression in terms of layer potentials. Note that the divergence-free term can split further by Hodge theory on Lipschitz orientable Riemannian manifolds [38].

*Remark 3.12.* The definition of the projections  $P_{\text{D}}$  and  $P_{\text{G}}$  can be extended to the whole of  $L^2(\partial\Omega; \mathbb{R}^d)$  as follows: for  $f = \eta f_{\eta} + f_{\text{T}} \in L^2(\partial\Omega; \mathbb{R}^d)$  put

$$P_{\text{D}} f = P_{\text{D}} f_{\text{T}}, \quad \mathbb{P}_{\text{D}}^{\perp} f = \eta f_{\eta} + P_{\text{G}} f_{\text{T}}. \quad (49)$$



**Corollary 3.13.** *Let  $f = \eta f_\eta + f_T$  be in  $\mathcal{N}[\mathcal{P}_i]$ . Then,  $f_\eta$  determines  $f_T$  uniquely up to a divergence-free tangent field.*

*Proof.* For  $f \in \mathcal{N}[\mathcal{P}_i]$ , the relation between  $f_\eta$  and  $f_T$  from Theorem 3.3 reads as

$$\left(\frac{1}{2} + K\right) f_\eta = -(\nabla_T S)^* f_T. \quad (50)$$

The divergence-free component of  $f_T$  does not contribute to this relation as it satisfies  $(\nabla_T S)^* P_D f_T = 0$ . Therefore, we can assume without loss of generality that  $f_T$  is a gradient field. To prove the corollary, we have to invert the right hand side of (50). The difficulty is that the left hand side of (50) may not have zero mean and thus, we cannot directly use Lemma 3.9.

To circumvent this, we split  $f_T$  into a field with zero mean plus  $\nabla_T S c$  for some constant  $c$ . Specifically, since  $f_T$  is a gradient field, there exists a constant  $c$  and a function  $\varphi \in L^2_{ZM}(\partial\Omega)$  such that  $f_T = \nabla_T S(c + \varphi)$ , and then  $\nabla_T S \varphi$  is in  $T_{ZM}(\partial\Omega)$  (this is simply the decomposition  $f_T = P_G f_T = P_S f_T + P_{ZM} A P_{ZM} f_T$  where we set  $\nabla_T S \varphi = P_{ZM} A P_{ZM} f_T$ , which is possible because both  $f_T$  and  $P_S f_T$  are gradients). Inserting this decomposition into (50) and rearranging terms, we get that

$$(\nabla_T S)^* \nabla_T S \varphi = -\left(\frac{1}{2} + K\right) f_\eta - c (\nabla_T S)^* \nabla_T S 1. \quad (51)$$

Since the left hand side is in  $L^2_{ZM}(\partial\Omega)$ , the right hand side also has zero mean. Therefore, either  $\nabla_T S 1 = 0$  and then Lemma 3.9 achieves the proof, or else

$$c = -\frac{1}{\|\nabla_T S 1\|^2} \left\langle \left(\frac{1}{2} + K\right) f_\eta, 1 \right\rangle. \quad (52)$$

In the latter case, applying  $(R_{ZM} (\nabla_T S)^* \nabla_T S)^{-1}$  gives us

$$\varphi = (R_{ZM} (\nabla_T S)^* \nabla_T S)^{-1} \left[ \left\langle \left(\frac{1}{2} + K\right) f_\eta, 1 \right\rangle \frac{(\nabla_T S)^* \nabla_T S 1}{\|\nabla_T S 1\|^2} - \left(\frac{1}{2} + K\right) f_\eta \right], \quad (53)$$

whence the constant  $c$  and the function  $\varphi$  are both determined by  $f_\eta$ . Thus, so is the tangent field  $f_T$ , as desired.  $\blacksquare$

We turn to the inverse of  $B_i B_i^*$ .

**Lemma 3.14.** *Let  $B_i$  be the inner boundary operator from definition 3.2 and let  $B_i^*$  denote its adjoint from Lemma 3.5, then  $B_i B_i^*: L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)$  is invertible.*

*Proof.* Define a symmetric continuous bilinear form,  $M$ , such that,

$$M(f, g) \doteq \langle B_i B_i^* f, g \rangle \quad (f, g \in L^2(\partial\Omega)). \quad (54)$$

Since  $(\frac{1}{2} + K^*)$  is invertible, there exists a positive constant  $C$  such that,

$$M(f, f) = \|B_i^* f\|^2 \geq \left\| \left( \frac{1}{2} + K^* \right) f \right\|^2 \geq C \|f\|^2. \quad (55)$$

Thus,  $M$  is coercive and  $B_i B_i^*$  is invertible by the Lax-Milgram theorem.  $\blacksquare$

The proof of the Theorem 3.8 is now an easy consequence of the previous results.

*Proof of Theorem 3.8.* Let  $P_D$  be as above, and recall the operators introduced in the theorem:  $P_- = B_i^* (B_i B_i^*)^{-1} B_i$  and  $P_i = \mathbf{1} - P_D - P_-$ . We show the following: i)  $P_-$  is the orthogonal projection onto  $H^2(\partial\Omega)$ ; ii) the range of  $P_D$  is  $\{f \in \mathcal{N}[\mathcal{P}_i] : f_\eta = 0\}$ ; iii)  $P_i$  is the orthogonal projection onto  $I(\partial\Omega)$ .

i): that  $P_-$  is an orthogonal projection is readily verified, since it is self-adjoint and idempotent. From Corollary 3.3 we have that  $\mathcal{N}[P_i] = \mathcal{N}[B_i] = \mathcal{N}[P_-]$ , and from Lemma 3.5 we get that  $\mathcal{R}[P_-] = \mathcal{N}[P_i]^\perp = H^2(\partial\Omega)$ . This proves i).

ii): by Lemma 3.10,  $P_D$  projects onto  $D(\partial\Omega)$ . The rest of the assertion follows from (27) and the injectivity of  $(\frac{1}{2} + K)$ .

iii): we show the projection property first. The operator  $P_i$  is self-adjoint because  $\mathbf{1}$ ,  $P_D$ , and  $P_-$  are self-adjoint. For the idempotence we compute,

$$P_i P_i = \mathbf{1} - P_D + P_- - P_- (\mathbf{1} - P_D) - (\mathbf{1} - P_D) P_-. \quad (56)$$

By i) and ii) the space  $H^2(\partial\Omega)$  is orthogonal to  $D(\partial\Omega)$ , so that  $P_- P_D = P_D P_- = 0$  and

$$P_- (\mathbf{1} - P_D) = P_-, \quad (\mathbf{1} - P_D) P_- = P_-. \quad (57)$$

Using these identities, the right hand side of (56) becomes  $\mathbf{1} - P_D - P_- = P_i$ .

From the definition of  $P_i$ , it is clear that  $P_i L^2(\partial\Omega; \mathbb{R}^d)$  is the orthogonal complement of  $D(\partial\Omega)$  in  $\mathcal{N}[\mathcal{P}_i]$ . Further, if  $f$  lies in  $P_i L^2(\partial\Omega; \mathbb{R}^d)$  and is silent outside as well, then  $f \in \mathcal{N}[\mathcal{P}_i] \cap \mathcal{N}[\mathcal{P}_o]$ . That  $\mathcal{N}[\mathcal{P}_i] \cap \mathcal{N}[\mathcal{P}_o] = D(\partial\Omega)$  is shown in Lemma 4.2, further below. Assuming this result for now, it follows that the only field that is in  $P_i L^2(\partial\Omega; \mathbb{R}^d)$  and in the intersection of the null spaces is the zero field. This proves the assertion.

Finally, we have that  $P_- + P_i + P_D = \mathbf{1}$ , and thus,

$$L^2(\partial\Omega; \mathbb{R}^d) = P_i L^2(\partial\Omega; \mathbb{R}^d) \oplus P_- L^2(\partial\Omega; \mathbb{R}^d) \oplus P_D L^2(\partial\Omega; \mathbb{R}^d). \quad (58)$$

This completes the proof.  $\blacksquare$

### 3.3 Outer decomposition

In this section we derive an orthogonal decomposition of  $L^2(\partial\Omega; \mathbb{R}^d)$  based on the null space of the outer scalar potential.

**Definition 3.15.** Define the operator,  $B_o: L^2(\partial\Omega; \mathbb{R}^d) \rightarrow L^2(\partial\Omega)$ , as

$$B_o f \doteq - \left( \frac{1}{2} - K \right) f_\eta + (\nabla_{\mathbb{T}} S)^* f_{\mathbb{T}}; \quad (59)$$

it is linear and bounded. We call  $B_o$  the outer boundary operator.

As in the previous section, we can write the outer potential as a harmonic extension of the outer boundary operator. Contrary to that section, however, we need to take separate care of the constant mode, because the operators  $\frac{1}{2} - K$  and  $\frac{1}{2} - K^*$  are invertible on  $L^2_{\text{ZM}}(\partial\Omega)$  only.

**Theorem 3.16.** Let  $\nu_o \in L^2_{\text{ZM}}(\partial\Omega)$  denote the function from Lemma 2.9. The outer potential of  $f \in L^2(\partial\Omega; \mathbb{R}^d)$  can be expressed as

$$\begin{aligned} \mathcal{P}_o f(x) = & -\mathcal{H} \left( \frac{1}{2} - K \right)^{-1} (B_o f - \langle B_o f, 1 \rangle)(x) \\ & + \langle B_o f, 1 \rangle \frac{\mathcal{S}(1 - \nu_o)(x)}{|S(1 - \nu_o)|}, \quad (x \in \Omega^o). \end{aligned} \quad (60)$$

The null space of  $\mathcal{P}_i$  is given by

$$\mathcal{N}[\mathcal{P}_o] = \mathcal{N}[B_o] = \left\{ f \in L^2(\partial\Omega; \mathbb{R}^d) : \left( \frac{1}{2} - K \right) f_\eta = (\nabla_{\mathbb{T}} S)^* f_{\mathbb{T}} \right\}. \quad (61)$$

*Proof.* From (17) we know that  $S(1 - \nu_o)$  is a non-zero constant, hence the second term in (60) is well defined and converges to  $\langle B_o f, 1 \rangle$  a.e. as  $x$  tends to the boundary nontangentially, by (7). Consequently, the right hand side of (60) converges to  $B_o f$  under the same conditions. Taking nontangential limits on  $\partial\Omega$  from  $\Omega^o$  on both sides of (60) while using (11) and (14), we get that

$$\lim_{x \rightarrow q} \mathcal{P}_o f(x) = - \left( \frac{1}{2} - K \right) f_\eta(q) + (\nabla_{\mathbb{T}} S)^* f_{\mathbb{T}}(q) = B_o f(q). \quad (62)$$

It follows that both hand sides of (60) are harmonic functions on  $\Omega^o$  with  $L^2$ -bounded nontangential maximal function, whose nontangential limits coincide a.e. on  $\partial\Omega$ . So, by uniqueness of a solution to the Dirichlet problem, they are equal. It follows that  $\mathcal{N}[\mathcal{P}_o] = \mathcal{N}[B_o]$ ; the second equality in (61) is immediate from the definition of  $B_o$ . ■

**Corollary 3.17.** Let  $f = \eta f_\eta + f_{\mathbb{T}}$  be in  $\mathcal{N}[\mathcal{P}_o]$ . Then,  $f_\eta$  determines  $f_{\mathbb{T}}$  uniquely, up to a divergence-free tangent field.

*Proof.* The proof follows the same steps as the one of Corollary 3.13. ■

Like in the previous section, the  $L^2$ -adjoint of  $B_o$ , denoted as  $B_o^*$ , helps one to characterize the orthogonal complement of the null space for  $\mathcal{P}_o$ :

**Lemma 3.18.** *The operator  $B_o^*: L^2(\partial\Omega) \rightarrow L^2(\partial\Omega; \mathbb{R}^d)$  can be expressed as*

$$B_o^*g = -\eta \left( \frac{1}{2} - K^* \right) g + \nabla_{\mathbb{T}} Sg. \quad (63)$$

The range of  $B_o^*$  satisfies  $\mathcal{R}[B_o^*] = H_+^2(\partial\Omega) = \mathcal{N}[\mathcal{P}_o]^\perp$ .

*Proof.* Let  $f$  be in  $L^2(\partial\Omega; \mathbb{R}^d)$  with normal component  $f_\eta$ , and tangential part  $f_{\mathbb{T}}$ . Then, for every  $g \in L^2(\partial\Omega)$  we get that

$$\langle B_o f, g \rangle = \left\langle - \left( \frac{1}{2} - K \right) f_\eta + (\nabla_{\mathbb{T}} S)^* f_{\mathbb{T}}, g \right\rangle \quad (64)$$

$$= \left\langle f_\eta, - \left( \frac{1}{2} - K^* \right) g \right\rangle + \left\langle f_{\mathbb{T}}, \nabla_{\mathbb{T}} Sg \right\rangle = \langle f, B_o^*g \rangle. \quad (65)$$

Hence, using (15) we obtain:

$$B_o^*g(p) = -\eta \left( \frac{1}{2} - K^* \right) g(p) + \nabla_{\mathbb{T}} Sg(p) = \lim_{x \rightarrow p} \nabla \mathcal{S}g(x), \quad (66)$$

where the limit is taken from inside  $\Omega$ . The rest of the proof follows the same steps as in Lemma 3.5; the only difference is that  $(\frac{1}{2} - K^*)$  is merely invertible on functions with zero mean and thus, we can only define  $\varphi = \mathcal{S}(\frac{1}{2} - K^*) f_\eta$  if  $f_\eta$  is in  $L_{\text{ZM}}^2(\partial\Omega)$ . This last condition, however, is guaranteed by the Gauss theorem when  $f_\eta$  is the normal component of a harmonic gradient in  $\Omega$ . ■

**Corollary 3.19.** *The space  $L^2(\partial\Omega; \mathbb{R}^d)$  splits into an orthogonal sum as*

$$L^2(\partial\Omega; \mathbb{R}^d) = H_+^2(\partial\Omega) \oplus \mathcal{N}[\mathcal{P}_o]. \quad (67)$$

### 3.4 Orthogonal projections for the outer decomposition

Let  $\nu_o$  denote the function from Lemma 2.9. Define two complementary projections,

$$P_B^\perp f = (\mathbf{1} - P_B) f, \quad P_B f = \begin{cases} \langle f, B_o^* \mathbf{1} \rangle \frac{B_o^* \mathbf{1}}{\|B_o^* \mathbf{1}\|^2} & \text{if } \nu_o \neq 0, \\ 0 & \text{if } \nu_o = 0, \end{cases} \quad (68)$$

and recall that  $R_{\text{ZM}}$  denotes the projection of  $L^2(\partial\Omega)$  to  $L_{\text{ZM}}^2(\partial\Omega)$ . In this section we prove the following theorem.

**Theorem 3.20.** *The operator*

$$P_+ = P_B + P_B^\perp B_o^* (R_{\text{ZM}} B_o B_o^*)^{-1} B_o P_B^\perp \quad (69)$$

defines an orthogonal projection from  $L^2(\partial\Omega; \mathbb{R}^d)$  onto  $H_+^2(\partial\Omega)$ .

For  $P_D$  as in definition 3.7, the operator

$$P_o \doteq \mathbb{1} - P_D - P_+ \quad (70)$$

is the orthogonal projection from  $L^2(\partial\Omega; \mathbb{R}^d)$  onto  $O(\partial\Omega) = \mathcal{N}[\mathcal{P}_o] \cap \mathcal{N}[\mathcal{P}_i]^\perp$ . Moreover, the space  $L^2(\partial\Omega; \mathbb{R}^d)$  splits into an orthogonal direct sum as

$$L^2(\partial\Omega; \mathbb{R}^d) = H_+^2(\partial\Omega) \oplus O(\partial\Omega) \oplus D(\partial\Omega). \quad (71)$$

Compared with the previous section, the only additional ingredient needed for the proof of this theorem is the fact that the operator  $R_{\text{ZM}} B_o B_o^*$  is invertible. We show this in the following lemma.

**Lemma 3.21.**  $R_{\text{ZM}} B_o B_o^*: L_{\text{ZM}}^2(\partial\Omega) \rightarrow L_{\text{ZM}}^2(\partial\Omega)$  is self-adjoint and invertible.

*Proof.* The operator  $R_{\text{ZM}} B_o B_o^*$  is self-adjoint, for if  $f, g \in L_{\text{ZM}}^2(\partial\Omega)$  we have

$$\langle R_{\text{ZM}} B_o B_o^* f, g \rangle = \langle B_o B_o^* f, g \rangle = \langle f, B_o B_o^* g \rangle = \langle f, R_{\text{ZM}} B_o B_o^* g \rangle. \quad (72)$$

Hence, it defines a symmetric bounded functional,  $M$ , such that for  $f, g \in L_{\text{ZM}}^2(\partial\Omega)$ ,

$$M(f, g) = \langle R_{\text{ZM}} B_o B_o^* f, g \rangle. \quad (73)$$

Since  $\frac{1}{2} - K^*$  is invertible on  $L_{\text{ZM}}^2(\partial\Omega)$  by Lemma 2.8, there exists a constant  $C$  such that

$$M(f, f) = \|B_o^* f\|^2 \geq \left\| \left( \frac{1}{2} - K^* \right) f \right\|^2 \geq C \|f\|^2. \quad (74)$$

By the Lax-Milgram theorem,  $R_{\text{ZM}} B_o B_o^*$  has a bounded inverse. ■

*Proof of Theorem 3.20.* With  $P_D$  as above, write the operator  $P_+$  from (69) as

$$P_+ = P_B + P_B^\perp C P_B^\perp, \quad \text{with } C = B_o^* (R_{\text{ZM}} B_o B_o^*)^{-1} B_o,$$

and define  $P_o$  through (70). We only show that  $P_+$  is the orthogonal projection from  $L^2(\partial\Omega; \mathbb{R}^d)$  onto  $H_+^2(\partial\Omega)$ . The rest of the proof follows the same steps as the one of Theorem 3.8.

The operator  $P_+$  is self-adjoint, since  $P_B$  and  $P_B^\perp$  clearly are, and  $(R_{\text{ZM}} B_o B_o^*)^{-1}$  is self-adjoint by Lemma 3.21. Further,

$$P_+ P_+ = P_B + P_B^\perp C P_B^\perp C P_B^\perp. \quad (75)$$

The idempotence will follow, once we establish that  $C P_B^\perp C P_B^\perp = C P_B^\perp$ .

First we show that  $P_B^\perp C P_B^\perp = C P_B^\perp$ . To see this, observe from Lemma 2.9 and the definition of  $B_o^\star$  that  $B_o^\star 1 = B_o^\star \nu_o$ . For  $f \in L^2(\partial\Omega; \mathbb{R}^d)$ , we then get from the definition of  $C$  that

$$\langle C P_B^\perp f, B_o^\star 1 \rangle = \langle C P_B^\perp f, B_o^\star \nu_o \rangle = \langle P_B^\perp f, B_o^\star \nu_o \rangle = \langle P_B^\perp f, B_o^\star 1 \rangle = 0. \quad (76)$$

Thus,  $P_B C P_B^\perp f = 0$  and  $P_B^\perp C P_B^\perp f = (\mathbf{1} - P_B) C P_B^\perp f = C P_B^\perp f$ .

Next, we prove that  $C C P_B^\perp = C P_B^\perp$ . For this, observe that (76) yields  $\langle B_o C P_B^\perp f, 1 \rangle = 0$ , so that  $B_o C P_B^\perp f$  has zero mean. Hence,

$$C C P_B^\perp = B_o^\star (R_{\text{ZM}} B_o B_o^\star)^{-1} (R_{\text{ZM}} B_o B_o^\star) (R_{\text{ZM}} B_o B_o^\star)^{-1} B_o P_B^\perp = C P_B^\perp. \quad (77)$$

Consequently,  $C P_B^\perp C P_B^\perp = C P_B^\perp$ , and (75) says that  $P_+ P_+ = P_+$ .

As to the range of  $P_+$ , it is easy to see that  $\mathcal{N}[P_+] = \mathcal{N}[B_o]$ , and from Theorem 3.16 we get  $\mathcal{N}[B_o] = \mathcal{N}[P_o]$ . Thus, by Lemma 3.18, we arrive at  $\mathcal{R}[P_+] = \mathcal{N}[P_o]^\perp = H_+^2(\partial\Omega)$ .  $\blacksquare$

## 4 Skew-orthogonal decompositions of fields

In this section we prove two skew-orthogonal decompositions of  $L^2(\partial\Omega; \mathbb{R}^d)$ . The first is the Hardy-Hodge decomposition of  $L^2(\partial\Omega; \mathbb{R}^d)$ . The second is the splitting of  $L^2(\partial\Omega; \mathbb{R}^d)$  into the spaces  $I(\partial\Omega)$ ,  $O(\partial\Omega)$ , and  $D(\partial\Omega)$ .

We begin with the Hardy-Hodge decomposition, which is a simple consequence of the above analysis.

**Theorem 4.1.** *The space  $L^2(\partial\Omega; \mathbb{R}^d)$  splits into a not necessarily orthogonal direct sum as*

$$L^2(\partial\Omega; \mathbb{R}^d) = H_+^2(\partial\Omega) + H_-^2(\partial\Omega) + D(\partial\Omega). \quad (78)$$

*Proof.* Note from the definition of  $B_i^\star$  and  $B_o^\star$  that for every  $\varphi \in L^2(\partial\Omega)$ :

$$(B_i^\star - B_o^\star) \varphi = \eta \varphi \quad \frac{1}{2} (B_i^\star + B_o^\star) \varphi = \eta K^\star \varphi + \nabla_{\text{T}} S \varphi. \quad (79)$$

Now, let  $f$  be in  $L^2(\partial\Omega; \mathbb{R}^d)$ . We use the Hodge decomposition from Lemma 3.10 and write  $f = \mathbb{P}_D^\perp f + \mathbb{P}_D f$ , with  $\mathbb{P}_D^\perp f = \eta f_\eta + P_G f_{\text{T}}$ . Since  $P_G f_{\text{T}}$  is a gradient field, there exists a function  $\varphi \in L^2(\partial\Omega)$  such that  $\mathbb{P}_D^\perp f = \eta f_\eta + \nabla_{\text{T}} S \varphi$ . Choosing

$$h_+ = -B_o^\star \left( f_\eta - \left( \frac{1}{2} + K^\star \right) \varphi \right) \in H_+^2(\partial\Omega), \quad (80)$$

$$h_- = B_i^\star \left( f_\eta + \left( \frac{1}{2} - K^\star \right) \varphi \right) \in H_-^2(\partial\Omega), \quad (81)$$

and using (79), we obtain

$$h_+ + h_- = (B_i^* - B_o^*)(f_\eta - K^*\varphi) + \frac{1}{2}(B_i^* + B_o^*)\varphi = \eta f_\eta + \nabla_{\mathbb{T}} S\varphi = \mathbb{P}_{\mathbb{D}}^\perp f.$$

The desired decomposition is then given by  $f = h_+ + h_- + \mathbb{P}_{\mathbb{D}} f$ .

For uniqueness, we first show that  $H_-^2(\partial\Omega) \cap H_+^2(\partial\Omega) = \{0\}$ . To see this, assume that  $g$  is in  $H_-^2(\partial\Omega) \cap H_+^2(\partial\Omega)$ . Then, there exist two functions  $\phi, \chi \in L^2(\partial\Omega)$  such that,

$$B_i^* \phi = B_o^* \chi = g. \quad (82)$$

The normal and the tangent components of that equation read

$$\left(\frac{1}{2} + K^*\right)\phi = -\left(\frac{1}{2} - K^*\right)\chi \quad \text{and} \quad \nabla_{\mathbb{T}} S\phi = \nabla_{\mathbb{T}} S\chi. \quad (83)$$

The equation for the tangent part says that  $\phi$  differs from  $\chi$  by the null space of  $\nabla_{\mathbb{T}} S$ . By Remark 2.10, the null spaces of  $\left(\frac{1}{2} - K^*\right)$  and  $\nabla_{\mathbb{T}} S$  are the same and thus the equation for the normal component reads

$$\left(\frac{1}{2} + K^*\right)\phi = -\left(\frac{1}{2} - K^*\right)\phi, \quad (84)$$

leading to  $\phi = 0$ . It follows that  $g = B_i \phi = 0$ .

Now, assume that  $\mathbb{P}_{\mathbb{D}}^\perp f = g_- + g_+$ , for some  $g_- \in H_-^2(\partial\Omega)$  and  $g_+ \in H_+^2(\partial\Omega)$ , then we have that

$$\mathbb{P}_{\mathbb{D}}^\perp f = h_+ + h_- = g_+ + g_-.$$

Hence, the two functions  $g_- - h_-$  and  $g_+ - h_+$  are in  $H_-^2(\partial\Omega) \cap H_+^2(\partial\Omega)$ , and thus,  $h_- = g_-$  and  $h_+ = g_+$ .  $\blacksquare$

Next, we prove the skew-orthogonal decomposition involving  $I(\partial\Omega)$  and  $O(\partial\Omega)$ .

**Lemma 4.2.** *It holds that  $\mathcal{N}[\mathcal{P}_i] \cap \mathcal{N}[\mathcal{P}_o] = D(\partial\Omega)$ .*

*Proof.* Let  $h$  be in  $\mathcal{N}[\mathcal{P}_i] \cap \mathcal{N}[\mathcal{P}_o]$ . Then, by (27) and (61) we have that  $B_o h = 0 = B_i h$ , therefore

$$0 = (B_o + B_i)h = h_\eta. \quad (85)$$

By Corollary 3.13, the normal component uniquely determines  $h$  up to a divergence-free field, and thus,  $h \in D(\partial\Omega)$ .  $\blacksquare$

**Theorem 4.3.** *The space  $L^2(\partial\Omega; \mathbb{R}^d)$  splits into a not-necessarily orthogonal direct sum as*

$$L^2(\partial\Omega; \mathbb{R}^d) = I(\partial\Omega) + O(\partial\Omega) + D(\partial\Omega). \quad (86)$$

*Proof.* We will show that for every  $f \in L^2(\partial\Omega; \mathbb{R}^d)$  there exists a function  $h \in I(\partial\Omega)$  such that  $f - h \in \mathcal{N}[\mathcal{P}_o]$ , and since  $\mathcal{N}[\mathcal{P}_o] = O(\partial\Omega) \oplus D(\partial\Omega)$  the decomposition will follow as  $f = h + \mathbb{P}_D^\perp(f - h) + \mathbb{P}_D(f - h)$ .

Recall that a function  $g$  is in  $\mathcal{N}[\mathcal{P}_o]$  if and only if  $B_o(g) = 0$ . By Theorem 3.3, every  $h \in I(\partial\Omega)$  satisfies

$$\left(\frac{1}{2} + K\right) h_\eta = -(\nabla_{\mathbb{T}} S)^* h_{\mathbb{T}}, \quad (87)$$

and thus, we require in view of (59) that

$$B_o(f - h) = -\left(\frac{1}{2} - K\right) f_\eta + (\nabla_{\mathbb{T}} S)^* f_{\mathbb{T}} + h_\eta = 0. \quad (88)$$

Choose  $h_\eta = \left(\frac{1}{2} - K\right) f_\eta - (\nabla_{\mathbb{T}} S)^* f_{\mathbb{T}}$ . By the proof of Corollary 3.13, we can find  $h_{\mathbb{T}}$  such that  $h = \eta h_\eta + h_{\mathbb{T}}$  is in  $I(\partial\Omega)$  and  $f - h$  is in  $\mathcal{N}[\mathcal{P}_o]$  by construction.

As in the previous theorem, uniqueness follows since  $I(\partial\Omega) \cap O(\partial\Omega) = \{0\}$ . This concludes the proof.  $\blacksquare$

## 5 Special case—the sphere

In this section we show that if  $\partial\Omega$  is a sphere, then the decompositions in (38), (71), (78), and (86) become equal. To begin, we prove the following theorem.

**Theorem 5.1.** *The Hardy-spaces are orthogonal if and only if  $\partial\Omega$  is a sphere.*

It is important to keep in mind that  $\Omega$  is bounded, otherwise the statement is not true and a half-space whose boundary is a hyperplane is a counterexample (although a half-space is arguably a very large ball). The theorem is known for Clifford-analytic Hardy spaces [37, thm. 1.1], which comprise Clifford algebra-valued functions. In contrast to this, the present Hardy-spaces are smaller as they can be seen as vector valued Clifford-analytic functions only. Therefore, the above statement requires proof.

*Proof.* It is well known that Hardy-spaces on a sphere are orthogonal (for example, [39]). Thus, we only have to prove the opposite.

Assume that the Hardy-spaces are orthogonal. Then, by Lemma 3.5 and Lemma 3.18 we have for every  $\varphi, \chi \in L^2(\partial\Omega)$ ,

$$0 = \langle B_i^* \varphi, B_o^* \chi \rangle = \langle B_o B_i^* \varphi, \chi \rangle, \quad (89)$$

which implies the operator identity  $B_o B_i^* = 0$ . Then,  $B_i B_o^* = (B_o B_i^*)^* = 0$  and  $B_o B_i^* - B_i B_o^* = 0$ . Using the definition of operators  $B_i$ ,  $B_o$ , and their adjoints the latter identity yields

$$B_o B_i^* - B_i B_o^* = K - K^* = 0. \quad (90)$$

Consequently, the double layer potential must be self-adjoint. This, however, can only happen when  $\partial\Omega$  is a sphere by [37, thm. 4.23]. This concludes the proof.  $\blacksquare$



**Corollary 5.2.**  $I(\partial\Omega)$  and  $O(\partial\Omega)$  are orthogonal if and only if  $\partial\Omega$  is a sphere.

*Proof.* The spaces  $I(\partial\Omega)$  and  $H_-^2(\partial\Omega)$  are orthogonal by Theorem 3.8, and the spaces  $O(\partial\Omega)$  and  $H_+^2(\partial\Omega)$  are orthogonal by Theorem 3.20. Moreover, from (78) and (86), we also have  $I(\partial\Omega) + O(\partial\Omega) = \mathbb{P}_\mathbb{D}^\perp L^2(\partial\Omega; \mathbb{R}^d) = H_-^2(\partial\Omega) + H_+^2(\partial\Omega)$ . Thus,  $I(\partial\Omega)$  is orthogonal to  $O(\partial\Omega)$  if and only if  $H_-^2(\partial\Omega)$  is orthogonal to  $H_+^2(\partial\Omega)$ , and the assertion follows from Theorem 5.1. ■

**Corollary 5.3.** If  $\partial\Omega$  is a sphere then  $I(\partial\Omega) = H_+^2(\partial\Omega)$  and  $O(\partial\Omega) = H_-^2(\partial\Omega)$ .

*Proof.* On a sphere  $I(\partial\Omega)$  and  $H_+^2(\partial\Omega)$  are both orthogonal to  $H_-^2(\partial\Omega)$ . Moreover,  $I(\partial\Omega) \oplus H_-^2(\partial\Omega) = H_+^2(\partial\Omega) \oplus H_-^2(\partial\Omega)$ , by (38) and (78). Hence, we get that  $I(\partial\Omega) = H_+^2(\partial\Omega)$ . An analogous argument holds for  $O(\partial\Omega)$  and  $H_-^2(\partial\Omega)$ . ■

By the above corollary it is immediate that (38), (71), (78), and (86) are identical when  $\partial\Omega$  is a sphere.

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## A Appendix

The definition of strongly Lipschitz domains is standard and can be found in many textbooks, including [34, 40]. Nevertheless, the basic differential-geometric notions on Lipschitz manifolds are not easy to ferret out in the literature. For the convenience of the reader we therefore recall the concepts here.

In the following  $\mathbb{B}$  will be an open ball in  $\mathbb{R}^{d-1}$ , and  $U$  will denote a doubly truncated cylinder whose cross-section is  $\mathbb{B}$ .

**Strongly Lipschitz domain.** We call a bounded region  $\Omega$  in  $\mathbb{R}^d$  a strongly Lipschitz domain if for each  $x \in \partial\Omega$  there is a cylinder  $U$ , a rigid motion  $L$ , and a Lipschitz function  $\psi: \mathbb{B} \rightarrow \mathbb{R}$  such that,

$$U \cap \Omega = \{L(y, t) : y \in \mathbb{B}, 0 \leq t < \psi(y)\} \quad \text{and} \quad U \cap \partial\Omega = \{L(y, \psi(y)) : y \in \mathbb{B}\}.$$

Since  $\partial\Omega$  is compact, we can cover it with finitely many such cylinders  $U_j$  associated with  $\mathbb{B}_j$ ,  $L_j$ , and  $\psi_j$ , for  $j \in \{1, \dots, N\}$ . If in addition, we introduce the projection onto the first  $(d-1)$  components as  $P_{d-1}: \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$  then the maps  $\phi_j := P_{d-1} \circ L_j^{-1}: U_j \cap \partial\Omega \rightarrow \mathbb{B}_j$  define a system of charts on  $\partial\Omega$  with Lipschitz inverse  $\phi_j^{-1}: \mathbb{B}_j \rightarrow \partial\Omega \subset \mathbb{R}^d$  given by  $\phi_j^{-1}(y) = (y, \psi_j(y))$ ; this provides us with the bi-Lipschitz change of charts and makes  $\partial\Omega$  a Lipschitz manifold.

**Singular and regular points.** A point  $x \in \partial\Omega$  is called singular if there is a  $j \in \{1, \dots, N\}$  such that  $x \in U_j$  and  $\phi_j^{-1}$  is not differentiable at  $\phi_j(x)$ . A point which is not singular is called regular. We denote the set of regular points by  $\text{Reg}(\partial\Omega)$  and put  $\text{Reg}(\mathbb{B}_j) = \phi_j(\text{Reg}(\partial\Omega) \cap U_j)$ . Since  $\phi_j$  is bi-Lipschitz, at regular points, its derivative,  $D\phi_j^{-1}(y)$ , is injective there.

The set of singular points has  $\sigma$ -measure zero on  $\partial\Omega$ . One can see this as follows: by Rademacher's theorem [41, thm. 3.2] the set of singular points  $\mathbb{B}_j \setminus \text{Reg}(\mathbb{B}_j)$ , has Lebesgue measure zero for each  $j$ . But the Lebesgue measure on  $\mathbb{R}^{d-1}$  coincides with the  $(d-1)$ -dimensional Hausdorff measure,  $\mathcal{H}^{d-1}$ , and thus the set of singular points has  $\mathcal{H}^{d-1}$ -measure zero, which is preserved by Lipschitz functions  $\phi_j^{-1}$  [42].

With the above definition, the set of regular points depends on the atlas. Nevertheless, there is also an intrinsic atlas-independent definition (see for example [28]), as the set of those points for which the measure theoretic normal to  $\Omega$  exists.

**Tangent space.** At  $x \in U_j \cap \text{Reg}(\partial\Omega)$ , we define the tangent space  $T_x(\partial\Omega) \subset \mathbb{R}^d$  to be  $\mathcal{R}[D\phi_j^{-1}(\phi_j(x))]$ . Then, each  $X \in T_x(\partial\Omega)$  has a local representative in the chart  $(U_j, \phi_j)$ , which is the unique vector  $v \in \mathbb{R}^{d-1}$  such that  $X = D\phi_j^{-1}(\phi_j(x))v$ .

At each regular point the tangent space has dimension  $d - 1$ , and thus, we can define the outer unit normal  $\eta(x)$ , oriented such that, for small  $t > 0$  the vector  $x + t\eta(x)$  is in  $\Omega^\circ$ .

**Differentiability on  $\partial\Omega$ .** A map  $g: \partial\Omega \rightarrow \mathbb{R}^k$  is said to be differentiable at  $x \in \text{Reg}(\partial\Omega)$  if  $g \circ \phi_j^{-1}$  is differentiable at  $\phi_j(x)$ ; and if  $v$  is the local representative of  $X$  then the derivative  $Dg(x): T_x(\partial\Omega) \rightarrow \mathbb{R}^k$  is defined by  $Dg(x)(X) = D(g \circ \phi_j^{-1})(\phi_j(x))v$ . By the chain rule, the derivative is chart independent.

When  $g: \partial\Omega \rightarrow \mathbb{R}$  is differentiable at  $x \in \text{Reg}(\partial\Omega)$ , the map  $Dg(x)$  is a linear form on  $T_x(\partial\Omega)$ , and thus it can be represented as  $X \mapsto \langle X, Y \rangle_{\mathbb{R}^d}$  for some unique vector  $Y \in T_x(\partial\Omega)$  that we call the tangential gradient of  $g$  at  $x$  and denoted by  $\nabla_T g(x)$ .

If  $f: \partial\Omega \rightarrow \mathbb{R}$  is a Lipschitz function on  $\partial\Omega$  than  $f \circ \phi_j^{-1}: \mathbb{B}_j \rightarrow \mathbb{R}^d$  is also Lipschitz for each  $j$ . By Rademacher's theorem,  $f \circ \phi_j^{-1}$  is differentiable a.e. on  $\mathbb{B}_j$  and consequently  $f$  is differentiable a.e. on  $\partial\Omega$ . Moreover, the derivatives of a Lipschitz function are uniformly bounded by the Lipschitz constant.

In fact, we can extend  $f$  to a function  $F$ , defined on a small neighborhood around  $\partial\Omega$ , such that  $\nabla_T f$  reads as the Euclidean gradient of  $F$ . To verify this extend  $f$  as follows: for each  $j$ , define  $\tilde{f}_j: U_j \rightarrow \mathbb{R}$  by  $\tilde{f}_j(L_j(y, t)) = f(L_j(y, \psi_j(y)))$ . As the differential of  $\tilde{f}_j$  is independent of  $t$ , it is easily seen that  $\tilde{f}_j$  is differentiable a.e. on  $\partial\Omega \cap U_j$ . Using a smooth partition of unity relative to the  $U_j$ 's,  $\Omega$ , and  $\Omega^\circ$ , we can glue the  $\tilde{f}_j$ 's together into a single Lipschitz map  $F: \cup_j U_j \rightarrow \mathbb{R}^d$ . This map has the following properties: its restriction to  $\partial\Omega$  is  $f$ ; it is differentiable a.e. on  $\partial\Omega$ ; and there, its Euclidean gradient,  $\nabla F$ , coincides with  $\nabla_T f$ .

**Sobolev spaces.** The Sobolev space  $W^{1,2}(\partial\Omega)$  is the Hilbert space obtained as the completion of Lipschitz functions with respect to the norm,

$$\|\psi\|_{W^{1,2}(\partial\Omega)} = \left( \|\psi\|_{L^2(\partial\Omega)}^2 + \|\nabla_T \psi\|_{L^2(\partial\Omega; \mathbb{R}^d)}^2 \right)^{1/2}. \quad (91)$$

Equivalently, the function  $\psi$  is in  $W^{1,2}(\partial\Omega)$  if and only if, for each chart  $(U_j, \phi_j)$  the function  $\psi \circ \phi_j^{-1}$  is in the Euclidean Sobolev space  $W^{1,2}(\mathbb{B}_j)$ ; thus  $W^{1,2}(\partial\Omega)$  agrees with the space  $L_1^2(\partial\Omega)$  used in [24, Def. 1.7]. Each  $\psi \in W^{1,2}(\partial\Omega)$  has a well defined tangential gradient  $\nabla_T \psi \in T(\partial\Omega) \subset L^2(\partial\Omega; \mathbb{R}^d)$ , see [34, thm. 3.17].

**Lemma A.1.** *For  $f \in L^2(\partial\Omega)$ , it holds that  $\nabla_T S f$  defined in Lemma 2.5 is the tangential gradient of  $S f \in W^{1,2}(\partial\Omega)$ .*

*Proof.* Let  $F \in L^2(\partial\Omega, \mathbb{R}^d)$  be the nontangential limit of  $\nabla \mathcal{S} f$  on  $\partial\Omega$  from inside, which is known to exist a.e. We will show that the tangential component of  $F$  is equal to  $\nabla_T S f$ . The limit of  $\nabla \mathcal{S} f$  from outside can be handled similarly, and in view of the limiting relation in Lemma 2.5 this will achieve the proof. The statement being local, it is enough to proceed in a coordinate cylinder  $U_j$ , and we may assume

for simplicity that it has vertical axis so that  $\phi_j : U_j \cap \partial\Omega \rightarrow \mathbb{B}_j$  is the projection onto the first  $d - 1$  components and  $\phi_j^{-1} : \mathbb{B}_j \rightarrow U_j \cap \partial\Omega$  is given by  $\phi_j^{-1}(y) = (y, \Psi_j(y))$  for some Lipschitz function  $\Psi_j : \mathbb{B}_j \rightarrow \mathbb{R}$ . Set  $e_d := (0, \dots, 0, 1)^t$  and let  $C_{\theta_1, e_n}(\xi)$  be a natural cone of aperture  $2\theta_1$  at  $\xi \in U_j \cap \partial\Omega$  in the chart  $(U_j, \phi_j)$ , contained in the cone  $C_{\theta, Z(\xi)}(\xi)$  from the regular family of cones we have fixed. Recall that  $\theta_1$  can be taken independent of  $\xi$ . For  $\varepsilon > 0$  small enough that  $(y, \Psi_j(y) - \varepsilon) \in \Omega^+ \cap U_j$  when  $y \in \mathbb{B}_j$ , the smoothness of  $\mathcal{S}f$  in  $\Omega^+$  implies that  $h_\varepsilon(y) := \mathcal{S}f(y, \Psi_j(y) - \varepsilon)$  is Lipschitz in  $\mathbb{B}_j$  with gradient  $(\nabla h_\varepsilon(y))^t = (\nabla \mathcal{S}f(y, \Psi_j(y) - \varepsilon))^t D\phi_j^{-1}(y)$ . Let  $\varepsilon_k \rightarrow 0$  and observe that  $(y, \Psi_j(y) - \varepsilon_k)$  converges to  $\xi = (y, \Psi_j(y))$  from within  $C_{\theta, e_d}(\xi)$ , hence  $\nabla \mathcal{S}f(y, \Psi_j(y) - \varepsilon_k)$  converges for  $m_{d-1}$ -a.e.  $y \in \mathbb{B}_j$  to  $F(y, \Psi_j(y))$ , while being dominated pointwise in norm by  $(\nabla \mathcal{S}f)^M(y, \Psi_j(y))$ ; here,  $m_{d-1}$  indicates  $d - 1$ -dimensional Lebesgue measure. As

$$\int_{\mathbb{B}_j} ((\nabla \mathcal{S}f)^M(y, \Psi_j(y)))^2 \sqrt{1 + |\nabla \Psi_j|^2} dm_{d-1} = \int_{U_j \cap \partial\Omega} ((\nabla \mathcal{S}f)^M(\xi))^2 d\sigma(\xi)$$

because the image of  $d\sigma$  in local coordinates is  $\sqrt{1 + |\nabla \Psi_j|^2} dm_{d-1}$ , and since  $|\nabla \Psi_j|$  is uniformly bounded, we get that  $(\nabla \mathcal{S}f)^M$  composed with  $\phi_j^{-1}$  lies in  $L^2(\mathbb{B}_j)$ . Therefore, by dominated convergence,  $\nabla h_{\varepsilon_k}$  converges in  $L^2(\mathbb{B}_j)$  to  $(D\phi_j^{-1})^t F \circ \phi_j^{-1}$ . Likewise,  $h_{\varepsilon_k}$  converges pointwise a.e. to  $Sf(y, \Psi_j(y))$  while being dominated pointwise by  $(\mathcal{S}f)^M(y, \Psi_j(y))$  that lies in  $L^2(\mathbb{B}_j)$ , therefore  $h_{\varepsilon_k} \rightarrow Sf \circ \phi_j^{-1}$  in  $L^2(\mathbb{B}_j)$  by dominated convergence. Hence, if we pick a smooth function  $\varphi$  with compact support in  $\mathbb{B}_j$  and pass to the limit in the relations  $\int_{\mathbb{B}_j} \partial_{y_i} h_{\varepsilon_k} \varphi = - \int_{\mathbb{B}_j} h_{\varepsilon_k} \partial_{y_i} \varphi$ , we find that  $Sf \circ \phi_j^{-1}|_{B(y_0, \delta)}$  lies in  $W^{1,2}(\mathbb{B}_j)$  with

$$\nabla(Sf \circ \phi_j^{-1}) = (D\phi_j^{-1})^t F \circ \phi_j^{-1}. \quad (92)$$

Since the normal  $\eta(\xi)$  is orthogonal to the columns of  $D\phi_j^{-1}(\phi_j(\xi))$  that span the tangent space  $T_\xi(\partial\Omega)$ , a short computation shows that (92) is equivalent to saying that the tangential component of the field  $F$  is the gradient of  $Sf$ .  $\blacksquare$