The power-dual and the exponential-dual of a matrix

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Talk's content

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Original path

- The problem addressed here is the derivation of closed forms of the solution to linear systems of difference or differential equations which have
  - constant coefficients,
  - a "simple" or "regular" structure,
  - but that have, in some cases, infinite size.
- These types of systems appear in many areas, in particular in the transient analysis of Markovian queueing models.
- We developed a methodology based on the concept of duality in Markov chains as proposed by Sigmund and developed by Anderson[1], and used it on basic queueing models ($M/M/1$, $M/M/1/H$, $M/M/1$ and $M/M/1/H$ with failures or catastrophes, etc.).

[1] *Continuous-Time Markov Chains*, W. J. Anderson, published by Springer in 1991 (out of press).

Interest and extension

- The goal of having closed forms is, obviously, to get insight into the solution to the considered models.
- This was the motivation with those stochastic models. Transient analysis is necessary today, for instance in telecommunications, to know what happens on short finite time intervals, and not only in equilibrium.
- Later, we extended this duality idea to arbitrary differential systems, not necessarily coming from stochastic processes. This paper describes new work in this direction.

Using the dual to analyze Markovian queues

- From a given Markov process X to analyze, living on the nonnegative integers, (i) we build a new one, its Anderson's dual as we call it, X^* (if it exists, which is not always the case).
- The transformation is linear (the generators are linearly related), and its main property is that the solutions to both systems are linearly related, and in the two directions: $X \to X^*$ and $X^* \to X$.
- Then, the idea is to (ii) find the transient distribution of X^*, in general using Uniformization, to move to discrete time and thus analyzing the transient distribution of a Markov chain Y^* (typically counting paths by means of lattice path combinatorics tools).
- To finish, (iii) we come back at X using the inverse dual (linear) transformation.
- In cases with some regularity in the structure, this can produce closed-forms solutions, and it can also work when the dimension of the initial problem is infinite (e.g. for the $M/M/1$ queue).

Comments

- This procedure is useful when X^* is easier to analyze, in a way or another.
- As we see, there is an initial continuous time phase, then a discrete time part process, where a transient distribution is needed.
- In some cases, we have a discrete time problem for which we look for transients from the beginning (instead of ODEs we have then difference equations).
- In this talk we will briefly describe a general approach where transients in continuous time (algebraically speaking, matrix exponentials) or in discrete time (matrix powers) are needed. The former are built using the latter.
- To simplify the discussion, we only present here in the finite case.

Talk's content

The power-dual of a matrix

- Let M be a square matrix (of reals, of complex numbers, ...) with dimension n. Let us index it on $\{0, 1, \ldots, n-1\}$. The power-dual of M is a new matrix that we denote here $\widetilde{M}$, with dimension $n+1$ and indexed on $\{0, 1, \ldots, n-1, n\}$, defined as follows:

$$\widetilde{M}_{i,j} = \begin{cases} \sum_{k=i}^{n-1} M_{j,k} - \sum_{k=i}^{n-1} M_{j-1,k} & \text{if } j \leq n-1, \\ 1 - \sum_{k=0}^{n-1} M_{n-1,k} & \text{when } j = n, \end{cases}$$

  where $M_{-1,k} = 0$.

- Example, dimension 2:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \rightsquigarrow \quad \widetilde{M} = \begin{pmatrix} a+b & c+d-(a+b) & 1-(c+d) \\ b & d-b & 1-d \\ 0 & 0 & 1 \end{pmatrix}$$

- Example, dimension 3:

$$M = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & -2 \\ 1 & -3 & 3 \end{pmatrix} \quad \rightsquigarrow \quad \widetilde{M} = \begin{pmatrix} 2 & -2 & 1 & 0 \\ 1 & -3 & 2 & 1 \\ 2 & -4 & 5 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- A stochastic matrix:

$$P = \begin{pmatrix} 2/5 & 0 & 3/5 \\ 1/6 & 1/2 & 1/3 \\ 0 & 1/4 & 3/4 \end{pmatrix} \quad \rightsquigarrow \quad \widetilde{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3/5 & 7/30 & 1/6 & 0 \\ 3/5 & \boxed{-4/15} & 5/12 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

as we see, in this case the power-dual is not a stochastic matrix;

- Another stochastic matrix:

$$P = \begin{pmatrix} 2/5 & 3/5 \\ 1/6 & 5/6 \end{pmatrix} \quad \rightsquigarrow \quad \widetilde{P} = \begin{pmatrix} 1 & 0 & 0 \\ 3/5 & 1/15 & 1/3 \\ 0 & 0 & 1 \end{pmatrix};$$

here, the power-dual is also a stochastic matrix.

Some properties

- If $\widetilde{M}$ is the power-dual of matrix M, then
  - the sum of the elements of any row of $\widehat{M}$ is 1,
  - and the last row is composed of 0s except for its last element $= 1$.
- Inversion Lemma: for $0 \le i, j \le n-1$, we have

$$M_{i,j} = \sum_{k=0}^{i} \widetilde{M}_{j,k} - \sum_{k=0}^{i} \widetilde{M}_{j+1,k}.$$

- Central result: For any two square matrices A and B with the same dimension, we have

$$\widetilde{AB} = \widetilde{B}\widetilde{A}.$$

  Corollary: for any square matrix M and any nonnegative integer k,

$$\boxed{\widetilde{M^k} = \widetilde{M}^k.}$$

  In words, the power-dual of the kth power of M is the kth power of M's power-dual.

Path

Instead of

$$M \underset{\substack{\uparrow \\ \text{counting paths,} \\ \text{spectral analysis...}}}{\longrightarrow} M^H,$$

we can try

$$M \underset{\substack{\uparrow \\ \text{(linear)}}}{\longrightarrow} \widetilde{M} \underset{\substack{\uparrow \\ \text{counting paths,} \\ \text{spectral analysis...}}}{\longrightarrow} \widetilde{M}^H = \widetilde{M^H} \underset{\substack{\uparrow \\ \text{Inversion Lemma} \\ \text{(linear)}}}{\longrightarrow} M^H$$

The exponential-dual of a matrix

- Let M be a square matrix (of reals, of complex numbers, ...) with dimension n. Let us index it on $\{0, 1, \ldots, n-1\}$.
  The exponential-dual of M is a new matrix that we denote $\widehat{M}$, with dimension $n+1$ and indexed on $\{0, 1, \ldots, n-1, n\}$, defined as follows:

$$\widehat{M}_{i,j} = \sum_{k=i}^{n-1} M_{j,k} - \sum_{k=i}^{n-1} M_{j-1,k},$$

  where $M_{-1,k} = 0$.
- Example, dimension 2:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \rightsquigarrow \quad \widehat{M} = \begin{pmatrix} a+b & c+d-(a+b) & -(c+d) \\ b & d-b & -d \\ 0 & 0 & 0 \end{pmatrix}$$

- We can observe that if we denote by W a square matrix with the same dimension as M, composed of 0s except the last column composed of 1s, then we have $\widehat{M} = \widetilde{M} - W$.

- Example, dimension 3:

$$M = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & -3 \\ 0 & 2 & -2 \end{pmatrix} \quad \rightsquigarrow \quad \widetilde{M} = \begin{pmatrix} 6 & -8 & 2 & 0 \\ 5 & -6 & 1 & 0 \\ 3 & -6 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- An infinitesimal generator:

$$M = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix} \quad \rightsquigarrow \quad \widetilde{M} = \begin{pmatrix} 0 & 0 & 0 \\ \lambda & -(\lambda+\mu) & \mu \\ 0 & 0 & 0 \end{pmatrix};$$

  as we see, the exponential-dual is also an infinitesimal generator;
- Another infinitesimal generator:

$$M = \begin{pmatrix} -3 & 1 & 2 \\ 6 & -7 & 1 \\ 1 & 2 & -3 \end{pmatrix} \quad \rightsquigarrow \quad \widetilde{M} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & -9 & 5 & 1 \\ 2 & \boxed{-1} & -4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

  here, the exponential-dual is not an infinitesimal generator.

Some properties

- If $\widehat{M}$ is the exponential-dual of matrix M, then
  - the sum of the elements of any row of $\widehat{M}$ is 0,
  - and the last row of $\widehat{M}$ is composed of 0s.
- Inversion Lemma: $0 \le i, j \le n-1$, we have

$$M_{i,j} = \sum_{k=0}^{i} \widehat{M}_{j,k} - \sum_{k=0}^{i} \widehat{M}_{j+1,k}.$$

- Important result: for any two square matrices A and B with the same dimension, we have

$$\widehat{AB} = \widehat{B}\widehat{A}.$$

Corollary: for any square matrix M and any nonnegative integer k,

$$\widehat{M^k} = \widehat{M}^k.$$

In words, the exponential-dual of the kth power of M is the kth power of M's exponential-dual.

- Let A and B be two square matrices with the same dimension, and α be a scalar. We have

$$\widehat{A+B} = \widehat{A} + \widehat{B}, \qquad \widehat{\alpha A} = \alpha \widehat{A}.$$

- Let $\left(M_k\right)_{k\geq 0}$ a matrix sequence, with $M_k \longrightarrow M$ as $k \longrightarrow \infty$. Then,

$$\widehat{M_k} \longrightarrow \widehat{M} \text{ as } k \longrightarrow \infty.$$

If $\sum_{k\geq 0} M_k = M$, then

$$\widehat{\sum_{k\geq 0} M_k} = \sum_{k\geq 0} \widehat{M_k} = \widehat{M}.$$

Key result

- Let M be an arbitrary square matrix. Using previous properties,

$$\boxed{\mathrm{e}^{\widehat{M}t} = \widetilde{\mathrm{e}^{Mt}}.}$$

  In words, the exponential-dual of the exponential of Mt is the exponential of the **power**-dual of M times t.
- This result explains how the evaluation of the power-dual appears when we work on matrix exponentials.
- In this short presentation with skipped some technical results that are needed to prove previous theorem.

Path

Instead of

$$M \underset{\substack{\uparrow \\ \text{any algorithm}}}{\longrightarrow} e^{Mt},$$

we can try

$$M \underset{\substack{\uparrow \\ \text{(linear)}}}{\longrightarrow} \widehat{M} \underset{\substack{\uparrow \\ \text{any algorithm}}}{\longrightarrow} e^{\widehat{M}t} = \widetilde{e^{Mt}} \underset{\substack{\uparrow \\ \text{Inversion Lemma} \\ \text{of } \textbf{power-dual} \\ \text{(linear)}}}{\longrightarrow} e^{Mt}$$

Other general properties:

- If P is a stochastic matrix, then the first row of $\widetilde{P}$ is $(1\ 0\ 0 \ldots 0\ )$. Recall that $\widetilde{P}$ need not be stochastic.
- If A is an infinitesimal generator, then the first row of $\widehat{A}$ is $(0\ 0\ 0 \ldots 0\ )$. Recall that $\widehat{A}$ need not be an infinitesimal generator.
- Connection with the dual of a Markov process as defined by Sigmund in 1976 and developed in Anderson's book: if A is the infinitesimal generator of Markov process X and if $\widehat{A}$ is also an infinitesimal generator, the associated Markov process X^* is the dual of X.

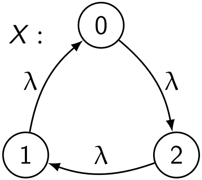
Talk's content

Overview

- This work generalizes the use of duality in the theory of Markov processes as proposed by Sigmund (see Anderson's book). The dual doesn't always exist.
- In previous work we proposed to define what we call here exponential-dual, adopting a purely algebraic approach, when we realized that the stochastic framework was not necessary for the interest we had in the concept. Many proofs needed to be rewritten.
- Here, we decompose explicitly the cases of discrete time and continuous time, and exhibit their connections, making the construction more complete and consistent.

On dual's existence (Markov processes)

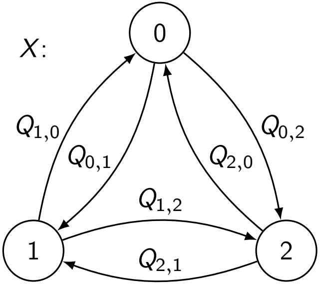
- This concerns the historical origin of this work, in the stochastic processes area.
- For instance, consider an homogeneous "ring" illustrated here for 3 states:



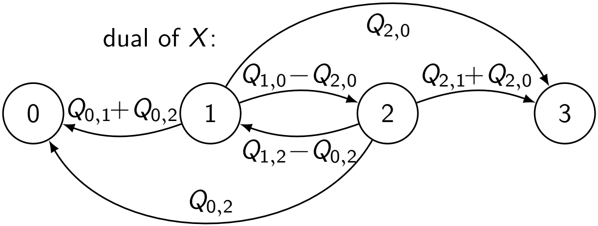
- Whatever the size of the ring and the value of λ, the associated Markov process has no dual (for those familiar with duality in Markov processes, the problem is the monotonicity condition impossible to satisfy here).

Consider now the general Markov process with 3 states:

X:



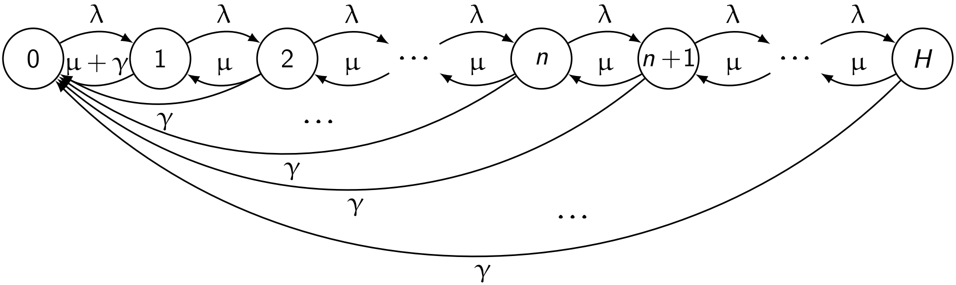
dual of X:



Here, the dual exists iff $Q_{1,0} > Q_{2,0}$ and $Q_{1,2} > Q_{0,2}$.

Zoomming on an example

Consider here the $M/M/1/H$ queue plus "catastrophes" (or failures) as shown in the following Markovian graph:



- Take $H = 4$, to shorten the slides. Denote $\Lambda = \lambda + \mu + \gamma$. We have

$$A = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 \\ \mu+\gamma & -\Lambda & \lambda & 0 & 0 \\ \gamma & \mu & -\Lambda & \lambda & 0 \\ \gamma & 0 & \mu & -\Lambda & \lambda \\ \gamma & 0 & 0 & \mu & -(\mu+\gamma) \end{pmatrix}$$

- If we move to discrete time using Uniformization, the computation of e^{At} translates into that of evaluating U^k where using the notation $p = \lambda/\Lambda$, $q = \mu/\Lambda$, $r = \gamma/\Lambda$, $U = I - A/\Lambda$ is

$$U = \begin{pmatrix} q+r & p & 0 & 0 & 0 \\ q+r & 0 & p & 0 & 0 \\ r & q & 0 & p & 0 \\ r & 0 & q & 0 & p \\ r & 0 & 0 & q & p \end{pmatrix}$$

The powers of U are hard to obtain in closed-form here.

- If we use the exponential-dual (plus Uniformization), we must compute the powers of

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ p & 0 & q & 0 & 0 & r \\ 0 & p & 0 & q & 0 & r \\ 0 & 0 & p & 0 & q & r \\ 0 & 0 & 0 & p & q+r & r \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

  Due to those '1' at the corners, the use of lattice path combinatorics is much easier here than with matrix U of previous slide. That's an example of situation where we can develop the process until its end.
- The infinite case can also be solved in the same way.

The formal solution of the ODEs (Chapman-Kolmogorov equations) is given below, using the notation $\rho = \lambda/\mu$, for the case of the queue starting empty at time 0 (the general case is similar while more complex):

$$p_j(t) = \pi_j - \frac{2\mu\rho^{j/2}}{H+1} \sum_{1 \le h \le H} \left\{ \frac{e^{-\left(\lambda+\mu+\gamma-2\sqrt{\lambda\mu}\cos\left(\frac{h\pi}{H+1}\right)\right)t}}{\lambda+\mu+\gamma-2\sqrt{\lambda\mu}\cos\left(\frac{h\pi}{H+1}\right)} \times \right.$$
$$\left. \times\sqrt{\rho}\sin\left(\frac{h\pi}{H+1}\right)\left[\sin\left(\frac{jh\pi}{H+1}\right) - \sqrt{\rho}\sin\left(\frac{(j+1)h\pi}{H+1}\right)\right]\right\}.$$

In the formula, π_j denotes $p_j(\infty)$ (steady-state), easy to evaluate symbolically.

Thanks for your time.