Symmetry in Multivariate Ideal Interpolation

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Abstract

An interpolation problem is defined by a set of linear forms on the (multivariate) polynomial ring and values to be achieved by an interpolant. For Lagrange interpolation the linear forms consist of evaluations at some nodes, while Hermite interpolation also considers the values of successive derivatives. Both are examples of ideal interpolation in that the kernels of the linear forms intersect into an ideal. For an ideal interpolation problem with symmetry, we address the simultaneous computation of a symmetry adapted basis of the least interpolation space and the symmetry adapted H-basis of the ideal. Beside its manifest presence in the output, symmetry is exploited computationally at all stages of the algorithm. For an ideal invariant, under a group action, defined by a Gröbner basis, the algorithm allows to obtain a symmetry adapted basis of the quotient and of the generators. We shall also note how it applies surprisingly but straightforwardly to compute fundamental invariants and equivariants of a reflection group.

Keywords: Interpolation; Symmetry; Representation Theory; Group Action; H-basis; Macaulay matrix; Vandermonde matrix; Invariants; Equivariants.

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1. Introduction

Preserving and exploiting symmetry in algebraic computations is a challenge that has been addressed within a few topics and, mostly for specific groups of symmetry; For instance interpolation and symmetric group (Krick et al., 2017), cubature (Collowald and Hubert, 2015; Gatermann, 1992), global optimisation (Gatermann and Parrilo, 2004; Riener et al., 2013), equivariant dynamical systems (Gatermann, 2000; Hubert and Labahn, 2013) and solving systems of polynomial equations (Faugere and Svartz, 2013; Gatermann, 1990; Gatermann and Guyard, 1999; Hubert and Labahn, 2012, 2016; Riener and Safey El Din, 2018; Verschelde and Gatermann, 1995). In (Rodriguez Bazan and Hubert, 2019, 2021) we tackled multivariate interpolation, laying down the essential principles to preserve and exploit symmetry. We provided a first algorithm, based on LU-factorisations to compute the basis of an interpolation space of minimal degree. In this article we go further with a new symmetry preserving and exploiting algorithm: its scope is extended. Based on QR-decompositions, it is more amenable to numerical computations. The algorithm computes, degree by degree, a symmetry adapted basis of the least interpolation space. In the case of an ideal interpolation problem the algorithm computes in addition a symmetry adapted H-basis of the associated ideal. In addition to being manifest in the output, symmetry is exploited all along the algorithm to reduce the sizes of the matrices involved, and avoid considerable redundancies.

Multivariate Lagrange, and Hermite, interpolation are examples of the encompassing notion of ideal interpolation, introduced in (Birkhoff, 1979). They are defined by linear forms consisting of evaluation at some nodes, and possibly composed with differential operators without *gaps*. More generally a space of linear forms Λ on the polynomial ring $\mathbb{K}[\mathrm{x}] = \mathbb{K}[x_1, \ldots, x_n]$ is an ideal interpolation scheme if

$$\mathcal{I} = \bigcap_{\lambda\in\Lambda} \ker \lambda = \{p \in \mathbb{K}[\mathrm{x}] : \lambda(p) = 0, \text{ for all } \lambda \text{ in } \Lambda\} \tag{1.1}$$

is an ideal in $\mathbb{K}[\mathrm{x}]$. In the case of Lagrange interpolation, $\mathcal{I}$ is the ideal of the nodes and is thus a radical ideal. If Λ is invariant under the action of a group $\mathcal{G}$, so is $\mathcal{I}$. An interpolation space for Λ is a subspace of the polynomial ring that has a unique interpolant for each instantiated interpolation problem. It identifies with the quotient space $\mathbb{K}[\mathrm{x}]/\mathcal{I}$. Hence a number of operations related to $\mathcal{I}$ can already be performed with a basis of an interpolation space for Λ: decide of membership to $\mathcal{I}$, determine normal forms of polynomials modulo $\mathcal{I}$ and compute matrices of multiplication maps in $\mathbb{K}[\mathrm{x}]/\mathcal{I}$. Yet it has also proved relevant to compute Gröbner bases or H-bases of $\mathcal{I}$.

Initiated in (Möller and Buchberger, 1982), for a set Λ of point evaluations, computing a Gröbner basis of $\mathcal{I}$ found applications in the design of experiments (Pistone and Wynn, 1996; Pistone et al., 2000). As pointed out in Marinari et al. (1991), one can furthermore interpret the FGLM algorithm (Faugere et al., 1993) as an instance of this problem. The alternative approach in (Faugère and Mou, 2017) can be understood similarly. The resulting algorithm then pertains to the Berlekamp-Massey-Sakata algorithm and is related to the multivariate version of Prony's problem to compute Gröbner bases or border bases (Berthomieu et al., 2017; Mourrain, 2017; Sauer, 2017, 2018). The above mentioned algorithms heavily depend on a term order and bases of monomials. These are notoriously ill suited for preserving symmetry. Our ambition in this paper is to showcase how symmetry can be embedded in the representation of both the interpolation space and the representation of the ideal. This is a marker for more canonical representations. Furthermore, we shall show, in a forthcoming paper Hubert and Rodriguez Bazan

(2021), how our symmetry preserving ideal interpolation algorithm applies, directly, to compute the fundamental equivariants and invariants of reflection groups: interpolating on an orbit in general position. This remarkable obervation lead us to develop new algorithms to compute the fundamental equivariants and invariants for all finite groups.

The *least interpolation space*, defined in (De Boor and Ron, 1990), and revisited in (Rodriguez Bazan and Hubert, 2019, 2021) is a canonically defined interpolation space. It serves here as the canonical representation of the quotient of the polynomial algebra by the ideal. It has great properties, even beyond symmetry, that cannot be achieved by a space spanned by monomials. In (Rodriguez Bazan and Hubert, 2019, 2021) we freed the computation of the least interpolation space from its reliance on the monomial basis by introducing *dual bases*. We pursue this approach here for the representation of the ideal by a H-basis (Macaulay, 1916; Möller and Sauer, 2000). Where Gröbner bases single out leading terms with a term order, H-bases work with leading forms and the orthogonality with respect to an inner product on the polynomial ring. When we consider the apolar product, the least interpolation space then reveals itself as the orthogonal complement of the ideal of leading forms. As loosely sketched in (De Boor, 1994), computing a H-basis of the interpolation ideal is achieved with linear algebra in subspaces of homogeneous polynomials of growing degrees. We shall first redefine the concepts at play in an intrinsic manner, a computation centered approach can be found in (Möller and Sauer, 2000; Sauer, 2001), and provide a precise algorithm. We shall then be in a position to incorporate symmetry in a natural way, refining the algorithm to exploit it; A totally original contribution.

Symmetry is preserved and exploited thanks to the block diagonal structure of the matrices at play in the algorithms. This block diagonalisation, with predicted repetitions in the blocks, happens when the underlying maps are discovered to be equivariant and expressed in the related *symmetry adapted bases*. The case of the Vandermonde matrix was settled in (Rodriguez Bazan and Hubert, 2019, 2021). In this paper, we also need the matrix of the Sylvester map, knowned in the monomial basis as the Macaulay matrix. Figuring out the equivariance of this map is one of the original key results of this paper.

The paper is organized as follows. In Section 2 we define ideal interpolation and explain the identification of an interpolation space with the quotient algebra. In Section 3 we review H-bases and discuss how they can be computed in the ideal interpolation setting. In Section 4 we provide an algorithm to compute simultaneously a basis of the least interpolation space and an orthogonal H-basis of the ideal. In Section 5 we exhibit the equivariance of the Sylvester map. The resulting block diagonalisation of its matrix is then applied in Section 6 to obtain an algorithm to compute simultaneously a symmetry adapted basis of the least interpolation space and a symmetry adapted H-basis of the ideal. In Section 7 we apply the above algorithm to compute, from aGröbner basis of an ideal that is invariant, a symmetry adapted basis of an invariant subspace that can be identified as the quotient.

A preliminary version of this article was presented at ISSAC'20 (Rodriguez Bazan and Hubert, 2020). We substantially reorganized, expanded, and made more precise, all the content of this previous paper so as to make the material more amenable and directly usable for readers. The last section is brand new, as well as Section 5.3 that discusses the computation of symmetry adapted bases of spaces of homogeneous polynomials. We added examples and included the case $\mathbb{K} = \mathbb{C}$ *all along the text.*

2. Ideal interpolation

In this section, we review somehow briefly the main notions for multivariate interpolation: interpolation problem and interpolation space. A more expository presentation can be found in (Rodriguez Bazan and Hubert, 2019, 2021). In the case of ideal interpolation we go through the identification of an interpolation space with the quotient algebra. The *least interpolation space* introduced in (De Boor and Ron, 1990), and revisited in (Rodriguez Bazan and Hubert, 2019, 2021), is then seen as the orthogonal complement of the leading form ideal.

$\mathbb{K}$ denotes either $\mathbb{C}$ or $\mathbb{R}$. $\mathbb{K}[\mathrm{x}] = \mathbb{K}[x_1, \ldots, x_n]$ denotes the ring of polynomials in the variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{K}$; $\mathbb{K}[\mathrm{x}]_{\leq d}$ and $\mathbb{K}[\mathrm{x}]_d$ the $\mathbb{K}-$vector spaces of polynomials of degree at most d and the space of homogeneous polynomials of degree d respectively. The *dual* of $\mathbb{K}[\mathrm{x}]$, the set of $\mathbb{K}-$linear forms on $\mathbb{K}[\mathrm{x}]$, is denoted by $\mathbb{K}[\mathrm{x}]^*$. A typical example of a linear form on $\mathbb{K}[\mathrm{x}]$ is the evaluation e_ξ at a point ξ of $\mathbb{K}^n$: $\mathrm{e}_\xi(p) = p(\xi)$.

2.1. Interpolation space

As the most common type, *Lagrange interpolation* starts with a set of points $\xi_1, \ldots, \xi_r$ in $\mathbb{K}^n$ and a set of values $\eta_1, \ldots \eta_r \in \mathbb{K}$, and consists in finding, a polynomial p such that $\mathrm{e}_{\xi_j}(p) = \eta_j$, $1 \leq j \leq r$. This can be generalized to other linear forms other than evaluations, as well as to vectors of value.

Definition 2.1. *An* interpolation problem *is a pair* (Λ, ϕ) *where* Λ *is a finite dimensional linear subspace of* $\mathbb{K}[\mathrm{x}]^*$ *and* $\phi : \Lambda \longrightarrow \mathbb{K}^m$ *is a* $\mathbb{K}$*-linear map. An interpolant, i.e., a solution to the interpolation problem, is a vector of polynomials* $[p_1, \ldots, p_m] \in \mathbb{K}[\mathrm{x}]^m$ *such that* $[\lambda(p_1), \ldots, \lambda(p_m)] = \phi(\lambda)$ for any $\lambda \in \Lambda$.

Other examples of linear forms on $\mathbb{K}[\mathrm{x}]$ that can be considered are given by compositions of evaluation and differentiations

$$\begin{array}{rccl} \lambda: & \mathbb{K}[\mathrm{x}] & \to & \mathbb{K} \\ & p & \mapsto & \sum_{j=1}^r \mathrm{e}_{\xi_j} \circ q_j(\partial)(p), \end{array} \qquad \text{with } \xi_j \in \mathbb{K}^n,\ q_j \in \mathbb{K}[\mathrm{x}] \text{ and } \partial^\alpha = \frac{\partial}{\partial x_1^{\alpha_1}} \cdots \frac{\partial}{\partial x_n^{\alpha_n}}.$$

In the univariate case, we speak of Hermite interpolation when Λ is a union of sets $\{\mathrm{e}_\xi, \mathrm{e}_\xi \circ \partial, \ldots, \mathrm{e}_\xi \circ \partial^d\}$ for some $d \in \mathbb{N}$.

Definition 2.2. *Consider* Λ *a subspace of* $\mathbb{K}[\mathrm{x}]^*$*, An* interpolation space for Λ *is a polynomial subspace* Q *of* $\mathbb{K}[\mathrm{x}]$ *such that there is a unique interpolant for any* $\mathbb{K}$*-linear map* $\phi : \Lambda \longrightarrow \mathbb{K}$.

If $\mathcal{L} = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ is a basis of Λ and $\mathcal{P} = \{p_1, p_2, \ldots, p_r\} \subset \mathbb{K}[\mathrm{x}]$, then $\mathcal{P}$ is a basis for an interpolation space of Λ if and only if the *Vandermonde matrix*

$$\mathrm{W}_{\mathcal{L}}^{\mathcal{P}} := \left[\lambda_i\left(p_j\right)\right]_{\substack{1\leq i\leq r \\ 1\leq j\leq r}} \tag{2.1}$$

is invertible. It is convenient to introduce the undelying linear map.

Definition 2.3. *The Vandermonde map* $w : \mathbb{K}[\mathrm{x}] \to \Lambda^*$ *is defined by that* $w(p)(\lambda) = \lambda(p)$*, for any* $p \in \mathbb{K}[\mathrm{x}]$ *and* $\lambda \in \Lambda$*. We denote by* $w_d : \mathbb{K}[\mathrm{x}]_{\leq d} \to \Lambda^*$ *the restriction of* w *to* $\mathbb{K}[\mathrm{x}]_{\leq d}$.

When Λ is finite dimensional w is a surjective map. Indeed, for any $\phi \in \Lambda^*$ take $p \in \mathbb{K}[\mathrm{x}]$ to be a solution of the interpolation problem (Λ, ϕ). Then $w(p) = \phi$.

Hence $\mathcal{P} = \{p_1, \ldots, p_r\}$ spans an interpolation space for Λ if the restriction of w to $\langle p_1, \ldots, p_r\rangle_{\mathbb{K}}$ is an isomorphism. $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$ is the matrix of this restriction and $r = \dim \Lambda$.

(Λ, ϕ) is an *ideal interpolation problem* if

$$\mathcal{I} = \ker w = \bigcap_{\lambda \in \Lambda} \ker \lambda$$

is an ideal in $\mathbb{K}[\mathrm{x}]$. When for instance $\Lambda = \langle \mathrm{e}_{\xi_1}, \ldots, \mathrm{e}_{\xi_r}\rangle_{\mathbb{K}}$ then $\mathcal{I}$ is the ideal of the points $\{\xi_1, \ldots, \xi_r\} \subset \mathbb{K}^n$.

With $Q = \{q_1, \ldots, q_r\} \subset \mathbb{K}[\mathrm{x}]$, we can identify $\mathbb{K}[\mathrm{x}]/\mathcal{I}$ with $\langle Q\rangle_{\mathbb{K}}$ if $\langle Q\rangle_{\mathbb{K}} \oplus \mathcal{I} = \mathbb{K}[\mathrm{x}]$. With a slight shortcut, we say that Q is then a basis for $\mathbb{K}[\mathrm{x}]/\mathcal{I}$.

Proposition 2.4. *If* $\mathcal{I} = \cap_{\lambda\in\Lambda} \ker \lambda$ *is an ideal in* $\mathbb{K}[\mathrm{x}]$, $Q = \{q_1, \ldots, q_r\} \subset \mathbb{K}[\mathrm{x}]$ *spans an interpolation space for* Λ *iff it is a basis for the quotient* $\mathbb{K}[\mathrm{x}]/\mathcal{I}$.

Proof. If $Q = \{q_1, \ldots, q_r\}$ is a basis of $\mathbb{K}[\mathrm{x}]/\mathcal{I}$ then for any $p \in \mathbb{K}[\mathrm{x}]$ there is a $q \in \langle q_1, \ldots, q_r\rangle_{\mathbb{K}}$ such that $p \equiv q \bmod \mathcal{I}$. Hence $\lambda(p) = \lambda(q)$ for any $\lambda \in \Lambda$ and thus $\langle Q\rangle_{\mathbb{K}}$ is an interpolation space for Λ. Conversely if $\langle q_1, \ldots, q_r\rangle_{\mathbb{K}}$ is an interpolation space for Λ then $\{q_1, \ldots, q_r\}$ are linearly independent modulo $\mathcal{I}$ and therefore a basis for $\mathbb{K}[\mathrm{x}]/\mathcal{I}$. Indeed if $q = a_1q_1 + \ldots + a_rq_r \in \mathcal{I}$ then any interpolation problem has multiple solutions in $\langle Q\rangle_{\mathbb{K}}$, i.e, if p is the solution of (Λ, ϕ) so is $p + q$, contradicting the interpolation uniqueness on $\langle Q\rangle_{\mathbb{K}}$. □

Hence $\dim \mathbb{K}[\mathrm{x}]/\mathcal{I} = r = \dim \Lambda$. Furthermore, for $p \in \mathbb{K}[\mathrm{x}]$ we can find its natural projection on $\mathbb{K}[\mathrm{x}]/\mathcal{I}$ by taking the unique $q \in \langle Q\rangle_{\mathbb{K}}$ that satisfies $\lambda(q) = \lambda(p)$ for all $\lambda \in \Lambda$. From a computational point of view, q is obtained by solving the Vandermonde system. If $Q = \{q_1, \ldots, q_r\}$ and $\mathcal{L} = \{\lambda_1, \ldots, \lambda_r\}$ is a basis of Λ then

$$q = a_1q_1 + \ldots + a_rq_r \ \text{ where } \ \mathrm{a} := \begin{bmatrix} a_1 \\ \vdots \\ a_r \end{bmatrix} = \left(\mathrm{W}_{\mathcal{L}}^{Q}\right)^{-1} \begin{bmatrix} \lambda_1(p) \\ \vdots \\ \lambda_r(p) \end{bmatrix} \tag{2.2}$$

Interpreting $\mathbb{Q}$ as a row vector we shall write $q = Q \cdot \mathrm{a}$

Similarly, the matrix of the multiplication map

$$\begin{array}{rccc} \mathfrak{m}_p : & \mathbb{K}[\mathrm{x}]/\mathcal{I} & \to & \mathbb{K}[\mathrm{x}]/\mathcal{I}, \ , \\ & [q] & \mapsto & [pq] \end{array}$$

in the basis Q, is obtained as $[\mathfrak{m}_p]_Q = \left(\mathrm{W}_{\mathcal{L}}^{Q}\right)^{-1} \mathrm{W}_{\mathcal{L}\circ\mathfrak{m}_p}^{Q}$ where $\mathcal{L} \circ \mathfrak{m}_p = \{\lambda_1 \circ \mathfrak{m}_p, \ldots, \lambda_r \circ \mathfrak{m}_p\}$.

2.2. Least interpolation

$\mathbb{K}[\mathrm{x}]^*$ can be identified with the ring of formal power series $\mathbb{K}[[\partial]] = \mathbb{K}[[\partial_1, \ldots, \partial_r]]$, with the understanding that $\partial^\beta(x^\alpha) = \alpha!$ or 0 according to whether $\alpha = \beta$ or not. Concomitantly $\mathbb{K}[\mathrm{x}]$ is equipped with the apolar product that is defined, for $p = \sum_\alpha p_\alpha x^\alpha$ and $q = \sum_\alpha q_\alpha x^\alpha$, by $\langle p, q\rangle := \overline{p}(\partial)q = \sum_\alpha \alpha! \overline{p}_\alpha q_\alpha \in \mathbb{K}$. Hence $\langle x^\beta, x^\alpha\rangle = \alpha!$ or 0 according to whether $\alpha = \beta$ or not. We can thus identify $\mathbb{K}[\mathrm{x}]$ to a subring of $\mathbb{K}[[\partial]]$ and, for $p \in \mathbb{K}[\mathrm{x}]$ we shall write $p(\partial)$

when we consider the polynomial as an element of $\mathbb{K}[[\partial]]$. If $\mathcal{P}$ is a (homogeneous) basis of $\mathbb{K}[\mathrm{x}]$ we denote $\mathcal{P}^\dagger$ its dual with respect to this scalar product. For $\lambda \in \mathbb{K}[\mathrm{x}]^*$ we can write $\lambda = \sum_{p\in\mathcal{P}} \lambda(p)\, \bar{p}^\dagger(\partial)$.

The least term $\lambda_\downarrow \in \mathbb{K}[\mathrm{x}]$ of a power series $\lambda \in \mathbb{K}[[\partial]]$ is the unique homogeneous polynomial for which $\lambda - \lambda_\downarrow(\partial)$ vanishes to highest possible order at the origin. Given a linear space of linear forms Λ, we define the *least interpolation space* $\Lambda_\downarrow$ as the linear span of all $\lambda_\downarrow$ with $\lambda \in \Lambda$.

Proposition 2.5. *(Atkinson and Han, 2012, Section 2.1) Consider $p, q \in \mathbb{K}[\mathrm{x}]$ and $a : \mathbb{K}^n \to \mathbb{K}^n$ a linear map. Then $\langle p, q \circ a\rangle = \langle p \circ \bar{a}^t, q\rangle$.*

Corollary 2.6. *Assume $\Lambda_\downarrow = \langle q_1, \ldots, q_r\rangle_{\mathbb{K}} \subset \mathbb{K}[x]$, with $q_1, \ldots, q_r$ homogeneous. For an invertible linear map $a : \mathbb{K}^n \to \mathbb{K}^n$, and $\lambda \in \Lambda$, define $\tilde{\lambda} \in \mathbb{K}[\mathrm{x}]^*$ by $\tilde{\lambda}(p) = \lambda(p \circ a^t)$ and $\tilde{\Lambda} = \{\tilde{\lambda} | \lambda \in \Lambda\}$. Then $\tilde{\Lambda}_\downarrow = \langle q_1 \circ a, \ldots, q_r \circ a\rangle_{\mathbb{K}} \subset \mathbb{K}[x]$.*

Hereafter we denote by p^0 the leading homogeneous form of p, i.e., the unique homogeneous polynomial such that $\deg\left(p - p^0\right) < \deg(p)$. Given a set of polynomials $\mathcal{P}$ we denote $\mathcal{P}^0 = \left\{p^0 \mid p \in \mathcal{P}\right\}$.

The following result was already presented by De Boor and Ron (1992, Theorem 4.8).

Proposition 2.7. *If $\cap_{\lambda\in\Lambda} \ker \lambda$ is an ideal in $\mathbb{K}[\mathrm{x}]$ then $\mathbb{K}[\mathrm{x}] = \Lambda_\downarrow \overset{\perp}{\oplus} \mathcal{I}^0$.*

Proof. We shall prove below that if Q be an interpolation space of minimal degree (Rodriguez Bazan and Hubert, 2019, 2021) for Λ then $Q + \mathcal{I}^0 = \mathbb{K}[\mathrm{x}]$. Then the result follows from the fact that: $\lambda(p) = 0 \Rightarrow \langle \lambda_\downarrow, p^0\rangle = 0$.

We proceed by induction on the degree, i.e, we assume that any polynomial p in $\mathbb{K}[\mathrm{x}]_{\leq d}$ can be written as $p = q + l$ where $q \in Q$ and $l \in \mathcal{I}^0$. Note that the hypothesis holds trivially when d is equal to zero.
Now let $p \in \mathbb{K}[\mathrm{x}]_{\leq d+1}$. Since $\mathbb{K}[\mathrm{x}] = Q \oplus \mathcal{I}$ there exist $q \in Q$ and $l \in \mathcal{I}$ such that $p = q + l$. Since Q is of minimal degree, q, and thus l, are in $\mathbb{K}[\mathrm{x}]_{\leq d+1}$. Writing $l = l^0 + l_1$ he have $p = q + l^0 + l_1$ with $l_1 \in \mathbb{K}[\mathrm{x}]_{\leq d}$. By induction $l_1 = q_1 + l_1^0$ with $q_1 \in Q$ and $l_1^0 \in \mathcal{I}^0$ and therefore $p = q + q_1 + l^0 + l_1^0 \in Q + \mathcal{I}^0$. ◻

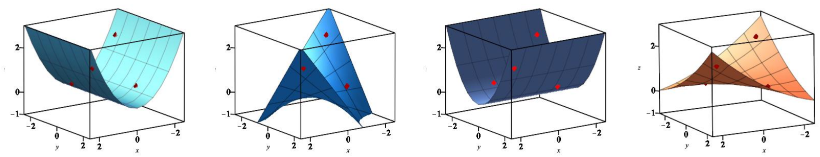
Example 2.8. *With $n = 2$, consider the nodes $\Xi = \{[a, b], [-a, -b], [c, d], [-c, -d]\} \subset \mathbb{R}^2$ and $\Lambda = \{\mathrm{e}_\xi,\ \xi \in \Xi\}$. If one looks for an interpolation space (of minimal degree) spanned by monomials one can choose*

- $\langle 1, x, y, x^2\rangle$ *when* $a\,d - b\,c \neq 0$ *and* $a^2 \neq c^2$
- $\langle 1, x, y, xy\rangle$ *when* $a\,d - b\,c \neq 0$ *and* $ab \neq cd$
- $\langle 1, x, y, y^2\rangle$ *when* $a\,d - b\,c \neq 0$ *and* $b^2 \neq d^2$

On the other hand, the least interpolation space is

$$\langle 1,\ x,\ y,\ (a^2-c^2)\, x^2 + 2(ab-cd)\, xy + (b^2-d^2)\, y^2\rangle \text{ when } a\,d - b\,c \neq 0 \text{ and } (a^2 \neq c^2 \text{ or } ab \neq bc \text{ or } b^2 \neq d^2).$$

One observes the continuous transition in the least interpolation space between the cases. Furthermore the least interpolation space is invariant under a central symmetry, as is Ξ. To make this more apparent we show the different interpolants obtained when we choose ϕ s.t. $\phi(\mathrm{e}_\xi) = \phi(\mathrm{e}_{-\xi})$.



Monomial bases for a quotient $\mathbb{K}[x]/\mathcal{I}$ are natural when we have a Gröbner basis of the ideal $\mathcal{I}$. The natural presentations for an ideal that is the complement of the least interpolation space on the other hand are H-bases.

3. H-bases

On one hand *Gröbner bases* (Buchberger, 1976; Cox et al., 2015) are the most established and versatile representations of ideals. They are defined after a term order is fixed and one then focuses on leading terms of polynomials and the initial ideal of $\mathcal{I}$. The basis of choice for $\mathbb{K}[x]/\mathcal{I}$ then consists of the monomials that do not belong to the initial ideal. We saw in previous section that such a basis was also a basis for an interpolation space for Λ when $\mathcal{I} = \bigcap_{\lambda\in\Lambda} \ker\lambda$ On the other hand a canonical interpolation space for a given Λ was introduced in (De Boor and Ron, 1992), and revisited in (Rodriguez Bazan and Hubert, 2019, 2021). When Λ defines an ideal $\mathcal{I} \in \mathbb{K}[x]$, the natural representation for a complement of this *least interpolation space* is given by a *H-basis*. These were introduced by Macaulay (1916), but for a different inner product on $\mathbb{K}[x]$. To define H-bases one focuses on the leading homogeneous forms instead of the leading terms. The use of H-basis in interpolation has been further studied in (Möller and Sauer, 2000; Sauer, 2001). In this section we review the definitions and present the sketch of an algorithm to compute the H-basis of $\mathcal{I} = \bigcap_{\lambda\in\Lambda} \ker\lambda$.

Definition 3.1. *A finite set $\mathcal{H} := \{h_1, \dots, h_m\} \subset \mathbb{K}[x]$ is a H-basis of the ideal $\mathcal{I} := \langle h_1, \dots, h_m\rangle$ if, for all $p \in \mathcal{I}$, there are $g_1, \dots g_m$ such that*

$$p = \sum_{i=1}^{m} h_i g_i \text{ and } \deg(h_i) + \deg(g_i) \leq \deg(p), i = 1, \dots, m.$$

Theorem 3.2. *(Möller and Sauer, 2000) Let $\mathcal{H} := \{h_1, \dots, h_m\}$ and $\mathcal{I} := \langle\mathcal{H}\rangle$. Then the following conditions are equivalent:*

1. *$\mathcal{H}$ is a H-basis of $\mathcal{I}$.*
2. *$\mathcal{I}^0 = \left\langle h_1^0, \dots, h_m^0\right\rangle$.*

Any ideal has a finite H-basis since Hilbert Basis Theorem ensures that $\mathcal{I}^0$ has a finite basis.

We shall now introduce the concepts of minimal, orthogonal and reduced H-basis. Our definitions somewhat differ from (Möller and Sauer, 2000) as we dissociate them from the computational aspect. The notion of orthogonality is considered here w.r.t the apolar product. Other choices of inner product can be pertinent, in particular when considering positive characteristic, but the choice of the apolar product is crucial to build upon the least interpolation space and later exploit symmetry. We need to introduce first the following vector spaces of homogeneous polynomials.

Definition 3.3. *Given a row vector* $\mathrm{h} = [h_1, \ldots, h_m] \in \mathbb{K}[\mathrm{x}]^m$ *of homogeneous polynomials and a degree d, we define the Sylvester map to be the linear map*

$$\begin{array}{rccc} \psi_{d,\mathrm{h}}: & \mathbb{K}[\mathrm{x}]_{d-d_1} \times \ldots \times \mathbb{K}[\mathrm{x}]_{d-d_m} & \rightarrow & \mathbb{K}[\mathrm{x}]_d \\ & \mathrm{f} = [f_1, \ldots, f_m]^t & \rightarrow & \sum_{i=1}^{m} f_i\, h_i = \mathrm{h} \cdot \mathrm{f} \end{array}$$

where $d_1, \ldots, d_m$ *are the respective degrees of* $h_1, \ldots, h_m$*. If* $\mathcal{H} = \langle h_1, \ldots, h_n \rangle_{\mathbb{K}}$ *then* $\Psi_d(\mathcal{H})$ *shall denote the image of* $\psi_{d,\mathrm{h}}$*, i.e.,*

$$\Psi_d(\mathcal{H}) = \left\{ \sum_{i=1}^{m} f_i\, h_i \;\middle|\; f_i \in \mathbb{K}[\mathrm{x}]_{d-\deg(h_i)} \right\} \subset \mathbb{K}[\mathrm{x}]_d.$$

We denote by $\mathrm{M}_{\mathcal{M}_d, \mathcal{P}_d}(\mathcal{H})$ the matrix of $\psi_{d,\mathrm{h}}$ in the bases $\mathcal{M}_d$ and $\mathcal{P}_d$ of $\mathbb{K}[\mathrm{x}]_{d-d_1} \times \ldots \times \mathbb{K}[\mathrm{x}]_{d-d_m}$ and $\mathbb{K}[\mathrm{x}]_d$ respectively. It is referred to as the Macaulay matrix when monomial bases are used.

If $\mathcal{H}$ is a set of polynomials, we shall use the notation $\mathcal{H}_d^0$ for the set of the degree d elements of $\mathcal{H}^0$. In other words $\mathcal{H}_d^0 = \mathcal{H}^0 \cap \mathbb{K}[\mathrm{x}]_d$.

Definition 3.4. *We say that a H-basis* $\mathcal{H}$ *is minimal if, for any* $d \in \mathbb{N}$*,* $\mathcal{H}_d^0$ *is linearly independent and*

$$\Psi_d\left(\mathcal{I}_{d-1}^0\right) \oplus \left\langle \mathcal{H}_d^0 \right\rangle_{\mathbb{K}} = \mathcal{I}_d^0. \tag{3.1}$$

When minimal, $\mathcal{H}$ *is said to be orthogonal if* $\left\langle \mathcal{H}_d^0 \right\rangle_{\mathbb{K}}$ *is the orthogonal complement of* $\Psi_d\left(\mathcal{I}_{d-1}^0\right)$ *in* $\mathcal{I}_d^0$*.*

Note that if h_i and h_j are two elements of an orthogonal H-basis with $\deg h_i > \deg h_j$ we have

$$\left\langle h_i^0, p h_j^0 \right\rangle = 0 \text{ for all } \; p \in \mathbb{K}[\mathrm{x}]_{\deg h_i - \deg h_j}.$$

The leading homogeneous forms of an orthogonal H-basis are unique up to linear transformations. Indeed, if $\mathcal{H}$ and $\mathcal{F}$ are two orthogonal H-basis then $\mathcal{H}_d^0$ and $\mathcal{F}_d^0$ are both bases of the orthogonal complement of $\Psi_d\left(\mathcal{I}_{d-1}^0\right)$ in $\mathcal{I}_d^0$. There thus exists a non singular matrix Q_d such that, interpreting for a moment $\mathcal{H}$ and $\tilde{\mathcal{H}}$ as row vectors, $\mathcal{H}_d^0 = \mathcal{F}_d^0 \cdot \mathrm{Q}_d$, and therefore

$$\mathcal{H}^0 = \mathcal{F}^0 \cdot \mathrm{Q}, \text{ with } \mathrm{Q} = \operatorname{diag}\left(\mathrm{Q}_d \,|\, \mathcal{H}_d^0 \neq \emptyset\right). \tag{3.2}$$

The concept of *reduced* H-basis allows to extend the uniqueness up to a linear transformation from the leading form to the complete H-basis.

Definition 3.5. *Let* $\mathcal{H} = \{h_1, \ldots, h_m\}$ *be an orthogonal H-basis of an ideal* $\mathcal{I}$*. The* reduced *H-basis associated to* $\mathcal{H}$ *is defined by*

$$\widetilde{\mathcal{H}} = \left\{ h_1^0 - \widetilde{h_1^0}, \ldots, h_m^0 - \widetilde{h_m^0} \right\} \tag{3.3}$$

where, for $h \in \mathbb{K}[x]$*,* $\tilde{h}$ *is the projection of h on the orthogonal complement of* $\mathcal{I}^0$ *parallel to* $\mathcal{I}$*.*

(Möller and Sauer, 2000, Lemma 6.2) show how $\tilde{h}$ can be computed given $\mathcal{H}$.

Proposition 3.6. *Let $\mathcal{H} = \{h_1, \ldots, h_m\}$ and $\mathcal{F} = \{f_1, \ldots, f_m\}$ be two reduced H−bases of $\mathcal{I}$. There exists a non singular matrix $\mathrm{Q} \in \mathbb{K}^{m \times m}$ such that*

$$\left[h_1, \ldots, h_m\right] = \left[f_1, \ldots, f_m\right] \cdot \mathrm{Q}.$$

Proof. By (3.2) there is a $m \times m$ matrix Q such that $\mathcal{H}^0 = \mathcal{F}^0 \cdot \mathrm{Q}$, thus $h_i^0 = \sum_{j=1}^m q_{ij} f_j^0$ for $1 \leq i \leq m$. By the uniqueness of $\widetilde{p}$ for any $p \in \mathbb{K}[\mathrm{x}]$ follows that $\widetilde{h_i^0} = \sum_{j=1}^m q_{ij} \widetilde{f_j^0}$ and therefore $h_i^0 - \widetilde{h_i^0} = \sum_{j=1}^m q_{ij} \left(f_i^0 - \widetilde{f_i^0}\right)$. Last equality implies that $\widetilde{\mathcal{H}} = \widetilde{\mathcal{F}} \cdot \mathrm{Q}$. □

In next section we present an algorithm to compute concomitantly the least interpolation space for a set of linear forms Λ and, when the kernels of the linear forms intersect into an ideal, a H-basis of this ideal. To ease in this algorithm, we conclude this section with a schematic algorithm proposed by De Boor (1994) to compute a H-basis until a given degree e. It is based on the asumption that we have access to a basis of $\mathcal{I}_d := \mathcal{I} \cap \mathbb{K}[\mathrm{x}]_{\leq d}$ for any d. When $\mathcal{I} = \cap_{\lambda \in \Lambda} \ker \lambda$, a basis of $\mathcal{I}_d$ can be computed for any d with linear algebra since $\mathcal{I}_d = \ker w_d$, where w_d is the restriction of the Vandermonde operator described in Section 2.1.

Algorithm 1 (De Boor, 1994) H-basis construction

Input: - a degree e;
- basis for $\mathcal{I}_d$ for $1 \leq d \leq e$.

Output : - a H-basis for $\mathcal{I}$ until degree e

1: $\mathcal{H}_0 \leftarrow \{\}$;
2: **for** $d = 0$ `to` e **do**
3: $\quad \mathcal{B}_d^0 \leftarrow$ a basis for the complement of $\Psi_d(\mathcal{H}^0)$ in $\mathcal{I}_d^0$;
4: $\quad \mathcal{B}_d \leftarrow$ projection of $\mathcal{B}_d^0$ in $\mathcal{I}_d$;
5: $\quad \mathcal{H} \leftarrow \mathcal{H} \bigcup \mathcal{B}_d$;
6: **return** $\mathcal{H}$;

The correctness of Algorithm 1 is shown by induction. Assume that $\mathcal{H}$ consists of the polynomials in a H-basis of $\mathcal{I}$ up to degree $d - 1$. Consider $p \in \mathcal{I}$ with $\deg(p) = d$. By Step 4 in Algorithm 1 we have

$$p^0 = \sum_{h_i \in \mathcal{H}} h_i^0 g_i + \sum_{b_i^0 \in \mathcal{B}_d^0} a_i b_i^0 \tag{3.4}$$

with $g_i \in \mathbb{K}[\mathrm{x}]_{d-\deg(h_i)}$ and $a_i \in \mathbb{K}$. From (3.4) we have that $p \in \mathcal{I}$ and $\sum_{h_i \in \mathcal{H}} h_i g_i + \sum_{b_i \in \mathcal{B}_d} a_i b_i \in \mathcal{I}$ have the same leading form. Thus

$$p - \sum_{h_i \in \mathcal{H}} h_i g_i - \sum_{b_i \in \mathcal{B}_d} a_i b_i \in \mathcal{I}_{d-1}$$

therefore using the induction hypothesis we get that

$$p = \sum_{h_i \in \mathcal{H}} h_i g_i + \sum_{b_i \in \mathcal{B}_d} a_i b_i + \sum_{h_i \in \mathcal{H}} h_i q_i$$

with $q_i \in \mathbb{K}[\mathrm{x}]_{\leq d-1-\deg(h_i)}$ and therefore $\mathcal{H} \cup \mathcal{B}_d$ consists of the polynomials in a H-basis of $\mathcal{I}$ up to degree d.

When the ideal is given by a set of generators it is also possible to compute a H-basis with linear algebra if you know a bound on the degree of the syzygies of the generators. A numerical approach, using singular value decomposition, was introduced in (Javanbakht and Sauer, 2019). Alternatively an extension of Buchberger's algorithm is presented in (Möller and Sauer, 2000). It relies, at each step, on the computation of a basis for the module of syzygies of a set of homogeneous polynomials.

4. Simultaneous computation of H-bases and least interpolation spaces

In this section we present an algorithm to compute both a basis of $\Lambda_\downarrow$ and an orthogonal H-basis $\mathcal{H}$ of the ideal $\mathcal{I} = \cap_{\lambda \in \Lambda} \ker \lambda$. We proceed degree by degree. At each iteration of the algorithm we compute a basis of $\Lambda_\downarrow \cap \mathbb{K}[\mathrm{x}]_d$ and the set $\mathcal{H}_d^0 = \mathcal{H}^0 \cap \mathbb{K}[\mathrm{x}]_d$. Recall from Corollary 2.7, Theorem 3.2, and Definition 3.4 that

$$\mathbb{K}[\mathrm{x}] = \Lambda_\downarrow \overset{\perp}{\oplus} \mathcal{I}^0, \quad \mathcal{I}^0 = \langle \mathcal{H}^0 \rangle, \quad \text{and} \quad \mathcal{I}_d^0 = \Psi_d\left(\mathcal{I}_{d-1}^0\right) \overset{\perp}{\oplus} \left\langle \mathcal{H}_d^0 \right\rangle_{\mathbb{K}}.$$

$\mathcal{I}$ is the kernel of the Vandermonde operator while $\Lambda_\downarrow$ can be inferred from a rank revealing form of the Vandermonde matrix. With orthogonality prevailing in the objects we compute it is natural that the QR-decomposition plays a central role in our algorithm.

For a matrix $\mathrm{M} \in \mathbb{K}^{m \times n}$, the QR-decomposition is $\mathrm{M} = \mathrm{QR}$ where, when $\mathbb{K} = \mathbb{R}$, Q is a $m \times m$ orthogonal matrix and R is a $m \times n$ upper triangular matrix. When $\mathbb{K} = \mathbb{C}$ the matrix Q is a unitary matrix and R is a complex upper triangular matrix. If r is the rank of M the first r columns of Q form an orthogonal basis of the column space of M and the remaining $m - r$ columns of Q form an orthogonal basis of the kernel of M^T (Golub and Van Loan, 1996, Theorem 5.2.1). We thus often denote the QR-decomposition of a matrix M as

$$[\mathrm{Q}_1 \mid \mathrm{Q}_2] \cdot \begin{bmatrix} \mathrm{R} \\ 0 \end{bmatrix} = \mathrm{M}$$

where $\mathrm{Q}_1 \in \mathbb{K}^{m \times r}, \mathrm{Q}_2 \in \mathbb{K}^{m \times (m-r)}$ and $\mathrm{R} \in \mathbb{K}^{r \times n}$. Algorithms to compute the QR-decomposition can be found for instance in (Golub and Van Loan, 1996).

In the Lagrange interpolation case, Fassino and Möller (2016) already used the QR-decomposition to propose a variant of the BM-algorithm (Möller and Buchberger, 1982) so as to compute a monomial basis of an interpolation space, the complement of the initial ideal for a chosen term order. They furthermore study the gain in numerical stability for perturbed data. We shall use QR-decomposition to further obtain a homogeneous basis of $\Lambda_\downarrow$ and an orthogonal H-basis of the ideal. Since this construction is done degree by degree, we shall introduce a number of subspaces indexed by this degree d.

Definition 4.1. *Given a space of linear forms Λ, we denote by $\Lambda_{\geq d}$ the subspace of Λ given by*

$$\Lambda_{\geq d} = \{\lambda \in \Lambda \mid \lambda_\downarrow \in \mathbb{K}[\mathrm{x}]_{\geqslant d}\} \cup \{0\}.$$

Hereafter we organize the elements of the bases of $\mathbb{K}[\mathrm{x}]$, Λ, or their subspaces, as row vectors. In particular $\mathcal{P} = \cup_{d \in \mathbb{N}} \mathcal{P}_d$ and $\mathcal{P}^\dagger = \cup_{d \in \mathbb{N}} \mathcal{P}_d^\dagger$ are dual homogeneous bases for $\mathbb{K}[\mathrm{x}]$ according to the apolar product; $\mathcal{P}_d$ and $\mathcal{P}_d^\dagger$ are dual bases of $\mathbb{K}[\mathrm{x}]_d$.

A basis $\mathcal{L}_{\geq d}$ of $\Lambda_{\geq d}$ can be computed inductively thanks to the following observation.

Proposition 4.2. *Assume $\mathcal{L}_{\geq d}$ is a basis of $\Lambda_{\geq d}$. Consider the QR-decomposition*

$$\mathrm{W}_{\mathcal{L}_{\geq d}}^{\mathcal{P}_d} = [\mathrm{Q}_1 \mid \mathrm{Q}_2] \cdot \begin{bmatrix} \mathrm{R}_d \\ 0 \end{bmatrix}$$

and the related change of basis $[\mathcal{L}_d \mid \mathcal{L}_{\geq d+1}] = \mathcal{L}_{\geq d} \cdot [\mathrm{Q}_1 \mid \mathrm{Q}_2]$.

Then

- *$\mathcal{L}_{\geq d+1}$ is a basis of $\Lambda_{\geq d+1}$;*
- *$\mathrm{R}_d = \mathrm{W}_{\mathcal{L}_d}^{\mathcal{P}_d}$ has full row rank;*
- *The components of $\mathcal{L}_{d\downarrow} = \mathcal{P}_d^{\dagger} \cdot \mathrm{R}_d^T$ form a basis of $\Lambda_{\downarrow} \cap \mathbb{K}[\mathrm{x}]_d$.*

We shall furthermore denote by $\mathcal{L}_{\leq d} = \bigcup_{i=0}^{d} \mathcal{L}_i$ the thus constructed basis of a complement of $\Lambda_{\geq d+1}$ in Λ.

Proof. It mostly follows from the fact that a change of basis $\mathcal{L}' = \mathcal{L}\mathrm{Q}$ of Λ implies that $\mathrm{W}_{\mathcal{L}'}^{\mathcal{P}} = \mathrm{Q}^T \mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$. In the present case $\mathrm{Q} = [\mathrm{Q}_1 \mid \mathrm{Q}_2]$ is orthogonal and hence $\mathrm{Q}^T = \mathrm{Q}^{-1}$. The last point simply follows from the fact that, for $\lambda \in \Lambda$, $\lambda = \sum_{p \in \mathcal{P}} \lambda(p) p^{\dagger}(\partial)$. Hence if $\mathrm{M} = \mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$ then the j-th component of $\mathcal{L}$ is $\sum_i m_{ji} p^{\dagger}(\partial)$. □

This construction gives us a basis of $\Lambda_{\downarrow} \cap \mathbb{K}[\mathrm{x}]_d$ in addition to a basis of $\Lambda_{\geq d+1}$ to pursue the computation at the next degree. Before going there, we need to compute a basis $\mathcal{H}_d^0$ for the complement of $\Psi_d(\mathcal{H}_{<d}^0)$ in $\mathcal{I}_d^0$. For that we shall use an additional QR-decomposition as explained in Proposition 4.5, after two preparatory lemmas.

Lemma 4.3. *Let $d \geq 0$ and let $\mathcal{P}_d$ be a basis of $\mathbb{K}[\mathrm{x}]_d$ then:*

$$\mathcal{I}_d^0 = \left\{ \sum_{i=1}^{|\mathcal{P}_d|} a_i p_i \mid (a_1, \ldots, a_{|\mathcal{P}_d|})^t \in \ker\left(\mathrm{W}_{\mathcal{L}_d}^{\mathcal{P}_d}\right) \text{ and } p_i \in \mathcal{P}_d \right\}.$$

Proof. Recall that $\mathcal{I}$ is the kernel of the Vandermonde operator, and $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$ is the matrix of this latter. The Vandermonde submatrix $\mathrm{W}_{\mathcal{L}_{\leq d}}^{\mathcal{P}_{\leq d}}$ can be written as follows

$$\mathrm{W}_{\mathcal{L}_{\leq d}}^{\mathcal{P}_{\leq d}} = \mathrm{W}_{[\mathcal{L}_{\leq d-1} \mid \mathcal{L}_d]}^{\mathcal{P}_{\leq d}} = \begin{pmatrix} \mathrm{W}_{\mathcal{L}_{\leq d-1}}^{\mathcal{P}_{\leq d-1}} & \mathrm{W}_{\mathcal{L}_{\leq d-1}}^{\mathcal{P}_d} \\ 0 & \mathrm{W}_{\mathcal{L}_d}^{\mathcal{P}_d} \end{pmatrix} \tag{4.1}$$

where $\mathrm{W}_{\mathcal{L}_{\leq d-1}}^{\mathcal{P}_{\leq d-1}}$ has full row rank.

Assume first that p is a polynomial in $\mathcal{I}_d^0$. Then there is $q \in \mathcal{I}$ of degree d such that $q^0 = p$. Let $\mathrm{q} = \begin{pmatrix} \mathrm{q}_{\leq d-1} \\ \mathrm{q}_d \end{pmatrix}$ and $\mathrm{p} = \mathrm{q}_d$ be the vectors of coefficients of q and p respectively in the basis $\mathcal{P}$. As $q \in \mathcal{I}_d$ we have that

$$\mathrm{W}_{\mathcal{L}_{\leq d}}^{\mathcal{P}_{\leq d}} \cdot \mathrm{q} = \begin{pmatrix} \mathrm{W}_{\mathcal{L}_{\leq d-1}}^{\mathcal{P}_{\leq (d-1)}} \cdot \mathrm{q}_{\leq d-1} + \mathrm{W}_{\mathcal{L}_{\leq d-1}}^{\mathcal{P}_d} \cdot \mathrm{q}_d \\ \mathrm{W}_{\mathcal{L}_d}^{\mathcal{P}_d} \cdot \mathrm{q}_d \end{pmatrix} = 0$$

and therefore $\mathrm{p} = \mathrm{q}_d$ is in kernel of $\mathrm{W}_{\mathcal{L}_d}^{\mathcal{P}_d}$. Now let v be a vector in the kernel of $\mathrm{W}_{\mathcal{L}_d}^{\mathcal{P}_d}$. A vector u such that $\begin{pmatrix} \mathrm{u} \\ \mathrm{v} \end{pmatrix} \in \mathbb{K}^{\binom{n+d}{d}}$ and $\mathrm{W}_{\mathcal{L}_{\leq d}}^{\mathcal{P}_{\leq d}} \cdot \begin{pmatrix} \mathrm{u} \\ \mathrm{v} \end{pmatrix} = 0$ can be found as the solution of the following equation.

$$\mathrm{W}_{\mathcal{L}_{\leq d-1}}^{\mathcal{P}_{\leq (d-1)}} \mathrm{u} = \mathrm{W}_{\mathcal{L}_d}^{\mathcal{P}_d} \mathrm{v} - \mathrm{W}_{\mathcal{L}_{\leq d-1}}^{\mathcal{P}_d} \mathrm{v}. \tag{4.2}$$

As $\mathrm{W}_{\mathcal{L}_{\leq d-1}}^{\mathcal{P}_{\leq d-1}}$ has full row rank, Equation 4.2 always has a solution. Then $\mathcal{P}_{\leq d} \cdot \begin{pmatrix} \mathrm{u} \\ \mathrm{v} \end{pmatrix} \in \mathcal{I}$ and therefore $\mathcal{P}_d \cdot \mathrm{v} \in \mathcal{I}_d^0$. □

Lemma 4.4. *Consider the row vector* q *of coefficients of a polynomial q of $\mathbb{K}[\mathrm{x}]_d$ in the basis $\mathcal{P}_d$. The polynomial q is in the orthogonal complement of $\Psi_d(\mathcal{H})$ in $\mathbb{K}[\mathrm{x}]_d$ if and only if the row vector* q *is in the left kernel of* $\mathrm{M}_{\mathcal{M}_d,\mathcal{P}_d^\dagger}(\mathcal{H})$.

Proof. The columns of $\mathrm{M}_{\mathcal{M}_d,\mathcal{P}_d^\dagger}$ are the vectors of coefficients, in the basis $\mathcal{P}_d^\dagger$, of polynomials that span $\Psi_d(\mathcal{H})$. The membership of q in the left kernel of $\mathrm{M}_{\mathcal{M}_d,\mathcal{P}_d^\dagger}(\mathcal{H})$ translates as the apolar product of q with these vectors to be zero. And conversely. □

Proposition 4.5. *Consider the QR-decomposition*

$$\left[\ \left(\mathrm{W}_{\mathcal{L}_d}^{\mathcal{P}_d}\right)^t \quad \mathrm{M}_{\mathcal{M}_d,\mathcal{P}_d^\dagger}(\mathcal{H})\ \right] = [\mathrm{Q}_1 \mid \mathrm{Q}_2] \cdot \begin{bmatrix} \mathrm{R} \\ 0 \end{bmatrix}$$

The components of the row vector $\mathcal{P}_d \cdot \mathrm{Q}_2$ span the orthogonal complement of $\Psi_d(\mathcal{H})$ in $\mathcal{I}_d^0$.

Proof. The columns in Q_2 span $\ker \mathrm{W}_{\mathcal{L}_d}^{\mathcal{P}_d} \cap \ker\left(\mathrm{M}_{\mathcal{M}_d,\mathcal{P}_d^\dagger}\right)^t$. The result thus follows from Lemmas 4.3 and 4.4. □

We are now able to show the correctness and termination of Algorithm 2.

Correctness. In the spirit of Algorithm 1, Algorithm 2 proceeds degree by degree. At the iteration for degree d we first compute a basis for $\Lambda_{\geq d+1}$ by splitting $\mathcal{L}_{\geq d}$ into $\mathcal{L}_{\geq d+1}$ and $\mathcal{L}_d$. As explained in Proposition 4.2, this is obtained through the QR-decomposition of $\mathrm{W}_{\mathcal{L}_{\geq d}}^{\mathcal{P}_d}$. From this decomposition we also obtain a basis $\mathcal{Q}_d$ for $\Lambda_\downarrow \cap \mathbb{K}[\mathrm{x}]_d$ as well as $\mathrm{W}_{\mathcal{L}_d}^{\mathcal{P}_d}$. We then go after $\mathcal{H}_d^0$, which spans the orthogonal complement of $\Psi_d(\mathcal{H}_{\leq d-1}^0)$ in $\mathcal{I}_d^0$. The elements of $\mathcal{H}_d^0$ are computed via intersection of $\ker \mathrm{W}_{\mathcal{L}_d}^{\mathcal{P}_d}$ and $\ker\left(\mathrm{M}_{\mathcal{M}_d,\mathcal{P}_d^\dagger}\right)^t$ as showed in Proposition 4.5. Algorithm 2 stops when we reach a degree δ such that $\mathcal{L}_{\geq\delta}$ is empty. Notice that for $d \geq \delta$ the matrix $\mathrm{W}_{\mathcal{L}_d}^{\mathcal{P}_d}$ is an empty matrix and therefore its kernel is the full space $\mathbb{K}[\mathrm{x}]_d$. Then as a consequence of Lemma 4.3, for all $d > \delta$ we have that $\Psi_d(\mathcal{I}_{d-1}^0) = \mathcal{I}_d^0$ hence $\langle \mathcal{H}_d^0 \rangle$ is an empty set. The latter implies that when the algorithm stops we have computed the full H-basis $\mathcal{H}^0$ for $\mathcal{I}^0$.

We then obtain a H-basis of $\mathcal{I}$ from the H-basis $\mathcal{H}^0$ of $\mathcal{I}^0$ as follows. For each $h \in \mathcal{H}$, let $\tilde{h}$ be its unique interpolant[1] in $\Lambda_\downarrow$, *i.e.,* the unique polynomial in $\Lambda_\downarrow$ such that $\lambda(\tilde{h}) = \lambda(h)$ for all $\lambda \in \Lambda$. Then the H-basis is $\mathcal{H} = \{h - \tilde{h} \mid h \in \mathcal{H}^0\}$.

---

[1] If $h = \sum_{p\in\mathcal{P}_d} a_p\, p \in \mathcal{H}^0$ then $\tilde{h} = \sum_{q\in\mathcal{Q}_{\leq d}} b_p\, q \in \mathcal{H}^0$ where $\mathrm{b} = \left(\mathrm{W}_{\mathcal{L}_{\leq d}}^{\mathcal{Q}_{\leq d}}\right)^{-1} \mathrm{a}$.

Algorithm 2

Input: - $\mathcal{L}$ a basis of Λ ($r = |\mathcal{L}| = \dim(\Lambda)$)
- $\mathcal{P}$ a homogeneous basis of $\mathbb{K}[\mathrm{x}]_{\leq r}$
- $\mathcal{P}^{\dagger}$ the dual basis of $\mathcal{P}$ w.r.t the apolar product.

Output: - $\mathcal{H}$ a reduced H-basis for $\mathcal{I} := \cap_{\lambda \in \Lambda} \ker \lambda$
- $\mathcal{Q}$ a basis of the least interpolation space of Λ.

$\mathcal{H}^0 \leftarrow \{\}, \ \mathcal{Q} \leftarrow \{\}$
$d \leftarrow 0$
$\mathcal{L}_{\leq 0} \leftarrow \{\}, \ \mathcal{L}_{\geq 0} \leftarrow \mathcal{L}$
while $\mathcal{L}_{\geq d} \neq \{\}$ **do**
  $\mathrm{Q} \cdot \begin{bmatrix} \mathrm{R}_d \\ 0 \end{bmatrix} = \mathrm{W}_{\mathcal{L}_{\geq d}}^{\mathcal{P}_d}$ ▷ QR-decomposition of $\mathrm{W}_{\mathcal{L}_{\geq d}}^{\mathcal{P}_d}$
  $\mathcal{Q} \leftarrow \mathcal{Q} \bigcup \mathcal{P}_d^{\dagger} \cdot \mathrm{R}_d^T$
  $[\mathcal{L}_d \mid \mathcal{L}_{\geq d+1}] \leftarrow \mathcal{L}_{\geq d} \cdot \mathrm{Q}^T$ ▷ Note that $\mathrm{R}_d = \mathrm{W}_{\mathcal{L}_d}^{\mathcal{P}_d}$
  $\mathcal{L}_{\leq d+1} \leftarrow \mathcal{L}_{\leq d} \cup \mathcal{L}_d$
  $[\mathrm{Q}_1 \mid \mathrm{Q}_2] \cdot \mathrm{R} = \begin{bmatrix} \mathrm{R}_d^T & \mathrm{M}_{\mathcal{M}_d, \mathcal{P}_d^{\dagger}}(\mathcal{H}) \end{bmatrix}$
  $\mathcal{H}^0 \leftarrow \mathcal{H}^0 \bigcup \mathcal{P}_d \cdot \mathrm{Q}_2$
  $d \leftarrow d + 1$
$\mathcal{H} \leftarrow \left\{ h - \tilde{h} \mid h \in \mathcal{H}^0 \right\}$ ▷ $\tilde{h}$ is the interpolant of h in $\Lambda_{\downarrow}$
return $(\mathcal{H}, \mathcal{Q})$

Termination. Considering $r := \dim(\Lambda)$ we have that $\mathcal{L}_{\geq r}$ is an empty set, this implies that in the worst case our algorithm stops after r iterations.

Complexity. The most expensive computational step in Algorithms 2 is the computation of the kernel of the matrix $\left[\left(\mathrm{W}_{\mathcal{L}_d}^{\mathcal{P}_d}\right)^T \ \mathrm{M}_{\mathcal{M}_d,\mathcal{P}_d^\dagger}(\mathcal{H})\right]$, with number of columns and rows given by

$$\begin{aligned} \mathrm{row}(d) &= \binom{d+n-1}{n-1} = \tfrac{d^{n-1}}{(n-1)!} + O\left(d^{n-1}\right) \\ \mathrm{col}(d) &= \sum_{i=1}^{|\mathcal{H}|} \binom{d-d_i+n-1}{n-1} + |\mathcal{L}_d| = \tfrac{|\mathcal{H}|d^{n-1}}{(n-1)!} + O\left(d^{n-1}\right) \end{aligned} \tag{4.3}$$

where $d_1, \ldots, d_{|\mathcal{H}|}$ are the degrees of the elements of the computed H-basis until degree d. Then the computational complexity of Algorithm 2 relies on the method used for the kernel computation of $\left[\left(\mathrm{W}_{\mathcal{L}_d}^{\mathcal{P}_d}\right)^T \ \mathrm{M}_{\mathcal{M}_d,\mathcal{P}_d^\dagger}(\mathcal{H})\right]$, which in our case is the QR-decomposition.

Algorithm 2 gives a framework for the simultaneous computation of the least interpolation space and a H-basis, but there is room for improvement. The structure of the Macaulay matrix might be taken into account to alleviate the linear algebra operations as for instance in (Berthomieu et al., 2017). We can also consider different variants of Algorithm 2. In Proposition 4.6 we show that orthogonal bases for $\mathbb{K}[\mathrm{x}]_d \cap \Lambda_\downarrow$ and $\mathcal{I}_d^0$ can be simultaneously computed by applying QR-decomposition in the Vandermonde matrix $(\mathrm{W}_{\mathcal{L}_{\geq d}}^{\mathcal{P}_d})^T$. Therefore we can split Step 9 in two steps. First do a QR-decomposition $(\mathrm{W}_{\mathcal{L}_{\geq d}}^{\mathcal{P}_d})^T$ to obtain orthogonal bases of $\mathbb{K}[\mathrm{x}]_d \cap \Lambda_\downarrow$ and $\mathcal{I}_d^0$. With these in hand, we obtain the elements of $\mathcal{H}_d^0$ as a basis of the complement in $\mathcal{I}_d^0$ of $\Psi_d(\mathcal{H}_{\leq d-1}^0)$, which is given by the column space of $\mathrm{M}_{\mathcal{M}_d,\mathcal{P}_d^\dagger}(\mathcal{H})$.

Proposition 4.6. *Let* $[\mathrm{Q}_1 \mid \mathrm{Q}_2] \cdot \begin{bmatrix} \mathrm{R}_d \\ 0 \end{bmatrix} = \left(\mathrm{W}_{\mathcal{L}_{\geq d}}^{\mathcal{P}_d}\right)^T$ *be a QR-decomposition of* $\left(\mathrm{W}_{\mathcal{L}_{\geq d}}^{\mathcal{P}_d}\right)^T$. *Let* r_d *be the rank of* $\left(\mathrm{W}_{\mathcal{L}_{\geq d}}^{\mathcal{P}_d}\right)^T$. *Let* $\{q_1 \ldots q_{r_d}\}$ *and* $\{q_{r_d+1} \ldots q_m\}$ *be the columns of* Q_1 *and* Q_2 *respectively. Then the following holds:*

1. $\mathcal{Q}_d = \left\{\mathcal{P}_d^\dagger \cdot q_1, \ldots, \mathcal{P}_d^\dagger \cdot q_{r_d}\right\}$ *is a basis of* $\mathbb{K}[\mathrm{x}]_d \bigcap \Lambda_\downarrow$.
2. $\mathcal{N}_d = \{\mathcal{P}_d \cdot q_{r_d+1}, \ldots, \mathcal{P}_d \cdot q_m\}$ *is a basis of* $\mathcal{I}_d^0$.
3. *If* $q \in \mathcal{Q}_d$ *and* $p \in \mathcal{N}_d$ *then* $\langle p, q\rangle = 0$, *i.e.,* $\mathbb{K}[\mathrm{x}] = (\Lambda_\downarrow \cap \mathbb{K}[\mathrm{x}]_d) \stackrel{\perp}{\oplus} \mathcal{I}_d^0$.

In the case where $\mathcal{P}$ *is orthonormal with respect to the apolar product, i.e.* $\mathcal{P} = \mathcal{P}^\dagger$, *then* $\mathcal{Q}_d$ *and* $\mathcal{N}_d$ *are also orthonormal bases.*

Proof. 1. Let e be the smallest integer such that $\mathcal{L}_{\geq e+1} = \{\}$ and let $\mathcal{L}_{\leq e} = \bigcup_{d \leq e} \mathcal{L}_d$ be a basis of Λ. Then the matrix $\mathrm{W}_{\mathcal{L}_{\leq e}}^{\mathcal{P}_{\leq e}}$ is block upper triangular with diagonal blocks of full row rank. Consider $\{a_1, \ldots a_r\} \in \mathbb{K}^{|\mathcal{P}_{\leq e}|}$ the rows of $\mathrm{W}_{\mathcal{L}_{\leq e}}^{\mathcal{P}_{\leq e}}$. By (Rodriguez Bazan and Hubert, 2019, 2021, Proposition 2.3) the least interpolation space $\Lambda_\downarrow$ admits as basis the lowest degree forms in the polynomials $\left\{\mathcal{P}_{\leq e}^\dagger \cdot a_1^t, \ldots, \mathcal{P}_{\leq e}^\dagger \cdot a_r^t\right\}$. Hence the degree d part of the least interpolation space, *i.e.,* $\Lambda_\downarrow \cap \mathbb{K}[\mathrm{x}]_d$, admits the basis $\mathcal{Q}_d = \bigcup_{d=1}^{e} \left\{\mathcal{P}_d^\dagger \cdot q_1^t, \ldots, \mathcal{P}_d^\dagger \cdot q_{r_d}^t\right\}$ where $\{q_1, \ldots, q_{r_d}\}$ is a basis of the row space of $\left(\mathrm{W}_{\mathcal{L}_d}^{\mathcal{P}_d}\right)$.

2. This is a direct consequence of Lemma 4.3 and the fact that the columns of Q_2 form a basis of the kernel of $\mathrm{W}_{\mathcal{L}_{\geq d}}^{\mathcal{P}_d}$.

3. Let now $p = \sum_{p_i \in \mathcal{P}_d} a_i p_i \in \mathcal{N}_d$ and $q = \sum_{q_i \in \mathcal{P}_d^\dagger} b_i q_i \in Q_d$ where $[a_1, \ldots, a_{|\mathcal{P}_d|}]^t$ is a column of Q_2 and $[b_1, \ldots, b_{|\mathcal{P}_d|}]^t$ is a column of Q_1. Then,

$$\langle p, q \rangle = \left\langle \sum_{p_i \in \mathcal{P}_d} a_i p_i, \sum_{q_i \in \mathcal{P}_d^\dagger} b_i q_i \right\rangle = \sum_{i=1}^{|\mathcal{P}_d|} a_i b_i = 0.$$

□

5. Symmetry reduction

Symmetry shall be described by the linear action of a finite group $\mathcal{G}$ on $\mathbb{K}^n$, where recall $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. It is no loss of generality to actually consider orthogonal (when $\mathbb{K} = \mathbb{R}$) or unitary (when $\mathbb{K} = \mathbb{C}$) representations[2]. It is thus given by a representation $\vartheta : \mathcal{G} \to \mathrm{O}_n(\mathbb{R})$ or $\vartheta : \mathcal{G} \to \mathrm{U}_n(\mathbb{C})$. It induces a representation ρ of $\mathcal{G}$ on $\mathbb{K}[x]$ given by

$$\rho(g)p(x) = p(\vartheta(g^{-1})x). \tag{5.1}$$

It also induces the dual representation ρ^* of ρ on the space of linear forms :

$$\rho^*(g)\lambda(p) = \lambda(\rho(g^{-1})p) = \lambda(p \circ \vartheta(g)), \; p \in \mathbb{K}[x] \text{ and } \lambda \in \mathbb{K}[x]^*. \tag{5.2}$$

From Proposition 2.5 we see that the apolar product is $\rho(\mathcal{G})$-invariant: $\langle \rho(g)p, \rho(g)q \rangle = \langle p, q \rangle$ for all $p, q \in \mathbb{K}[x]$ and $g \in \mathcal{G}$. Hence, the orthogonal complement of a $\rho(\mathcal{G})$-invariant subspace of $\mathbb{K}[x]$ is itself $\rho(\mathcal{G})$-invariant.

Our asumption is that the space Λ of linear forms is invariant: $\rho^*(g)\lambda \in \Lambda$ for all $g \in \mathcal{G}$ and $\lambda \in \Lambda$. Hence the restriction of ρ^* to Λ is a linear representation of $\mathcal{G}$ in Λ. In determining the least interpolation space, and a H-basis for $\mathcal{I} = \cap_{\lambda \in \Lambda} \ker \lambda$, when this latter is an ideal of $\mathbb{K}[x]$, Algorithm 2 relied on forming the matrices of the Vandermonde and Sylvester maps. The key to exploiting symmetry is to exhibit the equivariance of these maps since their matrices, in symmetry adapted bases, are then block diagonal. We shall then expand on how to compute economically the symmetry adapted bases for the auxiliary representations for which the Sylvester maps are equivariant.

5.1. Equivariance of the Vandermonde and Sylvester maps

We recall the equivariance of the Vandermonde map and exhibit the equivariance of the Sylvester map. This latter requires the introduction of auxiliary representations on the spaces $\mathbb{K}[x]_{d_1} \times \ldots \times \mathbb{K}[x]_{d_m}$.

Proposition 5.1. *Consider γ the restriction of ρ^* to the invariant subspace Λ of $\mathbb{K}[x]^*$, and γ^* its dual representation on Λ^*. The Vandermonde map $w : \mathbb{K}[x] \to \Lambda^*$ is $\rho-\gamma^*$ equivariant, meaning that $w \circ \rho(g) = \gamma^*(g) \circ w$ for any $g \in \mathcal{G}$.*

[2] From any inner product on $\mathbb{K}^n$ we obtain an $\mathcal{G}$-invariant inner product by summing over the group. In a basis of $\mathbb{K}^n$ that is orthonormal for this invariant inner product the matrices of the representation are orthogonal (resp. unitary).

Proof. We want to show that $w(\rho(g)(p)) = \gamma^*(g)\,(w(p))$. The left hand side applied to any $\lambda \in \Lambda$ is equal to $\lambda\,(\rho(g)(p)) = \left(\rho^*(g^{-1})(\lambda)\right)(p)$. The right handside applied to any $\lambda \in \Lambda$ is equal to $w(p)\left(\gamma(g^{-1})(\lambda)\right) = \left(\gamma(g^{-1})(\lambda)\right)(p)$. The conclusion follows since $\gamma(g^{-1})(\lambda) = \rho^*(g^{-1})(\lambda)$ by definition of γ. □

Consider now a set $\mathcal{H} = \{h_1, \dots, h_m\}$ of homogeneous polynomials of $\mathbb{K}[\mathrm{x}]$. We denote $d_1, \dots, d_m$ their respective degrees and $\mathrm{h} = [h_1, \dots, h_m]$ the row vector of $\mathbb{K}[x]^m$. Associated to h, and a degree d, is the Sylvester map introduced in Section 3

$$\begin{array}{rccc} \psi_{d,\mathrm{h}}: & \mathbb{K}[\mathrm{x}]_{d-d_1} \times \ldots \times \mathbb{K}[\mathrm{x}]_{d-d_m} & \to & \mathbb{K}[\mathrm{x}]_d \\ & \mathrm{f} = [f_1, \dots, f_m]^t & \to & \mathrm{h} \cdot \mathrm{f}. \end{array} \tag{5.3}$$

We assume that $\mathcal{H}$ forms a basis of an invariant subspace of $\mathbb{K}[\mathrm{x}]$ and we call θ the restriction of the representation ρ to this subspace, while $\Theta : \mathcal{G} \to \mathrm{GL}_m(\mathbb{K})$ is its matrix representation in the basis $\mathcal{H}$: $\Theta(g) = [\theta(g)]_{\mathcal{H}}$. Then

$$[\rho(g)(h_1), \dots, \rho(g)(h_\ell)] = \mathrm{h} \circ \vartheta(g^{-1}) = \mathrm{h} \cdot \Theta(g).$$

Proposition 5.2. *Consider* $\mathrm{h} = [h_1, \dots, h_m] \in \mathbb{K}[x]_{d_1} \times \ldots \times \mathbb{K}[x]_{d_l}$ *and assume that* $\mathrm{h} \circ \vartheta(g^{-1}) = \mathrm{h} \cdot \Theta(g)$*, for all* $g \in G$*. For any* $d \in \mathbb{N}$*, the map* $\psi_{d,\mathrm{h}}$ *is* $\tau - \rho$ *equivariant for the representation* τ *on* $\mathbb{K}[\mathrm{x}]_{d-d_1} \times \ldots \times \mathbb{K}[\mathrm{x}]_{d-d_m}$ *defined by* $\tau(g)(\mathrm{f}) = \Theta(g) \cdot \mathrm{f} \circ \vartheta(g^{-1})$.

Proof. $(\rho(g) \circ \psi_{d,\mathrm{h}})(\mathrm{f}) = \rho(g)(\mathrm{h} \cdot \mathrm{f}) = \mathrm{h} \circ \vartheta(g^{-1}) \cdot \mathrm{f} \circ \vartheta(g^{-1}) = \mathrm{h} \cdot \Theta(g) \cdot \mathrm{f} \circ \vartheta(g^{-1}) = (\psi_{d,\mathrm{h}} \circ \tau(g))\,(\mathrm{f})$. □

5.2. Symmetry adapted bases and equivariance

For any representation $\mathfrak{r} : \mathcal{G} \to \mathrm{GL}_n(V)$ of a group $\mathcal{G}$ on a $\mathbb{K}$-vector space V, a *symmetry adapted basis* $\mathcal{P}$ of V is characterized by the fact that the matrix of $\mathfrak{r}(g)$ in $\mathcal{P}$ is

$$[\mathfrak{r}(g)]_{\mathcal{P}} = \mathrm{diag}\left(\mathfrak{r}^{(1)}(g) \otimes \mathrm{I}_{m_1},\ \ldots,\ \mathfrak{r}^{(\mathfrak{n})}(g) \otimes \mathrm{I}_{m_\mathfrak{n}}\right).$$

where $\mathfrak{r}^{(\ell)} : \mathcal{G} \to \mathrm{GL}_{n_\ell}(\mathbb{K})$, for $1 \le \ell \le \mathfrak{n}$ are the matrices for the irreducible representations of $\mathcal{G}$ over $\mathbb{K}$ and m_ℓ is the multiplicity of $\mathfrak{r}^{(\ell)}$ in $\mathfrak{r}$.

A symmetry adapted basis of $\mathcal{P}$ thus splits into $\mathcal{P} = \bigcup_{\ell=1}^{\mathfrak{n}} \mathcal{P}^{(\ell)}$ where $\mathcal{P}^{(\ell)} = \bigcup_{k=1}^{n_\ell} \mathcal{P}^{(\ell,k)}$ spans the isotypic component $V^{(\ell)}$ associated to $\mathfrak{r}^{(\ell)}$, while $\mathcal{P}^{(\ell,k)}$ has cardinality m_ℓ. The component $\mathcal{P}^{(\ell)}$ of the symmetry adapted basis is nonetheless fully determined by a single of its component and many computations can be performed only on this component.

The construction of a symmetry adapted bases over $\mathbb{C}$ is basically given by (Serre, 1977, Chapter 2, Proposition 8) that we reproduce here for ease of reference.

Proposition 5.3. *Consider the base field to be* $\mathbb{C}$*. For* $1 \le \ell \le \mathfrak{n}$*, the isotypic component* $V^{(\ell)}$ *is the image of the projection*

$$\pi^{(\ell)} = \sum_{g \in \mathcal{G}} \mathit{Trace}(\mathfrak{r}^{(\ell)}(g^{-1}))\ \mathfrak{r}(g).$$

Furthermore the linear maps $\pi_{ij}^{(\ell)} : V \to V$ *defined by*

$$\pi_{ij}^{(\ell)}(v) = \sum_{g\in\mathcal{G}} [\mathfrak{r}^{(\ell)}(g^{-1})]_{ji}\ \mathfrak{r}(g)(v)$$

satisfy the following properties:

1. *For every* $1 \le i \le n_\ell$*, the map* $\pi_{ii}^{(\ell)}$ *is a projection; it is zero on the isotypic components* $V^{(k)}$*,* $k \neq \ell$*. Its image* $V^{(\ell,i)}$ *is contained in* $V^{(\ell)}$ *and*

$$V^{(\ell)} = V^{(\ell,1)} \oplus \cdots \oplus V^{(\ell,n_\ell)} \quad \textit{while} \quad \pi^{(\ell)} = \sum_{i=1}^{n_\ell} \pi_{ii}^{(\ell)}. \tag{5.4}$$

2. *For every* $1 \le i, j \le n_\ell$*, the linear map* $\pi_{ij}^{(\ell)}$ *is zero on the isotypic components* $V^{(k)}$*,* $k \neq \ell$*, as well as on the subspaces* $V^{(\ell,k)}$ *for* $k \neq j$ *; it defines an isomorphism from* $V^{(\ell,j)}$ *to* $V^{(\ell,i)}$*.*
3. *For any* $v \in V$ *and* $1 \le j \le n_\ell$ *consider* $v_i = \pi_{ij}^{(\ell)}(v) \in V^{(\ell,i)}$ *for all* $1 \le i \le n_\ell$*. If nonzero,* $v_1, \dots, v_{n_\ell}$ *are linearly independent and generate an invariant subspace of dimension* n_ℓ*. For each* $g \in \mathcal{G}$*, we have*

$$\mathfrak{r}(g)(v_j) = \sum_{i=1}^{n_\ell} \mathfrak{r}_{ij}^{(\ell)}(g)(v_i) \quad \forall i, j = 1, \dots, n_\ell.$$

Hence, $\mathcal{P}^{(\ell,1)} = \left\{p_1^{(\ell)},\ \dots,\ p_{m_\ell}^{(\ell)}\right\}$ shall be a basis of $\pi_{11}^{(\ell)}(V)$ and then $\mathcal{P}^{(\ell,k)} = \left\{\pi_{k1}^{(\ell)}\left(p_1^{(\ell)}\right),\ \dots,\ \pi_{k1}^{(\ell)}\left(p_{m_\ell}^{(\ell)}\right)\right\}$. Furhtermore, if the irreducible matrix representations $\mathfrak{r}^{(\ell)}$, $1 \le \ell \le \mathfrak{n}$, are chosen unitary, or orthogonal when defined over $\mathbb{R}$, then the resulting symmetry adapted bases is orthonormal.

The situation over $\mathbb{R}$ is slightly more involved. It is discussed in detail in (Rodriguez Bazan and Hubert, 2021) in the context of multivariate interpolation. In this paper, when dealing with $\mathbb{K} = \mathbb{R}$, we shall restrict to the case where all irreducible representations of $\mathcal{G}$ over $\mathbb{R}$ are absolutely irreducible, *i.e.,* remain irreducible over $\mathbb{C}$. This is the case of (real) reflection groups, but not of abelian groups. As was done in (Rodriguez Bazan and Hubert, 2021), all statements can be modified to work for any finite group but this requires some heavier notations and a number of case distinctions that would be detrimental to the readability of the main points of this article.

When necessary we shall spell out the elements of the symmetry adapted basis as $\mathcal{P}^{(\ell,k)} = \{p_{k1}^{(\ell)},\ \dots,\ p_{km_\ell}^{(\ell)}\}$, where $p_{ki}^{(\ell)} = \pi_{kt}^{(\ell)}(p_{ti}^{(\ell)})$ for all $1 \le i \le m_\ell$ and $1 \le k, t \le n_\ell$. It might seem more natural at first to decompose a basis of $V^{(\ell)}$ in into m_ℓ blocks $\mathrm{p}_1^{(\ell)}, \dots, \mathrm{p}_{m_\ell}^{(\ell)}$ whose components $\mathrm{p}_j^{(\ell)} = [p_{1j}^{(\ell)}, \dots, p_{n_\ell j}^{(\ell)}]$ span an invariant subspace with representation $\mathfrak{r}^{(\ell)}$. Yet the true value of symmetry adapted bases is revealed by this consequence of Schur's lemma.

Proposition 5.4. *(Fässler and Stiefel, 1992, Theorem 2.5) Let* ϑ *and* θ *be representations of* $\mathcal{G}$ *on the* $\mathbb{K}-$*vector space* V *and* W *respectively, with symmetry adapted bases* $\mathcal{P}$ *and* $\mathcal{Q}$*. Consider* $\psi : V \longrightarrow W$ *a* $\vartheta - \theta$ *equivariant map, i.e.,* $\psi \circ \vartheta(g) = \theta(g) \circ \psi$ *for all* $g \in \mathcal{G}$*. Then the matrix* Ψ *of* ψ *in the bases* $\mathcal{P}$ *and* $\mathcal{Q}$ *has the following structure*

$$\Psi = \mathrm{diag}\left(\mathrm{I}_{n_\ell} \otimes \Psi_\ell \mid \ell = 1 \dots \mathfrak{n}\right). \tag{5.5}$$

Thanks to this very general property the matrices of the Vandermonde and Sylvester maps will be block diagonal when considered in the appropriate symmetry adapted bases for Λ, $\mathbb{K}[\mathrm{x}]_d$ and $\mathbb{K}[\mathrm{x}]_{d-d_1} \times \dots \times \mathbb{K}[\mathrm{x}]_{d-d_\ell}$.

5.3. Polynomial bases

In this section we discuss the computation of symmetry adapted bases for homogeneous polynomial spaces $\mathbb{K}[\mathrm{x}]_d$ and for the product spaces $\mathbb{K}[\mathrm{x}]_{e_1} \times \ldots \times \mathbb{K}[\mathrm{x}]_{e_m}$ as they are needed to reveal the block diagonal structure of the Vandermonde and Sylvester map.

Like for any over invariant space, one can compute a symmetry adapted basis for $\mathbb{K}[\mathrm{x}]_d$ with linear algebra thanks to the projections introduced in Proposition 5.3. The growth of the dimensions of these vector spaces with d and n requires to have these precomputed. But one can also know them with a finite presentation. This is by exploiting the fact that they are formed by *equivariants*, and equivariants form $\mathbb{K}[x]^{\mathcal{G}}$-modules of finite rank. After making this fact explicit, we shall look into how to assemble the symmetry adapted bases for product spaces $\mathbb{K}[\mathrm{x}]_{e_1} \times \ldots \times \mathbb{K}[\mathrm{x}]_{e_m}$.

Spaces of homogeneous polynomials. For a row vector $\mathrm{q} = [q_1, \ldots, q_m] \in \mathbb{K}[\mathrm{x}]^m$ of polynomials we write $\rho(g)\mathrm{q} = \mathrm{q} \circ \vartheta(g^{-1})$ for the row vector $[\rho(g)q_1, \ldots, \rho(g)q_m] = \left[q_1 \circ \vartheta(g^{-1}), \ldots, q_m \circ \vartheta(g^{-1})\right] \in \mathbb{K}[\mathrm{x}]^m$. If $\mathfrak{r} : \mathcal{G} \to \mathrm{GL}_m(\mathbb{K})$ is a m-dimensional matrix representation of $\mathcal{G}$, an $\mathfrak{r}$-*equivariant* is a row vector $\mathrm{q} \in \mathbb{K}[\mathrm{x}]^m$ such that $\rho(g)\mathrm{q} = \mathrm{q} \cdot \mathfrak{r}(g)$, where the left handside is a vector-matrix multiplication.

We then observe that for $\mathcal{P} = \bigcup_{\ell=1}^{\mathfrak{n}} \mathcal{P}^{(\ell)}$ a symmetry adapted basis of $\mathbb{K}[\mathrm{x}]_d$ where

$$\mathcal{P}^{(\ell)} = \bigcup_{k=1}^{n_\ell} \mathcal{P}^{(\ell,k)} \text{ and } \mathcal{P}^{(\ell,k)} = \{p_{k1}^{(\ell)}, \ldots, p_{km_\ell}^{(\ell)}\},$$

each row vector $\mathrm{p}_j^{(\ell)} = [p_{1j}^{(\ell)}, \ldots, p_{n_\ell j}^{(\ell)}]$ is a $\mathfrak{r}^{(\ell)}$-equivariant. Conversely, the $\mathbb{K}$-linear bases of the $\mathfrak{r}^{(\ell)}$-equivariants of degree d, for $1 \leq \ell \leq \mathfrak{n}$, can be assembled into symmetry adapted bases of $\mathbb{K}[\mathrm{x}]_d$.

For any representation $\mathfrak{r} : \mathcal{G} \to \mathrm{GL}_m(\mathbb{K})$, the set of $\mathfrak{r}$-equivariants form a module over teh ring of invariants $\mathbb{K}[x]^{\mathcal{G}}$. The algebra of invariants $\mathbb{K}[x]^{\mathcal{G}}$ and the $\mathbb{K}[x]^{\mathcal{G}}$-module of $\mathfrak{r}$-equivariants $\mathbb{K}[x]_{\mathfrak{r}}^{\mathcal{G}}$ are finitely generated (Stanley, 1979). We can thus use generators for the $\mathbb{K}$-algebra $\mathbb{K}[x]^{\mathcal{G}}$, the *fundamental invariants*, and generators for each of the $\mathbb{K}[x]^{\mathcal{G}}$-module $\mathbb{K}[x]_{\mathfrak{r}^{(\ell)}}^{\mathcal{G}}$, the *fundamental equivariants*, to form symmetry adapted bases of higher degrees. Remains the question on how to compute these fundamental invariants and equivariants. This is actually the separate goal of (Hubert and Rodriguez Bazan, 2021) and Section **??** gives a preview for reflection groups. Some more explicit constructions are known for specific group. In particular, for some classical families of reflection groups, that include the symmetric group, one can obtain the fundamental invariants and equivariants by combinatorial means (Ariki et al., 1997; Specht, 1935).

Products of homogeneous polynomial spaces. We shall now discuss the symmetry adapted bases that make the matrices of the Sylvester maps block diagonal. For that let us first break down the Sylvester maps $\psi_{e,\mathrm{h}}$ and the representations τ that arose in Proposition 5.2.

For for $1 \leq \ell \leq \mathfrak{n}$ and $d \in \mathbb{N}$, let us first define on $(\mathbb{K}[\mathrm{x}]_d)^{n_\ell}$ the representation $\tau_d^{(\ell)} = \mathfrak{r}^{(\ell)} \otimes \rho_d$ where ρ_d is the restriction of ρ on $\mathbb{K}[\mathrm{x}]_d$. For $\mathrm{f} = [f_1, \ldots, f_{n_\ell}]^T \in (\mathbb{K}[\mathrm{x}]_d)^{n_\ell}$, $\tau_d^{(\ell)}(g)\mathrm{f} = \mathfrak{r}^{(\ell)} \cdot [\rho_d(g)f_1, \ldots, \rho_d(g)f_{n_\ell}]^T = \mathfrak{r}^{(\ell)} \cdot \mathrm{f} \circ \vartheta(g^{-1})$. We shall call $\mathcal{M}_d^{(\ell)}$ a symmetry adapted basis of $(\mathbb{K}[\mathrm{x}]_d)^{n_\ell}$ for $\tau_d^{(\ell)}$.

Within Algorithm 3 we construct, degree by degree, a set $\mathcal{H} = \mathcal{H}_1 \cup \ldots \cup \mathcal{H}_{e-1}$, where $\mathcal{H}_d$, $1 \leq d \leq e-1$, is a symmetry adapted basis of an invariant subspace of $\mathbb{K}[\mathrm{x}]_{e-1}$. Hence $\mathcal{H}_d$ can be decomposed into $\mathrm{r}^{(\ell)}$-equivariants $\mathrm{h}_{di}^{(\ell)} \in \mathbb{K}[\mathrm{x}]_d^{n_\ell}$, for i ranging between 1 and the multiplicity of $\mathrm{r}^{(\ell)}$ in $\langle \mathcal{H}_d \rangle_{\mathbb{K}}$. The overall subspace $\Psi_e(\mathcal{H})$ of $\mathbb{K}[\mathrm{x}]_e$ that needs to be computed, at each iteration of Algorithm 3, is thus the sum of the images of the Sylvester maps $\psi_{d,\mathrm{h}_{di}^{(\ell)}} : (\mathbb{K}[\mathrm{x}]_{e-d})^{n_\ell} \to \mathbb{K}[\mathrm{x}]_e$. We can hence assemble a symmetry adapted basis to block diagonalize any of the Sylvester maps that arise in Algorithm 3 from the bases $\mathcal{M}_d^{(\ell)}$ of $(\mathbb{K}[\mathrm{x}]_d)^{n_\ell}$ we described above.

Together with the symmetry adapted bases $\mathcal{P}_d$ of $\mathbb{K}[\mathrm{x}]_d$, the bases $\mathcal{M}_d^{(\ell)}$ are thus the building blocks for Algorithm 3. We shall conclude this section by mentioning how their computation can be optimized.

With $\mathcal{P}_d$ a symmetry adapted bases of $\mathbb{K}[\mathrm{x}]_d$, the matrix of ρ_d in $\mathcal{P}_d$ is thus $\mathrm{R}_d = \mathrm{diag}\,(\mathrm{r}^{(1)} \otimes I_{m_1}, \ldots, \mathrm{r}^{(\mathrm{n})} \otimes I_{m_\mathrm{n}})$ and the matrix of $\tau_d^{(\ell)}$ in the product basis of $(\mathbb{K}[\mathrm{x}]_d)^{n_\ell}$ is $\mathrm{r}^{(\ell)} \otimes \mathrm{R}_d$. We hence see that if we have precomputed symmetry adapted bases for the matrix representations $\mathrm{r}^{(\ell)} \otimes \mathrm{r}^{(k)}$ we can assemble the bases $\mathcal{M}_d^{(\ell)}$ from the bases $\mathcal{P}_d$ without resorting to the projections $\pi_{ij}^{(\ell)}$ of Proposition 5.3.

6. Constructing symmetry adapted H-bases

In this section we show that, when the space Λ is invariant under the action of $\mathcal{G}$, one can compute a symmetry adapted basis of the least interpolation space simultaneously to a symmetry adapted H-basis $\mathcal{H}$ of $\mathcal{I} = \cap_{\lambda\in\Lambda} \ker \lambda$, when this latter is an ideal of $\mathbb{K}[\mathrm{x}]$. For that we elaborate on Algorithm 2 in order to exploit the symmetry. This is possible by using symmetry adapted bases. The block diagonal structures of the matrices of the Vandermonde and Sylvester maps then allow to reduce the size of the matrices on which linear algebra operations are performed. Futhermore the output of the algorithm reflects the symmetry, despite any numerical inaccuracy.

Before anything we need to secure the invariance of some subspaces of $\mathbb{K}[\mathrm{x}]$. First, if Λ is $\rho^*(\mathcal{G})$-invariant, $\Lambda_\downarrow$ is $\rho(\mathcal{G})$-invariant by virtue of Corollary 2.6.

Proposition 6.1. *Let $\mathcal{I} = \cap_{\lambda\in\Lambda} \ker \lambda$ and $d \in \mathbb{N}$. If Λ is invariant under the action of the group $\mathcal{G}$, then so are $\mathcal{I}$, $\mathcal{I}^0$, $\mathcal{I}_d^0$, $\Psi_d\left(\mathcal{I}_{<d}^0\right)$.*

Proof. Let $p \in \mathcal{I}$ and $g \in \mathcal{G}$, since Λ is closed under the action of $\mathcal{G}$, $\lambda(\rho(g)(p)) = \rho^*(g)\circ\lambda(p) = 0$ for all $\lambda \in \Lambda$ therefore $\rho(g)(p) \in \mathcal{I}$ implying the invariance of $\mathcal{I}$. Considering d the degree of p we can write p as $p = p^0 + p_1$, with $p_1 \in \mathbb{K}[\mathrm{x}]_{<d}$. Then we have that $\rho(g)p = \rho(g)p^0 + \rho(g)p_1 \in \mathcal{I}$, as ρ is degree preserving then $\rho(g)p^0 \in \mathcal{I}_d^0$ and the invariance of $\mathcal{I}^0$ follows. Now for every $q = \sum_{h_i\in\mathcal{I}_{d-1}^0} q_i h_i \in \Psi_d\left(\mathcal{I}_{d-1}^0\right)$, it holds that $\rho(g)q = \sum_{h_i\in\mathcal{I}_{d-1}^0} \rho(g)q_i\rho(g)h_i \subset \Psi_d\left(\mathcal{I}_{d-1}^0\right)$, thus $\Psi_d\left(\mathcal{I}_{<d}^0\right) = \Psi_d\left(\mathcal{I}_{d-1}^0\right)$ is an invariant subspace. □

In the following proposition orthogonality in $\mathbb{K}[\mathrm{x}]$ needs to be understood w.r.t. the apolar product as it relies on the invariance of this inner product.

Proposition 6.2. *If $\mathcal{H}$ is an orthogonal H-basis of $\mathcal{I} = \cap_{\lambda\in\Lambda} \ker \lambda$, where Λ is $\mathcal{G}$-invariant, then $\langle \mathcal{H}_d^0 \rangle_{\mathbb{K}}$ is invariant.*

Proof. By (3.1) $\langle \mathcal{H}_d^0 \rangle_{\mathbb{K}}$ is the orthogonal complement of $\Psi_d(\mathcal{I}_{d-1}^0)$ in $\mathcal{I}_d^0$. Both $\Psi_d(\mathcal{I}_{d-1}^0)$ and $\mathcal{I}_d^0$ are invairant by previous property. Hence, thanks to the property of the apolar product (Proposition 2.5) so is $\langle \mathcal{H}_d^0 \rangle_{\mathbb{K}}$. □

We are thus in a position to define precisely the object we shall compute, in addition to a symmetry adapted basis of the least interpolation space.

Definition 6.3. *Let $\mathcal{I}$ be a $\mathcal{G}$-invariant ideal of $\mathbb{K}[\mathrm{x}]$. A reduced H-basis $\mathcal{H}$ of $\mathcal{I}$ is symmetry adapted if $\mathcal{H}_d^0$ is a symmetry adapted basis of the orthogonal complement of $\Psi_d(\mathcal{I}_{d-1}^0)$ in $\mathcal{I}_d^0$, for all d that is the degree of a polynomial in $\mathcal{H}$.*

This structure is obtained degree by degree. Assuming that the elements of $\mathcal{H}_{<d}^0$ form a symmetry adapted basis it follows from Proposition 5.1 and 5.2 that the matrices $\mathrm{W}_{\mathcal{L}_{\geq d}}^{\mathcal{P}_d}$ and $\mathrm{M}_{\mathcal{M}_d,\mathcal{P}_d}(\mathcal{H}_{<d}^0)$ have the following structure:

$$\begin{aligned}\mathrm{W}_{\mathcal{L}_{\geq d}}^{\mathcal{P}_d} &= \mathrm{diag}\left(\mathrm{I}_{n_1}\otimes \mathrm{W}_{\mathcal{L}_{\geq d}^{(1,1)}}^{\mathcal{P}_d^{(1,1)}},\ \dots,\ \mathrm{I}_{n_{\mathfrak{n}}}\otimes \mathrm{W}_{\mathcal{L}_{\geq d}^{(\mathfrak{n},1)}}^{\mathcal{P}_d^{(\mathfrak{n},1)}}\right),\\ \mathrm{M}_{\mathcal{M}_d,\mathcal{P}_d}(\mathcal{H}_{<d}^0) &= \mathrm{diag}\left(\mathrm{I}_{n_1}\otimes \mathrm{M}_d^{(1)}(\mathcal{H}_{<d}^0),\ \dots,\ \mathrm{I}_{n_{\mathfrak{n}}}\otimes \mathrm{M}_d^{(\mathfrak{n})}(\mathcal{H}_{<d}^0)\right).\end{aligned} \tag{6.1}$$

Computations over the symmetry blocks lead to the symmetry adapted structure of $\mathcal{H}_d^0$. For any degree d we only need to consider the matrices $\mathrm{W}_{\mathcal{L}_{\geq d}^{(\ell,1)}}^{\mathcal{P}_d^{(\ell,1)}}$ and $\mathrm{M}_d^{(\ell)}(\mathcal{H}_{<d}^0)$, i.e., only one block per irreducible representation.

Once we have in hand $\mathcal{H}^0 = \left[h_{11}^1,\dots,h_{1n_1}^1,\dots,h_{c_{\mathfrak{n}}n_{\mathfrak{n}}}^{\mathfrak{n}}\right]^T$ and a symmetry adapted basis for $\Lambda_\downarrow$, we compute $\mathcal{H}$ by interpolation. Since $\mathcal{H}^0 \in \mathbb{K}[\mathrm{x}]_\vartheta^\theta$, by (Rodriguez Bazan and Hubert, 2019, 2021, Proposition 3.4), its interpolant in $\Lambda_\downarrow$ is also $\vartheta-\theta$ equivariant. Therefore

$$\mathcal{H} = \left[h_{11}^1-\widetilde{h_{11}^1},\dots,h_{1n_1}^1-\widetilde{h_{1n_1}^1},\dots,h_{c_{\mathfrak{n}}n_{\mathfrak{n}}}^{\mathfrak{n}}-\widetilde{h_{c_{\mathfrak{n}}n_{\mathfrak{n}}}^{\mathfrak{n}}}\right]^T \in \mathbb{K}[\mathrm{x}]_\vartheta^\theta.$$

The set $\mathcal{H}$ of its component is thus a symmetry adapted basis.

The correctness and termination of Algorithm 3 follow from the same arguments exposed for Algorithm 2. Note that both the matrices of the Vandermonde and Sylvester maps split in $\sum_{\ell=1}^{\mathfrak{n}} n_\ell$ blocks. Thanks to (Serre, 1977, Proposition 5) we can approximate the dimensions of the blocks by

$$\frac{\dim \mathrm{M}^\ell(\mathcal{H}^0)}{\dim \mathrm{M}(\mathcal{H}^0)} \approx \frac{\dim \mathrm{W}_{\mathcal{L}^{(\ell)}}^{\mathcal{P}^{(\ell)}}}{\dim \mathrm{W}_{\mathcal{L}}^{\mathcal{P}}} \approx \frac{n_\ell^2}{|\mathcal{G}|}.$$

Therefore depending on the size of $\mathcal{G}$ the dimensions of the matrices to deal with in Algorithm 3 can be considerably reduced.

Example 6.4. *The group here is O_h, the subgroup of the orthogonal group $\mathbb{R}^3$ that leaves the cube invariant. It has order 48 and 10 inequivalent irreducible representations whose dimensions are $(1,1,1,1,2,2,3,3,3,3)$. Consider $\Xi\subset\mathbb{R}^3$ the invariant set of 26 points illustrated on Figure 1a. They are grouped in three orbits O_1, O_2, and O_3 of O_h. The points in O_1 are the vertices of a cube with the center at the origin and edge length $\sqrt{3}$. The points in O_2 and in O_3 are the centers of the faces and edges of a cube with center at the origin and edge length 1. Consider $\Lambda = \left\langle\left\{\mathrm{e}_\xi \mid \xi\in\Xi\right\}\right\rangle_{\mathbb{R}}$. Λ is an invariant subspace and therefore $\mathcal{I} = \bigcap_{\lambda\in\Lambda}\ker\lambda$ is an invariant ideal under the action of O_h. Applying Algorithm 3 to Λ we get the following orthogonal symmetry adapted H-basis $\mathcal{H}$ of $\mathcal{I}$.*

Algorithm 3

Input: - $\mathcal{L} = \cup_{\ell=1}^{\mathfrak{n}} \mathcal{L}^{(\ell)}$ a s.a.b of Λ

- $\mathcal{P} = \bigcup_{\ell=1}^{\mathfrak{n}} \cup_{d=1}^{r} \mathcal{P}_d^{(\ell)}$ an orthonormal graded s.a.b of $\mathbb{K}[\mathrm{x}]_{\leq r}$

- $\mathcal{M}_d^{(\ell)}$ the s.a.b for the representations $\tau_d^{(\ell)}$ on $(\mathbb{K}[\mathrm{x}]_d)^{n_\ell}$, $1 \leq \ell \leq \mathfrak{n}$, $1 \leq d \leq r-1$

Output: - $\mathcal{H}$ a symmetry adapted H-basis for $\mathcal{I} := \bigcap_{\lambda \in \Lambda} \ker \lambda$

- $\mathcal{Q} = \bigcup_{\ell=1}^{\mathfrak{n}} \mathcal{Q}^{(\ell)}$ a s.a.b of the least interpolation space for Λ.

$\mathcal{H}^0 \leftarrow \{\}, \ \mathcal{Q} \leftarrow \{\}$

$d \leftarrow 0$

$\mathcal{L}_{\leq 0} \leftarrow \{\}, \ \mathcal{L}_{\geq 0} \leftarrow \mathcal{L}$

while $\mathcal{L}_{\geq d} \neq \{\}$ **do**

  for $\ell = 1$ `to` $\mathfrak{n}$ such that $\mathcal{L}_{\geq d}^{(\ell,1)} \neq \{\}$ **do**

    $\mathrm{Q} \cdot \begin{bmatrix} \mathrm{R}_d^{(\ell)} \\ 0 \end{bmatrix} = \mathrm{W}_{\mathcal{L}_{\geq d}^{(\ell,1)}}^{\mathcal{P}_d^{(\ell,1)}}$ ▹ QR-decomposition of $\mathrm{W}_{\mathcal{L}_{\geq d}^{(\ell,1)}}^{\mathcal{P}_d^{(\ell,1)}}$

    $\left[\mathcal{L}_d^{(\ell,1)} \mid \mathcal{L}_{\geq d+1}^{(\ell,1)}\right] \leftarrow \mathcal{L}_{\geq d}^{(\ell,1)} \cdot \mathrm{Q}^T$

    $\mathcal{L}_{\leq d+1}^{(\ell,1)} \leftarrow \mathcal{L}_{\leq d}^{(\ell,1)} \cup \mathcal{L}_d^{(\ell,1)}$

    $[\mathrm{Q}_1 \mid \mathrm{Q}_2] \cdot \mathrm{R} = \left[\ \left(\mathrm{R}_d^{(\ell)}\right)^T \quad \mathrm{M}_d^{(\ell)}(\mathcal{H}^0) \ \right]$

    $\mathcal{Q}^{(\ell)} \leftarrow \mathcal{Q}^{(\ell)} \cup \left\{\mathcal{P}_d^{(\ell,1)} \cdot \left(\mathrm{R}_d^{(\ell)}\right)^T, \ldots, \mathcal{P}_d^{(\ell,n_\ell)} \cdot \left(\mathrm{R}_d^{(\ell)}\right)^T\right\}$

    $\mathcal{H}_\ell^0 \leftarrow \mathcal{H}_\ell^0 \bigcup \left\{\mathcal{P}_d^{(\ell,1)} \cdot \mathrm{Q}_2, , \ldots, \mathcal{P}_d^{(\ell,n_\ell)} \cdot \mathrm{Q}_2\right\}$

  $d \leftarrow d+1$

for $\ell = 1$ `to` $\mathfrak{n}$ **do**

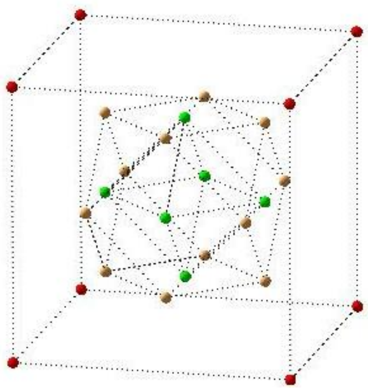
  for all $p \in \mathcal{H}_\ell^0$ **do**

    $\mathcal{H} \leftarrow \mathcal{H} \bigcup \{p - \tilde{p}\}$ ▹ $\tilde{p}$ is the interpolant of p in $\Lambda_\downarrow$

return $(\mathcal{H}, \mathcal{Q})$

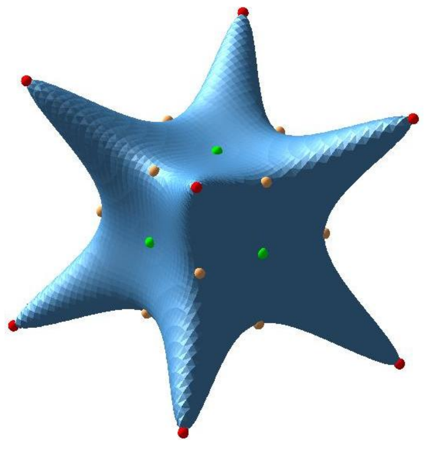
$$
\mathcal{H} = \begin{cases}
\mathcal{H}^{(1)} = \left\{p := -19\left(x^4+y^4+z^4\right) + 18\left(x^2y^2+x^2z^2+y^2z^2\right) + x^2+y^2+z^2+18\right\} \\
\mathcal{H}^{(5)} = \left\{x^4 - 2y^4 + z^4 - x^2 + 2y^2 - z^2,\ \sqrt{3}\left(x^4 - z^4 - x^2 + z^2\right)\right\} \\
\mathcal{H}^{(7)} = \left\{yz\left(y^2-z^2\right), -xz\left(x^2-z^2\right), xy\left(x^2-y^2\right)\right\} \\
\mathcal{H}^{(9)} = \left\{-yz\left(4x^2-3y^2-3z^2+6\right), xz\left(3x^2-4y^2+3z^2-6\right), xy\left(3x^2+3y^2-4z^2-6\right)\right\}
\end{cases}
$$

From the structure of $\mathcal{H}$ it follows that p is the minimal degree invariant polynomial of $\mathcal{I}$. In Figure 1b we show the zero surface of p which is O_h invariant.



	#Nodes	A node per orbit	value
O_1	8	$\xi_2 = (-\sqrt{3}, -\sqrt{3}, -\sqrt{3})$	$\phi(\mathbb{e}_{\xi}) = 0$
O_2	6	$\xi_{10} = (-1, 0, 0)$	$\phi(\mathbb{e}_{\xi}) = 0$
O_3	12	$\xi_{16} = (0, -1, -1)$	$\phi(\mathbb{e}_{\xi}) = 0$

(a) Points in Ξ divided in orbits



(b) Lowest degree invariant algebraic surface through an invariant set of the points Ξ.

Figure 1: Interpolation data and variety of the interpolant p that goes through the points in $O_2 \cup O_3 \cup O_4$.

Example 6.5. *The subgroup of the orthogonal group $\mathbb{R}^3$ that leaves the regular tetrahedron invariant is commonly called T_h. It has order 24 and 5 inequivalent irreducible representations,* *all absolutely irreducible, whose dimensions are* $(1,1,2,3,3)$. *We consider the the following action of* T_h *in* $\mathbb{R}^3$

$$T_h = \left\{\delta^i\sigma^j\alpha^k\beta^\ell \mid 0 \le i \le 1,\ 0 \le j \le 2,\ 0 \le k \le 1,\ 0 \le \ell \le 1\right\}$$

which is defined by the matrices.

$$\delta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \alpha = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Consider $\Xi \subset \mathbb{R}^3$ *the invariant set of 14 points illustrated on Figure 2a. They are grouped in three orbits* O_1 *(violet points),* O_2 *(brown points) and* O_3 *(red points). Consider the space of linear forms* Λ *given by*

$$\Lambda = \left\langle \left\{ \mathrm{e}_\xi \mid \xi \in \Xi \right\} \cup \left\{ \mathrm{e}_\xi \circ D_{\vec{\xi}} \mid \xi \in O_3 \right\} \right\rangle_{\mathbb{R}}$$

The fact that $\mathcal{I} = \cap_{\lambda\in\Lambda}\lambda$ *is an ideal in* $\mathbb{R}[x]$ *can be easily deduce from the fact that*

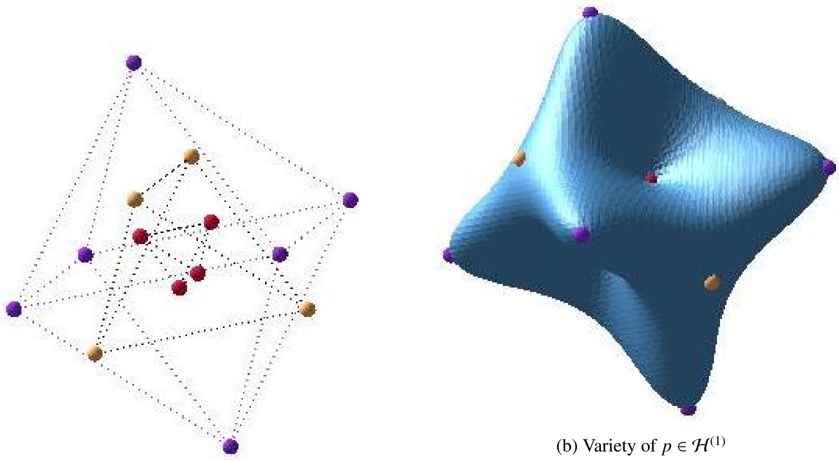
$$\mathrm{e}_\xi \circ D_{\vec{\xi}}(f\cdot g) = f(\xi)\cdot \mathrm{e}_\xi \circ D_{\vec{\xi}}(g) + g(\xi)\cdot \mathrm{e}_\xi \circ D_{\vec{\xi}}(f)$$

so if $\xi \in O_3$ *then for any* $f \in \mathcal{I}$ *and* $g \in \mathbb{R}[x]$, $\mathrm{e}_\xi \circ D_{\vec{\xi}}(f\cdot g) = 0$. *Applying Algorithm 3 to* $\mathcal{I}$*, we get a symmetry adapted H-basis* $\mathcal{H}$

$$\mathcal{H} = \begin{cases} \mathcal{H}^{(1)} = \left\{ \frac{2152}{1875}\left(x^4+y^4+z^4\right) + \frac{25973}{1875}\left(x^2y^2+x^2z^2+y^2z^2\right) + \frac{45000}{25973}xyz - \frac{13750}{1875}\left(x^2+y^2+z^2\right) + 1 \right\} \\ \mathcal{H}^{(3)} = \left\{ (4x^2+4z^2-25)(x-z)(x+z), (4y^2+4z^2-25)(y-z)(y+z) \right\} \\ \mathcal{H}^{(4)} = \begin{cases} 270x^2yz + 90y^3z + 90yz^3 - 8x^3 + 79xy^2 + 79xz^2 - 250yz + 50x, \\ 90x^3z + 270xy^2z + 90xz^3 + 79x^2y - 8y^3 + 79yz^2 - 250xz + 50y, \\ 90x^3y + 90xy^3 + 270xyz^2 + 79x^2z + 79y^2z - 8z^3 - 250xy + 50z \end{cases} \\ \mathcal{H}^{(5)} = \begin{cases} \mathcal{H}^{(5,1)} = \left\{ xy^2 - xz^2, yz\left(y^2-z^2\right) \right\} \\ \mathcal{H}^{(5,2)} = \left\{ -x^2y + yz^2, -xz\left(x^2-z^2\right) \right\} \\ \mathcal{H}^{(5,3)} = \left\{ x^2z - y^2z, xy\left(x^2-y^2\right) \right\} \end{cases} \end{cases}$$

and a symmetry adapted basis Q *of the least interpolation space*

$$Q = \begin{cases} Q^{(1)} = \left\{ 1, x^2+y^2+z^2, xyz, 25973\left(x^4+y^4+z^4\right) - 12912\left(x^2y^2+x^2z^2+y^2z^2\right) \right\} \\ Q^{(3)} = \left\{ 2x^2 - y^2 - z^2, y^2 - z^2 \right\} \\ Q^{(4)} = \begin{cases} Q^{(5,1)} = \left\{ x, yz, x^3, x\left(y^2+z^2\right) \right\} \\ Q^{(5,2)} = \left\{ y, xz, y^3, y\left(x^2+z^2\right) \right\} \\ Q^{(5,3)} = \left\{ z, yx, z^3, z\left(x^2+y^2\right) \right\} \end{cases} \end{cases}$$



(a) Points in Ξ divided in orbits

(b) Variety of $p \in \mathcal{H}^{(1)}$

Figure 2: Interpolation data and variety of the interpolant p that goes through the points in Ξ and with zero directional derivative in O_3.

7. From a Gröbner basis to a symmetry adapted basis of the quotient

Gröbner bases are the most versatile presentation for ideals in $\mathbb{K}[x]$. From the Gröbner basis of an ideal $\mathcal{I}$, with a certain term order, many information on the ideal can be deduced. The dimension and Hilbert polynomial of $\mathcal{I}$, a monomial basis for the quotient $\mathbb{K}[x]/\mathcal{I}$, the multiplication matrices when this latter quotient is finite dimensional. And most importantly, a Gröbner basis can be computed from any set of generators of $\mathcal{I}$. How to preserve symmetry within the computation or in the output has nonetheless remained a challenge, with some successes for specific group actions. The goal of this section is to show, when $\mathcal{I}$ is an invariant ideal, how to apply Algorithm 3 to obtain a symmetry adapted H-basis of the ideal and a symmetry adapted basis of the complement of the space of leading form of $\mathcal{I}$ from any Gröbner bases of $\mathcal{I}$. This goal is similar in spirit to (Faugère and Rahmany, 2009), and more particularly to (Faugere and Svartz, 2013) that deals with diagonal representations of finite abelian groups. Algorithm 3 can indeed be seen as a generalization of the Diagonal-FGLM algorithm proposed therein to any orthogonal representation of any finite group.

If, for some term ordering on $\mathbb{K}[x]$, we have a Groebner basis $\mathcal{F}$, of a zero dimensional ideal $\mathcal{I}$, then the associated normal set $\mathcal{B} = \{b_1, \ldots, b_r\}$ is a set of monomials such that $\mathbb{K}[x] = \mathcal{I} \oplus \langle b_1, \ldots, b_r \rangle$. Furthermore the associated linear forms $\lambda_1, \ldots, \lambda_r$ such that $p \equiv \lambda_1(p)b_1 + \ldots + \lambda_r(p)b_r \mod \mathcal{I}$, for any $p \in \mathbb{K}[x]$, are then computable using the Hironaka division w.r.t. $\mathcal{F}$. We can thus apply Algorithm 2 to $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ to compute a reduced H-basis of $\mathcal{I}$ and a basis of the orthogonal complement $\mathcal{Q}$ of $\mathcal{I}^0$ in $\mathbb{K}[x]$. If $\mathcal{I}$ is $\mathcal{G}$-invariant, so is $\mathcal{Q}$ (Proposition 6.1 and 6.2). The following proposition allows us in this situation to take advantage of symmety in all the intermediate computation and retrieve a symmetry adapted output.

Proposition 7.1. *Assume $q_1, \ldots, q_r \in \mathbb{K}[x]$ is a basis of a direct complement of $\mathcal{I}$ in $\mathbb{K}[x]$. Define $\lambda_1, \ldots, \lambda_r \in \mathbb{K}[x]^*$ by $p \equiv \lambda_1(p)\, q_1 + \ldots + \lambda_r(p) q_r \mod \mathcal{I}$ so that $\mathcal{I} = \bigcap_{i=1}^r \ker \lambda_i$. If $\mathcal{I}$ is $\mathcal{G}$-invariant then the subspace $\Lambda = \langle \lambda_1, \ldots, \lambda_r \rangle_{\mathbb{K}}$ of $\mathbb{K}[x]^*$ is $\mathcal{G}$-invariant.*

Proof. If $\lambda \in \mathbb{K}[x]^*$ is such that $\lambda(p) = 0\ \forall p \in \mathcal{I}$ then $\lambda = \sum_{i=1}^r \lambda(q_i)\lambda_i$. Hence $\Lambda = \mathcal{I}^\perp = \{\lambda \in \mathbb{K}[x]^* \,|\, \lambda(p) = 0\ \forall p \in \mathcal{I}\}$.

For $\lambda \in \Lambda$ and any $p \in \mathcal{I}$ we have $\rho^*(g)(\lambda)(p) = \lambda\left(\rho(g^{-1})p\right) = 0$ since $\rho(g^{-1})p \in \mathcal{I}$. Hence $\rho^*(g)(\lambda) \in \Lambda$. □

Hence when $\mathcal{I}$ is $\mathcal{G}$-invariant, Proposition 7.1 allows us to apply *Algorithm* 3 instead of Algorithm 2 so as to draw a computational advantage from the symmetry. The output will furthermore reflect then this symmetry: the H-basis of $\mathcal{I}$ and of Q are symmetry adapted. This brings some information on the arrangements of the zeros of the ideal in orbits. Indeed, just like the dimension of the quotient $\mathbb{K}[x]/\mathcal{I}$ allows to determine the number of zeros, one can infer the orbit types of the zeros from the dimensions of the isotypic components of $\mathbb{K}[x]/\mathcal{I}$ as explained in (Collowald and Hubert, 2015, Section 8.2).

Example 7.2. *Let us consider the symmetric group $\mathfrak{S}_3$ acting on $\mathbb{R}^3$ by permutaiton of the coordinates. $\mathfrak{S}_3$ has 3 inequivalent absolutely irreducible representation, two of dimension 1 and one of dimension 2. We examine the nonconstant Lotka-Volterra equations (Noonburg, 1989), which appear in the context of neural network modeling. The system is defined by the polynomials:*

$$LV(3) = \begin{cases} 1 - cx - xy^2 - xz^2 \\ 1 - cy - yx^2 - yz^2 \\ 1 - cz - zx^2 - zy^2 \end{cases}$$

The associated ideal $\mathcal{I} = \langle LV(3)\rangle$ of $LV(3)$ is invariant under $\mathfrak{S}_3$. Let Λ be the space of linear forms spanned by the coefficient forms for the normal forms w.r.t a Gröbner basis of $\mathcal{I}$ w.r.t. a reverse lexicographic order. Applying Algorithm 3 to Λ, we obtain a symmetry adapted H-basis

$$\mathcal{H} = \begin{cases} \mathcal{H}^{(1)} = \left\{x^2(y+z) + y^2(z+x) + z^2(x+y) + (x+y+z) - 3\right\} \\[2ex] \mathcal{H}^{(3,1)} = \begin{Bmatrix} x^2(2z-y) + y^2(2z-x) - (z^2+c)(x+y) + 2cz, \\ 2c\left(x^4+y^4-2z^4\right) + 2c^2\left(x^2+y^2-2z^2\right) - 2\left(x^3+y^3-2z^3\right) + 3(x+y)(xy-z^2) - c\,(x+y-2z) \end{Bmatrix} \\[2ex] \mathcal{H}^{(3,2)} = \begin{Bmatrix} (x-y)\left(-xy+z^2+c\right), \\ (x-y)\left(2c(x^3+y^3) + 2(cxy+z+c^2)(y+x) - 2x^2 - 2y^2 + z^2 - xy - c\right) \end{Bmatrix} \end{cases}$$

as well as a symmetry adapted representation of the quotient

$$Q = \begin{cases} Q^{\mathcal{G}} = \begin{Bmatrix} 1,\ x+y+z,\ x^2+y^2+z^2,\ xy+xz+yz,\ x^3+y^3+z^3,\ xyz,\ x^4+y^4+z^4, \\ x^3y + x^3z - 3x^2yz + xy^3 - 3xy^2z - 3xyz^2 + xz^3 + y^3z + yz^3 \end{Bmatrix} \\[2ex] Q^{(2)} = \{(y-z)(x-z)(x-y)\} \\[2ex] \mathcal{P}^{(3,1)} = \begin{Bmatrix} x+y-2z,\ x^2+y^2-2z^2,\ 2xy-xz-yz,\ x^3+y^3-2z^3,\ (x+y)(xy-z^2) \\ 2x^3y - x^3z + 3x^2yz + 2xy^3 + 3xy^2z - 6xyz^2 - xz^3 - y^3z - yz^3 \end{Bmatrix} \\[2ex] Q^{(3,2)} = \begin{Bmatrix} x-y,\ x^2-y^2,\ z(x-y),\ x^3-y^3,\ (x-y)\left(xy+2xz+2yz+z^2\right) \\ z(x-y)\left(x^2+4xy+y^2+z^2\right) \end{Bmatrix} \end{cases}$$

The dimension of the quotient, 21*, is the number of zeros of the system, counting multiplicities (Cox et al., 2006). Furthermore from the dimensions of the isotypic components of this quotient we can infer the orbit types of the zeros of the system. There are 3 types of orbits. The orbits with* 6 *elements live outside of the planes $x = y$, $y = z$ and $z = x$. These have trivial isotropy. The orbits with a single element live on the line $x = y = z$. These have isotropy $\mathcal{G}$. The other orbits have 3 elements. Let a_k be the number of orbits with k elements. By (Collowald and Hubert, 2015, Section 7.4.1 and Theorem 8.5) we know that*

$$a_1 + a_3 + a_6 = |Q^{\mathcal{G}}|, \quad a_3 = |Q^{(2)}|, \quad a_3 + 2a_6 = |Q^{(3,1)}|.$$

Hence $a_1 = 3$, $a_3 = 4$ and $a_6 = 1$.

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