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Abstract

Fixed-time and finite-time stabilities of non-linear systems are investigated using the introduced notions of sub- and sup-homogeneity. These concepts allow the systems to be analyzed using the homogeneity, even if they do not admit homogeneous approximations. Finite-time and fixed-time stability properties can be established using homogeneity degree of sub- and sup-homogeneous extensions.

Keywords: Finite/Fixed-time stability, Asymptotic stability, Nonlinear systems.

1. Introduction

Finite-time stability (FTS) and stabilization is rather popular research direction in the last decades (see, e.g., [6, 21, 23, 26, 9, 8, 10]). Lyapunov function method is the main tool for FTS analysis. Two possible approaches for its application are known today. The first one is to find a strict Lyapunov function, which derivative satisfies a very specific estimate [6], [21]. Design of such a Lyapunov function is a nontrivial task even in two dimensional case [30]. The second option is to consider the so-called homogeneous approximation [1] of the system provided that this approximation exists. Negative homogeneity degree and an asymptotic stability of the homogeneous approximation at zero implies FTS of the original system [1]. The second approach is much more simple and useful for control systems analysis and design. Indeed, the equations describing the approximated dynamics are more elementary and the homogeneity of this system can be utilized for the asymptotic stability analysis [7]. The aim of this paper is to study a possibility to use the same homogeneity-based methodology for systems, which do not admit homogeneous approximations.

By definition, the homogeneity is a dilation symmetry. The notion of a homogeneous function was first introduced by Leonhard Euler in 18th century. Nowadays, the homogeneity is used in systems analysis and control design. Many generalizations of this concept can be found for ordinary differential equations [35, 19, 16, 17], time-delay systems [14], discrete-time systems [32] and partial differential

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equations [28]. The homogeneity is utilized for FTS and fixed-time stability (FxTS) analysis of nonlinear systems [22], [7], [23], [20]. Local homogeneity and homogeneous approximations [1] allow the mentioned analysis to be applied for a larger class of systems. Homogeneous differential inclusions (DIs) were studied as well [15, 23, 20]. In this paper we introduce a new property of DIs, which is called as sup/sub-homogeneity.

Sup/sub-homogeneity is a certain relaxation of the usual homogeneity (dilation symmetry) of DIs. A system may not admit a homogeneous approximation being, at the same time, sup- or sub-homogeneous in the proposed sense. Below we show that the usual homogeneity-based arguments can be utilized for FTS/FxTS analysis in this case. This work proves the existence of a homogeneous Lyapunov function for a globally asymptotically stable (GAS) sup/sub-homogeneous DI. This result is a generalization of the converse Lyapunov theorem for homogeneous systems [31], [4], [2], [5], [3]. The property of weak FTS/FxTS is also investigated for the considered class of DIs.

To summarize, the main contribution of this work is a new simple approach to analyze FTS or FxTS of nonlinear systems, which do not admit local homogeneous approximations. The principal novelty is the concept of sub/sup - homogeneous extension, which can be constructed for any nonlinear system (in contrast to homogeneous approximation).

The paper is organized as follows. After introducing notations and definitions of asymptotic, finite-time, fixed-time stability and homogeneity in Section 2, we investigate the FTS of sub-homogeneous DIs (in Subsection 3.1) and FxTS of sup-homogeneous DIs (in Subsection 3.2). In Subsection 3.3 we construct examples of sup- and sub-homogeneous extensions, which are used next for the FTS and FxTS analysis of some classes of nonlinear systems. In Section 4, to illustrate the obtained results, an example of finite-time control design is considered.

2. Preliminaries

2.1. Stability of differential equations and inclusions

Let us consider a system:

$$\begin{cases} \dot{x}(t) & = f(x(t)), \quad t > 0, \\ x(0) & = x_0 \in \mathbb{R}^n \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector field, $f(0) = 0$. This system has (probably non-unique) solutions being continuously differentiable functions satisfying (1).

Below we deal only with the so-called uniform stability analysis of differential equations and inclusions (see e.g. [12, 6] and [29] for more details), so we omit the word "uniform" for shortness.

Definition 2.1. *The origin of the system* (1) *is globally*

- ***Lyapunov stable****, if* $\exists \alpha \in \mathcal{K}_\infty$ *such that* $\|x(t)\| \leq \alpha(\|x_0\|),\ \forall t \geq 0,$
- ***asymptotically stable****, if there exists* $\beta \in \mathcal{KL}$ *such that* $\|x(t)\| \leq \beta(\|x_0\|, t), \forall t \geq 0,$

- ***FTS***, *if it is globally* ***Lyapunov stable*** *and there exists a locally bounded settling-time function* $T : \mathbb{R}^n \to [0, +\infty)$ *such that* $x(t) = 0, \forall t \geq T(x_0)$,
- ***FxTS***, *if it is globally* ***FTS*** *and the settling-time function* T *is uniformly bounded,*
- ***nearly FxTS*** *if it is globally* ***Lyapunov stable*** *and any neighborhood of the origin is globally fixed-time attractive, i.e., for any nonempty neighborhood* $M \subset \mathbb{R}^n$ *of the origin there exists a positive number* $0 < T_M < +\infty$ *such that* $x(t) \in M, \forall t \geq T_M$,

for any $x_0 \in \mathbb{R}^n$ *and any solution* $t \to x(t)$ *of the system* (1)*, where the classes of functions* $\mathcal{K}^\infty$ *and* $\mathcal{KL}$ *are defined in the usual way (see e.g. [18]).*

Notice that usually the asymptotic stability is a combination of two properties: Lyapunov stability and the asymptotic convergence $\lim_{t\to+\infty} \|x(t)\| \to 0$. However, in the case of autonomous system the latter is equivalent to the existence of the function $\beta \in \mathcal{KL}$ satisfying Definition 2.1 (see [12], [18] for more details). More details about FTS and FxTS stability and illustrative examples can be found in the survey [29].

Let us consider the following differential inclusion

$$\dot{x}(t) \in F(x(t)), \ \forall t \geq 0, \tag{2}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping.

By definition, a solution of (2) is an absolutely continuous function satisfying (2) almost everywhere. Such solutions exist, for example, if F is nonempty-valued, compact-valued, convex-valued and upper-semi-continuous [15]. This condition is referred below as the *standard assumption.* In this paper, we deal also with DI, which do not satisfy the standard assumption. Such DIs are also studied in the literature (see, e.g., [34], [25]).

For each $\Omega \subseteq \mathbb{R}^n$, we denote the set $\mathbf{S}(\Omega) \subset \mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$ of all forward complete solutions[1] $t \to x(t)$ of (2) satisfying $x(0) \in \Omega$. If Ω is a singleton $\{x_0\}$, we write $\mathbf{S}(x_0)$. We denote $\mathbf{S} = \mathbf{S}(\mathbb{R}^n)$ as the set of all forward complete solutions. The next definition concerns GAS and FTS/FxTS properties of DIs (see, e.g., [12], [29] for more details).

Definition 2.2. *The origin of the system* (2) *is*

- ***strongly asymptotically stable*** *(resp.,* ***FTS/FxTS****) if all solutions of* (2) *satisfy the asymptotic stability (resp., FTS/FxTS) property given in Definition 2.1;*
- ***weakly asymptotically stable*** *(resp., FTS/FxTS) if for any* $x_0 \in \mathbb{R}^n$ *there exists a non-empty subset of solutions* $\hat{\mathbf{S}}^\beta(x_0) \subset \mathbf{S}(x_0)$ *such that all solutions from the set* $\hat{\mathbf{S}}^\beta = \bigcup_{x_0 \in \mathbb{R}^n} \hat{\mathbf{S}}^\beta(x_0)$ *satisfy the asymptotic stability (resp., FTS/FxTS) property given in Definition 2.1.*

[1]A solution of (2) said to be forward complete if it is defined on $[0, +\infty)$ and it does not blow up in a finite time.

Example: The DI $\dot{x} \in [-2, 2]x$ is weakly GAS, since its set of solutions contains the set $\hat{\mathbf{S}}^{\beta}$ of all solutions of the DI $\dot{x} \in [-2, -1]x$ which obviously satisfy $|x(t)| \leq \beta(|x(0)|, t)$ with $\beta(r, t) = re^{-t}$ for $r, t \geq 0$.

The strong GAS means that $\exists \beta \in \mathcal{KL}$ such that $\|x(t)\| \leq \beta(\|x_0\|, t)$ for all $t \geq 0$, all $x_0 \in \mathbb{R}^n$ and all $x \in \mathbf{S}(x_0)$. In the case of the weak GAS, the latter holds for solutions from a nonempty set $\hat{\mathbf{S}}^{\beta}$, which may depend on the function $\beta \in \mathcal{KL}$ (see the example above). That is why β is indicated in the notation $\hat{\mathbf{S}}^{\beta}$.

2.2. Homogeneity

In control and systems theory, homogeneity simplifies qualitative analysis of nonlinear systems. Being a dilation symmetry, it allows a local properties (e.g., stability) of dynamical systems to be extended globally. The homogeneity can be introduced using various types of dilations [35, 19, 17]. In this paper we deal with the so-called linear dilation [27] in $\mathbb{R}^n$ given by

$$\mathbf{d}(s) = e^{G_{\mathbf{d}} s} = \sum_{i=0}^{+\infty} \frac{s^i G_{\mathbf{d}}^i}{i!}$$

where $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$ is an anti-Hurwitz[2] matrix called as the generator of the dilation group [24]. The definitions of $\mathbf{d}$-homogeneous functions, single-valued and multi-valued vector fields can be found in [17], [7], [20], [23].

Definition 2.3. *A multi-valued vector field $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ (a function $h : \mathbb{R}^n \to \mathbb{R}$) is said to be $\mathbf{d}$-homogeneous of degree $\nu \in \mathbb{R}$ if*

$$F(\mathbf{d}(s)x) = e^{\nu s}\mathbf{d}(s)F(x) \quad (\textit{resp. } h(\mathbf{d}(s)x) = e^{\nu s}h(x)) \quad \textit{for all} \quad s \in \mathbb{R}, \quad x \in \mathbb{R}^n.$$

The homogeneous ball of the radius $r > 0$ is denoted as

$$B_{\mathbf{d}}(r) = \{x \in \mathbb{R}^n : \ \|\mathbf{d}(-\ln(r))x\| \leq 1\}.$$

For any $r > 0$ the set $B_{\mathbf{d}}(r)$ is a compact in $\mathbb{R}^n$ and $\mathbf{d}(s)B_{\mathbf{d}}(r) = B_{\mathbf{d}}(e^s r), \forall s \in \mathbb{R}$. For more details about geometric structures induced by linear dilations we refer the reader to [27, Chapter 6]. For $n = 1$, the dilation $\mathbf{d}$ can be chosen as follows: $\mathbf{d}(s) = e^s$.

3. Main Results

3.1. Sub-homogeneous DI

In this section, we define sub-homogeneous DI and we investigate its FTS.

[2] A matrix $G \in \mathbb{R}^{n \times n}$ is anti-Hurwitz if $-G$ is Hurwitz.

Definition 3.1. *A set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be* $\mathbf{d}$*-sub-homogeneous of degree* $\nu \in \mathbb{R}$*, if for all* $x \in \mathbb{R}^n$ *and all* $s \geq 0$ *we have*

$$e^{\nu s}\mathbf{d}(s)F(x) \subseteq F(\mathbf{d}(s)x),$$

where $\mathbf{d}$ *is a linear dilation in* $\mathbb{R}^n$*.*

Obviously, the sub-homogeneity implies the inclusion $F(\mathbf{d}(s)x) \subseteq e^{\nu s}\mathbf{d}(s)F(x)$, $\forall s \leq 0$, $\forall x \in \mathbb{R}^n$ but not the equality as in the conventional case. Notice that $\mathbf{d}(s)$ has different properties for positive and negative values the group parameter $s \in \mathbb{R}$. Indeed, $\mathbf{d}(s)$ is a strong contraction $\|\mathbf{d}(s)\| < 1$ for $s < 0$ and a strong expansion $\|\mathbf{d}(s)\| > 1$ for $s > 0$ provided that the matrix norm $\|\mathbf{d}(s)\|$ is properly defined (see, e.g., [27] for more details). Therefore, the sub-homogeneity introduced above (the sup-homogeneity introduced below) is a symmetry with respect to a semi-group of contractions/expansions.

Below we show that such DIs may appear as extensions of some dynamical systems, which do not have homogeneous approximation. The sub-homogeneity simplifies the finite-time stability analysis in the latter case.

The following proposition ensures a symmetry of solutions of sub-homogeneous DI (2)

Proposition 3.1. *Let F be* $\mathbf{d}$*-sub-homogeneous of degree* $\nu \in \mathbb{R}$*. If* $t \to x(t)$ *is a solution of* (2) *then for each* $s \geq 0$ *the function* $t \to \mathbf{d}(s)x(e^{\nu s}t)$ *is a solution of* (2) *as well.*

Proof of Proposition 3.1: Consider a solution $t \to x(t)$ of (2). The curve $t \to \mathbf{d}(s)x(e^{\nu s}t)$ is continuous for all $s \geq 0$. Moreover, for almost all $t \in \mathbb{R}$ we have:

$$\tfrac{d}{dt}\left[\mathbf{d}(s)x(e^{\nu s}t)\right] \overset{a.e.}{=} e^{\nu s}\mathbf{d}(s)\dot{x}(e^{\nu s}t) \overset{a.e.}{\in} e^{\nu s}\mathbf{d}(s)F(x(e^{\nu s}t)).$$

Since F is $\mathbf{d}$-sub-homogeneous with degree ν, one gets

$$\frac{d}{dt}\left[\mathbf{d}(s)x(e^{\nu s}t)\right] \overset{a.e.}{\in} F(\mathbf{d}(s)x(e^{\nu s}t)),$$

i.e., $t \to \mathbf{d}(s)x(e^{\nu s}t)$ is a solution of (2) for any $s \geq 0$. ■

The latter proposition obviously implies that the function $t \to \mathbf{d}(s)x(e^{\nu s}t)$ belongs to $\mathbf{S}(\mathbf{d}(s)x_0)$ for any $x \in \mathbf{S}(x_0)$ and any $s \geq 0$. Notice that this conclusion may not hold for a set of weakly GAS solutions. Let us introduce this as an assumption.

Assumption 3.1. *There exists a set* $\hat{\mathbf{S}}^{\beta_{\mathbf{d}}}$ *of weakly GAS solutions of* (2) *(see Definition 2.2) such that the function* $t \to \mathbf{d}(s)x(e^{\nu s}t)$ *belongs to* $\hat{\mathbf{S}}^{\beta_{\mathbf{d}}}(\mathbf{d}(s)x_0)$ *for any* $x \in \hat{\mathbf{S}}^{\beta_{\mathbf{d}}}(x_0)$*, any* $x_0 \in \mathbb{R}^n$ *and any* $s \geq 0$*.*

The next corollary shows that the latter assumption is fulfilled under certain conditions.

Corollary 3.1. *Let F be* $\mathbf{d}$*-sub-homogeneous of degree* $\nu \in \mathbb{R}$ *and* $\hat{\mathbf{S}}^{\beta} \subset \mathbf{S}$ *be a set of weakly GAS solutions of* (2) *with some* $\beta \in \mathcal{KL}$*. If*

$$\exists \beta_{\mathbf{d}} \in \mathcal{KL} : \beta_{\mathbf{d}}(\rho,t) \geq \sup_{s\geq 0}\|\mathbf{d}(-s)\|\beta\big(\|\mathbf{d}(s)\|\rho, e^{-\nu s}t\big), \tag{3}$$

for all $t \geq 0$ *and all* $\rho \geq 0$*, then there exists* $\hat{\mathbf{S}}^{\beta_{\mathbf{d}}} \subset \mathbf{S}$ *satisfying Assumption 3.1 such that* $\hat{\mathbf{S}}^{\beta} \subset \hat{\mathbf{S}}^{\beta_{\mathbf{d}}}$*.*

Proof of Corollary 3.1: If

$$\hat{\mathbf{S}}^{\beta_{\mathbf{d}}}(x_0):=\bigcup_{s\geq 0}\left\{\tilde{x}\in\mathcal{C}(\mathbb{R}_+,\mathbb{R}^n) : \tilde{x}(t)=x(e^{\nu s}t), t\geq 0,\ x\in\hat{\mathbf{S}}^{\beta}(\mathbf{d}(-s)x_0)\right\},$$

where $\hat{\mathbf{S}}^{\beta}$ is the set of weakly GAS solutions with $\beta\in\mathcal{KL}$, then $\hat{\mathbf{S}}^{\beta}\subset\hat{\mathbf{S}}^{\beta_{\mathbf{d}}}$ and the properties mentioned in Assumption 3.1 are fulfilled by construction and $\hat{\mathbf{S}}^{\beta_{\mathbf{d}}}\subset\mathbf{S}$ due to Proposition 3.1. Let us prove that $\hat{\mathbf{S}}^{\beta_{\mathbf{d}}}$ is a set of weakly GAS solutions with $\beta^{\mathbf{d}}\in\mathcal{KL}$.

By construction, for any $x_0\in\mathbb{R}^n$ the inclusion $\tilde{x}\in\hat{\mathbf{S}}^{\beta_{\mathbf{d}}}(x_0)$ means that for any $s\geq 0$ there exists $x\in\hat{\mathbf{S}}^{\beta}(\mathbf{d}(s)x_0)$ such that

$$\tilde{x}(t)=\mathbf{d}(-s)x(e^{-\nu s}t),\quad x(0)=\mathbf{d}(s)x_0.$$

Hence, we derive

$$\|\tilde{x}(t)\|=\|\mathbf{d}(-s)x(e^{-\nu s}t)\|=\|\mathbf{d}(-s)\|\cdot\|x(e^{-\nu s}t)\|\leq$$

$$\|\mathbf{d}(-s)\|\beta(\|\mathbf{d}(s)\|\cdot\|x_0\|,e^{-\nu s}t)\leq\beta_{\mathbf{d}}(\|x_0\|,t),$$

for all $t\geq 0$, all $x_0\in\mathbb{R}^n$ and all $\tilde{x}\in\hat{\mathbf{S}}^{\beta_{\mathbf{d}}}$. ■

Taking into account that any function $\beta\in\mathcal{KL}$ admits the estimate [33, Lemma 8]

$$\beta(\rho,t)\leq\sigma_1(\rho)\sigma_2(e^{-t}),\qquad\text{for some } \sigma_1,\sigma_2\in\mathcal{K}_\infty,$$

the condition (3) can be represented as follows

$$\exists\sigma_{\mathbf{d}}\in\mathcal{K}_\infty\quad:\quad\sigma_{\mathbf{d}}(\rho)\geq\sup_{s\geq 0}\|\mathbf{d}(-s)\|\sigma_1(\|\mathbf{d}(s)\|\rho),$$

provided that $\nu\leq 0$. Indeed, since $e^{-e^{-\nu s}t}\leq e^{-t},\forall s\geq 0,\forall\nu\leq 0,\forall t\geq 0$ then we can select

$$\beta_{\mathbf{d}}(\rho,t)=\sigma_{\mathbf{d}}(\rho)\sigma_2(e^{-t}),\quad\rho,t\geq 0.$$

Theorem 3.1. *Let F be $\mathbf{d}$-sub-homogeneous of degree $\nu\in\mathbb{R}$ and satisfy the standard assumption. Let p be an arbitrary natural number. The origin of (2) is GAS if and only if there exists a pair (V,W) of continuous functions such that*

1) *$V\in\mathcal{C}^p(\mathbb{R}^n,\mathbb{R}_+)$, V is positive definite and $\mathbf{d}$-homogeneous with degree $k>0$ such that the matrix $pG_{\mathbf{d}}-kI_n$ is Hurwitz;*
2) *$W\in\mathcal{C}^\infty(\mathbb{R}^n\setminus\{0\},\mathbb{R}_+)$, W is positive definite and $\mathbf{d}$-homogeneous with degree $k+\nu$;*
3) *$\max\limits_{h\in F(x)}\langle\nabla V(x),h\rangle\leq -W(x)$ for all $x\neq 0$,*

where $\nabla V(x)=\left(\frac{\partial V(x)}{\partial x_1},\cdots,\frac{\partial V(x)}{\partial x_n}\right)^\top$ stands for the gradient of the function $V:\mathbb{R}^n\to\mathbb{R}$ at point $x\in\mathbb{R}^n$ and $\langle\nabla V(x),h\rangle=\sum\limits_{i=1}^{n}\frac{\partial V(x)}{\partial x_i}h$ defines the directional derivative of a continuously differentiable function V in the direction $h\in\mathbb{R}^n$.

Proof of Theorem 3.1: Let us prove the necessity, since the sufficiency is trivial. For non-homogeneous V_0 and W_0 the claimed result is proven in [11], i.e., there exists a pair (V_0, W_0) of continuous functions (see [11]), such that:

1) $V_0 \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}_+)$, V_0 is positive definite;
2) $W_0 \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R}_+)$, W_0 is positive definite;
3) $\max\limits_{h\in F(x)} \langle \nabla V_0(x), h\rangle \leq -W_0(x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

Since $V_0(0) = 0$, $V_0(x) \to +\infty$ as $\|x\| \to +\infty$ and $\mathbf{d}$ is a dilation then there exist $\gamma > 0$ and $0 < v_1 < v_2$ such that

$$V_0(\mathbf{d}(s)x) \leq v_1 \text{ for any } x : \|x\| = 1 \text{ and } \forall s \leq 0,$$
$$V_0(\mathbf{d}(s)x) \geq v_2 \text{ for any } x : \|x\| = 1 \text{ and } \forall s \geq \gamma.$$

Following Rosier's theorem [31] to build a homogeneous Lyapunov function V we consider a $\mathcal{C}^\infty$ function $a : [0, +\infty) \to [0, 1]$ such that

$$a(t) = \begin{cases} 0, & \text{if } t \leq v_1, \\ 1, & \text{if } t \geq v_2, \end{cases}$$

and $\dot{a}(t) > 0$ for all $t \in (v_1, v_2)$. Take $k > \max\{-\nu, 0\}$ then the function

$$x \to V(x) := \int_{\mathbb{R}} e^{-ks} a \circ V_0(\mathbf{d}(s)x) ds$$

is well defined, continuous on $\mathbb{R}^n$ and $\mathcal{C}^\infty$ on $\mathbb{R}^n \setminus \{0\}$. Moreover, V is $\mathbf{d}$-homogeneous $V(\mathbf{d}(\tau)x) = e^{k\tau} V(x)$, and for any $x : \|x\| = 1$ and any $h \in F(x)$ we have

$$\langle \nabla V(x), h\rangle = \int_{\mathbb{R}} e^{-ks} \dot{a}\left(V_0(\mathbf{d}(s)x)\right) \langle \nabla V_0(\mathbf{d}(s)x), \mathbf{d}(s)h\rangle ds.$$

and

$$\max_{h\in F(x)} \langle \nabla V(x), h\rangle = \int_{\mathbb{R}} e^{-ks} \dot{a}\left(V_0(\mathbf{d}(s)x)\right) \max_{h\in F(x)} \langle \nabla V_0(\mathbf{d}(s)x), \mathbf{d}(s)h\rangle ds$$

using sub-homogeneity of F, one gets

$$\begin{aligned} \max_{h\in F(x)} \langle \nabla V(x), h\rangle &\leq \int_0^\gamma e^{-ks} \dot{a}\left(V_0(\mathbf{d}(s)x)\right) \max_{h\in e^{-\nu s}\mathbf{d}(-s)F(\mathbf{d}(s)x)} \langle \nabla V_0(\mathbf{d}(s)x), \mathbf{d}(s)h\rangle ds \\ &= \int_0^\gamma e^{-(k+\nu)s} \dot{a}\left(V_0(\mathbf{d}(s)x)\right) \max_{h\in F(\mathbf{d}(s)x)} \langle \nabla V_0(\mathbf{d}(s)x), h\rangle ds \\ &\leq -\int_0^\gamma e^{-(k+\nu)s} \dot{a}\left(V_0(\mathbf{d}(s)x)\right) W_0(\mathbf{d}(s)x) ds = -W(x) \leq 0. \end{aligned}$$

These inequalities are derived by using the sub-homogeneity of F. W is a $\mathbf{d}$-homogeneous positive definite function with the degree $(\nu + k) > 0$. This proves that V is a $\mathbf{d}$-homogeneous Lyapunov function for (2). Now, let $k > \max\{-\nu, 0\}$ be such that the matrix $pG_{\mathbf{d}} - kI_n$ is Hurwitz. Then for $s \in \mathbb{R}$ and $x : \|x\| = 1$ we have

$$\frac{d^p}{d(\mathbf{d}(s)x)^p} V(\mathbf{d}(s)x) = e^{ks}\mathbf{d}(-ps) \frac{d^p}{dx^p} V(x).$$

To guarantee that $V \in \mathcal{C}^p(\mathbb{R}^n, \mathbb{R}_+)$ it is enough to show

$$\lim_{s\to-\infty} \left\| \tfrac{d^p V(\mathbf{d}(s)x)}{d(\mathbf{d}(s)x)^p} \right\| = \lim_{s\to-\infty} \left\| e^{ks}\mathbf{d}(-ps) \tfrac{d^p}{dx^p} V(x) \right\| = 0.$$

This is true, because the matrix $pG_{\mathbf{d}} - kI_n$ is Hurwitz and

$$e^{ks}\mathbf{d}(-ps) = e^{s(kI_n - pG_{\mathbf{d}})}.$$

We use the same argument to prove that $W \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R}_+)$. ■

The latter result generalizes the Rosier's theorem [31], [4], [2], [5] to sub-homogeneous DI.

Theorem 3.2. *If the set $\hat{\mathbf{S}}^\beta \subset \boldsymbol{S}$ of weakly GAS solutions satisfies Assumption 3.1 for $\nu < 0$ then $\hat{\mathbf{S}}^\beta$ is a set of weakly globally FTS solutions.*

Proof of Theorem 3.2: Let $\hat{\mathbf{S}}^\beta$ be the set of solutions of (2) satisfying Assumption 3.1.

Let $R > 0$ be an arbitrary real number and let us define a time τ_R for weakly GAS solutions from $\hat{\mathbf{S}}^\beta$ which start from the set $B_{\mathbf{d}}(2R)$ and converge into the set $B_{\mathbf{d}}(R)$:

$$\tau_R = \sup_{x \in \hat{\mathbf{S}}^\beta(B_{\mathbf{d}}(2R))} \inf\{T > 0 : \ x(t) \in B_{\mathbf{d}}(R), \forall t \geq T\}.$$

Let us denote

$$\overline{\rho}(R) = \sup_{z \in B_{\mathbf{d}}(2R)} \|z\| \quad \text{and} \quad \underline{\rho}(R) = \inf_{z \in \mathbb{R}^n \setminus B_{\mathbf{d}}(R)} \|z\|.$$

Notice that $\overline{\rho}(R) > 0, \underline{\rho}(R) > 0$ for any $R > 0$ and

$$\|x(t)\| \leq \beta(\|x(0)\|, t) \leq \beta(\overline{\rho}(R), t), \quad \forall t \geq 0$$

for any $x \in \hat{\mathbf{S}}^\beta(B_{\mathbf{d}}(2R))$, where monotonicity of $\beta \in \mathcal{KL}$ with respect to the first argument is taken into account. Since $\beta(\bar{\rho}(R), t)$ monotonically tends to 0 as $t \to +\infty$ then there exists a finite number $T_R^\beta > 0$ such that $\beta(\bar{\rho}(R), t) \leq \underline{\rho}(R), \forall t \geq T_R^\beta$. This means $\tau_R < +\infty$ for any $R > 0$.

On the other hand, we derive

$$\begin{aligned}
&\tau_{R/2} = \sup_{x \in \hat{\mathbf{S}}^\beta(B_{\mathbf{d}}(R))} \inf\{T > 0 : \ x(t) \in B_{\mathbf{d}}(R/2), \forall t \geq T\} \\
&= \sup_{x \in \hat{\mathbf{S}}^\beta(B_{\mathbf{d}}(R))} \inf\{T > 0 : \ \mathbf{d}(\ln(2))x(t) \in B_{\mathbf{d}}(R), \ \forall t \geq T\} \\
&= 2^\nu \sup_{x \in \hat{\mathbf{S}}^\beta(B_{\mathbf{d}}(R))} \inf\left\{2^{-\nu}T > 0 : \begin{array}{l} \mathbf{d}(\ln(2))x(2^\nu t) \in B_{\mathbf{d}}(R), \\ \forall t \geq 2^{-\nu}T \end{array}\right\} \\
&\leq 2^\nu \sup_{y \in \hat{\mathbf{S}}^\beta(\mathbf{d}(\ln(2))B_{\mathbf{d}}(R))} \inf\left\{\tilde{T} > 0 : \ y(t) \in B_{\mathbf{d}}(R), \ \forall t \geq \tilde{T}\right\} \\
&= 2^\nu \sup_{y \in \hat{\mathbf{S}}^\beta(B_{\mathbf{d}}(2R))} \inf\left\{\tilde{T} > 0 : \ y(t) \in B_{\mathbf{d}}(R), \ \forall t \geq \tilde{T}\right\} = 2^\nu \tau_R,
\end{aligned}$$

where $y(t) = \mathbf{d}(\ln(2))x(2^\nu t)$. Hence, the convergence time of $x \in \hat{\mathbf{S}}^\beta(B_{\mathbf{d}}(2R))$ to zero admits the estimate

$$\tau_R \sum_{i=1}^{+\infty} 2^{\nu i} < +\infty$$

provided that $\nu < 0$. ■

Theorem 3.2 generalizes the results from [23], [20] about FTS of homogeneous DI with negative degree.

Proposition 3.1 implies that Assumption 3.1 is fulfilled for the set of all solutions of (2) provided that (2) is strongly GAS.

Corollary 3.2. *Let F be $\mathbf{d}$-sub-homogeneous of degree $\nu < 0$. If (2) is strongly GAS then it is strongly globally FTS.*

3.2. Sup-homogeneous DI

Definition 3.2. *The set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be $\mathbf{d}$-sup-homogeneous of degree $\nu \in \mathbb{R}$ if it satisfies*

$$e^{\nu s}\mathbf{d}(s)F(x) \subseteq F(\mathbf{d}(s)x), \quad \forall s \leq 0, \quad \forall x \in \mathbb{R}^n,$$

where $\mathbf{d}$ is a linear dilation in $\mathbb{R}^n$.

The following proposition shows the symmetry of solutions to the differential inclusion (2) provided that F satisfies the sup-homogeneity property.

Proposition 3.2. *Let F be $\mathbf{d}$-sup-homogeneous of degree $\nu \in \mathbb{R}$. If $x(\cdot)$ is a solution of (2), then for any $s \leq 0$ the function $t \to \mathbf{d}(s)x(e^{\nu s}t)$ is a solution of (2).*

Proof of Proposition 3.2: Proof of proposition uses the sup-homogeneity property and follows the same steps as the proof of Proposition 3.1.■

Let us introduce the assumption

Assumption 3.2. *The property in Assumption 3.1 is fulfilled for $s \leq 0$.*

Similarly to Corollary 3.1, it can be shown that Assumption 3.2 is fulfilled if there exists a set $\hat{\mathbf{S}}^\beta \subset \mathbf{S}$ of weakly stable solutions and

$$\exists \beta_{\mathbf{d}} \in \mathcal{KL} : \beta_{\mathbf{d}}(\rho, t) \geq \sup_{s \leq 0} \|\mathbf{d}(-s)\| \beta\big(\|\mathbf{d}(s)\|\rho, e^{-\nu s}t\big), \tag{4}$$

for all $\forall t \geq 0$ and all $\rho \geq 0$, where $\beta \in \mathcal{KL}$ corresponds to $\hat{\mathbf{S}}^\beta$.

Taking into account that any function $\beta \in \mathcal{KL}$ admits the estimate [33, Lemma 8]

$$\beta(\rho, t)) \leq \sigma_1(\rho)\sigma_2(e^{-t}), \qquad \text{for some } \sigma_1, \sigma_2 \in \mathcal{K}_\infty,$$

the condition (4) can be represented as follows

$$\exists \sigma_{\mathbf{d}} \in \mathcal{K}_\infty \quad : \quad \sigma_{\mathbf{d}}(\rho) \geq \sup_{s \leq 0} \|\mathbf{d}(-s)\| \sigma_1(\|\mathbf{d}(s)\|\rho),$$

provided that $\nu \geq 0$. Indeed, since $e^{-e^{-\nu s}t} \leq e^{-t}, \forall s \leq 0, \forall \nu \geq 0, \forall t \geq 0$ then we can select

$$\beta_{\mathbf{d}}(\rho, t) = \sigma_{\mathbf{d}}(\rho)\sigma_2\left(e^{-t}\right), \quad \rho, t \geq 0.$$

The following theorem shows the existence of a homogeneous Lyapunov function for an asymptotically stable sup-homogeneous differential inclusion.

Theorem 3.3. *Let F be $\mathbf{d}$-sup-homogeneous of degree $\nu \in \mathbb{R}$ and satisfy the standard assumption. Let p be an arbitrary natural number. The origin of (2) is GAS if and only if there exists a pair (V, W) of continuous functions*

1) *$V \in \mathcal{C}^p(\mathbb{R}^n, \mathbb{R}_+)$, V is positive definite and $\mathbf{d}$-homogeneous of a degree $k > 0$ such that the matrix $pG_{\mathbf{d}} - kI_n$ is Hurwitz;*
2) *$W \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R}_+)$, W is positive definite and $\mathbf{d}$-homogeneous of the degree $k + \nu$;*
3) *$\max\limits_{h \in F(x)} \langle \nabla V(x), h \rangle \leq -W(x)$ for all $x \in \mathbb{R}^n \backslash \{0\}$.*

Proof of Theorem 3.3: We follow the arguments of the proof of Theorem 3.1 and we construct the so-called cut-off function a. For any constant $\gamma > 0$, by the definition of the dilation $\mathbf{d}$ and since $V_0(0) = 0$ and $V_0(x) \to +\infty$ as $\|x\| \to +\infty$, there exist two numbers $0 < v_1 < v_2$, such that

$$\begin{aligned} &V_0(\mathbf{d}(s)x) \leq v_1, \text{ for } \|x\| = 1 \text{ and } \forall s \leq -\gamma, \\ &V_0(\mathbf{d}(s)x) \geq v_2, \text{ for } \|x\| = 1 \text{ and } \forall s \geq 0. \end{aligned} \tag{5}$$

This is the difference with the proof of Theorem 3.1 (the property (5) holds for a different interval of s). The cut-off function a will have the same form that we defined in the proof of Theorem 3.1, then for $k > \max\{-\nu, 0\}$ The Lyapunov function V is given by $V(x) = \int_{\mathbb{R}} e^{-ks} a \circ V_0(\mathbf{d}(s)x) ds$ and the function $W(x) = \int_{\mathbb{R}} e^{-(k+\nu)s} \dot{a}\left(V_0(\mathbf{d}(s)x)\right) W_0(\mathbf{d}(s)x) ds$. ■

Theorem 3.4. *If Assumption 3.2 is fulfilled for $\nu > 0$ then (2) is weakly globally nearly FxTS.*

Proof of Theorem 3.4: Let $\hat{\mathbf{S}}^\beta$ be the set of solutions of satisfying Assumption 3.2. For a given $R > 0$ and let us define

$$\tau_R = \sup_{x \in \hat{\mathbf{S}}^\beta(B_{\mathbf{d}}(2R))} \inf\left\{T > 0 : \; x(t) \in B_{\mathbf{d}}(R), \forall t \geq T\right\}.$$

If $\nu > 0$ then repeating the proof of Theorem 3.2 we derive $\tau_{2^i R} \leq 2^{-i\nu}\tau_R$ and the time of convergence of any solution from $\hat{\mathbf{S}}^\beta$ to a $\mathbf{d}$-homogeneous ball of the radius $R > 0$ is finite

$$\sum_{i=0}^{+\infty} \tau_{2^i R} \leq \tau_R \sum_{i=0}^{+\infty} 2^{-i\nu} = \frac{2^\nu \tau_R}{2^\nu - 1} < +\infty$$

independently of the initial value $x(0)$. ■

Corollary 3.3. *Let F be $\mathbf{d}$-sub-homogeneous of degree $\nu > 0$. If (2) is strongly GAS then it is strongly globally nearly FxTS.*

3.3. FTS/FxTS of non-homogeneous systems

In this section, we consider the dynamical system (1). The goal of this section is to study the FTS and FxTS of globally asymptotically stable nonlinear systems which are not homogeneous and do not allow a homogeneous approximations at the zero or/and at infinity. To present a homogeneity-based analysis we construct sub- or sup-homogeneous extension of the function f. Homogeneous extensions were introduced in [28] to simplify a stability analysis of non-homogeneous PDEs. In this paper, we deal with finite-dimensional models and prove that, under certain conditions on f, the GAS nonlinear system (1) is globally FTS or globally nearly FxTS, according to the degree of the sub/sup-homogeneous extension.

3.3.1. *Sub-homogeneous extension*

Studying the finite-time convergence of a dynamical system by construction of a Lyapunov function is not easy and sometimes it is quite impossible. To overpass this difficulty, we consider the DI (2) with

$$F(x) = \bigcup_{s\leq 0} \left\{ e^{-\nu s}\mathbf{d}(-s) f(\mathbf{d}(s)x) \right\}, \tag{6}$$

where $\mathbf{d}$ is a linear dilation and $\nu \in \mathbb{R}$. Indeed, the set valued mapping F is $\mathbf{d}$-sub-homogeneous with the degree of homogeneity $\nu \in \mathbb{R}$. The set $F(x)$ is nonempty for every $x \in \mathbb{R}^n$. Since, by construction, $f(x) \in F(x)$ for all $x \in \mathbb{R}^n$ then any solution of (1) is a solution of (2) with the right-hand side (6) and Corollary 3.2 immediately implies

Corollary 3.4. *Let f be continuous and the system* (2) *with F given by* (6) *for some $\nu < 0$ is GAS then* (1) *is globally FTS.*

Consider the following example: $f(x) = -x^{1/3}(2 - \cos\left(\frac{1}{|x|}\right)$. The function f does not have a homogeneous approximation at the origin, but it has a sub-homogeneous extension, which is given by

$$F(x) = -x^{1/3} \bigcup_{s\leq 0} \left\{ \left(2 - \cos\left(\frac{1}{|e^s x|} \right) \right) \right\} = -x^{1/3}[1,3].$$

Where $\mathbf{d}(s) = e^s$ and $\nu = -2/3$.

The following corollary deals with the FTS of the dynamical system (1).

Corollary 3.5. *If* (1) *is GAS with $\beta \in \mathcal{KL}$ satisfying the condition* (3) *for some $\nu < 0$ then it is globally FTS.*

Proof of Corollary 3.5: The set of solutions of (1) is a subset $\hat{\mathbf{S}}^\beta$ of weakly asymptotically stable solutions of (2) with F given (6). Using Corollary 3.1 we conclude that all conditions of Theorem 3.2 are fulfilled and the origin of (1) is globally FTS.■

3.3.2. *Sup-homogeneous extension*

In this section we define the sup-homogeneous extension of a vector field f as follows

$$F(x) = \bigcup_{s\geq 0} \left\{ e^{-\nu s}\mathbf{d}(-s) f(\mathbf{d}(s)x) \right\}, \tag{7}$$

where $\mathbf{d}$ is a linear dilation and $\nu \in \mathbb{R}$.

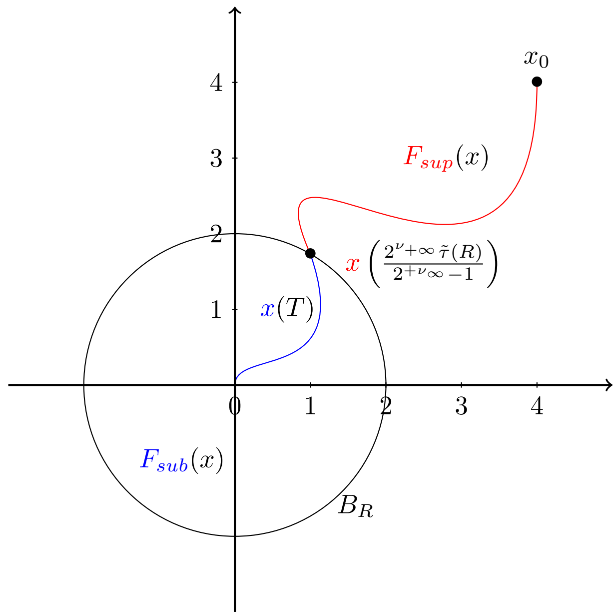

Figure 1: Fixed-time stable solution.

Example: Let $x \to f(x)$ given by $f(x) = x(3-\cos(|x|))$. The function f does not have a homogeneous approximation at infinity, but it has a homogeneous extension, which is defined by

$$F(x) = x \bigcup_{s\geq 0} \{(3 - \cos(|e^s x|))\} = x[2,4],$$

where $\mathbf{d}(s) = e^s$ and $\nu = 0$.

Similarly to the previous section, the following condition of the nearly FxTS of the system (1) can be derived.

Corollary 3.6. *Let f be continuous and the system (2) with F given by (7) for some $\nu > 0$ is GAS then (1) is globally nearly FxTS.*

Corollary 3.7. *If (1) is GAS with $\beta \in \mathcal{KL}$ satisfying the condition (4) then it is globally nearly FxTS.*

Combining the above results we derive the condition of global FxTS.

Corollary 3.8. *If (1) is GAS with $\beta \in \mathcal{KL}$ satisfying both (3) and (4) with some degrees $\nu_- < 0$ and $\nu_+ > 0$, respectively, then (1) is globally FxTS.*

Proof of Corollary 3.8: Using Corollary 3.7 we deduce that the system (1) is nearly fixed-time stable and all trajectories arrive to a ball B_R in a fixed time. In addition, Corollary 3.5 implies that the system (1) is FTS for all $x_0 \in B_R$. These two facts imply that the state of the system (1) converges to the origin in a fixed time (see Fig. 1).■

4. Example

Let us consider the system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -k_1(x_1) - k_2(x_2), \end{cases} \tag{8}$$

where $x_1(t), x_2(t) \in \mathbb{R}$ are components of the state vector $x = (x_1, x_2)^\top$ and $k_i \in \mathcal{C}(\mathbb{R}, \mathbb{R}),\ k_i(\rho)\rho > 0, \forall \rho \in \mathbb{R}, i = 1, 2$. Let us denote by f the vector field, which defines the right-hand side of the considered system. It is easy to see that f is continuous on $\mathbb{R}^2$. The system (8) can be interpreted as a mechanical model with the total energy (i.e., the sum of kinetic and potential energy) given by

$$U(x) = \int_0^{x_1} k_1(\sigma)d\sigma + \frac{x_2^2}{2}.$$

The mechanical system is dissipative since

$$\langle \nabla U(x), f(x) \rangle = -k_2(x_2)x_2 < 0 \quad \text{ for } x_2 \neq 0.$$

Applying LaSalle invariance principle (Barbashin-Krasovski Theorem) we conclude that the origin of the considered system is GAS. Its FTS and nearly FxTS can be studied using the sub/sup-homogeneous extensions. Taking

$$\mathbf{d}(s) = \text{diag}\{e^s, e^{(1+\nu)s}\}, s \in \mathbb{R}, \quad \text{ and } \quad \nu > -1$$

we derive

$$g(s, x) := e^{-\nu s}\mathbf{d}(-s)f(\mathbf{d}(s)x) = \begin{pmatrix} x_2 \\ -e^{-(1+2\nu)s}(k_1(e^s x_1) + k_2(e^{(1+\nu)s}x_2)) \end{pmatrix}.$$

Let the functions $k_1,\ k_2$ and the degree ν be such that $e^{-(1+2\nu)s}k_1(e^s x_1) = k_1(x_1),\ \forall s, x_1 \in \mathbb{R}$. Then

$$\langle \nabla U(x), g(s, x) \rangle = -e^{-(1+2\nu)s}x_2 k_2(e^{(1+\nu)s}x_2) < 0$$

and take sub-homogeneous extension

$$F(x) = \overline{co} \bigcup_{s \leq 0} \{g(s, x)\}, \tag{9}$$

where $\overline{\text{co}}A$ denotes the closed convex hull of $A \subset \mathbb{R}^n$. Then, we have

$$\sup_{h \in F(x)} \langle \nabla U(x), h \rangle \leq 0,\ x_2 \neq 0.$$

If the corresponding number ν is negative and

$$0 < \liminf_{s \to -\infty} e^{-(1+2\nu)s}x_2 k_2(e^{(1+\nu)s}x_2) < \limsup_{s \to -\infty} e^{-(1+2\nu)s}x_2 k_2(e^{(1+\nu)s}x_2) < +\infty,\ \forall x_2 \neq 0,$$

then, obviously, F is nonempty-, compact-, convex-valued and upper-semi-continuous. Moreover, the sub-homogeneous extension (2),(9) is strongly GAS due to LaSalle invariance principle (see Theorem 14 [13] for an extension of the LaSalle invariance principle to DI). Hence, from Corollary (3.6) we conclude that (8) is globally FTS. For example, if

$$k_1(x_1) = -x_1^{\frac{1}{3}},\ k_2(x_2) = \frac{-x_2}{\sqrt{|x_2|}}\left(1 + |x_2|(1 + \cos(1/x_2)) + 0.5\sin(1/x_2)\right),$$

the system (8) does not have an asymptotically stable $\mathbf{d}$-homogeneous approximation at 0, but the system is globally FTS since the conditions given above are fulfilled for $\nu = -1/3$.

Similarly if $\nu > 0$ and

$$0 < \liminf_{s\to+\infty} e^{-(1+2\nu)s} x_2 k_2(e^{(1+\nu)s} x_2) < \limsup_{s\to+\infty} e^{-(1+2\nu)s} x_2 k_2(e^{(1+\nu)s} x_2) < +\infty, \ \forall x_2 \neq 0,$$

then the sup-homogeneous extension

$$F(x) = \overline{co} \bigcup_{s\geq 0} \{g(s,x)\} \tag{10}$$

is nonempty-, compact-, convex-valued and upper-semi-continuous. Moreover, (2),(10) is GAS, which implies the globally nearly FxTS of the origin for the system (8). For example, if $k_1(x_1) = -x_1^3$, $k_2(x_2) = -x_2|x_2|(1 + 0.5\sin(x_2))$.

Notice also that a strict Lyapunov function $V(x) = U^{\gamma}(x) + \varepsilon x_1 x_2$ with $\varepsilon, \gamma > 0$ can be utilized for FTS (nearly FxTS) analysis of (8). However, in this case, to prove FTS we need to show $\dot{V}(x) \leq -cV^{1-\alpha}(x)$ with $\alpha \in (0,1)$, $c > 0$ at least close to the origin (to infinity with $\alpha > 1$, respectively). The corresponding derivations are much more complicated (see e.g. [30]) than the given above sup/sub-homogeneous extension-based analysis.

5. Conclusion and Discussion

The notions of sup- and sub-homogeneity are introduced. It is shown that GAS of sub/sup-homogeneous DIs satisfying the standard assumptions are characterized by a homogeneous Lyapunov function. The strong and weak FTS/FxTS of sub/sup-homogeneous GAS DIs are investigated. Sub/sup-homogeneous extensions for non-homogeneous systems are constructed.

The suggested example illustrates how the introduced notions allow us to use the conventional homogeneity-based FTS analysis for a larger class of systems. The second order system considered in the example was shown to be GAS by using the classical non-strict Lyapunov function and the LaSalle invariance principle. However, this Lyapunov function cannot be used for FTS/FxTS analysis of the system. The homogeneity-based tools can be utilized in this case (see, e.g., [1]) provided that the system admits GAS homogeneous approximations. We showed that the latter assumption can be relaxed to the existence of GAS sup- or sub-homogeneous extensions of the considered system. In the studied example, the GAS of the corresponding extensions has been easily proven using the same non-strict Lyapunov function. Hence, we deduced the FTS and nearly FxTS only regarding the signs of the homogeneity degrees of the extensions. The presented example showed the usefulness of the proposed notions for simplification of FTS/FxTs analysis of some non-homogeneous systems, which do not have homogeneous approximations at zero or at infinity. Further development of this concept seems to be a promising direction of the research for finite and infinite-dimensional systems. The sup/sub-homogeneity-based robustness (input-to-state stability analysis) can be mentioned as an interesting problem for future investigation.

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