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Summary

In this paper, continuous and discontinuous integral controllers for MIMO systems are designed for a large class of nonlinear systems, which are (partially) feedback linearizable. These controllers of arbitrary positive or negative degree of homogeneity are derived by combining a Lyapunov function obtained from the Implicit Lyapunov Function (ILF) method with some extra explicit terms. Discontinuous integral controllers are able to stabilize an equilibrium or track a time-varying signal in finite time, while rejecting vanishing uncertainties and non-vanishing Lipschitz matching perturbations. Continuous integral controllers achieve asymptotic stabilization despite non-vanishing constant perturbations in finite-time, exponentially or nearly fixed-time for negative, zero or positive homogeneity degree, respectively. The design method and the properties of the different classes of integral controllers are illustrated by means of a simulation example.

KEYWORDS:

Homogeneous controller, nonlinear control design, robustness, high order sliding mode control, implicit Lyapunov function

1 | INTRODUCTION

The presence of external perturbations or internal model uncertainties constitutes a fundamental obstacle to attain good point stabilization or trajectory tracking. This makes the design of robust controllers an essential control task. If the effect of the disturbances/uncertainties does not vanish at the equilibrium point, i.e. they are non-vanishing, it is well-known that a continuous and static feedback controller is unable to stabilize it, and only practical stability can be achieved. However, a memoryless discontinuous (at the equilibrium) feedback controller can fully compensate for non-vanishing bounded matched perturbations. This fact makes the well-known (High-Order) Sliding-Mode Control (HOSMC)[1,2,3] a popular strategy to achieve such insensitivity to exogenous disturbances or internal uncertainties for Single-Input Single-Output (SISO) systems employing discontinuity. These controllers have been extended to Multiple-Input Multiple-Output (MIMO) systems in[4,5,6]. On the other hand, an undesirable effect of discontinuous controllers is the "Chattering" phenomenon[2].

An alternative approach to deal with non-vanishing perturbations is the use of dynamic controllers, being the integral control[7] the most classical one. But it is only able to fully compensate constant perturbations, which is by far a smaller class than a discontinuous controller can counterbalance. Therefore, an interesting idea is to design nonlinear integral controllers, which are able to reach the advantages of both approaches, i.e. using a continuous control signal to attenuate chattering while rejecting a large family of perturbations.

Since homogeneous systems posses several advantages[8], imposing homogeneity (at least locally) on the closed-loop dynamics has been used frequently for the design of nonlinear controllers and observers. Homogeneous continuous or discontinuous controllers can be designed by different methods. In[9] the design is based on explicit Lyapunov functions, while in[10] the controllers are designed through an implicitly defined Lyapunov function. One attractive feature of this latter method is that the gain design can be reduced to an LMI problem.

For SISO systems with relative degree one, a classical discontinuous integral controller is the well-known Super-Twisting Algorithm (see e.g.[11,3]). In[12,13] explicit Lyapunov functions have been proposed for the stability analysis and the gain design of the Super-Twisting. Recently, nonlinear continuous and homogeneous integral controllers have been proposed for mechanical systems[14,15] but without a formal stability proof. Distinct Continuous and Discontinuous homogeneous integral controllers have been presented for systems with different relative degrees in[16,17,18,19,20,21,22,23,24], showing also diverse structural properties. In these works, various explicit Lyapunov functions have been developed to assure convergence and robustness on the one side, and to design the gains on the other side. For instance, the basic idea of the construction of the (non-smooth) Lyapunov function for relative degree one used in[12] has been extended for an arbitrary relative degree in[19] to obtain continuous and discontinuous integral controllers. All these methods have the drawback, that the gain selection becomes a highly nonlinear problem, making its application more difficult.

In[25] an integral controller for SISO systems is presented, which is based on the Implicit Lyapunov Function (ILF) method. The ILF method was proposed originally in[10] to design static state-feedback controllers. We considered in[25] homogeneous controllers of non-positive homogeneity degree, affected solely by time-varying matched perturbations. In the current paper, we extend this result to MIMO systems, of arbitrary (negative or positive) degree of homogeneity, and considering vanishing not matched perturbations and non-vanishing matched perturbations, which can both depend on state and time. The design of these homogeneous continuous or discontinuous integral controllers - for SISO and MIMO systems - is developed around the ILF method[10].

An important advantage of the proposed solution is the obtainment of constructive rules for tuning the control gains formulated in the form of LMIs, similar to linear time-invariant systems. Since a direct application of the ILF idea does not lead to a usable integral controller, we combine the ILF method for the design of a (rational) state feedback controller and an explicit Lyapunov function for the calculation of the integral part. This resembles the idea used for the Super-Twisting in[12], and which is generalized for arbitrary order in[19]. This leads to a very useful method for designing homogenous integral controllers of an arbitrary positive or negative degree.

It is important to stress that continuous integral controllers can compensate only for *constant* perturbations (as for the classical integral action), while discontinuous controllers can reject *Lipschitz* perturbations. It is also worth noticing here that, since the control signal in the discontinuous integral action is obtained through the (time) integration of the signal produced by a discontinuous term, it is continuous and the effect of chattering can be in principle strongly reduced. The class of systems considered includes minimum phase partially feedback linearizable MIMO nonlinear systems, which are transformed to the Byrnes-Isidori normal form to make use of homogeneity.

Overview

The current work is organized as follows. In Section 2 we present some background on the required mathematical tools. In Section 3 we formulate the problem to be solved and the class of systems considered. In Section 4, we present the main result that consists of the homogeneous integral controller. In Section 5, we prove the main result by devising a strong Lyapunov function, resulting as a combination of the ILF proposed in[10,26] and some extra terms, that ensures the robust global stability of the origin for the closed-loop system, and allows the design of the gains. Finally, in Section 6, we show some simulations using the controllers presented here to illustrate their properties.

Notation

$\mathbb{R}$, $\mathbb{R}_{0+} = \{z \in \mathbb{R} \mid z \geq 0\}$ and $\mathbb{R}_{+} = \{z \in \mathbb{R} \mid z > 0\}$ stand for the real, the non negative and the positive real numbers, respectively. For a real variable $z \in \mathbb{R}$ and a real number $p \in \mathbb{R}$ the symbol $\lceil z \rfloor^{p} = |z|^{p} \operatorname{sign}(z)$ represents the signed power p of z. According to this $\lceil z \rfloor^{0} = \operatorname{sign}(z)$. $\frac{d}{dz}\lceil z \rfloor^{m} = m|z|^{m-1}$ and $\frac{d}{dz}|z|^{m} = m\lceil z \rfloor^{m-1}$. Note that $\lceil z \rfloor^{2} = |z|^{2} \operatorname{sign}(z) \neq z^{2}$, and if p is an odd number then $\lceil z \rfloor^{p} = z^{p}$ and $|z|^{p} = z^{p}$ for any even integer p. Moreover $\lceil z \rfloor^{p}\lceil z \rfloor^{q} = |z|^{p+q}$, $\lceil z \rfloor^{p}\lceil z \rfloor^{0} = |z|^{p}$ and $\lceil z \rfloor^{0}\lceil z \rfloor^{p} = |z|^{p}$.

Notice that we usually consider $\lceil z\rfloor^0 = \text{sign}(z)$ as a multivalued function, i.e. $\lceil z\rfloor^0 = 1$ if $z>0$, $\lceil z\rfloor^0 \in [-1,+1]$ if $z=0$, $\lceil z\rfloor^0 = -1$ if $z<0$. Likewise, let f a vector field, $L_f(z)$ represents the Lie derivative of z with respect to f.

For a vector $\mathbf{v}\in\mathbb{R}^n$, with components $v_i, i=1,...,n$, $\lceil \mathbf{v}\rfloor^p$ and $|\mathbf{v}|$ represent the $\mathbb{R}^n$ vectors defined by $\lceil \mathbf{v}\rfloor^p = [\lceil v_1\rfloor^p,...,\lceil v_n\rfloor^p]^T$ and $|\mathbf{v}| = [|v_1|,...,|v_n|]^T$, respectively. For $n\in\mathbb{N}$, $\mathbb{I}_n$ is the $n\times n$ identity matrix in $\mathbb{R}^{n\times n}$, i.e. $\mathbb{I}_n = \text{diag}\{1,\cdots,1\}$. Finally, $\text{diag}\{\mathbf{v}\}$ stands for the diagonal $\mathbb{R}^{n\times n}$ matrix

$$\text{diag}\{\mathbf{v}\} = \begin{bmatrix} v_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & v_n \end{bmatrix}.$$

2 | PRELIMINARIES

For a given vector $x=(x_1,...,x_n)^T\in\mathbb{R}^n$ and any positive scalar $\varepsilon>0$, the family of linear *dilation* operators is defined as $\Lambda_{\mathbf{r}}(\varepsilon)x := (\varepsilon^{r_1}x_1,...,\varepsilon^{r_n}x_n)$, where $r_i>0$ are the weights of the coordinates and $\mathbf{r}=(r_1,...,r_n)$ is the vector of weights.

A function $V:\mathbb{R}^n\to\mathbb{R}$ (respectively, a vector field $f:\mathbb{R}^n\to\mathbb{R}^n$, or a vector-set field $F\subset\mathbb{R}^n$) is called $\mathbf{r}$-homogeneous of degree $m\in\mathbb{R}$ if the identity $V(\Lambda_{\mathbf{r}}(\varepsilon)x) = \varepsilon^m V(x)$ holds for every $x\in\mathbb{R}^n$ and $\varepsilon>0$ (resp., $f(\Lambda_{\mathbf{r}}(\varepsilon)x) = \varepsilon^m\Lambda_{\mathbf{r}}(\varepsilon)f(x)$, or $F(\Lambda_{\mathbf{r}}(\varepsilon)x) = \varepsilon^m\Lambda_{\mathbf{r}}(\varepsilon)F(x)$)[3,27,8]. A system is called homogeneous if its vector field (or vector-set field) is $\mathbf{r}$-homogeneous of some degree. For a vector field f the Lie derivative of V along f is represented by $L_fV = \frac{\partial V}{\partial x}f$. For a vector-set field F we denote by $L_FV(x) = \{y\in\mathbb{R} \mid y = \partial_x V(x)\cdot v,\, v\in F(x)\}$ the set of values taken by the Lie derivative of V along all vector fields contained in F. If V and F are $\mathbf{r}$-homogeneous, then so is L_FV.

The homogeneous norm is defined by $\|x\|_{\mathbf{r},p} := \left(\sum_{i=1}^n |x_i|^{\frac{p}{r_i}}\right)^{\frac{1}{p}}$, $\forall x\in\mathbb{R}^n$, for any $p\geq\max\{r_i\}$, and it is $\mathbf{r}$-homogeneous of degree 1. The set $\mathbb{S} = \{x\in\mathbb{R}^n : \|x\|_{\mathbf{r},p}=1\}$ is the corresponding unit homogeneous sphere.

Stability of homogeneous systems can be studied by means of homogeneous Lyapunov functions (HLFs)[28,8,29,30,31]. Any continuous homogeneous system $\dot{x}=f(x)$, with globally asymptotically stable (GAS) equilibrium point, admits a $C^p(\mathbb{R}^n)$ HLF of degree m if $m>p\cdot\max\{r_i\}$ for any $p\in\mathbb{N}$[8].

For continuous homogeneous systems the type of convergence is characterized completely by the homogeneity degree of the system. The following Lemma is well-known[30,8]

Lemma 1. Let f be an $\mathbf{r}$-homogeneous continuous vector field of degree l_f. Let $\|\cdot\|_{\mathbf{r},p}$ be any homogeneous norm. Let $x=0$ be an equilibrium point of $\dot{x}=f(x)$, and assume that it is GAS. Then

(i) If $l_f=0$ then $x=0$ is *Exponentially Stable*, i.e., there exist positive constants $b_1,b_2>0$ such that $\forall x_0\in\mathbb{R}^n$ all solutions $x(t,x_0)$ satisfy
$$\|x(t,x_0)\|_{\mathbf{r},p} \leq b_1\exp(-b_2t)\,\|x_0\|_{\mathbf{r},p},\, \forall t\geq 0.$$

(ii) If $l_f>0$ then $x=0$ is *nearly Fixed-time Stable (nFxTS)*, i.e., there exists a positive constant $b>0$ such that $\forall x_0\in\mathbb{R}^n$ all solutions $x(t,x_0)$ satisfy
$$\|x(t,x_0)\|_{\mathbf{r},p} \leq \frac{1}{\left(1+l_fb\,\|x_0\|_{\mathbf{r},p}^{l_f}\,t\right)^{\frac{1}{l_f}}}\,\|x_0\|_{\mathbf{r},p},\, \forall t\geq 0.$$
Moreover, for any $\rho>0$ there exists $T_\rho\in(0,+\infty)$ such that $\|x(t,x_0)\|\leq\rho$ for all $t\geq T_\rho$ and all $x_0\in\mathbb{R}^n$.

(iii) If $l_f<0$ then $x=0$ is *Globally Finite-Time Stable (GFTS)*, i.e., $x(t,x_0)=0$ for all $t\geq T(x_0), \forall x_0\in\mathbb{R}^n$, where $T:\mathbb{R}^n\to\mathbb{R}_+$ is a settling-time function. Moreover, the settling-time $T(x_0)$ can be estimated, using an $\mathbf{r}$-homogeneous Lyapunov function of degree l_V, as follows
$$T(x_0) \leq \frac{1}{\kappa\left(\frac{-l_f}{l_V}\right)}V^{\frac{-l_f}{l_V}}(x_0).$$

□

Let $F:\mathbb{R}_{0+}\times\mathbb{R}^n\to\mathbb{R}^n$ be a vector-set field (multivalued map). It is said to satisfy *standard assumptions* if it fulfills the following conditions[32,30,3]: (i) $F(t,x)$ is a non-empty, compact, convex subset of $\mathbb{R}^n$ for each $t\geq 0$ and each $x\in\mathbb{R}^n$. (ii) $F(t,x)$, as a set-valued map of x, is upper semi-continuous for each $t\geq 0$. (iii) $F(t,x)$, as a set-valued map of t, is Lebesgue measurable for each $x \in \mathbb{R}^n$. (iv) $F(t,x)$ is locally bounded. For a differential inclusion (DI) $\dot{x} \in F(t,x)$ satisfying these hypotheses (the solutions are understood in the sense of Filippov[32]), if $0 \in F(t,0)$ for almost all $t \geq 0$, then the definitions of strong stability notions coincide with ones given in Lemma 1 (strong means that all solutions issued in an initial condition possess the property). In addition, $x = 0$ is strongly GAS, iff there exists a $C^p(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\})$ (homogeneous) strong LF[29,31,27]. Moreover, if F is $\mathbf{r}$-homogeneous of degree $l < 0$, then $x = 0$ is strongly globally finite-time stable (GFTS) and the settling time is continuous at zero and locally bounded[3,31,27].

Finally, for the Implicit Lyapunov Function Method the following characterization of ILFs[10] is crucial.

Theorem 1. Consider a system described by a DI

$$\dot{x}(t) \in F(t,x(t)), \quad t \in \mathbb{R}_+, \quad x(0) = x_0 \in \mathbb{R}^n, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $F : \mathbb{R}_{0+} \times \mathbb{R}^n \to \mathbb{R}^n$ is a multivalued map satisfying standard assumptions

$$0 \in F(t,0), \quad \text{for almost every } t \geq 0.$$

If there exists a continuous function

$$\begin{aligned} Q : \mathbb{R}_+ \times \mathbb{R}^n &\to \mathbb{R} \\ (V,x) &\to Q(V,x) \end{aligned}$$

satisfying the conditions:

C1) Q is continuously differentiable outside the origin;

C2) for any $x \in \mathbb{R}^n \setminus \{0\}$ there exists $V \in \mathbb{R}_+$ such that

$$Q(V,x) = 0;$$

C3) let $\Omega = \{(V,x) \in \mathbb{R}_+ \times \mathbb{R}^n : Q(V,x) = 0\}$ and

$$\lim_{x \to 0} V = 0, \qquad \lim_{V \to 0^+} \|x\| = 0, \qquad \lim_{\|x\| \to \infty} V = +\infty; \quad \forall (V,x) \in \Omega;$$

C4) $\frac{\partial Q(V,x)}{\partial V} < 0 \qquad \forall V \in \mathbb{R}_+$ and $x \in \mathbb{R}^n \setminus \{0\}$;

C5) $\sup_{t \in \mathbb{R}_+, y \in F(t,x)} \frac{\partial Q(V,x)}{\partial x} y < 0 \qquad \forall (V,x) \in \Omega$;

then the origin of system (1) is globally uniformly asymptotically stable.

Remark 1. It is important to stress that under the restrictions imposed on the function Q in Theorem 1 a Lyapunov function V exists, that is defined implicitly by the equation $Q(V(x),x) = 0$. This can be seen from the required conditions. Condition C2 requires V to be positive definite for any $x \in \mathbb{R}^n \setminus \{0\}$ and, together with C4, implies that V is a function. Condition C3 says that V is positive definite and radially unbounded. Finally, conditions C4 and C5 imply that $\dot{V}(x)$ is negative definite. The ILF $V(x)$ is continuously differentiable for every $x \in \mathbb{R}^n \setminus \{0\}$. However, it is not assured to be differentiable at $x = 0$.

3 | PROBLEM STATEMENT

Consider the following MIMO non-linear system

$$\begin{aligned} \dot{\xi} &= f(\xi) + \sum_{i=1}^{m} g_i(\xi) v_i + \psi(t,\xi) = f(\xi) + g(\xi) v + \psi(t,\xi), \\ \sigma_1 &= h_1(\xi), \\ &\vdots \\ \sigma_m &= h_m(\xi), \end{aligned} \tag{2}$$

where $\xi \in \mathbb{R}^s$ are the states, σ_i are the outputs and v_i are the control inputs. f and g_i are smooth vector fields, h_i are smooth real-valued output functions, and ψ is a smooth time dependent vector field representing some parameter or model uncertainties and/or external perturbations acting on the system. We write $\sigma = [\sigma_1, \cdots, \sigma_m]$ and $v = [v_1, \cdots, v_m]$, for the vector of outputs and control inputs, respectively. The outputs σ_i can represent tracking errors or sliding variables in sliding-mode control as e.g.[3,33,34]. The aim of the control is to render the outputs $\sigma_i = 0$ asymptotically or in finite-time despite of the acting perturbations/uncertainties ψ. When the (unperturbed) system (2) has a well-defined vector relative degree $\rho = [\rho_1, \cdots, \rho_m]$, with nonsingular matrix

$$b(\xi) = \begin{bmatrix} L_{g_1}L_f^{\rho_1-1}h_1(\xi) & \cdots & L_{g_m}L_f^{\rho_1-1}h_1(\xi) \\ L_{g_1}L_f^{\rho_2-1}h_2(\xi) & \cdots & L_{g_m}L_f^{\rho_2-1}h_2(\xi) \\ \vdots & \ddots & \vdots \\ L_{g_1}L_f^{\rho_m-1}h_m(\xi) & \cdots & L_{g_m}L_f^{\rho_m-1}h_m(\xi) \end{bmatrix} \in \mathbb{R}^{m\times m}, \tag{3}$$

and some regularity assumptions on the distributions, it is well-known[35] that the system can be transformed by a diffeomorphism of the states and a regular feedback $v = \alpha(\xi) + \beta(\xi)u$, with matrix $\beta(\xi) \in \mathbb{R}^{m\times m}$ invertible, to the (Byrnes-Isidori) Normal form

$$\dot{\eta} = q(\eta, x) + \mu_0(t, x, \eta), \tag{4}$$

$$\dot{x} = Ax + B(u + \mu_1(t, x, \eta)) + \mu_2(t, x, \eta) \tag{5}$$

$$y_1 = x_{1,1},$$

$$\vdots$$

$$y_m = x_{m,1},$$

where the vector $x = \mathrm{col}(x_1, \cdots, x_m) \in \mathbb{R}^n$ is composed of the partial state vectors $x_i \in \mathbb{R}^{\rho_i}$, $n = \sum_{i=1}^{m} \rho_i$, and $u \in \mathbb{R}^m$ is the transformed control vector. We have also included the uncertainties/perturbations from $\psi(t, \xi)$. Subsystem (4) corresponds to the zero dynamics, with state $\eta \in \mathbb{R}^{s-n}$. Matrices A and B have the Brunovsky canonical form, i.e.

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix} \in \mathbb{R}^{n\times n}, \qquad A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{\rho_i\times\rho_i}, \quad i = 1, \ldots, m.$$

$$B = \begin{bmatrix} \bar{b}_1 & 0 & \cdots & 0 \\ 0 & \bar{b}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{b}_m \end{bmatrix} \in \mathbb{R}^{n\times m}, \qquad \bar{b}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^{\rho_i\times 1}, \quad i = 1, \ldots, m.$$

The decoupling matrix $b(\xi)$ (3) becomes in these coordinates the identity matrix, i.e.

$$b = \mathbb{I}_m. \tag{6}$$

The uncertainties/perturbation vector $\psi(t, \xi)$ is decomposed in three terms: $\mu_0(t, x, \eta)$ affects the zero-dynamics, $\mu_1(t, x, \eta)$ is a matched term, acting on the control channel, and $\mu_2(t, x, \eta)$ is a not matched term, acting on the main dynamics.

In general[35], the representation (4)-(5) is valid only locally. But for simplicity in the presentation we will assume that it is globally valid, and we will state the results in a global form. However, it is easy to recover the local statements, when the representation is valid only locally. We will also assume that the (perturbed) zero dynamics (4) is well-behaved, in the sense that its trajectories are globally bounded for all times, whenever $(x(t), \eta(t))$ is bounded. This allows us to concentrate on the main dynamics (5) to solve the output-zeroing problem.

Thus, in this set-up, the *control objective* is to robustly stabilize the origin of system (5), asymptotically or in finite time, despite the perturbations μ_1, μ_2. When both terms μ_1 and μ_2 are vanishing, i.e. $\mu_1(t, 0, \eta) = 0$ and $\mu_2(t, 0, \eta) = 0$, it is possible to design a memoryless state feedback control law $u = \phi(x)$ to stabilize the origin of the plant (5). However, when $\mu_1(t, x, \eta)$ is non-vanishing, what is a reasonable assumption since it contains e.g. the derivatives of the signals to be tracked or external disturbances, a continuous controller is not able to render the origin $x = 0$ asymptotically stable. In this case, a *discontinuous* control law for u can solve the problem for any *bounded* but otherwise arbitrary perturbation $\mu_1(t, x, \eta)$. This is the main ingredient of High-Order Sliding-Mode (HOSM) controllers[3,33,34,9], and such a design can be obtained using the ILF Method, as it is presented in[10,26]. The disadvantage of a discontinuous control law for u resides in the undesirable *chattering* effect, which is a high-frequency oscillation of the control signal, that causes the excitation of unmodeled fast dynamics in the plant and the rapid deterioration of the actuators.

4 | MAIN RESULT: INTEGRAL CONTROLLER DESIGN USING THE IMPLICIT LYAPUNOV FUNCTION METHOD

In this paper, instead of using a discontinuous memoryless state-feedback we propose the use of the following *dynamic* control structure

$$\begin{aligned} u &= u_1(x) + z\,, \\ \dot{z} &= u_2(x)\,, \end{aligned} \tag{7}$$

which resembles the classical PI-controller. This controller is composed of a (continuous) state feedback u_1, which is able to stabilize the origin of the nominal system in the absence of the non-vanishing perturbation μ_1, and an *integral* term u_2, which is allowed to be discontinuous, and with the aim to estimate and compensate the perturbation term μ_1. Note that even when the function u_2 is discontinuous, the control signal u is *continuous*, since it is the addition of the signal generated by the continuous state feedback u_1 and the time integral of the discontinuous signal u_2. This fact may help in reducing the chattering effect.

Due to the multiple properties of homogeneous systems, as e.g. achieving in a simple form finite-time stability and making use of a powerful mathematical apparatus, we want to design the controller (7) such that the closed-loop system (without perturbations) is homogeneous of degree ν, for positive or negative values of ν. The weights of the vectors $(x_1,...,x_m)$ are given by $\mathbf{r}_i = (r_{i,1},...,r_{i,\rho_i})$, for $i = 1,...,m$, with components $r_{i,j+1} = r_{i,j} + \nu$, $j = 1,...,\rho_i$. The vector of weights can be written as $\mathbf{r} = [\mathbf{r}_1,...,\mathbf{r}_m]$, and the dilation matrix is given by

$$\Lambda_{\mathbf{r}}(\lambda) = \mathrm{diag}\{\lambda^{\mathbf{r}}\} = \begin{bmatrix} \mathrm{diag}\{\lambda^{\mathbf{r}_1}\} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{diag}\{\lambda^{\mathbf{r}_m}\} \end{bmatrix} = \begin{bmatrix} \lambda^{r_{1,1}} & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda^{r_{1,\rho_1}} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & \lambda^{r_{m,1}} & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & \lambda^{r_{m,\rho_m}} \end{bmatrix}.$$

We also fix (without loss of generality) all the weights of the components of vector z to be equal to 1, i.e. $r_{z,i} = r_z = 1$ for $i = 1,\cdots,m$. From their relationship with the weights of x, given by $r_{z,i} = r_{i,\rho_i} + \nu$, we conclude that

$$r_{z,i} = r_z = 1\,, \quad r_{i,\rho_i} = 1-\nu\,, \quad r_{i,j} = 1-(\rho_i+1-j)\,\nu\,, \quad -1 \le \nu < \frac{1}{\max \rho_i}\,, \qquad i = 1,\cdots,m,\ j = 1,\cdots,\rho_i\,. \tag{8}$$

For homogeneity of the closed-loop system, function u_1 requires to be homogeneous of degree 1, while function u_2 needs to be homogeneous of degree $1+\nu$. The homogeneity degree ν of the closed-loop system can be selected in the interval given in (8), because of the non negativity of the weights.

On the one extreme of the interval, when $\nu = -1$, function u_2 is discontinuous, with homogeneity degree zero. In this case the right-hand side of the closed-loop system is discontinuous, the trajectories are to be understood in the sense of Filippov[32], and convergence is in finite-time (recall Lemma 1). A linear state feedback with linear integral control has homogeneity degree $\nu = 0$, the right-hand-side is globally Lipschitz, and convergence is exponential. When $\nu > 0$ the right-hand side of the closed-loop system is locally but not globally Lipschitz and convergence is nearly fixed-time.

In this section, we present the main result of the paper: we provide a procedure, based on the Implicit Lyapunov Function Method introduced in[10,26], to design a homogeneous integral controller (7) that robustly stabilizes the origin of system (5), despite *non vanishing* matched perturbations μ_1, and *vanishing* unmatched perturbations μ_2. Selection of the homogeneity degree ν determines the type of convergence (in finite-time, exponential or asymptotic) and the size of the allowable perturbations. The convergence is nearly fixed-time for $\nu > 0$, exponential if $\nu = 0$ and in finite-time when $\nu < 0$. For $\nu > -1$ the matched perturbation μ_1 needs to be *constant* to be fully rejected. Otherwise, if it is slowly time-varying, just practical stability will be achieved, i.e. the trajectories will reach after some finite time a neighborhood of the origin and will remain in it for all future times. This is the usual and expected behavior of a PI controller with respect to the perturbation. In contrast, and this is a quite surprising property, when $\nu = -1$ the control law u_2 is *discontinuous*, the convergence is in finite-time and the matched perturbation μ_1 can be an *arbitrary Lipschitz* signal, i.e. μ_1 can be a growing signal, with bounded time derivative, and it will be *fully rejected* in the closed-loop. For μ_2 it is required in all cases, that it is vanishing, i.e. $\mu_2(t,x,z)=0$ when $x=0$. Its growth with respect to x will depend on ν, as it will be presented below. In general, the time derivative of μ_1 can also grow with the states, as detailed in the next subsection.

4.1 | The ILF homogeneous integral controller

We first *select* some value of the homogeneity degree ν in the interval $-1 \leq \nu < \frac{1}{\max \rho_i}$, what determines the corresponding weights, as given by (8). By solving the following matrix inequalities, for some $\varepsilon > 0$ and some positive definite and constant matrix $R \in \mathbb{R}^{n\times n} > 0$,

$$\begin{aligned} H_r P + P H_r &< 0\,, & H_r &= -\mathrm{diag}\{\mathbf{r}_i\} \\ P(A-BK) + (A-BK)^T P &\leq \varepsilon\,(H_r P + P H_r) - R\,, & \varepsilon &> 0\,,\ R > 0\,, \end{aligned} \tag{9}$$

we find a constant, symmetric and positive definite matrix $P = P^T \in \mathbb{R}^{n\times n}$ and a constant matrix $K \in \mathbb{R}^{m\times n}$. It is shown in [10,26] that inequalities (9) have always a solution $P > 0$ and K for any ν. Moreover, these matrix inequalities can be transformed to an LMI, which is manageable by using standard software. Furthermore, given P, it is also shown in [10,26] that the equation

$$Q(V,x) \triangleq x^T \Lambda_{\mathbf{r}}\left(V^{-1}\right) P \Lambda_{\mathbf{r}}\left(V^{-1}\right) x - 1 = 0\,, \tag{10}$$

defines *implicitly* a unique, continuous, homogeneous of degree 1, radially unbounded and positive definite function[1] $V:\mathbb{R}^n \to \mathbb{R}_+$, such that $Q(V(x),x)=0$ (see Theorem 1). Assume now that the perturbations μ_1 and μ_2 satisfy globally the following conditions:

$$\left|\left|\frac{d\mu_1}{dt}\right|\right|_\infty = ||\dot{\mu}_1||_\infty \leq D_1 + D_2 \|x\|_{\mathbf{r},p}^{1+\nu}\,, \qquad D_1, D_2 \geq 0 \tag{11}$$

$$\mu_2^T \Lambda_{\mathbf{r}}\left(V^{-1}\right) P R^{-1} P \Lambda_{\mathbf{r}}\left(V^{-1}\right) \mu_2 \leq \beta \varepsilon V^{2\nu} x^T \Lambda_{\mathbf{r}}\left(V^{-1}\right)\left(-H_r P - P H_r\right) \Lambda_{\mathbf{r}}\left(V^{-1}\right) x\,, \qquad 0 \leq \beta < 1\,. \tag{12}$$

Define also the following function (which is the partial derivative of Q with respect to V)

$$Q_V(V,x) := V^{-1} x^T \Lambda_{\mathbf{r}}\left(V^{-1}\right)\left(H_r P + P H_r\right) \Lambda_{\mathbf{r}}\left(V^{-1}\right) x\,.$$

Theorem 2. Consider the system (5) and the homogeneous integral controller

$$\begin{aligned} u &= u_1(x) + z = -V(x) K \Lambda_{\mathbf{r}}\left(V^{-1}(x)\right) x + z\,, \\ \dot{z} &= u_2(x) = \gamma \frac{V^\nu(x)}{Q_V(V(x),x)} B^T P \Lambda_{\mathbf{r}}\left(V^{-1}(x)\right) x\,, \quad \gamma > 0\,, \end{aligned} \tag{13}$$

for some $-1 \leq \nu < \frac{1}{\max \rho_i}$. Let $\zeta := z + \mu_1(t,x,\eta)$. Then for any $\beta < 1$, any $D_2 \geq 0$ and a sufficiently large $\gamma > 0$, the point $(x,\zeta)=0$ of the closed-loop system is

1. GFTS if $\nu=-1$, $D_1 > 0$ and $\frac{D_1+D_2}{\gamma}$ sufficiently small.
2. GFTS if $-1 < \nu < 0$ and $D_1 = 0$.
3. Globally Exponentially Stable if $\nu = 0$ and $D_1 = 0$.
4. nFxTS if $0 < \nu < \frac{1}{\max \rho_i}$ and $D_1 = 0$.

□

The proof of this result is deferred until the next Section 5. To show the convergence of the closed-loop with the integral controller (13) a (strong) Lyapunov Function will be used, which combines the ILF V obtained from (10) with an explicit term depending on the integral variable. The devised Lyapunov Function has the form (see (19) below)

$$\mathscr{V}(x,z) = \theta\left(\frac{1}{2}V^2(x) + \frac{1}{\gamma} z^T z\right)^{\frac{3-\nu}{2}} - \frac{1}{\gamma}\left[x_{1,\rho_1}, \cdots, x_{m,\rho_m}\right]\lceil z \rfloor^2\,, \quad \gamma > 0\,, \theta > 0\,. \tag{14}$$

Using this Lyapunov function and Lemma 1 (see the proof in Section 5) it is possible to estimate for $\nu \neq 0$ the transit time

[1] As noted in Remark 1, the ILF V is not assured to be differentiable at $x=0$. For example, for $\nu=0$ the ILF obtained from (10) is $V(x)=\sqrt{x^T P x}$, which is clearly not differentiable at $x=0$.

1. $\nu=-1$ and $D_1>0$ or $-1<\nu<0$ and $D_1=0$: The convergence time from an initial condition (x_0,ζ_0), where $\zeta_0=z_0+\mu_1(0)$ includes the initial value of the perturbation, is upper bounded by
$$T_{i\to 0}(x_0,\zeta_0)=\frac{3-\nu}{|\nu|\eta}\mathscr{V}^{\frac{|\nu|}{3-\nu}}(x_0,\zeta_0)\,,$$
for some $\eta>0$ depending on the parameters of the problem.

2. $0<\nu<\frac{1}{\max\rho_i}$ and $D_1=0$: The transit time from an arbitrarily large initial condition $(x_0,\zeta_0)\to\infty$ to a final condition different from zero $\left(x_f,\zeta_f\right)\neq 0$, is upper bounded by
$$T_{\infty\to f}\left(x_f,\zeta_f\right)=\frac{3-\nu}{\nu\eta}\frac{1}{\mathscr{V}^{\frac{\nu}{3-\nu}}\left(x_f,\zeta_f\right)}\,,$$
for some $\eta>0$ depending on the parameters of the problem.

In conclusion, for $\nu<0$ the convergence to zero from an arbitrary initial condition is in finite-time. For $\nu>0$ the convergence to zero happens only asymptotically, but the convergence to an, e.g., small ball around the origin from "infinity" occurs in finite-time.

Note from the Theorem 2 that (assuming for simplicity that $D_2=0$) if $D_1>0$, i.e. the perturbation μ_1 is not a *constant*, then convergence to zero will be only possible if $\nu=-1$, i.e. the controller u_2 is *discontinuous*. From Item (1) of the Theorem 2 it follows that a perturbation of any size $D_1>0$ can be fully compensated in this case by choosing a sufficiently large integral gain γ.

However, if $\nu>-1$ and $D_1>0$ the trajectories of the system are *globally uniformly ultimately bounded with bound b*, that is, (see[7]) for every initial condition there is a finite time T (independent of the initial time) such that the trajectories will enter a neighborhood of zero of radius b and remain there for all future times. This is also equivalent to saying that for the closed-loop system the map $\dot{\mu}_1\to(x,\zeta)$ is ISS. This is basically the content of the following Lemma, also proved in the next Section 5. The first part of this result can be derived from[36].

Lemma 3. Consider the closed-loop system of Theorem 2 under the same hypothesis.

(i) If $-1<\nu<\frac{1}{\max\rho_i}$ and $D_1>0$ the closed-loop system is Input-to-State Stable (ISS) from the input $\dot{\mu}_1$ to the state (x,ζ).

(ii) Let b be the ultimate bound. If $\frac{D_1+D_2}{\gamma}$ is sufficiently small, then
$$\lim_{\nu\to-1^+} b=0\,.$$
□

The second item in the Lemma 3 is interesting, since it shows that the nearer the homogeneity degree ν is to the discontinuous case $\nu=-1$, the smaller is also the effect of the perturbation μ_1 when it is not constant. This is, in some sense, intuitively appealing. Note however, that for this to be true it is required to have the ratio $\frac{D_1+D_2}{\gamma}$, between the size of $\dot{\mu}_1$ and the integral gain γ, small. If this is not the case, then the conclusion is false. Note that smallness of $\frac{D_1+D_2}{\gamma}$ can always be achieved by selecting γ sufficiently large.

4.2 | Discussion of the results

We present some observations with respect to the results.

- For the implementation of the controller (13) it is necessary to find the actual value of $V(x)$ by solving the (implicit) equation $Q(V,x)=0$. This can be hardly done analytically, so that it has to be obtained numerically on-line. A numerical procedure is proposed in[10].
- The asymptotic stability of the closed-loop, and moreover the existence of a smooth and strong Lyapunov Function[37], implies that the asymptotic stability is *robust* under rather general perturbations to the vector field, as e.g. small discretization errors, small delays in states or small noises acting on the variables. For homogeneous systems this robustness has some interesting forms (see e.g.[3,33] and[36]). In particular, for the implementation of the control law derived using the ILF method, it is shown in[10,26] that the discretization and numerical errors induced by the on-line solution of the implicit equation $Q(V,x)=0$ does not destroy the stability properties, and ultimate boundedness of the solutions is attained.

- For $-1 < \nu < \frac{1}{\max \rho_i}$ both control functions u_1 and u_2 in (13) are continuous everywhere. However, for $\nu = -1$ although u_1 is continuous everywhere, u_2 is continuous for $x \in \mathbb{R}^n \setminus \{0\}$. At $x = 0$ it is discontinuous, but its value is bounded (see remark 13 in[26, Remark 13]). This kind of controllers is usually named *Quasi-Continuous* controllers in the High-Order Sliding-Mode literature[38,39], in contrast to the *Discontinuous* ones, which have discontinuities also outside from $x = 0$.

- Note that in (11) the bound can also depend on z, i.e. $||\dot{\mu}_1||_\infty \leq D_1 + D_2 \|(x,z)\|_{\mathbf{r},p}^{1+\nu}$, without any change in the proof. Moreover, when $\nu = -1$ the bound becomes simply $||\dot{\mu}_1||_\infty \leq D_1 + D_2$.

- The matched perturbation, i.e. the perturbation entering through the same channel as the control variable, has two terms: the component $\mu_{2,n}$ of the vector μ_2, which is vanishing at $x = 0$, and μ_1, which is non vanishing. If μ_1 depends only on an external perturbation, what we can represent as an exogenous time-varying signal $\mu_1(t)$, then it has to be constant ($\dot{\mu}_1(t) \equiv 0$) for a continuous integral term ($\nu > -1$), but it can be an arbitrary *Lipschitz continuous* signal, i.e. $|\dot{\mu}_1(t)| \leq D_1 + D_2$, for the discontinuous integral term ($\nu = -1$). In this latter case the exogenous signal $\mu_1(t)$ can be time-varying and it does not require to be bounded, but its derivative has to be bounded. This is a much larger class of perturbations that can be fully compensated. When μ_1 is also a function of the states x, $\dot{\mu}_1$ may also depend on u and z. If condition (11) is satisfied only locally instead of globally, then the stability result will be also local.

- The bound (12) for μ_2 imposes to each of the components of the vector μ_2 to satisfy $|\mu_{2,i,j}(t,x,\eta)| \leq \delta_{i,j} \|x\|_{\mathbf{r},p}^{r_{i,j}+\nu}$, for some $\delta_{i,j} \geq 0$, and where the weights $r_{i,j}$ are given in (8). This requires $\mu_{2,i,j}$ has to be vanishing when $x = 0$, and to grow with the homogeneity degree corresponding to the variable $x_{i,j+1}$, i.e. of the component of the vector field of its channel. Since (12) depends on the value of the function V, which can be determined numerically on-line, the actual value of the allowed size of $\delta_{i,j}$, which also depends on K, can also be calculated on-line. This calculation is part of the process of design of the gain matrix K, which becomes a matrix inequality. Due to homogeneity it suffices to do that on the unit homogeneous sphere. In any case, asymptotic stability is preserved for sufficiently small values of $\delta_{i,j}$ due to its intrinsic robustness.

- The problem of how to check the implicitely defined inequality (12) has been already addressed in the paper[26]. Proposition 16 in[26] provides a simple sufficient condition to verify (12) without the necessity of calculating the ILF V. It requires only the value of matrix P.

- In the proof of the Theorem 2 it is also shown that the integral variable z converges (nearly fixed-time, exponentially or in finite-time) to the value of μ_1 if $D_1 = 0$ for $-1 < \nu < \frac{1}{\max \rho_i}$, or when $D_1 > 0$ if $\nu = -1$. This shows, as it is well-known for the classical case, that the integral part of the controller *reconstructs* the perturbation and thus is able to fully compensate for it.

- For this controller, if stability is achieved for some value of the integral gain γ, say γ^*, the stability is preserved for any $\gamma \geq \gamma^*$, without changing the gain K. This property can be understood from the passivity interpretation given in the proof of the Theorem in Section 5. The controller proposed in[19] posses this property, but it is not shared by other integral controllers presented in the literature as e.g.[21,22].

- In general, the gain selection here is easier than for the integral controllers designed using explicit Lyapunov functions, as e.g. the ones presented in[21,40].

- We note some other differences between the controller (13) and other integral controllers presented e.g. in[21,20,22,23]:

  (i) In (13) the integral action u_2 depends on the full state x, while u_2 in[21,20,22] can be a function of x_1 alone, or x_1 and a homogeneous function of any other states.

  (ii) For $\nu = -1$ the integral controller in (13) is discontinuous *only* at $x = 0$, so that it is of the quasi-continuous form. However, u_2 in[21,20] can be discontinuous on homogeneous varieties larger than the set $\{x = 0\}$.

  (iv) Since the ILF V is not smooth at $x = 0$ (see Remark 1), the Lyapunov function (14)(see next Section) for (13) is not smooth. In contrast, the Lyapunov functions for the controllers in[21,20,22] are smooth. Moreover, the basic idea of the proof is completely different, and the specific properties are different.

5 | PROOF OF MAIN RESULT: A LYAPUNOV FUNCTION APPROACH

We divide the proof of Theorem 2, including also Lemma 3, in three parts: i) First we consider the design of the state feedback controller u_1 using the ILF method. For this we assume that the matched perturbation μ_1 is absent. ii) Then we complete the ILF with an extra (explicit) term to build a weak Lyapunov Function to design the integral term u_2. Since the previous weak LF does not allow to assert robustness with respect to e.g. perturbation μ_1, we complete the weak LF with an extra cross-term in order to obtain a strong LF, which allows us to assure robustness and the type of convergence.

Recall that for a *weak* Lyapunov function V, $\sup_{t\in\mathbb{R}_+, y\in f(t,x)} \frac{\partial V(x)}{\partial x} y \leq 0, \forall x \neq 0$, so that using standard Lyapunov arguments[8,41] only uniform stability can be assured. In contrast, for a *strong* Lyapunov function $\sup_{t\in\mathbb{R}_+, y\in f(t,x)} \frac{\partial V(x)}{\partial x} y < 0, \forall x \neq 0$ and uniform asymptotic stability follows. Note also that we are considering in this work only the concept of *strong* stability for a differential inclusion. This means roughly that the property is satisfied for *all* solutions, in opposition to the concept of *weak* stability, which applies for *some* solution[8].

We start by showing that the ILF Lyapunov function V, obtained as solution of (10), and the controller functions u_1 and u_2 are homogeneous of degrees 1, 1 and $1+\nu$, respectively. Using the definition of the dilation matrix it is easy to see that $\Lambda_{\mathbf{r}}\left((\lambda V)^{-1}\right) = \Lambda_{\mathbf{r}}\left(V^{-1}\right)\Lambda_{\mathbf{r}}^{-1}(\lambda)$. This implies that function Q, as defined in (10), is homogeneous of degree 0 with weight 1 for V, i.e.

$$Q(\lambda V, \Lambda_{\mathbf{r}}(\lambda)x) = x^T \Lambda_{\mathbf{r}}(\lambda)\Lambda_{\mathbf{r}}^{-1}(\lambda)\Lambda_{\mathbf{r}}\left(V^{-1}\right)P\Lambda_{\mathbf{r}}\left(V^{-1}\right)\Lambda_{\mathbf{r}}^{-1}(\lambda)\Lambda_{\mathbf{r}}(\lambda)x - 1 = Q(V,x)\,.$$

This implies that the function V, which is solution of the equation (10), is homogeneous of degree 1, i.e. $V(\Lambda_{\mathbf{r}}(\lambda)x) = \lambda V(x)$.

The homogeneity degree of $u_1(x)$ is $r_z = 1$, since

$$u_1(\Lambda_{\mathbf{r}}(\lambda)x) = -V(\Lambda_{\mathbf{r}}(\lambda)x)K\Lambda_{\mathbf{r}}\left(V^{-1}(\Lambda_{\mathbf{r}}(\lambda)x)\right)\Lambda_{\mathbf{r}}(\lambda)x = -\lambda V(x)K\Lambda_{\mathbf{r}}\left(V^{-1}(x)\right)x = \lambda u_1(x)\,.$$

It follows also that function Q_V (since it is the partial derivative of Q with respect to V) is homogeneous of degree -1, i.e.

$$Q_V(\lambda V, \Lambda_{\mathbf{r}}(\lambda)x) = (\lambda V)^{-1}x^T\Lambda_{\mathbf{r}}(\lambda)\Lambda_{\mathbf{r}}\left((\lambda V)^{-1}\right)(H_rP+PH_r)\Lambda_{\mathbf{r}}\left((\lambda V)^{-1}\right)\Lambda_{\mathbf{r}}(\lambda)x = \lambda^{-1}Q_V(V,x)\,,$$

while $u_2(x)$ is homogeneous of degree $1+\nu$, that is

$$\begin{aligned} u_2(\Lambda_{\mathbf{r}}(\lambda)x) &= \gamma V^\nu(\Lambda_{\mathbf{r}}(\lambda)x)Q_V^{-1}(\lambda V, \Lambda_{\mathbf{r}}(\lambda)x)B^TP\Lambda_{\mathbf{r}}\left(V^{-1}(\Lambda_{\mathbf{r}}(\lambda)x)\right)\Lambda_{\mathbf{r}}(\lambda)x \\ &= \gamma\lambda^{1+\nu}V^\nu(Q_V(V,x))^{-1}B^TP\Lambda_{\mathbf{r}}\left(V^{-1}\right)x = \lambda^{1+\nu}u_2(x)\,. \end{aligned}$$

5.1 | Design of the static feedback control $u_1(x)$ using an ILF

We first design a state feedback controller $u_1(x)$ for the system

$$\dot{x} = Ax + Bu_1(x) + \mu_2(t,x,z)\,.$$

using the ILF Method, developed in[10,26]. The LF V is defined implicitly by equation (10), i.e.

$$Q(V,x) = x^T\Lambda_{\mathbf{r}}\left(V^{-1}\right)P\Lambda_{\mathbf{r}}\left(V^{-1}\right)x - 1, \quad P = P^T > 0\,.$$

Q satisfies all conditions C1-C5 of Theorem 1 (the details are given in[10,26]). Derivating Q with respect to time, we obtain

$$\dot{Q}(V,x) = Q_V(V,x)\dot{V} + \frac{\partial Q(V,x)}{\partial x}(Ax + Bu_1 + \mu_2) = 0$$

where

$$\begin{aligned} Q_V(V,x) &\triangleq \frac{\partial Q(V,x)}{\partial V} = V^{-1}x^T\Lambda_{\mathbf{r}}\left(V^{-1}\right)(H_rP+PH_r)\Lambda_{\mathbf{r}}\left(V^{-1}\right)x\,, \\ \frac{\partial Q(V,x)}{\partial x} &= 2x^T\Lambda_{\mathbf{r}}\left(V^{-1}\right)P\Lambda_{\mathbf{r}}\left(V^{-1}\right)\,, \end{aligned}$$

and $H_r = -\mathrm{diag}\{\mathbf{r}_i\}$. By hypothesis $H_rP + PH_r < 0$, so that condition C4 in Theorem 1 is satisfied, and we can obtain the derivative $\dot{V}$ from $\dot{Q}$ as

$$\dot{V} = -2(Q_V(V,x))^{-1}x^T\Lambda_{\mathbf{r}}\left(V^{-1}\right)P\Lambda_{\mathbf{r}}\left(V^{-1}\right)(Ax + Bu_1 + \mu_2)\,.$$

It can be easily shown that matrices A and B satisfy the following properties

$$\lambda^\nu\Lambda_{\mathbf{r}}(\lambda)A = A\Lambda_{\mathbf{r}}(\lambda)\,, \quad \Lambda_{\mathbf{r}}(\lambda)B = \lambda^{1-\nu}B\,.$$

Using them in the previous expression of $\dot{V}$, we achieve

$$\begin{aligned}\dot{V} &= -2\left(Q_V(V,x)\right)^{-1} x^T \Lambda_{\mathbf{r}}\left(V^{-1}\right) P\left[V^{\nu} A \Lambda_{\mathbf{r}}\left(V^{-1}\right) x + V^{-1+\nu} B u_1 + \Lambda_{\mathbf{r}}\left(V^{-1}\right)\mu_2\right] \\ &= -\left(Q_V(V,x)\right)^{-1} x^T \Lambda_{\mathbf{r}}\left(V^{-1}\right)\left[V^{\nu}\left(PA + A^T P\right)\Lambda_{\mathbf{r}}\left(V^{-1}\right) x + 2V^{-1+\nu} PBu_1 + 2P\Lambda_{\mathbf{r}}\left(V^{-1}\right)\mu_2\right].\end{aligned}$$

Selecting the controller u_1 as

$$u_1(x) = -VK\Lambda_{\mathbf{r}}\left(V^{-1}\right) x,$$

we obtain

$$\dot{V} = -\left(Q_V(V,x)\right)^{-1}\begin{bmatrix}\Lambda_{\mathbf{r}}\left(V^{-1}\right) x \\ \Lambda_{\mathbf{r}}\left(V^{-1}\right)\mu_2\end{bmatrix}^T \begin{bmatrix} V^{\nu}\left(P(A-BK) + (A-BK)^T P\right) & P \\ P & 0\end{bmatrix}\begin{bmatrix}\Lambda_{\mathbf{r}}\left(V^{-1}\right) x \\ \Lambda_{\mathbf{r}}\left(V^{-1}\right)\mu_2\end{bmatrix}. \tag{15}$$

We assume that the perturbation $\mu_2(t,x,z)$ satisfies the bound (12), that can be written as

$$\begin{bmatrix}\Lambda_{\mathbf{r}}\left(V^{-1}\right) x \\ \Lambda_{\mathbf{r}}\left(V^{-1}\right)\mu_2\end{bmatrix}^T \begin{bmatrix} -\beta\varepsilon V^{\nu}\left(H_r P + PH_r\right) & 0 \\ 0 & -V^{-\nu} PR^{-1} P\end{bmatrix}\begin{bmatrix}\Lambda_{\mathbf{r}}\left(V^{-1}\right) x \\ \Lambda_{\mathbf{r}}\left(V^{-1}\right)\mu_2\end{bmatrix} \geq 0, \tag{16}$$

where $0 \leq \beta < 1$. To show that when (16) is fulfilled then (15) is also satisfied we can use the well-known S-lemma. Alternatively, we use the following argumentation. Adding inequality (16) to the one of $\dot{V}$, i.e. (15), we arrive at

$$\dot{V} \leq -\left(Q_V(V,x)\right)^{-1}\begin{bmatrix}\Lambda_{\mathbf{r}}\left(V^{-1}\right) x \\ \Lambda_{\mathbf{r}}\left(V^{-1}\right)\mu_2\end{bmatrix}^T \begin{bmatrix} V^{\nu}\left(P(A-BK) + (A-BK)^T P - \beta\varepsilon\left(H_r P + PH_r\right)\right) & P \\ P & -V^{-\nu} PR^{-1} P\end{bmatrix}\begin{bmatrix}\Lambda_{\mathbf{r}}\left(V^{-1}\right) x \\ \Lambda_{\mathbf{r}}\left(V^{-1}\right)\mu_2\end{bmatrix}.$$

If the matrix of this quadratic form is negative definite, then we can conclude that $\dot{V} < 0$. Using Shur's complement this is the case if

$$V^{\nu}\left(P(A-BK) + (A-BK)^T P - \beta\varepsilon\left(H_r P + PH_r\right) + R\right) \leq 0.$$

And thus, condition (9), together with the definition of Q_V, implies that

$$\dot{V} \leq -(1-\beta)\varepsilon V^{\nu}\left(Q_V(V,x)\right)^{-1} x^T \Lambda_{\mathbf{r}}\left(V^{-1}\right)\left(H_r P + PH_r\right)\Lambda_{\mathbf{r}}\left(V^{-1}\right) x \leq -(1-\beta)\varepsilon V^{\nu+1}.$$

5.2 | Design of the dynamic control feedback $u_2(x)$ using an explicit (control) Lyapunov Function

Consider now the full closed-loop system, where we define as state variable $\zeta = z + \mu_1(t,x,\eta)$, which is the addition of the integral state z and the matched perturbation μ_1. Since μ_1 is unknown, this implies that the variable ζ is not available for feedback. Using as states (x,ζ) the dynamics of the closed-loop system are given by

$$\begin{aligned}\dot{x} &= Ax + B\left(u_1 + \zeta\right) + \mu_2(t,x,\eta), \\ \dot{\zeta} &= u_2(x) + d(t,x,\eta) \\ d(t,x,\eta) &:= \frac{d}{dt}\mu_1(t,x,\eta).\end{aligned}$$

The perturbation d is the total time derivative of the matched perturbation μ_1. According to (11), it is assumed to be bounded as $||d||_\infty \leq D_1 + D_2\|x\|_{\mathbf{r},p}^{1+\nu}$. Our aim in this section will be to design $u_2(x)$.

We note first that a direct utilization of the ILF method proposed in [10,26] is unfeasible for the design of a usable integral term $u_2(x)$, since the ILF method would lead to a controller as $u_2(x,\zeta)$, and since ζ is not measurable, the control would not be implementable. Moreover, a function u_2 depending on (x,ζ) is not a "true" integral action. For these reasons we have to combine the ILF V with some other (explicit) terms to arrive at a LF appropriate for the design of the integral controller. The following development is based on the idea used in [12,42] to obtain a LF for the Super-Twisting algorithm, which has been generalized to an arbitrary order in [19]. Since the ILF V is not smooth at $x = 0$ (see Remark 1), it leads to a non-smooth Lyapunov function. This technical issue does not cause any serious problems with the proof, since the lack of differentiability can be overcome by using e.g. the idea presented in [42] for the Super-Twisting (for more details see [23]). We will not repeat the argumentation in what follows.

5.2.1 | A weak Lyapunov Function

We first construct a homogeneous and smooth (except at $x = 0$) but weak Lyapunov function for the integral control, of degree $2r_z = 2$

$$W(x,\zeta) = \frac{1}{2}V^2(x) + \frac{1}{\gamma}\zeta^T\zeta, \quad \gamma > 0. \tag{17}$$

Using the results of the previous section for the function V and the perturbation μ_2, we arrive at the following expression for its derivative

$$\dot{W} \leq V\left\{-(1-\beta)\varepsilon V^{1+\nu} - 2V^{-1+\nu}\left(Q_V(V,x)\right)^{-1} x^T \Lambda_{\mathbf{r}}\left(V^{-1}\right) PB\zeta\right\} + \frac{2}{\gamma}\left(u_2+d\right)^T \zeta$$
$$= -(1-\beta)\varepsilon V^{2+\nu} - 2V^{\nu}\left(Q_V(V,x)\right)^{-1} x^T \Lambda_{\mathbf{r}}\left(V^{-1}\right) PB\zeta + \frac{2}{\gamma}u_2^T\zeta + \frac{2}{\gamma}d^T\zeta\,.$$

Note that the first term is negative (in x), while the second one is a cross-term, without definite sign. The third term depends on the selection of u_2, and the last one is the effect of the perturbation d. If we forget about the term due to the perturbation, we can render $\dot{W}$ at least negative semi-definite by selecting $u_2(x)$ such that the cross-term is cancelled, i.e. with

$$u_2(x) = \gamma V^{\nu}\left(Q_V(V,x)\right)^{-1} B^T P \Lambda_{\mathbf{r}}\left(V^{-1}\right) x\,, \tag{18}$$

where the parameter $\gamma > 0$ can be selected arbitrarily. With this selection we obtain

$$\dot{W} \leq -(1-\beta)\varepsilon V^{2+\nu} + \frac{2}{\gamma}d^T\zeta\,,$$

which is negative semi-definite for $d \equiv 0$, and therefore W is a weak Lyapunov function. Using the extended LaSalle's invariance theorem, which is presented in[43,44], we conclude that the origin $(x,\zeta) = 0$, in the absence of perturbation $d = 0$, is GAS. For $\nu = 0$ the convergence is exponential, since the homogeneity degree is zero, while for $\nu < 0$ the convergence is in finite-time, due to negative homogeneity degree of the vector field, and for $\nu > 0$ convergence is nearly fixed-time, due to the positive homogeneity degree (see Lemma 1).

A Passivity Interpretation

The weak Lyapunov function (17) has a simple passivity interpretation: The system is a negative feedback interconnection of two passive systems, subsystem x and subsystem ζ. Subsystem x is a (strictly) passive system with V as storage function, input u_1 and output u_2, as given in (18). Subsystem ζ is also passive, with storage function $\zeta^T\zeta$, input u_2 and output ζ. W is the storage function of the interconnected system.

5.2.2 | A strong Lyapunov function

The weak Lyapunov function W does neither allow us to establish the robustness of the closed-loop with respect to the perturbation d nor to estimate its convergence time, for example. It is advantageous to have a strong Lyapunov function, i.e. one with a negative definite derivative, instead of only a negative semi-definite one. In this section we obtain a strong LF by adding a cross-term to W as

$$\mathscr{V}(x,\zeta) = \theta W^{\alpha}(x,\zeta) - \frac{1}{\gamma}x_\rho^T\lceil\zeta\rfloor^{\omega}\,,\quad \gamma>0\,,\theta>0\,,\alpha = \frac{1+\omega-\nu}{2}\,,\omega = 2\,. \tag{19}$$

Here, the vector $x_\rho = (x_{1,\rho_1},\cdots,x_{m,\rho_m})^T$ is composed of the ρ_i-th components of the state x, with time derivative given by $\dot{x}_\rho = u_1 + \zeta + \bar{\mu}_2$ (since the decoupling matrix b presented in (6) is the identity), and the vector $\bar{\mu}_2 = (\mu_{2,\rho_1},\cdots,\mu_{2,\rho_m})$ contains the ρ_i-th components of the perturbation vector μ_2. Due to homogeneity we can conclude that $\mathscr{V}$ is positive definite for $\theta > 0$ sufficiently large.

Its time derivative along the closed-loop system is

$$\dot{\mathscr{V}}(x,\zeta,d) \leq \alpha\theta W^{\alpha-1}\left(-(1-\beta)\varepsilon V^{2+\nu} + \frac{2}{\gamma}d^T\zeta\right) - \frac{1}{\gamma}\left(u_1(x)+\zeta+\bar{\mu}_2\right)^T\lceil\zeta\rfloor^{\omega} - \frac{1}{\gamma}\omega x_\rho^T \mathrm{diag}\{|\zeta|^{\omega-1}\}\left(u_2(x)+d\right)$$
$$= -\mathscr{W}(x,\zeta) + \left(2\alpha\theta W^{\alpha-1}\zeta^T - \omega x_\rho^T\mathrm{diag}\{|\zeta|^{\omega-1}\}\right)\frac{1}{\gamma}d - \frac{1}{\gamma}\bar{\mu}_2^T\lceil\zeta\rfloor^{\omega}\,, \tag{20}$$

where

$$\mathscr{W}(x,\zeta) \triangleq \theta\alpha(1-\beta)\varepsilon W^{\alpha-1}(x,\zeta)V^{2+\nu}(x) + \frac{1}{\gamma}\omega x_\rho^T\mathrm{diag}\{|\zeta|^{\omega-1}\}u_2(x) + \frac{1}{\gamma}u_1^T(x)\lceil\zeta\rfloor^{\omega} + \frac{1}{\gamma}\zeta^T\lceil\zeta\rfloor^{\omega}\,.$$

Note that $\mathscr{V}(x,\zeta)$ is homogeneous of degree $\delta_{\mathscr{V}} = 1+\omega-\nu$, function $\mathscr{W}$ is homogeneous of degree $\delta_{\mathscr{W}} = 1+\omega$, while the term $\left(2\alpha\theta W^{\alpha-1}\zeta^T - \omega x_\rho^T\mathrm{diag}\{|\zeta|^{\omega-1}\}\right)$ is homogeneous of degree $\delta_d = \omega-\nu$. Note that $\delta_{\mathscr{W}} = \delta_d$ if $\nu = -1$, i.e. when $u_2(x)$ is a discontinuous function of homogeneity degree 0.

The derivative of the LF $\mathscr{V}$ in (20) has three terms. The second and third ones depend on the perturbations d and μ_2, respectively. In absence of these perturbations, the derivative of $\mathscr{V}$ is negative definite.

Lemma 4. $\mathscr{W}(x,\zeta) > 0$ for $\theta > 0$ large enough.

Proof. We note first that, although function u_2 is discontinuous at $x = 0$ for $\nu = -1$, the function $\mathscr{W}$ is continuous (and homogeneous). Recall the following well-known property of continuous homogeneous functions (see e.g.[9, Lemma 12]):

Let $\eta : \mathbb{R}^n \to \mathbb{R}$ and $\varphi : \mathbb{R}^n \to \mathbb{R}_+$ be two continuous homogeneous functions, with weights $\mathbf{r} = (r_1,...,r_n)$ and degrees m, with $\varphi(x) \geq 0$, such that it holds $\{x \in \mathbb{R}^n \backslash \{0\} : \varphi(x) = 0\} \subseteq \{x \in \mathbb{R}^n \backslash \{0\} : \eta(x) > 0\}$. Then, there exists a real number λ^* such that, for all $\lambda \geq \lambda^*$ for all $x \in \mathbb{R}^n \setminus \{0\}$, and some $c > 0$, $\eta(x) + \lambda\varphi(x) > c\|x\|^m_{\mathbf{r},p}$.

The claim of the Lemma is a simple consequence of this property. The first term in $\mathscr{W}$ is non negative and it vanishes only when $x = 0$. The value of $\mathscr{W}$ for $x = 0$ is $\mathscr{W}(0,\zeta) = \frac{1}{\gamma}\zeta^T\lceil\zeta\rfloor^{\omega}$, which is positive for $\zeta \neq 0$. And therefore $\mathscr{W}$ can be rendered positive definite selecting $\theta > 0$ sufficiently large (for any $\gamma > 0$). ☐

Due to homogeneity, there exist positive constants $0 < \underline{\eta}_{\mathscr{W}} < \bar{\eta}_{\mathscr{W}}$ and $\eta_d > 0$ such that

$$\underline{\eta}_{\mathscr{W}}\mathscr{V}^{\frac{\delta_{\mathscr{V}}+\nu}{\delta_{\mathscr{V}}}}(x,\zeta) \leq \mathscr{W}(x,\zeta) \leq \bar{\eta}_{\mathscr{W}}\mathscr{V}^{\frac{\delta_{\mathscr{V}}+\nu}{\delta_{\mathscr{V}}}}(x,\zeta),$$
$$\left|2\alpha\theta W^{\alpha-1}\zeta^T - \omega x_\rho^T \mathrm{diag}\{|\zeta|^{\omega-1}\}\right| \leq \eta_d\mathscr{V}^{\frac{\delta_d}{\delta_{\mathscr{V}}}}(x,\zeta).$$

Moreover, from inequality (12) it follows that each component $\mu_{2,i,j}$, for $i = 1,\cdots,m$, $j = 1,\cdots,\rho_i$, of the the vector μ_2 is bounded by $|\mu_{2,i,j}| \leq \beta\varepsilon\delta V^{r_{i,j}+\nu}$, for some $\delta > 0$. Since $r_{i,\rho_i} = 1-\nu$, this implies for the term $\bar{\mu}_2^T\lceil\zeta\rfloor^{\omega}$

$$|\bar{\mu}_2^T\lceil\zeta\rfloor^{\omega}| \leq \beta\varepsilon\bar{\delta}\mathscr{V}^{\frac{\delta_{\mathscr{V}}+\nu}{\delta_{\mathscr{V}}}}(x,\zeta),$$

for some $\bar{\delta} > 0$. Besides, from inequality (11) it also follows that (for some $\tilde{\eta}_d > 0$)

$$\left|\left(2\alpha\theta W^{\alpha-1}\zeta^T - \omega x_\rho^T\mathrm{diag}\{|\zeta|^{\omega-1}\}\right)\frac{1}{\gamma}d\right| \leq \frac{1}{\gamma}\left(\eta_d D_1\mathscr{V}^{\frac{\delta_d}{\delta_{\mathscr{V}}}}(x,\zeta) + \tilde{\eta}_d D_2\mathscr{V}^{\frac{\delta_d+1+\nu}{\delta_{\mathscr{V}}}}(x,\zeta)\right).$$

Accordingly, if we define $\tilde{\beta} = \beta\varepsilon\bar{\delta} + \tilde{\eta}_d D_2$, inequality (20) can be written as

$$\dot{\mathscr{V}}(x,\zeta) \leq -\left(\underline{\eta}_{\mathscr{W}} - \frac{1}{\gamma}\tilde{\beta}\right)\mathscr{V}^{\frac{3}{3-\nu}}(x,\zeta) + \eta_d\mathscr{V}^{\frac{2-\nu}{3-\nu}}(x,\zeta)\frac{1}{\gamma}D_1. \tag{21}$$

Remark 2. Note that the solution to the scalar differential equation $\dot{\xi} = -\kappa\xi^{\frac{3}{3-\nu}}$, with $\xi \geq 0$ $\kappa > 0$ and $-1 \leq \nu < \frac{1}{\max\rho_i}$, is given by

$$\xi^{-\frac{\nu}{3-\nu}}(t) = \xi^{-\frac{\nu}{3-\nu}}(t_0) - \frac{\nu}{\nu-3}\kappa(t-t_0),\ \text{if } \nu \neq 0, \qquad \text{or} \qquad \xi(t) = \exp(-\kappa(t-t_0))\xi(t_0),\ \text{if } \nu = 0.$$

From these expressions we can calculate that the transit time $T_{i\to f}$ to go from an initial value ξ_i to a final one ξ_f, for $\nu \neq 0$, is given by

$$T_{i\to f} = \frac{\nu-3}{\nu\kappa}\left(\xi_i^{-\frac{\nu}{3-\nu}} - \xi_f^{-\frac{\nu}{3-\nu}}\right).$$

If $\nu < 0$ and $\xi_f = 0$, then $T_{i\to 0} = \frac{\nu-3}{\nu\kappa}\xi_i^{-\frac{\nu}{3-\nu}}$ is finite. In contrast, if $\nu > 0$ and $\xi_0 \to \infty$, then $T_{\infty\to f} = \frac{3-\nu}{\nu\kappa}\xi_f^{-\frac{\nu}{3-\nu}}$ is finite. Note that this is related to the results of Lemma 1. ☐

From inequality (21) we come to the following conclusions:

1. In the absence of the non-vanishing part of the perturbation d, i.e. $D_1 \equiv 0$, or equivalently, if μ_1 is an arbitrary constant plus a term vanishing with the state x: the origin $(x,\zeta) = 0$ is Globally Asymptotically Stable (GAS) for any value of $-1 \leq \nu < \frac{1}{\max\rho_i}$ if the perturbation satisfies (12) with $\frac{\tilde{\beta}}{\gamma}$ sufficiently small, i.e. $0 \leq \frac{\tilde{\beta}}{\gamma} \leq \underline{\eta}_{\mathscr{W}}$. This can be always achieved selecting γ sufficiently large.

   Using the comparison principle and the Remark 2, we can obtain from (21) the convergence time from an initial condition (x_0,ζ_0) to the origin for $\nu < 0$ to be

$$T_{i\to 0} = \frac{\nu-3}{\nu\left(\underline{\eta}_{\mathscr{W}} - \frac{\tilde{\beta}}{\gamma}\right)}\mathscr{V}^{-\frac{\nu}{3-\nu}}(x_0,\zeta_0),$$

   or from an initial condition at infinity to a final condition (x_f,ζ_f) when $\nu > 0$ to be

$$T_{\infty\to f} = \frac{3-\nu}{\nu\left(\underline{\eta}_{\mathscr{W}} - \frac{\tilde{\beta}}{\gamma}\right)}\mathscr{V}^{-\frac{\nu}{3-\nu}}(x_f,\zeta_f).$$

2. In presence of a non-vanishing time-varying matched perturbation μ_1, i.e. $D_1 > 0$, we consider two situations, associated to the relation between the powers of $\mathscr{V}$ in (21):

   (a) $\nu = -1$**:** In this case the two powers are equal, (21) becomes $\dot{\mathscr{V}} \leq -\left(\underline{\eta}_{\mathscr{W}} - \frac{\tilde{\beta}}{\gamma} - \eta_d \frac{1}{\gamma} D_1\right) \mathscr{V}^{\frac{3}{4}}$ and the origin $(x, \zeta) = 0$ is Globally Finite-Time Stable if the perturbation is sufficiently small (or γ sufficiently large), i.e.

$$\frac{1}{\gamma} D_1 < \frac{\underline{\eta}_{\mathscr{W}} - \frac{\tilde{\beta}}{\gamma}}{\eta_d} . \tag{22}$$

   The convergence time from an initial condition is given by

$$T_{i\to 0} = \frac{4}{\left(\underline{\eta}_{\mathscr{W}} - \frac{\tilde{\beta}}{\gamma} - \eta_d \frac{1}{\gamma} D_1\right)} \mathscr{V}^{\frac{1}{4}} (x_0, \zeta_0) .$$

   (b) $-1 < \nu < \frac{1}{\max \rho_i}$**:** In this case, the powers in (21) satisfy $\frac{2-\nu}{3-\nu} < \frac{3}{3-\nu}$, so that the term due to the perturbation d dominates near the origin $(x, \zeta) = 0$ and $\dot{\mathscr{V}}$ can be positive in a neighborhood of zero. However, far from the origin the negative term is dominating, and $\dot{\mathscr{V}} < 0$ at points far from zero, i.e. choosing some $0 < \lambda < 1$,

$$\begin{aligned}
\dot{\mathscr{V}} &\leq -\lambda \left(\underline{\eta}_{\mathscr{W}} - \frac{\tilde{\beta}}{\gamma}\right) \mathscr{V}^{\frac{3}{3-\nu}} - (1-\lambda) \left(\underline{\eta}_{\mathscr{W}} - \frac{\tilde{\beta}}{\gamma}\right) \mathscr{V}^{\frac{3}{3-\nu}} + \eta_d \mathscr{V}^{\frac{2-\nu}{3-\nu}} \frac{1}{\gamma} D_1 \\
&= -\lambda \left(\underline{\eta}_{\mathscr{W}} - \frac{\tilde{\beta}}{\gamma}\right) \mathscr{V}^{\frac{3}{3-\nu}} - \left[(1-\lambda) \left(\underline{\eta}_{\mathscr{W}} - \frac{\tilde{\beta}}{\gamma}\right) \mathscr{V}^{\frac{1+\nu}{3-\nu}} - \eta_d \frac{1}{\gamma} D_1\right] \mathscr{V}^{\frac{2-\nu}{3-\nu}} \\
&\leq -\lambda \left(\underline{\eta}_{\mathscr{W}} - \frac{\tilde{\beta}}{\gamma}\right) \mathscr{V}^{\frac{3}{3-\nu}} , \quad \forall \mathscr{V}(x, \zeta) \geq \left(\frac{\eta_d \frac{1}{\gamma} D_1}{(1-\lambda)\left(\underline{\eta}_{\mathscr{W}} - \frac{\tilde{\beta}}{\gamma}\right)}\right)^{\frac{3-\nu}{1+\nu}} .
\end{aligned}$$

   From this latter inequality we conclude that the trajectories of the closed-loop system are ultimately and uniformly bounded, and that they will arrive at the set $\mathscr{V}(x, \zeta) \leq \left(\frac{\eta_d \frac{1}{\gamma} D_1}{(1-\lambda)\left(\underline{\eta}_{\mathscr{W}} - \frac{\tilde{\beta}}{\gamma}\right)}\right)^{\frac{3-\nu}{1+\nu}}$ in finite-time, and they will remain there for all future times. This is also equivalent to saying that the system is ISS with respect to d.

   Note that when D_1 satisfies (22), we can select λ such that $\frac{\eta_d \frac{1}{\gamma} D_1}{(1-\lambda)\left(\underline{\eta}_{\mathscr{W}} - \frac{\tilde{\beta}}{\gamma}\right)} < 1$. Since $\lim_{\nu \to -1^+} \frac{3-\nu}{1+\nu} = +\infty$ we conclude that the final bound for the trajectories shrinks to zero as $\nu \to -1$, i.e. $\lim_{\nu \to -1^+} \left(\frac{\eta_d \frac{1}{\gamma} D_1}{(1-\lambda)\left(\underline{\eta}_{\mathscr{W}} - \frac{\tilde{\beta}}{\gamma}\right)}\right)^{\frac{3-\nu}{1+\nu}} = 0$.

6 | SIMULATION RESULTS

In this section we illustrate the behavior of the integral controllers of different homogeneity degrees developed in the paper. For this we perform a simulation study on the following academic example

$$\begin{aligned}
\dot{x} = \begin{bmatrix} \dot{x}_{1,1} \\ \dot{x}_{2,1} \\ \dot{x}_{2,2} \\ \dot{x}_{3,1} \\ \dot{x}_{3,2} \\ \dot{x}_{3,3} \end{bmatrix} &= \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \end{bmatrix} x + \begin{bmatrix} 10 & -5 & 3 \\ 0 & 0 & 0 \\ 2 & 7 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 8 \end{bmatrix} (u + \mu_1) + \mu_2 \\
y &= \begin{bmatrix} x_{1,1} \\ x_{2,1} \\ x_{3,1} \end{bmatrix} .
\end{aligned} \tag{23}$$

This is a MIMO system, having 6 states, 3 control inputs and 3 outputs. It is assumed that is already in the normal form. Although the matrix B is not in the Brunovsky form, it can be easily transformed to it by just multiplying the control input by an invertible matrix M, i.e. $v = Mu$. The subsystems have relative degrees 1, 2 and 3, respectively. For the simulation we use as non-vanishing matching perturbation μ_1, given by

$$\mu_1(t) = \begin{bmatrix} 0.5 + 0.05\sin(t) \\ 0.25 + 0.1\cos(t) \\ 0.1t \end{bmatrix}.$$

μ_1 is time-varying and the third component is a ramp. The vanishing non matched perturbation μ_2 is state-dependent and is given for the simulation as

$$\mu_2(x) = \begin{bmatrix} \mu_{2,1} \\ \mu_{2,2} \\ \mu_{2,3} \\ \mu_{2,4} \\ \mu_{2,5} \\ \mu_{2,6} \end{bmatrix} = \begin{bmatrix} 0.3\lceil x_{1,1} \rfloor^{\frac{1}{r_{1,1}}} + 0.2\lceil x_{2,2} \rfloor^{\frac{1}{r_{2,2}}} \\ 0.2\lceil x_{2,1} \rfloor^{\frac{r_{2,2}}{r_{2,1}}} + 0.1x_{2,2} \\ 0.2\lceil x_{1,1} \rfloor^{\frac{1}{r_{1,1}}} + 0.2\lceil x_{2,1} \rfloor^{\frac{1}{r_{2,1}}} \\ 0.5\lceil x_{3,1} \rfloor^{\frac{r_{3,2}}{r_{3,1}}} \\ 0.1\lceil x_{3,1} \rfloor^{\frac{r_{3,3}}{r_{3,1}}} + 0.1x_{3,3} \\ 0.3\lceil x_{3,2} \rfloor^{\frac{1}{r_{3,2}}} \end{bmatrix}.$$

Note that the powers correspond to the weights of homogeneity associated to the variables. This is chosen in this way, because the growth of the vanishing perturbation which can be compensated depends on the homogeneity degree of the integral control designed.

We design 4 integral controllers, given by equation (13), with homogeneity degrees $\nu = \{-1, -\frac{1}{2}, 0, \frac{1}{4}\}$. In all cases we find the Lyapunov matrix P and the state-feedback gain matrix K by solving an LMI problem derived from equations (9). For this YALMIP of MATLAB$^{\text{TM}}$ with the SeDuMi solver, is used. A value of $\varepsilon = 0.5$ was used. The following matrices are obtained, for each of the controllers

$$P_L = \begin{bmatrix} 0.1653 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5053 & 0.1685 & 0 & 0 & 0 \\ 0 & 0.1685 & 0.1685 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.2731 & 1.0042 & 0.2162 \\ 0 & 0 & 0 & 1.0042 & 1.1071 & 0.2677 \\ 0 & 0 & 0 & 0.2162 & 0.2677 & 0.1338 \end{bmatrix}, \quad K_L^T = \begin{bmatrix} -1.4613 & -2.5934 & 0.8822 \\ 2.3072 & 0.2229 & -1.0588 \\ 2.2188 & 0.0806 & -1.0011 \\ 0.1116 & 0.3246 & 0.6270 \\ 0.0965 & 0.3744 & 0.8683 \\ 0.1136 & 0.2308 & 0.2890 \end{bmatrix},$$

$$P_H = \begin{bmatrix} 0.0327 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.3230 & 0.0680 & 0 & 0 & 0 \\ 0 & 0.0680 & 0.0340 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5.0131 & 1.9358 & 0.2048 \\ 0 & 0 & 0 & 1.9358 & 0.9107 & 0.1122 \\ 0 & 0 & 0 & 0.2048 & 0.1122 & 0.0249 \end{bmatrix}, \quad K_H^T = \begin{bmatrix} 1.3902 & 1.0852 & -2.3251 \\ 1.3689 & 2.2945 & -3.0996 \\ 0.5738 & 0.9693 & -1.4777 \\ 8.6435 & 3.0146 & 4.7366 \\ 4.7411 & 1.6549 & 2.5864 \\ 1.1533 & 0.4343 & 0.3531 \end{bmatrix},$$

$$P_D = \begin{bmatrix} 0.0094 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1921 & 0.0285 & 0 & 0 & 0 \\ 0 & 0.0285 & 0.0095 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8.0861 & 1.9738 & 0.1345 \\ 0 & 0 & 0 & 1.9738 & 0.5780 & 0.0457 \\ 0 & 0 & 0 & 0.1345 & 0.0457 & 0.0065 \end{bmatrix}, \quad K_D^T = \begin{bmatrix} -1.5880 & -5.2796 & -2.8393 \\ 17.5418 & 1.0609 & -7.4199 \\ 5.6924 & 0.1046 & -2.3723 \\ 42.9179 & -6.2731 & 20.9095 \\ 14.6293 & -2.1035 & 7.0112 \\ 2.2693 & -0.1809 & 0.6029 \end{bmatrix},$$

$$P_{PH} = \begin{bmatrix} 0.3411 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4060 & 0.1709 & 0 & 0 & 0 \\ 0 & 0.1709 & 0.3419 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1597 & 0.1778 & 0.0817 \\ 0 & 0 & 0 & 0.1778 & 0.6238 & 0.2103 \\ 0 & 0 & 0 & 0.0817 & 0.2103 & 0.2804 \end{bmatrix}, \quad K_{PH}^T = \begin{bmatrix} 1.1855 & 1.3700 & -1.4182 \\ -0.2230 & 0.2721 & -0.0904 \\ -0.5456 & 0.3841 & -0.1159 \\ 0.1363 & -0.0087 & 0.0292 \\ 0.2764 & -0.0720 & 0.2400 \\ 0.4796 & -0.0219 & 0.0731 \end{bmatrix}.$$

Subindex L stands for the linear integral controller (with homogeneity degree $\nu = 0$), H represents the homogeneous continuous controller (with homogeneity degree $\nu = -\frac{1}{2}$), D corresponds to discontinuous case (with homogeneity degree $\nu = -1$), while PH symbolize the integral controller with positive homogeneity degree ($\nu = \frac{1}{4}$).

As integration method for the simulation a fourth-order Runge-Kutta method of fixed step is used. The sampling time was 1×10^{-5}[s]. In all cases an integral gain of $\gamma = 5$ is implemented. During the simulation, the actual value of the (implicit) Lyapunov function is obtained numerically on-line using the method presented in [10].

The simulation results are organized in two groups.

(i) Figures 1 to 3, present the results for the integral controllers L, H and D. They illustrate mainly the behavior in steady-state, since negative homogeneity degrees are particularly good performing near the equilibrium point. In particular, we emphasize the high precision of the discontinuous controller D, despite of time-varying perturbations. For these simulations a small initial condition $x_0 = x(0) = \begin{bmatrix} 0.2 & 0.2 & 0 & 0.2 & 0 & 0 \end{bmatrix}^T$ is selected.

Figure 1 contains the time behavior of the states for the linear L (Figure 1a), the homogeneous H (Figure 1b) and the discontinuous D (Figure 1c) integral controllers, respectively. As expected, the steady-state behavior of the discontinuous D controller is much better, i.e. the error is smaller, than that of the homogeneous H and of the linear L ones. Moreover, the smaller the homogeneity degree, the smaller also the final error. Since the initial condition is small, it is also noticeable that the D controller converges faster than the other ones.

Figure 2 shows the three control signals generated by the L, H and D integral controllers. As is characteristic of the integral action, all are continuous. Note, moreover, that in steady-state they all converge to the inverse of the perturbation, since they aim to compensate for it. As shown in the previous figures, the lower the homogeneity degree, the better is the compensation and the nearer the control signal is to the perturbation.

It has been shown in the main Theorem, that the signal $z + \mu_1$ converges to zero. This is a characteristic of the integral action, being able to estimate the non-vanishing perturbation in order to counteract its influence in the system. The time evolution of $z + \mu_1$ is depicted in Figure 3 for the three integral controllers L, H and D. Again, the discontinuous controller D is able to force this signal to zero in finite-time, showing that it can estimate exactly the perturbation, while the H and the L controllers are only capable to perform an approximate estimation. Once again, the smaller the homogeneity degree the smaller is also the estimation error.

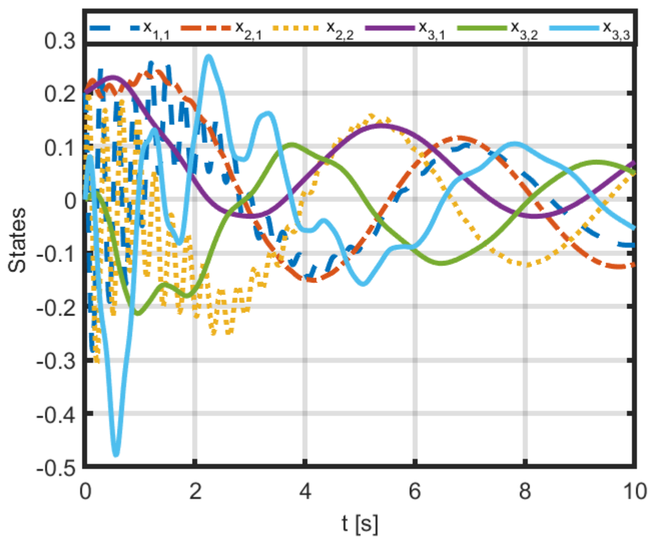
(ii) Fig. 4 shows the behavior of the PH integral controller, compared to the linear (L) and the discontinuous (D) ones. Since a remarkable characteristic of controllers with positive homogeneity degree is its high velocity of convergence for large initial conditions, a much larger initial state $x(0) = 3,000 \times [1,1,1,1,1,1]$ was selected. For these simulations we also eliminated the perturbations, i.e. $\mu_1 = 0$, $\mu_2 = 0$, since they are more relevant for the steady-state behavior.

Figure 4 presents the Euclidean norm $\|x(t)\|_2$ of the states for the integral controller with positive homogeneity degree PH. For comparison, the corresponding norms for the linear L and the discontinuous D integral controllers are also shown. It is apparent that the convergence velocity to a neighborhood of the equilibrium for the PH controller is much higher than that for the linear and the discontinuous controllers.

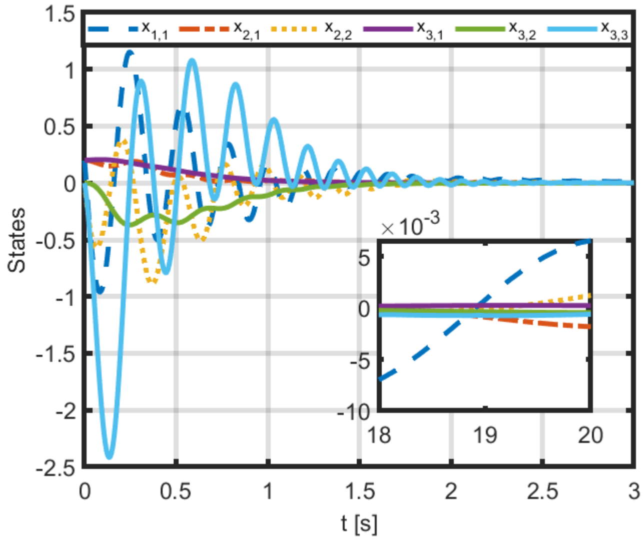
7 | CONCLUSIONS

We have shown that it is possible to combine an Implicit Lyapunov function, obtained using the ILF method, with some extra terms to design integral controllers of arbitrary positive or negative degree. This includes the discontinuous integral action, which stands out because it is able to fully compensate non-vanishing Lipschitz matched perturbations. Continuous integral controllers are in contrast able to counteract just constant perturbations. They reach the equilibrium point in finite time when the homogeneity degree is negative, exponentially for homogeneity degree zero or nearly fixed-time, when the homogeneity degree is positive. This latter type of convergence is very weak near the equilibrium, but it is very strong for initial conditions very far from it. They reach a neighborhood of the equilibrium in a time independent of the initial condition. A natural task to do is to combine controllers of different degrees of homogeneity. For integral controllers this combination is not so straightforwardly as for memoryless state feedback.

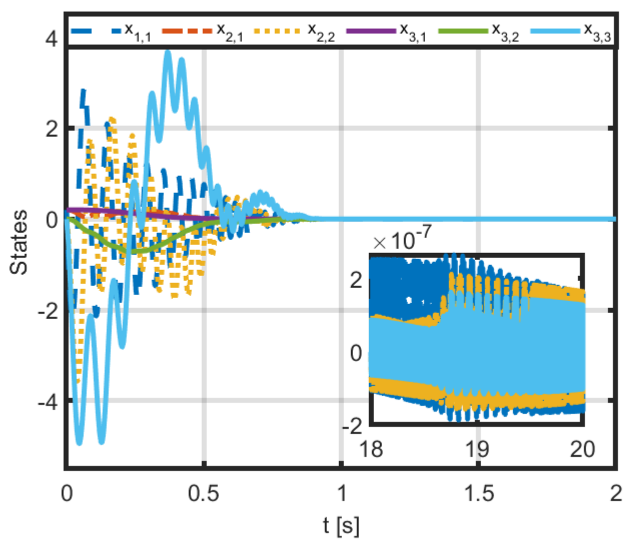
The class of systems considered includes MIMO nonlinear systems which are partially feedback linearizable and minimum phase. They are transformed to the Byrnes-Isidori normal form and (partially) linearized in order to take full advantage of the



(a) Homogeneity degree $\nu = 0$.



(b) Homogeneity degree $\nu = -\frac{1}{2}$.



(c) Homogeneity degree $\nu = -1$.

FIGURE 1 Time evolution of all states for Integral controllers L (1a), H (1b) and D (1c)

weighted homogeneity property, which is coordinate dependent. In the original coordinates, although the homogeneity (in strict sense) is lost, the convergence and robustness properties remain. Compared to other methods to design integral controllers the ILF-based method proposed here is attractive because the gain design can be converted to an LMI problem. This is in contrast to other methods, where the finding of appropriate gains is a highly nonlinear problem.

8 | ACKNOWLEDGEMENTS

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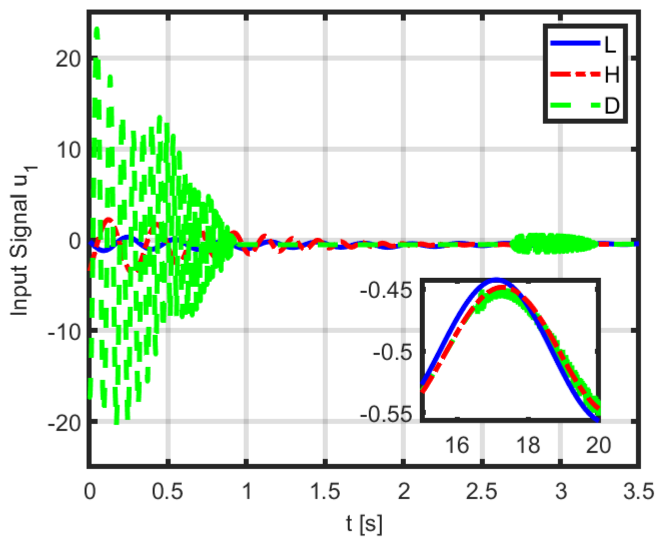
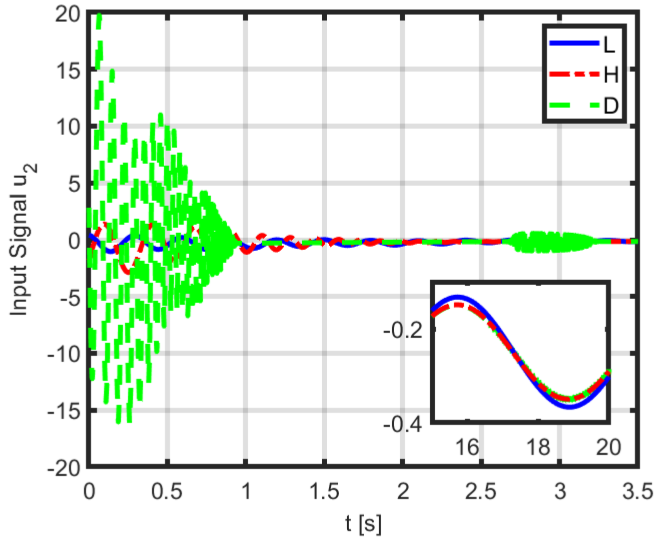
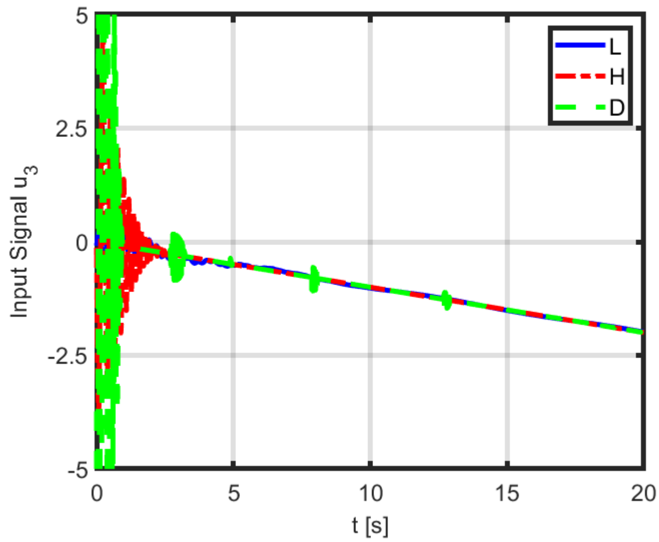


(a) Input signal u_1.



(b) Input signal u_2.



(c) Input signal u_3.

FIGURE 2 Time evolution of the three control signals u_1 (2a), u_2 (2b) and u_3 (2c), generated by the Integral controllers L, H and D.

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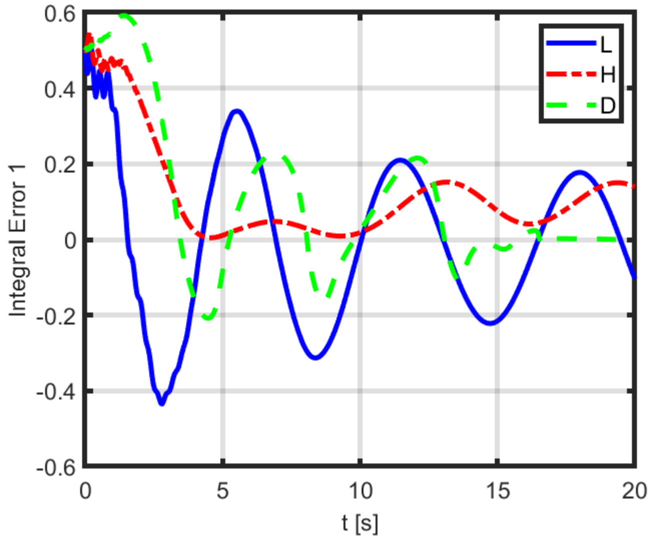
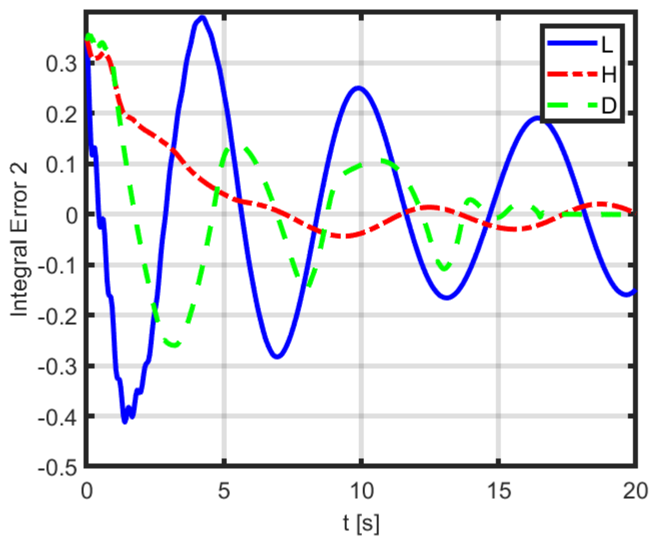
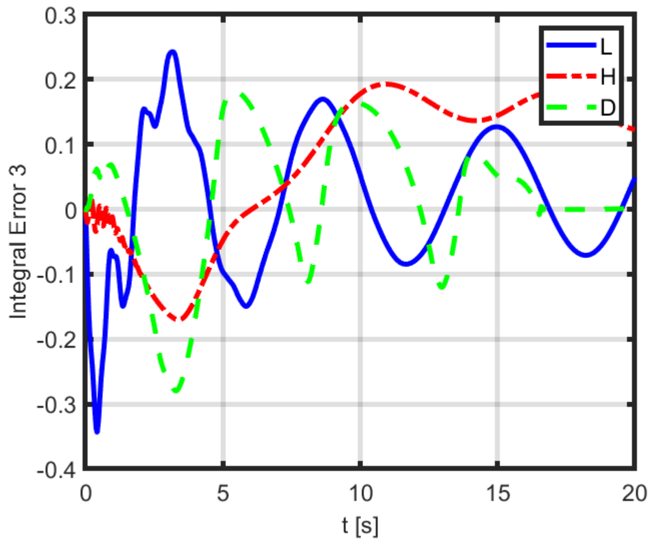


(a) First integral error signal $z_1 + \mu_{1,1}$.



(b) Second integral error signal $z_2 + \mu_{1,2}$.



(c) Third integral error signal $z_3 + \mu_{1,3}$.

FIGURE 3 Time evolution of the three integral error signals $z + \mu_1$, i.e. $z_1 + \mu_{1,1}$ (3a), $z_2 + \mu_{1,2}$ (3b), and $z_3 + \mu_{1,3}$ (3c), for the Integral controllers L, H and D.

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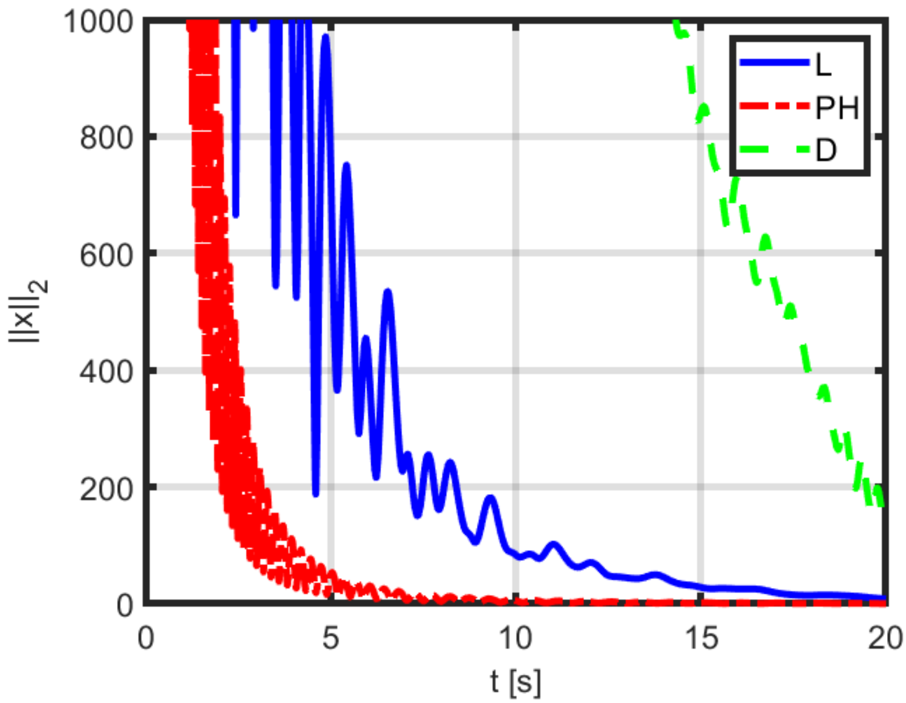

FIGURE 4 Time evolution of the Euclidean norm $\|x(t)\|_2$ of the states for the Integral controllers *L*, *PH* and *D*.

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