How to Take a Function Apart with SboxU

(Also Featuring some New Results on Ortho-Derivatives)

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Boolean **F**unctions and their **A**pplications 2020

A wild vectorial Boolean function appears!

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What do you do?

Outline

Plan of this Section

1. Basic Functionalities
   - Installation
   - Core Functionalities
2. CCZ-Equivalence
3. Ortho-Derivative
4. Conclusion

Plan of this Section

How to

- You need to have SAGE installed
- Then head to `https://github.com/lpp-crypto/sboxU`

Demo

Sbox from SAGE vs. sboxU

There are already many functions for investigating vectorial boolean functions in SAGE:

- Class SBox from `sage.crypto.sbox` (or `sage.crypto.mq.sbox` in older versions)
- Module `boolean_function` from `sage.crypto`

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SAGE SBox

- Supports output size $\neq$ input size
- Sub-routines written in Python or Cython
- Built-in SAGE

sboxU

- **Assumes** output size $=$ input size
- Sub-routines written in Python or multi-threaded C++
- Cutting edge functionalities

Plan of this Section

Some Tools

1. DDT/LAT (+ Pollock representation thereof)

Demo

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4. Linear mappings

   Demo

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CCZ- and EA-equivalence

Definition (CCZ-Equivalence)

$F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ and $G : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ are *C(arlet)-C(harpin)-Z(inoviev) equivalent* if

$$\Gamma_G = \left\{ (x, G(x)), \forall x \in \mathbb{F}_2^n \right\} = L\left(\left\{ (x, F(x)), \forall x \in \mathbb{F}_2^n \right\} \right) = L(\Gamma_F) ,$$

where $L : \mathbb{F}_2^{n+m} \rightarrow \mathbb{F}_2^{n+m}$ is an affine permutation.

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Definition (EA-Equivalence; EA-mapping)

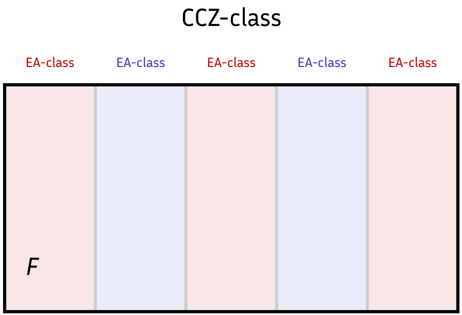
F and G are *E(xtented) A(ffine) equivalent* if $G(x) = (B \circ F \circ A)(x) + C(x)$, where A, B, C are affine and A, B are permutations; so that

$$\left\{(x, G(x)), \forall x \in \mathbb{F}_2^n\right\} = \begin{bmatrix} A^{-1} & 0 \\ CA^{-1} & B \end{bmatrix} \left(\left\{(x, F(x)), \forall x \in \mathbb{F}_2^n\right\}\right) .$$

Some Algorithmic Problems with CCZ-Equivalence

CCZ-class

Some Algorithmic Problems with CCZ-Equivalence



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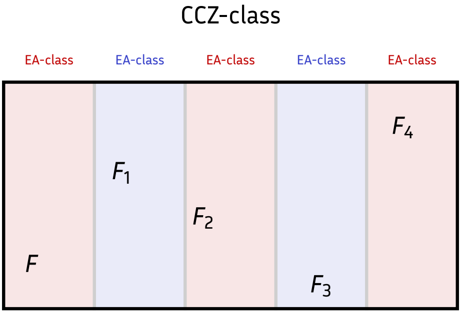

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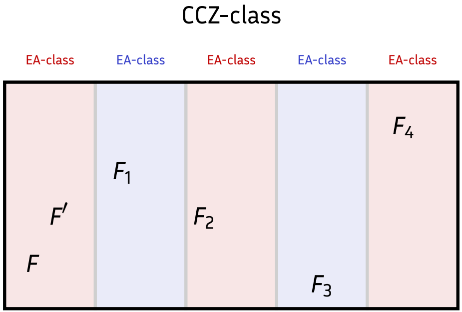

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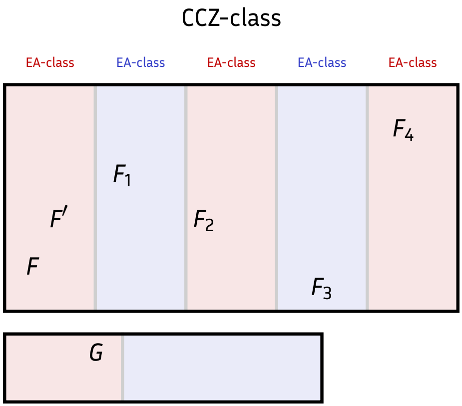

Plan of this Section

Exploring a CCZ-class

`Demo`

Class Invariants

Definition (Differential spectrum)

Recall that $DDT_F[a, b] = \#\{x, F(x + a) + F(x) = b\}$. The differential spectrum is the number of occurrences of each number in the DDT.

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- Differential and extended Walsh spectra are constant in a **CCZ**-class.
- The algebraic degree and the **thickness spectrum** are constant in an **EA**-class.

Demo

Plan of this Section

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Ortho-Derivative Let F be a quadratic function of $\mathbb{F}_2^n$. The ortho-derivatives of F are the functions of $\mathbb{F}_2^n$ such that

$$\forall x \in \mathbb{F}_2^n,\ \pi_F(a) \cdot \Big(\underbrace{F(x+a) + F(x)}_{\Delta_a F(x)} + F(a) + F(0) \Big) = 0\ .$$

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- $\pi_F(a)$ is orthogonal to the linear part of the hyperplane $\mathrm{Im}(\Delta_a F)$
- π_F can take any value in 0.

Basic Properties

Lemma (Ortho-derivatives of APN functions)

F is APN if and only if $\pi_F(a)$ is uniquely defined for all $a \in (\mathbb{F}_2^n)^*$.

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If $G = B \circ F \circ A$ where A and B are linear permutations, then

$$\pi_F = (B^T)^{-1} \circ F \circ A$$

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It seems like[1] the algebraic degree of the ortho-derivative of an APN function is always $n - 2$.

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Preimages of the Ortho-Derivative

Theorem

Linear Structures (APN case) If

$$T_F(b) = \left\{x \in \mathbb{F}_2^n : \pi_F(x) = b\right\} ,$$

then $T_F(b) = \mathrm{LS}(x \mapsto b \cdot F(x))$.

Corollary

For any b, $T_F(b)$ *is a linear subspace of* $\mathbb{F}_2^n$ *whose dimension has the same parity as* n. *Furthermore,*

$$\left(\mathcal{W}_F[a, b]\right)^2 \in \left\{0,\ 2^{n+\dim T_F(b)}\right\}$$

Plan of this Section

1. Basic Functionalities
2. CCZ-Equivalence
3. Ortho-Derivative
   - Definition and Basic Theorems
   - Algorithmic Uses
   - Inverting the DDT of a Quadratic Function
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Identifying EA- and CCZ-classes

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The differential and ***extended Walsh spectra*** *of the ortho-derivative of an APN function is the same within an EA-class.*

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In practice, these spectra differ from one EA-class to the next!

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In practice, these spectra differ from one EA-class to the next!

We can use this to very efficiently sort large numbers of quadratic functions into distinct EA-classes.

Demo

Plan of this Section

Principle

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We can write the scalar product $x \cdot y$ as $(\vec{x})^T \times \vec{y}$, where $\times$ is a matrix operation.

We represent F as a vector of $\mathbb{F}_2^{n2^n}$ by concatenating the n-bit representation of each of the 2^n values $F(x)$:

$$\text{vec}(F) = \begin{bmatrix} F_0(0) \\ F_1(0) \\ \cdots \\ F_{n-1}(0) \\ F_0(1) \\ \cdots \\ F_{n-1}(2^n - 1) \end{bmatrix} .$$

Re-Defining Ortho-Derivatives

Let G be a function and $\zeta_a(G)$ be a matrix defined by

1. $\zeta_G(a)[x, x] = \overrightarrow{G(a)}^T$, $\quad \zeta_G(a)[x, x + a] = \overrightarrow{G(a)}^T$,
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so that

$$\zeta_G(a) \times \mathrm{vec}(F) = \begin{bmatrix} G(a) \cdot \big(F(0) + F(0+a) + F(a) + F(0)\big) \\ G(a) \cdot \big(F(1) + F(1+a) + F(a) + F(1)\big) \\ \dots \\ G(a) \cdot \big(F(2^n - 1) + F(2^n - 1 + a) + F(a) + F(2^n - 1)\big) \end{bmatrix},$$

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from which we deduce that if π_F is an ortho-derivative of F then

$$\mathrm{vec}(F) \in \ker\left(\zeta(\pi_F)\right) \text{ where } \zeta(\pi_F) = \begin{bmatrix} \zeta_0(\pi_F) \\ \cdots \\ \zeta_{2^n-1}(\pi_F) \end{bmatrix}.$$

Inverting the DDT of a Quadratic Function

1. Find a DDT,
2. deduce the corresponding π,
3. build $\zeta(\pi)$,
4. find $\ker(\zeta(\pi))$,
5. obtain $\mathrm{vec}(F)$!

[2] Tricks are used to get rid of redundancies in ζ, and trivial solutions.

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5. obtain vec(F)!

In practice, starting from "cleverly" built functions π yields $\zeta(\pi)$ with empty[2] kernels...

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Plan of this Section

Conclusion

Go an use `sboxU`!

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Thank you!