Bézout Identity in Pseudoratoinal Transfer Functions

Yutaka Yamamoto, Catherine Bonnet

Bézout Identity in Pseudoratoinal Transfer Functions

Yutaka Yamamoto [*,1]

[*] *Professor Emeritus, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan (e-mail: yy@i.kyoto-u.ac.jp).*

Catherine Bonnet [**]

[**] *Inria, Université Paris-Saclay, L2S–CentraleSupélec, 3 rue Joliot Curie 91192 Gif-sur-Yvette cedex France.* Catherine.Bonnet@inria.fr.

Abstract: Coprime factorizations of transfer functions play various important roles, e.g., minimality of realizations, stabilizability of systems, etc. This paper studies the Bézout condition over the ring $\mathscr{E}'(\mathbb{R}_-)$ of distributions of compact support and the ring $\mathfrak{M}(\mathbb{R}_-)$ of measures with compact support. These spaces are known to play crucial roles in minimality of state space representations and controllability of behaviors. We give a detailed review of the results obtained thus far, as well as discussions on a new attempt of deriving general results from that for measures. It is clarified that there is a technical gap in generalizing the result for $\mathfrak{M}(\mathbb{R}_-)$ to that for $\mathscr{E}'(\mathbb{R}_-)$. A detailed study of a concrete example is given.

Keywords: Bézout identity, pseudorationality, distributions, Gel'fand representation, delay-differential systems
AMS subject classification: 46F10, 46J15

1. INTRODUCTION

This short note studies the issue of coprimeness for a certain class of systems.

We first consider the input/output relation described by

$$q \cdot y = p \cdot u \tag{1}$$

where u and y denote the input and output, respectively, and q and p are given elements belonging to a ring R that acts on inputs and outputs and describes the type of input/output relations. The multiplication in (1) is interpreted according to the context it is considered.

With varied choices of the ring R, we can describe a variety of different classes of systems. For example, when R is the ring of polynomials $\mathbb{R}[s]$, it will give the class of finite-dimensional linear systems with (1) interpreted in the Laplace transform domain. Likewise, $\mathbb{R}[z]$ corresponds to the class of finite-dimensional discrete-time linear systems with (1) considered in the sense of the z-transform.

When $R = \mathbb{R}[s,z]$ with s being the Laplace transform variable and z a finite-time delay, (1) can be interpreted as the class of delay-differential systems. Extensive studies have been conducted in this context in the literature; see, e.g., Rocha and Willems (1997); Glüsing-Lüerssen (1997).

When R is taken to be the ring of stable rational functions $\mathbb{R}_{stable}(s)$, it can be used to study stabilization with compensators over this ring (Vidyasagar (1985)). More generally, when we take R to be H^∞, (1) can describe a certain class of distributed parameter systems as well. This class of systems is often adequate for studying stabilization/stabilizability over the compensators constructed in the ring H^∞. This class has also been studied in depth in the literature. The authors have also conducted some research concerning a coprimeness condition over H^∞ (Bonnet and Yamamoto (2016)).

In all these classes, the notion of coprimeness has played crucial roles depending on the context where (1) is interpreted. For example, for $R = \mathbb{R}[s]$ the coprimeness corresponds to the minimality of the representation (i.e., realization) naturally associated to (1); likewise for $\mathbb{R}[z]$. For $R = \mathbb{R}_{stable}(s)$ or H^∞, the coprimeness yields stabilizability with compensators constructed over these rings.

In all these studies, a strong notion of coprimeness, i.e., the Bézout identity (or Bézout condition)

$$px + qy = 1 \tag{2}$$

plays a critical role in deriving desired results, i.e., minimality or stabilizability, in the respective contexts.

For the study of distributed parameter systems, the first author has introduced the class of *pseudorational* impulse responses or transfer functions, and developed realization theory, various spectral analysis, and coprimeness conditions Yamamoto (1988, 1989).

In this note, we will give a brief overview for this class of systems, the background and motivation for the study of this

[1] This author was supported in part by the Japan Society for the Promotion of Science under Grants-in-Aid for Scientific Research No. 19H02161. The author also wishes to thank DIGITEO and Laboratoire des Signaux et Systemes (L2S, UMR CNRS), CNRS-CentraleSupelec-University Paris-Sud and Inria Saclay for their financial support while part of this research was conducted.

class, and then proceed to the study of Bézout identity over this class of systems.

The present article intends to give a new attempt to derive a condition for the Bézout identity (2) in the class of systems described by pseudorational impulse responses. We will provide a Banach algebra approach for a small class of impulse responses described by the space of measures, and then try to generalize it to the general class of distributions. To this end, we need to give some background materials, explain their motivations, and show how this approach is important for some distributed parameter systems. While we have once given a brief review of this background material Yamamoto (2007b), it is important to further motivate more deeply into the need of developing actions induced by distributions and minimal representations there. This will lead naturally into the minimality of the state space representation and also controllability of behaviors over such a ring, e.g., the ring $\mathscr{E}'(\mathbb{R})$ or $\mathscr{E}'(\mathbb{R}_-)$ of distributions of compact support (see below for the definitions). The Bézout identity over $\mathscr{E}'(\mathbb{R})$ has been studied in Yamamoto and Willems (2008); Yamamoto (2016).

The paper is organized as follows: In Section 2, we review some basics for the study of pseudorational impulse responses. We will motivate and give the definition of pseudorationality (Yamamoto (1988)) and state the basic problem here. In Section 3, we show how the Bézout identity translates to an invertibility condition in a quotient algebra. In Section 4, we confine our attention to the algebra $\mathfrak{M}(\mathbb{R}_-)$ consisting of measures with compact support contained $(-\infty, 0]$. This restriction leads us to the study of a Gel'fand algebra. In Section 5, we attempt to generalize the result to the general case in $\mathscr{E}'(\mathbb{R}_-)$, and also show that how a simple generalization fails. While a general result is available, a straightforward generalization is shown to be not possible because there exists a pair in $\mathfrak{M}(\mathbb{R}_-)$ that satisfies a Bézout identity over $\mathscr{E}'(\mathbb{R}_-)$ but is never coprime over $\mathfrak{M}(\mathbb{R}_-)$. In the final section we will disscuss how this example suggests a further approach in this direction.

2. PSEUDORATIONALITY AND ITS BACKGROUND

Back in the early 1980s, the first author was trying to find a suitable class of transfer functions that accommodates at least delay-differential systems, and also enables us to study infinite-dimensional systems with algebraic structures. The ring $\mathbb{R}[s,z]$ is one such example, but it has a fatal defect in that it is not closed under pole-zero cancellations. For example,

$$\frac{1-e^{-s}}{s}$$

has a common zero at $s=0$, but when we cancel this common zero between the numerator and the denominator, the resulting entire (holomorphic on the whole complex plane) function

$$1-\frac{s}{2}+\frac{s^2}{3!}-\cdots$$

is no longer expressible as the ratio of two-variable polynomials in s and z. This defect becomes critical when we want to establish a framework that allows us to freely form a minimal (canonical, i.e., reachable and observable) realization. This naturally raises the following question:

What is the smallest algebra that includes the differentiation and delay operator?

From here on, let us turn our discussion into the time domain with convolution that gives a multiplication structure. Then, the differentiation is expressed by the derivative δ' of the Dirac delta, and time-delay operator is expressed as the convolution with δ_a with $a>0$. Likewise, the time advance operator of length is δ_a with $a<0$.

We now invoke an analogy with realization theory of the discrete-time finite-dimensional systems. Note here that the ring we consider there is $\mathbb{R}[z]$ where z is the time-advance operator, *not* the delay operator. This formalism is particularly suited for realization theory.

In analogy with this, we consider the convolution actions induced by the finite sum of elements of δ' and δ_{a_i} as follows:

$$x(t)\mapsto\sum_i\alpha_i\delta^{(k_i)}\delta_{a_i}x=\sum_i\alpha_i x^{(k_i)}(t-a_i)\qquad(3)$$

where $\alpha_i\in\mathbb{R}$ and $\delta^{(k_i)}$ denotes the k_i-th order derivative of δ.

However, the finite sum (3) does not constitute a closed space convenient for analytical handling. We thus want to take a suitable completion of the elements of form (3) under a suitable topology.

One natural choice of such a topology is that of Schwartz distributions (Schwartz (1966)). We thus introduce the following notions.

Let $\mathscr{D}'$ denotes the space of distributions on $\mathbb{R}$. Let $\mathscr{E}'(\mathbb{R})$ be its subspace consisting of those having compact support. $\mathscr{E}'(\mathbb{R}_-)$ is also its subspace with support contained in the negative half line $(-\infty,0]$. $\mathscr{D}'_+$ denotes the subspace of $\mathscr{D}'$ consisting of elements having support bounded on the left. Distributions such as Dirac's delta δ_a placed at $a\in\mathbb{R}$, its derivative δ'_a are examples of elements in $\mathscr{E}'(\mathbb{R})$. If $a\leq 0$, then they belong to $\mathscr{E}'(\mathbb{R}_-)$. If we take the closure of elements of type (3) in $\mathscr{D}'$, it is $\mathscr{E}'(\mathbb{R})$. If we confine (3) to those with nonpositive a_i, it will be $\mathscr{E}'(\mathbb{R}_-)$. (See Schwartz (1966, 1961) for more detail.)

We thus consider fraction representations over $\mathscr{E}'(\mathbb{R}_-)$ (relying on the analogy with $\mathbb{R}[z]$ and $\mathscr{E}'(\mathbb{R}_-)$).

Definition 2.1. An impulse response function G ($\operatorname{supp} G\subset[0,\infty)$) is said to be *pseudorational* (Yamamoto (1988)) if there exist $q,p\in\mathscr{E}'(\mathbb{R}_-)$ such that

(1) $G=q^{-1}*p$ where the inverse is taken with respect to convolution and belongs to $\mathscr{D}'_+$;
(2) $\operatorname{ord} q^{-1}=-\operatorname{ord} q$, where $\operatorname{ord} q$ denotes the order of a distribution q (Schwartz (1966)). [2]

If this condition is satisfied, we call (p,q) a *pseudorational pair*. The Laplace transform $\hat{q}^{-1}\hat{p}$ is called a *pseudorational transfer function*.

The delay-differential equation:

$$\dot{x}(t)=x(t-1)+u(t)$$
$$y(t)=x(t),$$

admits the representation

$$y=(\delta'_{-1}-\delta)^{-1}*\delta_{-1}*u,$$

and hence it is pseudorational.

The main problem that concerns us here is the following:
Problem *Given a pseudorational pair* $(p,q)\in\mathscr{E}'(\mathbb{R}_-)\times$

[2] Roughly speaking, the order of a distribution α is the least integer r such that $\alpha=(d/dt)^r\beta$ for some measure β.

$\mathscr{E}'(\mathbb{R}_-)$, characterize a condition under which p and q satisfy the Bézout identity:

$$p * x + q * y = \delta \tag{4}$$

for some $x, y \in \mathscr{E}'(\mathbb{R}_-)$.

If we consider $\mathscr{E}'(\mathbb{R})$ instead of $\mathscr{E}'(\mathbb{R}_-)$, it gives a necessary and sufficient condition for the behavior defined over $\mathscr{D}'$ (Yamamoto (2016)). Actually, the Bézout condition over $\mathscr{E}'(\mathbb{R})$ is in close relationship with that in $\mathscr{E}'(\mathbb{R}_-)$ (Yamamoto (2016)).

3. COPRIMENESS IN $\mathscr{E}'(\mathbb{R}_-)$

We first translate (4) to a divisibility condition by considering the principal ideal $(q) = q * \mathscr{E}'(\mathbb{R}_-)$ generated by q in $\mathscr{E}'(\mathbb{R}_-)$. Note first that (4) is easily seen to be equivalent to

$$p * \phi = \delta \mod q. \tag{5}$$

for some $\phi \in \mathscr{E}'(\mathbb{R}_-)$. In other words,

$$[p] * [\phi] = [\delta] \tag{6}$$

in $\mathscr{E}'(\mathbb{R}_-)/(q)$. This means that the equivalence class $[p]$ is invertible in the quotient algebra $\mathscr{E}'(\mathbb{R}_-)/(q)$.

Condition (6) by itself is not so easy to handle because of the intricate topology of $\mathscr{E}'(\mathbb{R}_-)$. However, because q has compact support, the following remarkable property holds:

Proposition 3.1. Take any $T > 0$ such that $\operatorname{supp} q \subset (-T, 0]$. Then

$$\mathscr{E}'(\mathbb{R}_-)/(q) \cong \mathscr{E}'([-T,0])/(q) \tag{7}$$

Proof Let π be the projection operator

$$\pi : \mathscr{D}' \to \mathscr{D}'_{(0,\infty)} : \psi \mapsto \psi|(0,\infty) \tag{8}$$

where $\mathscr{D}'_{(0,\infty)}$ is the space of distributions with support contained in $(0,\infty)$. Given $\psi \in \mathscr{D}'$, define the following operator π^q as

$$\pi^q : \mathscr{E}'(\mathbb{R}_-) \to \mathscr{E}'(\mathbb{R}_-) : \psi \mapsto q * \pi(q^{-1} * \psi). \tag{9}$$

Now for a distribution $\psi \in \mathscr{D}'_+$, define $\ell(\psi)$ as

$$\ell(\psi) := \inf\{t \in \operatorname{supp} \psi\} \tag{10}$$

where $\operatorname{supp} \psi$ denotes the support of ψ.

Take any $x \in \mathscr{E}'(\mathbb{R}_-)$ along with $\pi^q x$. We claim that $\pi^q x$ belongs to $\mathscr{E}'(\mathbb{R}_-)$ (hence (9) is well defined as a map from $\mathscr{E}'(\mathbb{R}_-)$ into itself) and that $x \cong \pi^q x \mod q$. We have

$$q^{-1} * (x - \pi^q x) = q^{-1} * x - q^{-1} * q * \pi(q^{-1} x) = q^{-1} * x - \pi(q^{-1} * x).$$

The last term $\phi := q^{-1} * x - \pi(q^{-1} * x)$ belongs to $\mathscr{E}'(\mathbb{R}_-)$ because $q^{-1} * x - \pi q^{-1} * x$ must be zero on $(0,\infty)$. That is to say,

$$x - \pi^q x = q * \phi \in q * \mathscr{E}'(\mathbb{R}_-) = (q).$$

This also shows that $\pi^q x = x - q * \phi \in \mathscr{E}'(\mathbb{R}_-)$. In other words, $[x] = [\pi^q x]$ in $\mathscr{E}'(\mathbb{R}_-)/(q)$. Moreover, since $\ell(\pi(q^{-1} * x)) \geq 0$ and $\ell(q) \geq -T$, the support of $\pi^q x = q * \pi(q^{-1} * x)$ must be contained in $[-T, 0]$ by Lemma A.1. That is, for every $x \in \mathscr{E}'(\mathbb{R}_-)$, there always exists an element $\pi^q x$ such that $\operatorname{supp} \pi^q x \subset [-T, 0]$, and $x \cong \pi^q x \mod q$. This proves (7). □

Remark 3.2. Proposition 3.1 claims that as far as a pseudorational impulse response is concerned, we can confine our attention to those inputs with support contained in $[-T, 0]$ with $-T < \ell(q)$. This result is not so surprising if we pay proper attention to the compact-support property of q. Since q has bounded support, its maximum length should determine the maximum length of memory needed to reconstruct the state or future outputs. This can be easily guessed once we resort to the analogy with realization theory for discrete-time linear systems: The degree of the denominator polynomial $q(z)$ determines the dimension of the state in the standard reachable realization, and the degree here exactly corresponds to the length of the support of q here. The projection scheme used above is an analogy to the finite-dimensional theory developed by Fuhrmann (1976).

4. GEL'FAND ALGEBRA STRUCTURE OF THE SPACE OF MEASURES

We have seen that the existence of the Bézout condition reduces to the invertibility of $[p]$ in the quotient algebra $\mathscr{E}'(\mathbb{R}_-)/(q)$. It is also seen that this space $\mathscr{E}'(\mathbb{R}_-)/(q)$ is isomorphic to $\mathscr{E}'([-T,0])/(q)$ for some $T > 0$ so that its structure is quite simplified. However, the space $\mathscr{E}'(\mathbb{R}_-)/(q)$ is still not that easy to tackle due to a rather complex topological structure of $\mathscr{E}'(\mathbb{R}_-)/(q)$.

We now choose to confine ourselves to the subspace $\mathfrak{M}(\mathbb{R}_-)$ that is the subspace of $\mathscr{E}'(\mathbb{R}_-)$ consisting of measures, i.e., those with elements of order 0. As shown in Proposition 3.1, $\mathfrak{M}(\mathbb{R}_-)/(q) \cong \mathfrak{M}([-T,0])/(q)$ for some $T > 0$. (Proposition 3.1 claims this fact for $\mathscr{E}'(\mathbb{R}_-)$, but the proof remains essentially the same. Note that $\operatorname{ord} q^{-1} = -\operatorname{ord} q = 0$ by condition (2) of Definition 2.1, so that q^{-1} is also a measure.) We here observe that the space $\mathfrak{M}([-T,0])/(q)$ has a remarkable advantage over $\mathscr{E}'([-T,0])/(q)$ in that it can be regarded as a Banach space with respect to the strong dual topology as the dual space of the space of continuous functions $C[-T,0]$. Furthermore, it inherits a natural algebra structure induced from $\mathfrak{M}(\mathbb{R}_-)$ (with respect to convolution) with the unity element $[\delta]$. In other words, it is a Gel'fand algebra (Gel'fand et al. (1964); Berberian (1973)).

A Gel'fand algebra is known to have a remarkable property in that the invertibility of an element can be well tested by characterizing the space of its maximal ideals (Berberian (1973); Gel'fand et al. (1964)). This fact is best suited to study the invertibility condition (6).

Let us now make the following Assumption:

Assumption 1 There exists $\sigma \in \mathbb{R}$ such that $p(s)$ and $q(s)$ do not vanish on $\{s \mid \operatorname{Re} s \geq \sigma\}$.

Remark 4.1. Pseudorationality assumes the existence of $q^{-1} \in \mathscr{D}'_+$, hence the above condition is automatically satisfied by Lemma A.4. If $p^{-1} \in \mathscr{D}'_+$ is assumed, it also satisfies Assumption 1. If Assumption 1 is satisfied we may assume that σ can be taken to be zero, without loss of generality. For if necessary, we can always shift the complex variable as $s \mapsto s - \sigma$, and this clearly does not affect the coprimeness relationship.

The following theorem was first given in Yamamoto (2007a), but we here give a more complete proof for the sake of completeness.

Theorem 4.2. Let $p, q \in \mathfrak{M}(\mathbb{R}_-)$, and satisfy Assumption 1. Suppose that there exists $c > 0, a \in \mathbb{R}$ such that

$$|p(s)| + |q(s)| \geq c > 0 \tag{11}$$

for every $s \in \mathbb{C}_- = \{s \in \mathbb{C} \mid \operatorname{Re} s \leq 0\}$. Then the (p,q) is a Bézout pair, i.e., satisfies the Bézout identity (4).

For the proof, we need some preliminaries. The question here is to find a condition under which $[p]$ is invertible in $\mathfrak{M}([-T,0])/(q)$. By Gel'fand representation theory (Berberian

(1973); Gel'fand et al. (1964)), an element $[p]$ is invertible if and only if it belongs to no maximal ideals.

Consider the Laplace transform of elements in $\mathfrak{M}(\mathbb{R}_-)$. It is easy to see that this is a subalgebra of $H^\infty(\mathbb{C}_-)$. Then, as in Hoffman (1962), we see that the correspondence

$$\psi \mapsto \hat{\psi}(s)$$

considered for $s \in \mathbb{C}_-$ gives the Gel'fand representation.

What is then a maximal ideal in $\mathfrak{M}(\mathbb{R}_-)$*?* Take any $\lambda \in \mathbb{C}_-$, and consider the point evaluation

$$\phi_\lambda : f \mapsto \hat{f}(\lambda). \qquad (12)$$

It is easy to see that ϕ_λ is a complex homomorphism (i.e., homomorphism from $\mathfrak{M}(\mathbb{R}_-)$ to $\mathbb{C}$), and hence $\ker \phi_\lambda$ is a maximal ideal of $\mathfrak{M}(\mathbb{R}_-)$. Observe however that this does not necessarily yield a maximal ideal in $\mathfrak{M}(\mathbb{R}_-)/(q)$, because in order to be an ideal in this space, this ideal should contain (q). In other words, $\hat{q}$ should vanish there. If M is given by

$$M_\lambda = \{f | \hat{f}(\lambda) = 0\},$$

then this means that λ should be a zero of $\hat{q}$ for $M_\lambda \supset (q)$. Now let

$$\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots \qquad (13)$$

be the set of zeros of $\hat{q}$. Then we have maximal ideals

$$M_{\lambda_1}, M_{\lambda_2}, \ldots, M_{\lambda_n}, \ldots$$

of $\mathfrak{M}(\mathbb{R}_-)/(q)$. But these are not all. There are other maximal ideals that are centered at "infinity".

To see this, let us first start with the following proposition:

Proposition 4.3. Let $f \in \mathfrak{M}(\mathbb{R}_-)$, and suppose that $\phi(f) = 0$ for some complex homomorphism, i.e., f belongs to a maximal ideal $\ker \phi$. Suppose also that ϕ does not agree with any of M_{λ_n} as given above. Then there exists a sequence μ_n such that

- $\mu_n \to \infty$ and
- $\hat{f}(\mu_n) \to 0$ as $n \to \infty$

Proof Suppose there exists no such μ_n. Then there exists $\delta > 0$ and $R > 0$ such that $|\hat{f}(s)| \geq \delta$ for $|s| \geq R$. In view of the Hadamard factorization (A.3) of q, it follows that either

(1) $\hat{f}(s)$ has infinitely many zeros, or
(2) $\hat{f}(s)$ has only finitely many zeros.

The first case is clearly impossible by $|\hat{f}(s)| \geq \delta$. Hence $\hat{f}$ has only finitely many zeros. But this yields $\hat{f}(s) = e^{\alpha s}P(s)$ where P is a polynomial. Note that $\alpha \geq 0$ because the inverse Laplace transform of $\hat{f}$ is a measure in $\mathfrak{M}(\mathbb{R}_-)$. Since $\alpha = 0$ just corresponds to a constant, we assume $\alpha \neq 0$, so that $\alpha > 0$. But then $e^{\alpha s}$ can have infinitely many zeros along the imaginary axis, and this contradicts $|\hat{f}(s)| \geq \delta$ for $|s| \geq R$. Hence $\hat{f}$ must be a polynomial. But this is again impossible unless $\hat{f}$ is a (nonzero) constant because the inverse Laplace transform of $\hat{f}$ must be a measure. Therefore $\hat{f}$ must be a constant. But this yields $\phi(1) = 0$, which clearly means that ϕ annihilates the whole space, and this contradicts the fact that ϕ is a nontrivial complex homomorphism (or $\ker \phi$ is a maximal ideal). □

In particular, this holds also for q. Then if M is a maximal ideal of $\mathfrak{M}(\mathbb{R}_-)/(q)$, then $\pi^{-1}(M)$ is clearly a maximal ideal of $\mathfrak{M}(\mathbb{R}_-)$, and this should contain (q).

We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2 Suppose (11) holds, but p belongs to a maximal ideal in $\mathfrak{M}(\mathbb{R}_-)/(q)$. If p belongs to one of M_{λ_n}, then this would clearly contradict (11). Hence assume that $\hat{p}$ vanishes at no λ_n, $n = 1, 2, \ldots$. Then by Proposition 4.3, there exists μ_n such that $\mu_n \to \infty$ and $\hat{p}(\mu_n) \to 0$. Since this maximal ideal should contain q, $\hat{q}$ should also vanish there, and hence a suitable subsequence of $\hat{q}(\mu_n)$ should go to 0. This clearly contradicts (11). □

Here are some examples:

Example 4.4. The pair $(e^{s/2} - 1, e^s - 1)$ is not a Bezout pair. The pair possesses infinitely many common zeros.

Example 4.5. The pair $(e^s, e^{s/2} - 1)$ is a Bezout pair. It is easy to check (11). This can also be directly verified by $e^s - (e^{s/2} - 1)(e^{s/2} + 1) = 1$.

Remark 4.6. Condition (11) is the same as that in the celebrated Corona theorem by Carleson for H^∞ (Duren (1970); Garnett (1981)). One should of course be careful not to confuse the present result with the Corona theorem, because such conditions crucially depend on the choice of a ring. The proof here is good deal simpler than that of the monstrous Corona theorem (Duren (1970); Garnett (1981)). This is because the algebra $\mathfrak{M}(\mathbb{R}_-)/(q)$ is much "smaller" than H^∞, and the way it yields "cancellation at infinity" is quite much restricted by the discrete zeros $\{\lambda_n\}$ whereas in the case of the Corona theorem, there are almost arbitrary ways in which such sequences go to infinity.

5. EXTENSION TO $\mathscr{E}'(\mathbb{R}_-)$

Let us first make the following assumption:

Assumption 2: The algebraic multiplicity of each zero λ_n of $\hat{q}(s)$ is globally bounded.

The Gel'fand algebra structure and the characterization of the maximal ideal space in $\mathfrak{M}([-T,0])/(q)$ is fairly appealing, and not so difficult as the case for H^∞. It is thus quite tempting to try to generalize the above result to the general case of $\mathscr{E}'(\mathbb{R}_-)$ or $\mathscr{E}'(\mathbb{R})$.

We first note the following Theorem 5.1 obtained in Yamamoto (2007a, 2016).

Theorem 5.1. Let $q^{-1} * p$ be pseudorational, and suppose that there exists a nonnegative integer m such that

$$|\lambda_n^m \hat{p}(\lambda_n)| \geq c > 0, n = 1, 2, \ldots \qquad (14)$$

Then the pair (p, q) satisfies the Bézout identity (4) for some $\phi, \psi \in \mathscr{E}'(\mathbb{R}_-)$.

The proof given in Yamamoto (2007a, 2016) is fairly complicated and highly technical. It does involve some elaborate analysis of complex analytic functions of exponential type, and some deep facts of their growth orders.

It is thus tempting to try to give a proof by using Theorem 4.2, extending the result for $\mathfrak{M}(\mathbb{R}_-)$ to $\mathscr{E}'(\mathbb{R}_-)$.

Let us first prepare some pertinent facts on the structure of $\mathscr{E}'(\mathbb{R}_-)$. Since every element of $\mathscr{E}'(\mathbb{R}_-)$ has compact support, it is of finite order (Schwartz (1966)). That is, for every $\psi \in \mathscr{E}'(\mathbb{R}_-)$, there exists $r \geq 0$ such that

$$\psi = (d/dt)^r \psi_0 \qquad (15)$$

for some $\psi_0 \in \mathfrak{M}(\mathbb{R}_-)$ and $r \geq 0$. This readily implies

$$\mathscr{E}'(\mathbb{R}_-) = \cup_{r=0}^\infty (d/dt)\mathfrak{M}(\mathbb{R}_-). \qquad (16)$$

In other words, the algebra $\mathscr{E}'(\mathbb{R}_-)$ is derived as the differentiated union of measures.

We now suppose that we are given a pseudorational pair (p,q) belonging to $\mathscr{E}'(\mathbb{R}_-)$. Since $\mathscr{E}'(\mathbb{R}_-)$ is the nested union of differentiated measures, we may hope that we can reduce the coprimeness problem of $\mathscr{E}'(\mathbb{R}_-)$ into that of $\mathfrak{M}(\mathbb{R}_-)$. A procedure like the Euclid division algorithm can be a hint for this.

Suppose for the moment that p is of order 0 and q is of order 1. Suppose also that $\hat{q}(s)$ has one real zero, say, λ. Then the inverse Laplace transform of $\hat{q}(s)/(s-\lambda)$ should be of order zero because the division by $s-\lambda$ should act as an integration. Therefore, both p and $L^{-1}[\hat{q}(s)/(s-\lambda)]$ should be of order zero, i.e., measure.

Then it is naturally expected that the coprimeness of (p,q) should reduce to that of $(p,L^{-1}[\hat{q}(s)/(s-\lambda)])$.

In fact, if (p,q_0*q_1) is coprime in a ring R (p,q_1) is coprime and vice versa. So it is natural to expect that the Bézout condition of (p,q) is translated to that of $(p,L^{-1}[\hat{q}(s)/(s-\lambda)])$ where the latter belong to the space of measures $\mathfrak{M}(\mathbb{R}_-)$, where Theorem 4.2 is available.

However, this seemingly reasonable idea unfortunately does not work. The following counterexample shows why.

Example 5.2. Consider the pair $(\delta'_{-1}-\delta,\delta_{-1})$. This pair is clearly pseudorational. The element $\delta'_{-1}-\delta$ has order 1, and δ_{-1} has order 0, i.e., measure. They admit Laplace transforms se^s-1 and e^s, respectively. They satisfy the Bézout identity

$$(se^s-1)\cdot(-1)+s\cdot e^s=1, \tag{17}$$

or

$$(\delta'_{-1}-\delta)*(-\delta)+\delta'*\delta_{-1}=\delta, \tag{18}$$

and hence the pair is coprime over $\mathscr{E}'(\mathbb{R}_-)$.

The former element se^s-1 has one positive zero, say α. This means that

$$\frac{se^s-1}{s-\alpha}$$

(or its inverse Laplace transform) has order 0 because division by $s-\alpha$ entails in integration of $\delta'_{-1}-\delta$ once, whereby yielding an element of order zero, i.e., a measure.

In other words, the pair (or the respective inverse Laplace transforms)

$$\left(\frac{se^s-1}{s-\alpha},e^s\right) \tag{19}$$

belongs to $\mathfrak{M}(\mathbb{R}_-)$, and they are coprime over $\mathscr{E}'(\mathbb{R}_-)$. However, this does not guarantee that this pair admits a coprime factorization over $\mathfrak{M}(\mathbb{R}_-)$ in the sense of Theorem 4.2.

To see this, observe that se^s-1 admits infinitely many zeros λ_n such that $\mathrm{Re}\,\lambda_n\to-\infty$. (This can easily be seen by noting that it is the characteristic function of the retarded delay-differential equation $\dot{x}=x(t-1)+u$.) Indeed, $\lambda_n e^{\lambda_n}=1$ admits infinitely many solutions such that $e^{\lambda_n}=1/\lambda_n, n=1,2,\ldots$. This also implies that $\hat{p}(\lambda_n)\to0$ as $n\to\infty$. That is, it contradicts condition (11) of Theorem 4.2, and cannot be a Bézout pair in $\mathfrak{M}(\mathbb{R}_-)$.

In other words, the pair can admit a Bézout identity over $\mathscr{E}'(\mathbb{R}_-)$ with $x,y\in\mathscr{E}'(\mathbb{R}_-)$, but it cannot satisfy a Bézout condition over the algebra of $\mathfrak{M}(\mathbb{R}_-)$ because the latter algebra is much smaller than $\mathscr{E}'(\mathbb{R}_-)$ and does not give as much freedom as that induced by $\mathscr{E}'(\mathbb{R}_-)$. This can be more directly seen by noting the identity

$$s\cdot e^s+\left(\frac{se^s-1}{s-\alpha}\right)\cdot(\alpha-s)=1.$$

This looks trivial and not any different from (17). The difference here is that the multiplying factor $\alpha-s$ that makes the pair (19) satisfy the Bézout identity does not belong to (the Laplace transform of) $\mathfrak{M}(\mathbb{R}_-)$. To cover this situation, we do need Theorem 5.1, which cannot be, unfortunately, covered as a natural variant of Theorem 4.2.

In fact, at the zeros λ_n of $\hat{q}$, $e^{\lambda_n}=1/\lambda_n$ holds, so that the $\hat{p}(\lambda_n)$ clearly satisfy condition (14) for $m=1$. This condition can also be rewritten as

$$|s\hat{p}(s)|+|s\hat{q}(s)|\geq c>0,\forall s\in\mathbb{C}_-. \tag{20}$$

6. CONCLUDING REMARKS

We have seen that the space $\mathfrak{M}(\mathbb{R}_-)$ of measures admits a Gel'fand algebra structure, and it yields a concrete Corona-like condition (11) for the Bézout identity for a pseudorational pair (p,q). We have also pursued to derive the general condition (14) for the Bézout identity over $\mathscr{E}'(\mathbb{R}_-)$, but also seen that a straightforward reduction idea does not work. The modified generalized Corona-like condition (20) may, however, suggest that there could still be a possibility of generalizing (11) to a more general context in $\mathscr{E}'(\mathbb{R}_-)$.

Appendix A. PRELIMINARY MATERIALS

For $\ell(\alpha)$ defined by (10), the following lemma holds:

Lemma A.1. Let $\alpha,\beta\in\mathscr{D}'_+$. Then

$$\ell(\alpha*\beta)=\ell(\alpha)+\ell(\beta) \tag{A.1}$$

Proof The proof is immediate from the local version of Titchmarsh's theorem on convolution (Donoghue (1969)). Actually, if $\alpha,\beta\neq0$, then $\alpha*\beta\neq0$. Hence $\ell(\alpha*\beta)\geq\ell(\alpha)+\ell(\beta)$ readily follows. However, it can also be deduced from the local version of Titchmarsh's theorem that if α and β are nonzero in some neighborhoods of $\ell(\alpha)$ and $\ell(\beta)$, respectively, then $\alpha*\beta$ is nonzero in a neighborhood of $\ell(\alpha)+\ell(\beta)$. This proves (A.1). □

We now need some basic properties for the Laplace transform of elements in $\mathscr{E}'(\mathbb{R}_-)$:

Theorem A.2. (Paley-Wiener, (Schwartz (1966))). A complex analytic function $f(s)$ is the Laplace transform of a distribution $\phi\in\mathscr{E}'(\mathbb{R}_-)$ if and only if $f(s)$ is an entire function that satisfies the following growth estimate for some $C>0,a>0$ and integer $m\geq0$: if and only if it satisfies the estimate

$$\begin{aligned}|\hat{f}(s)|&\leq C(1+|s|)^m e^{a\mathrm{Re}\,s},\mathrm{Re}\,s\geq0,\\&\leq C(1+|s|)^m,\mathrm{Re}\,s\leq0\end{aligned} \tag{A.2}$$

for some $C>0,a>0$ and integer $m\geq0$. In this case, the support of ϕ is contained in $[-a,0]$

This implies the following important lemma:

Lemma A.3. Let $\hat{f}(s)$ is an entire function that satisfies the above Paley-Wiener estimate (A.2). Suppose $\lambda\in\mathbb{C}$ is a zero of $\hat{f}(s)$. Then $\hat{f}(s)/(s-\lambda)$ also satisfies (A.2) (with different C,a,m, of course), and hence it is the Laplace transform of a distribution $\phi\in\mathscr{E}'(\mathbb{R}_-)$.

Proof Draw a unit circle around λ. Outside of this circle, $\hat{f}(s)/(s-\lambda)$ should satisfy the same estimate of type (A.2). On the other hand, inside this circle, $\hat{f}(s)/(s-\lambda)$ is a continuous function, and hence bounded. Combining these two facts easily yields the conclusion. □

For a distribution $\phi \in \mathscr{E}'(\mathbb{R}_-)$, that $\hat{\phi}(s)$ is an entire function of exponential type as in (A.2) yields the fact that it allows the so-called Hadamard factorization as the infinite-product consisting of its zeros (Boas (1954)):

$$\hat{\phi}(s) = cs^k e^{as} \prod_{n=1}^{\infty} \left(1 - \frac{s}{\lambda_n}\right) \exp\left(\frac{s}{\lambda_n}\right). \tag{A.3}$$

The following lemma is of relevance to Assumption 1:

Lemma A.4. (Schwartz (1961)). An analytic function $\phi(s)$ is the Laplace transform of a distribution $f \in \mathscr{D}'_+$ if and only if there exists $c \in \mathbb{R}$ such that $|\phi(s)|$ is bounded by a polynomial in s.

Suppose that $f \in \mathscr{D}'_+$ is invertible in $\mathscr{D}'_+$. Then $1/\hat{f}(s)$ satisfies a polynomial estimate for some half plane $\{s | \operatorname{Re} s > c\}$. This means that there exist no zeros of $\hat{f}(s)$ for $\operatorname{Re} s > c$.

REFERENCES

Berberian, Sterling, K. (1973). *Lectures in Functional Analysis and Operator Theory*. Springer.

Boas, Ralph, P.J. (1954). *Entire Functions*. Academic Press.

Bonnet, C. and Yamamoto, Y. (2016). Coprimeness of fractional representations. In *Proc. 55th IEEE CDC*, 1399–1404.

Donoghue, William, F.J. (1969). *Distributions and Fourier Transforms*. Academic Press.

Duren, P.L. (1970). *Theory of H^p Spaces*. Academic Press, San Diego.

Fuhrmann, Paul, A. (1976). Algebraic system theory: an analyst's point of view. *J. Franklin Inst.*, 301, 521–540.

Garnett, J.B. (1981). *Bounded Analytic Functions*. Academic Press, San Diego.

Gel'fand, I., M., Raikov, D., A., and Shilov, G., E. (1964). *Commutative Normed Rings*. Chelsea.

Glüsing-Lüerssen, H. (1997). A behavioral approach to delay-differential systems. *SIAM J. Contr. Optimiz.*, 35(2), 480–499.

Hoffman, K. (1962). *Banach Spaces of Analytic Functions*. Prentice-Hall, Englewood Cliffs, reprinted by Dover Publications.

Rocha, P. and Willems, J.C. (1997). Behavioral controllability of delay-differential systems. *SIAM J. Contr. Optimiz.*, 35(1), 254–264.

Schwartz, L. (1961). *Méthodes Mathématiques pour les Sciences Physiques*. Hermann.

Schwartz, L. (1966). *Théorie des Distribution*. Hermann.

Vidyasagar, M. (1985). *Control System Synthesis*. MIT Press.

Yamamoto, Y. (1988). Pseudo-rational input/output maps and their realizations: a fractional representation approach to infinite-dimensional systems. *SIAM J. Control & Optimiz*, 26, 1415–1430.

Yamamoto, Y. (1989). Reachability of a class of infinite-dimensional linear systems: an external approach with applications to general neutral systems. *SIAM J. Control & Optimiz*, 27, 217–234.

Yamamoto, Y. (2007a). Coprimeness in the ring of psedorational transfer functions. In *Proc. 15th Mediterranean Conf. on Control and Automation*.

Yamamoto, Y. (2007b). Pseudorational transfer functions—a survey of a class of infinite-dimensional systems. In *Proc. 46th IEEE CDC*, 848–853.

Yamamoto, Y. (2016). Behavioral controllability and coprimeness for pseudorational transfer functions. *Systems & Control Letters*, 95, 20–26.

Yamamoto, Y. and Willems, J.C. (2008). Behavioral controllability and coprimeness for a class of infinite-dimensional systems. In *Proc. 47th IEEE CDC*, 1513–1518.