

Certified numerical algorithm for isolating the singularities of the plane projection of generic smooth space curves

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Abstract

Isolating the singularities of a plane curve is the first step towards computing its topology. For this, numerical methods are efficient but not certified in general. We are interested in developing certified numerical algorithms for isolating the singularities. In order to do so, we restrict our attention to the special case of plane curves that are projections of smooth curves in higher dimensions. This type of curve appears naturally in robotics applications and scientific visualization. In this setting, we show that the singularities can be encoded by a regular square system whose solutions can be isolated with certified numerical methods. Our analysis is conditioned by assumptions that we prove to be generic using transversality theory. We also provide a semi-algorithm to check their validity. Finally, we present experiments, some of which are not reachable by other methods, and discuss the efficiency of our method.

Keywords: Transversality, Generic Singularities, Certified Numerical Algorithms, Interval Analysis, Singular Curve Topology

1. Introduction

The problem of computing the topology of a real plane curve consists of computing a piecewise-linear plane graph that can be deformed continuously into that curve. Such a problem is critical for drawing plane curves with the correct topology. One of the main challenges is to isolate the singular points efficiently and correctly. The aim of this paper is to do so with certified numerical methods and we show that this could be achieved for the specific class of plane curves that are projections of C^∞ smooth curves in higher dimension.

Although this class of curves seems specific, it appears naturally in visualization and robotic applications and the curves from this class often contain singularities. When visualizing a curve given by $n - 1$ implicit equations in n dimensions for instance, we compute its projection in 2D to display it on a screen. This class of curves also appears in robotic applications. For instance given a robot with two degrees of freedom that moves in the plane, the set of points it can reach is bounded by a curve. In this case, computing the correct topology of this curve is often needed for deciding if a specific position is reachable. This curve is usually the projection of a smooth curve embedded in a space of higher dimension, and it often contains singular points.

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14 By certified algorithms, we refer to algorithms that always output mathematically correct results in a given
15 model of computation; for instance, randomized Las-Vegas algorithms are (usually) certified, but randomized
16 Monte-Carlo algorithms are not; numerical methods that may miss solutions or output spurious solutions are not
17 certified. We consider in this paper the RAM model of computation. Recall that the singular points of a plane
18 curve, defined by the equation $f(x, y) = 0$, are the solutions of the system defined by $f(x, y) = \frac{\partial f}{\partial x}(x, y) =$
19 $\frac{\partial f}{\partial y}(x, y) = 0$; it should be stressed that this system is over-determined, i.e., it has more equations than variables,
20 which prevents the use of certified numerical methods such as interval Newton methods [MKC09]. On the other
21 hand, symbolic methods can solve such over-determined systems but they are restricted to algebraic systems and
22 their complexity is high with respect to the degree of the equations.

23 *Main contributions.* In this paper, we present a square and regular system that encodes the singularities of the
24 plane projection of a C^∞ smooth curve in \mathbb{R}^n (Theorems 11 & 27). Our approach does not use elimination
25 theory to compute the equation of the projected curve and it is not restricted to the algebraic case: it applies to the
26 larger class of C^∞ smooth curves. Being square and regular, this system can thus be solved with state-of-the-art
27 certified numerical methods based on interval arithmetic or certified homotopy tracking. However it encodes the
28 singularities of the plane projection only if some assumptions, defined in Section 2.4, are satisfied. Our second main
29 result is that those assumptions are satisfied generically, which we prove using transversality theory (Section 7).
30 We also present Semi-algorithm 4 that checks whether a given curve satisfies our assumptions, that is, an algorithm
31 that stops if and only if the assumptions are satisfied. The combination of these results provides a method that is
32 both numerical and certified for isolating the singularities of the plane projection of a generic curve. Finally,
33 we present several experiments and discuss the efficiency of our algorithm in Section 6. Our contribution is a
34 generalization of [IMP16b] that only considers the 3-dimensional case and is in the same spirit as the work of
35 Delanoue et al. [DL14].

36 We also address the case of curves that are the silhouettes of smooth surfaces in \mathbb{R}^n (the silhouette being the
37 set of points on the surface where the tangent plane projects on the plane of projection in a line or a point). Such
38 curves naturally appear in parametric systems since they partition the parametric space with respect to the number
39 of solutions of the system. For such curves, we were only able to prove some partial results on their genericity
40 (see Section 2.4, Proposition 61 and Conjecture 62), but our other main results hold (Theorems 11 & 27 and our
41 semi-algorithms).

42 *State of the art.* The problem of isolating the singularities of a plane curve is a special case of the problem of
43 isolating the solutions of a zero-dimensional system in \mathbb{R}^2 . We give a concise summary of the state of the art of
44 certified methods for these two problems, organized in two main classes.

45 *Symbolic methods.* Symbolic methods are widely used for solving in a certified way zero-dimensional alge-
46 braic systems. Classical such methods are based on Gröbner bases, resultant theory and univariate representations
47 (see e.g., [CLO92, BPR06]). In this context, methods dedicated to the bivariate case have also been designed (see

48 [Hon96, GVK96, BLM⁺16, vdHL18] and references therein). Compared to numerical methods, these methods are
 49 adapted to over-constrained or non-regular systems. On the other hand, they suffer several drawbacks. They are
 50 not adaptive in the sense that solving in a small region is not easier than solving for all solutions. They are limited
 51 to algebraic systems and their complexity is high with respect to the degree of the system.

52 *Certified numerical methods.* When a zero-dimensional system is regular (Definition 1), its solutions can be
 53 isolated in a certified way using interval-arithmetic subdivision methods [Neu91, MKC09] or homotopy approaches
 54 with certified path tracking (see [BL13] and references therein). However, these methods do not directly work for
 55 isolating the singularities of a plane curve given by the equation $f(x, y) = 0$ because the system $f(x, y) =$
 56 $\frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial y}(x, y) = 0$ that encodes the singularities is neither square nor regular. On the other hand, the curve
 57 may not be given by its implicit equation and computing this representation may not be required nor desirable.
 58 There are only few contributions designing certified numerical algorithms, even with additional restrictions on the
 59 curve and its singularities. When the curve is algebraic, Burr et al. [BCGY12] use separation bounds to isolate the
 60 singularities via a subdivision algorithm but the worst-case values of such bounds make it inefficient in practice.
 61 Lien et al. [LSVY14, LSVY20] study the special case of a singular curve that is a union of non-singular ones
 62 such that the singularities are only transverse intersections between them. They propose a subdivision algorithm
 63 using the Moore-Kioustelidis interval test for isolating the square and regular system defined by two curves. No
 64 implementation is available but such a subdivision scheme in two dimensions is expected to be efficient. In the case
 65 where the plane curve is defined as a projection, only two contributions present certified numerical approaches for
 66 isolating the singularities: Delanoue and Lagrange [DL14] consider the apparent contours of smooth surfaces in \mathbb{R}^4
 67 and Imbach et al. [IMP16b] handle the plane projections of smooth curves in \mathbb{R}^3 using a subdivision scheme locally
 68 in four dimensions. Even though subdivision approaches may suffer in practice from the curse of dimensionality,
 69 Imbach et al. observe experimentally that, for algebraic curves, their approach is more efficient than computing the
 70 implicit equation of the projected plane curves and its singularities using symbolic methods.

71 The rest of the paper is organized as follows: In Section 2, we introduce notation and the assumptions we
 72 consider in our approach. In Section 3, we introduce the so-called *Ball system* that characterizes the singularities
 73 of the plane projection and we prove, in Section 4, that it is regular at its solutions. In Section 5, we provide a semi-
 74 algorithm to check the assumptions introduced in Section 2. Experiments are presented in Section 6. Finally, in
 75 Section 7, we prove the genericity of our assumptions, with a focus on the case of silhouette curves in Section 7.3.

76 2. Notation and assumptions

77 The main technical notation is summarized in Table 3 at the end of the paper. For a positive integer n , a
 78 closed (resp. open) n -box is the Cartesian product of n closed (resp. open) intervals. Assume that $n \geq 3$ and
 79 let B be an open n -box and \bar{B} be the topological closure of B with respect to the usual topology in \mathbb{R}^n . Let
 80 $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ denote the set of smooth functions (i.e., differentiable infinitely many times) from \mathbb{R}^n to \mathbb{R}^{n-1} .
 81 Consider the function $P = (P_1, \dots, P_{n-1}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$. We denote by \mathfrak{C} (resp. $\bar{\mathfrak{C}}$) the solution set of the

82 system $\{P_1(x) = \dots = P_{n-1}(x) = 0\}$, with $x = (x_1, \dots, x_n) \in B$ (resp. with $x \in \overline{B}$). Also, consider the
83 projection $\pi_{\mathfrak{C}}$ (resp. $\pi_{\overline{\mathfrak{C}}}$) from \mathfrak{C} (resp. $\overline{\mathfrak{C}}$) to the (x_1, x_2) -plane. Unless otherwise stated, the plane projection of
84 a point $x \in \mathbb{R}^n$ is (x_1, x_2) . Our main goal is to compute the cusp points and nodes of $\pi_{\mathfrak{C}}$. If $\overline{\mathfrak{C}}$ is a smooth curve
85 (see the definition below), define \mathfrak{L}_c (resp. $\overline{\mathfrak{L}}_c$) as the set of points q in \mathfrak{C} (resp. $\overline{\mathfrak{C}}$) where the tangent line, denoted
86 by $T_q\mathfrak{C}$, (resp. $T_q\overline{\mathfrak{C}}$) is orthogonal to the (x_1, x_2) -plane. We also define the set \mathfrak{L}_n (resp. $\overline{\mathfrak{L}}_n$) to be the set of points
87 q in \mathfrak{C} (resp. $\overline{\mathfrak{C}}$) such that the cardinality of the pre-image of $\pi_{\mathfrak{C}}(q)$ under $\pi_{\mathfrak{C}}$ (resp. $\pi_{\overline{\mathfrak{C}}}$) is at least two without
88 counting multiplicities. We will see later that, under some generic assumption, \mathfrak{L}_c (resp. \mathfrak{L}_n) is the set of points in
89 \mathfrak{C} that project to cusps (resp. nodes), which explains the subscript c (resp. n).

90 2.1. Regular and singular points

91 Let $m \geq 1$ be an integer, V be a subset of \mathbb{R}^m and $p \in V$. We call p a regular (or smooth) point of V if V is
92 a sub-manifold at p , that is, there exist a neighborhood W of p in \mathbb{R}^m , an integer $k > 0$ and k smooth functions
93 $\varphi_1, \dots, \varphi_k$ defined over W , such that $V \cap W$ is the set of solutions of $\{\varphi_1(x) = \dots = \varphi_k(x) = 0\}$ in W and
94 the rank of the matrix $\begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & \dots & \frac{\partial \varphi_1}{\partial x_m} \\ \vdots & \vdots & \vdots \\ \frac{\partial \varphi_k}{\partial x_1} & \dots & \frac{\partial \varphi_k}{\partial x_m} \end{pmatrix}$ evaluated at q is k [Dem00, Definition 2.2.2]. We call this matrix the
95 Jacobian matrix of the system $\{\varphi_1(x) = \dots = \varphi_k(x) = 0\}$ and we denote it by $J_{(\varphi_1, \dots, \varphi_k)}$. If q is not a regular
96 point of V , we call it a singular point. If all points in V are regular, then V is called regular or smooth. Otherwise,
97 V is called singular.

98 For $\varphi = (\varphi_1, \dots, \varphi_k) \in C^\infty(\mathbb{R}^n, \mathbb{R}^k)$, we denote by $T_q\varphi$ its derivative (also known as the tangent map) at the
99 point q . Note that the Jacobian matrix $J_\varphi = J_{(\varphi_1, \dots, \varphi_k)}$ is the expression of the derivative in the canonical bases
100 of \mathbb{R}^n and \mathbb{R}^k .

101 **Definition 1.** *let $F = (f_1, \dots, f_n)$ be a vector of smooth real-valued functions that are defined in \mathbb{R}^n and let*
102 *$a \in \mathbb{R}^n$ be a solution of the system $\{F = 0\}$. We say that the latter system is regular at $a \in \mathbb{R}^n$ if the determinant*
103 *of its Jacobian matrix, evaluated at a , does not vanish. We call $\{F = 0\}$ regular if it is regular at all of its*
104 *solutions.*

105 2.2. Multiplicity in zero-dimensional systems

106 We recall the definition of multiplicity in the univariate case before generalizing it to higher dimensions.

107 **Definition 2.** *Let f be a real smooth function at $a \in \mathbb{R}$. The multiplicity of f at a is the integer $\text{mult}_a(f(x)) =$
108 $\min\{k \in \mathbb{N} \mid \frac{\partial^k f}{\partial x^k}(a) \neq 0\}$ if it exists, otherwise $\text{mult}_a(f(x)) = \infty$. For the case $a = 0$, we write $\text{mult}(f) =$
109 $\text{mult}_a(f)$ for simplicity.*

110 **Definition 3** ([CLO05, Definition 4.2.1]). *For integers $m \geq n \geq 1$, let $G = (g_1(x), \dots, g_m(x))$ be a polynomial*
111 *function from \mathbb{R}^n to \mathbb{R}^m and q be a solution of the system $\{G = 0\}$. Let $\mathbb{R}[x]$ be the ring of polynomials with n*
112 *variables and define $\mathbb{R}[x]_q = \{\frac{h_1}{h_2} \mid h_1, h_2 \in \mathbb{R}[x], h_2(q) \neq 0\}$ to be the localization of $\mathbb{R}[x]$ at q . Define the*

113 intersection multiplicity of q in the system $\{G = 0\}$ (or equivalently the multiplicity of the system $\{G = 0\}$ at q) to
 114 be the dimension of the real vector space $\frac{\mathbb{R}[x]_q}{I_G}$, where I_G is the ideal generated by the set $\{\frac{g_1}{1}, \dots, \frac{g_m}{1}\}$ in $\mathbb{R}[x]_q$.

115 The previous definition is classical for the algebraic case. However, in our paper, we are interested in curves
 116 defined as the zero locus of smooth functions. For this goal, we consider a more general definition for a system
 117 $S = \{f_1(x) = \dots = f_m(x) = 0\}$ with $f_i \in C^\infty(\mathbb{R}^n, \mathbb{R})$. Let a be a solution of S and k be a non-negative
 118 integer, we define the dual space of rank k , denoted by $D_a^k[S]$, to be the vector space of all linear combinations c
 119 of differential functionals $\frac{\partial^{k_1+\dots+k_n}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}$ with $k_1 + \dots + k_n \leq k$ such that:

- 120 (a) $D_a^0[S] = \text{span}(\{\frac{\partial^0}{\partial x_1^0 \dots \partial x_n^0}\})$,
- 121 (b) c in $D_a^k[S]$ applied to f_i , evaluated at a is zero for all integers $1 \leq i \leq m$, and
- 122 (c) for all $i \in \{1, \dots, n\}$, the anti-differentiation transformation ϕ_j applied to c in $D_a^k[S]$ is in $D_a^{k-1}[S]$.

123 The anti-differentiation transformation ϕ_j is the linear operator mapping the order h differential functional
 124 $\frac{\partial^h}{\partial x_1^{h_1} \dots \partial x_j^{h_j} \dots \partial x_n^{h_n}}$ to the order $(h-1)$ differential functional $\frac{\partial^{h-1}}{\partial x_1^{h_1} \dots \partial x_j^{h_j-1} \dots \partial x_n^{h_n}}$ if $h_j > 0$ or to the order 0
 125 differential functional $\frac{\partial^0}{x_j^0}$ otherwise, where $h = \sum_{i=1}^n h_i$.

126 **Definition 4** ([DLZ11, Definition 1]). Let $F \in C^\infty(\mathbb{R}^n, \mathbb{R}^k)$ such that $F^{-1}(0)$ is a finite set and let $a \in \mathbb{R}^n$
 127 be a solution of the system $S = \{F = 0\}$. Consider the ascending chain of dual spaces $D_a^0[F] \subseteq D_a^1[F] \subseteq$
 128 $\dots \subseteq D_a^h[F] \subseteq \dots$. If there exists an integer α such that $D_a^\alpha[F] = D_a^{\alpha+1}[F]$, then the dimension of the vector space
 129 $D_a^\alpha[F]$ is called the multiplicity of a in the system S . If such an α does not exist, the multiplicity is, by convention,
 130 infinite.

131 For polynomial systems, the two definitions are equivalent [DLZ11, Theorem 2] and in addition the following
 132 proposition shows that algebraic tools can be used in the smooth case.

133 **Proposition 5** ([DLZ11, Corollary 3]). For an integer $k \geq n$, let $F = (f_1, \dots, f_k) \in C^\infty(\mathbb{R}^n, \mathbb{R}^k)$ and let
 134 $a \in \mathbb{R}^n$ be a solution of the system $\{F = 0\}$. Suppose that the multiplicity of a in $\{F = 0\}$ is $m < \infty$, then the
 135 intersection multiplicity at a of the polynomial system $\{G = (g_1, \dots, g_k) = 0\}$ is also m , where g_i is equal to the
 136 Taylor expansion of f_i at a up to degree at least m .

137 2.3. Singularities of plane curves, nodes and ordinary cusps

138 **Definition 6** ([AGZV12, §17.1]). For $i \in \{1, 2\}$, let C_i be a plane curve defined in a neighborhood $U_i \subset \mathbb{R}^2$ of
 139 p_i by the 0-set of a smooth function f_i . The pairs (p_1, C_1) and (p_2, C_2) are equivalent, and thus define the same
 140 plane curve singularity, if there exists a diffeomorphism φ from U_1 to U_2 such that $f_1 = f_2 \circ \varphi$ and $\varphi(p_1) = p_2$.

141 In particular, a singularity is of type A_k if the curve is locally defined at the origin by the 0-set of the function
 142 $x^2 - y^{k+1}$. As important special cases, A_1 is called a node singularity and A_2 is called an ordinary cusp singularity,
 143 see Figure 1.



Figure 1: Left: At an A_1 singularity, two branches of the curve intersect transversally. Right: At an A_{2k+1} singularity with $k > 1$, the tangent lines of the two branches at the intersection point coincide.

144 **Remark 7.** *It is worthwhile to notice that a curve C is an ordinary cusp at a point p if C can be locally parame-*
 145 *terized with (z^2, z^3) and p corresponds to the value $z = 0$. This remark is helpful to characterize ordinary cusps*
 146 *in Section 3.*

147 2.4. Assumptions

148 Our goal is to design a numerical algorithm for isolating the singularities that appear in the plane projection
 149 of a curve \mathcal{C} in \mathbb{R}^n . Numerical algorithms usually cannot handle degenerate cases, that is, singularities in our
 150 context. However, under some assumptions on \mathcal{C} , we succeed to isolate in a certified way some singularities of the
 151 projection. Namely, we require that the singularities are “generic”, that is, only nodes can appear in the projection
 152 (Assumption \mathcal{A}_5). Our other assumptions on \mathcal{C} are, roughly speaking, that it is smooth (\mathcal{A}_1), that its projection
 153 only has a discrete set of singularities (\mathcal{A}_4), and that at most two points of \mathcal{C} project on each singularity (\mathcal{A}_3);
 154 see Figure 2. We will prove in Section 5 that our numerical algorithm is certified and terminates under these
 155 assumptions and in Section 7 that these assumptions are generically satisfied.

156 \mathcal{A}_1 – For all $q \in \overline{\mathcal{C}}$, $\text{rank}(J_P(q)) = n - 1$. In particular, $\overline{\mathcal{C}}$ is a smooth curve.¹

157 \mathcal{A}_2 – The set $\overline{\Sigma_c}$ is discrete and does not intersect the boundary of B .

158 \mathcal{A}_3 – For all points $p = (\alpha, \beta) \in \pi_{\overline{\mathcal{C}}}(\overline{\mathcal{C}})$, the pre-image of p under $\pi_{\overline{\mathcal{C}}}$ consists of at most two points in \overline{B} counted
 159 with multiplicities in the system $\{P(x) = 0 \in \mathbb{R}^{n-1}, x_1 - \alpha = x_2 - \beta = 0\}$.

160 \mathcal{A}_4 – The set $\overline{\Sigma_n}$ is discrete and does not intersect the boundary of B .

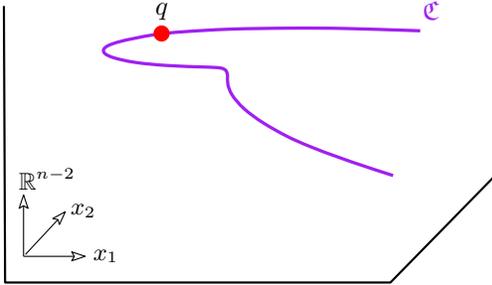
161 \mathcal{A}_5 – The singular points of $\pi_{\mathcal{C}}(\mathcal{C})$ are only nodes (see Definition 6).

162 We also consider a weaker version of Assumption \mathcal{A}_5 in which ordinary cusps can also appear in the projection:

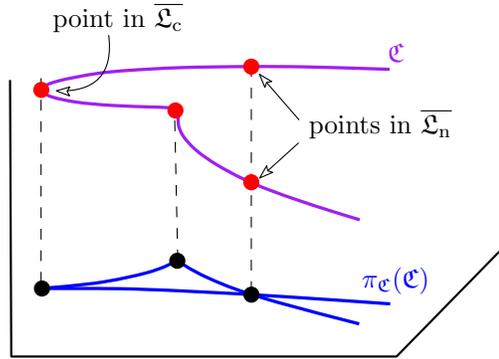
163 \mathcal{A}_5^- – The singular points of $\pi_{\mathcal{C}}(\mathcal{C})$ are only ordinary cusps or nodes (see Definition 6).

164 **Definition 8.** *Assumptions \mathcal{A}_{1-5} are called the strong assumptions and Assumptions \mathcal{A}_{1-4} and \mathcal{A}_5^- are called the*
 165 *weak assumptions.*

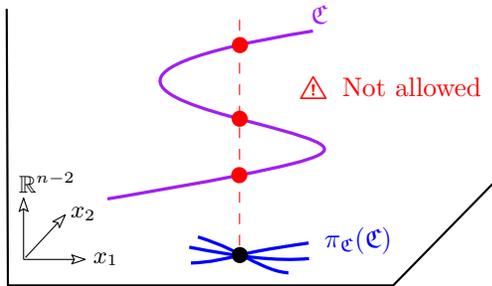
¹Note that the converse is not true as the vertical (double) line defined by $x_1^2 = x_2 = 0$ in \mathbb{R}^3 is smooth but the rank of its Jacobian is never full.



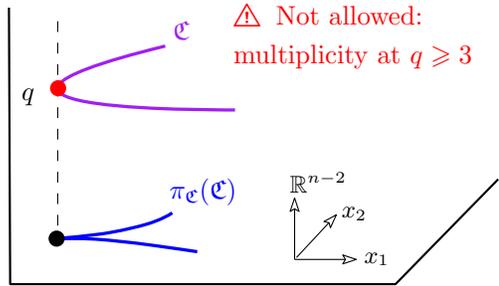
(a) Assumption \mathcal{A}_1 : $\bar{\mathcal{C}}$ is a smooth; the rank of the Jacobian of P at q is full.



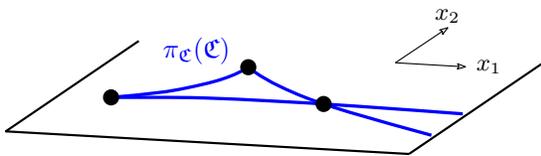
(b) Assumptions \mathcal{A}_2 and \mathcal{A}_4 : the sets $\bar{\mathcal{L}}_c$ and $\bar{\mathcal{L}}_n$ are finite and do not intersect the boundary of B .



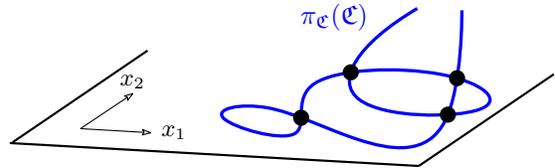
(c) Assumption \mathcal{A}_3 : No three points (counted with multiplicity) have the same projection.



(d) Assumption \mathcal{A}_3 : No three points (counted with multiplicity) have the same projection.



(e) Assumption \mathcal{A}_5^- : points in $\pi_{\mathcal{C}}(\mathcal{C})$ are either smooth, nodes or ordinary cusps.



(f) Assumption \mathcal{A}_5 : points in $\pi_{\mathcal{C}}(\mathcal{C})$ are only smooth or nodes.

Figure 2: Illustration of the assumptions.

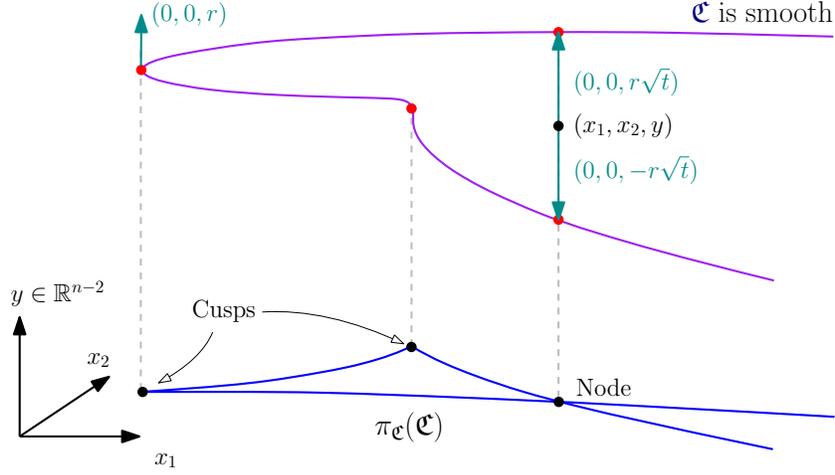


Figure 3: Illustration of a node and cusps in the plane projection of a smooth curve.

166 Our motivation for also considering these weak assumptions is dual. First, our certified algorithm for isolating
 167 the singularities of the projection of curves satisfying the strong assumptions also works, to some extent, if only
 168 the weak assumptions hold: namely, it outputs a *superset* of the isolating boxes of the singularities. Second, we
 169 conjecture that our weak assumptions are satisfied by silhouette curves of generic surfaces (see Proposition 61 and
 170 Conjecture 62).

171 3. Modelling system

172 Our goal in this section is, under the weak assumptions, to encode the singularities of the projected curve
 173 $\pi_{\mathcal{C}}(\mathcal{C})$ by a square and regular (see Definition 1) system so that it is solvable with certified numerical methods. In
 174 Section 3.1, we first define this system $\text{Ball}(P)$ and state the first main result of this section, Theorem 11, showing
 175 that the Ball system exactly encodes the singularities of $\pi_{\mathcal{C}}(\mathcal{C})$. In Sections 3.2 and 3.3, we study the singularities
 176 obtained as the projections of the points in \mathcal{L}_n and \mathcal{L}_c , respectively. Theorem 11 is proved in Section 3.4. In
 177 Section 4, we will prove our second main result, Theorem 27, stating necessary and sufficient conditions for the
 178 Ball system to be regular.

179 3.1. Encoding the singular points of the plane projection

180 By Assumption \mathcal{A}_5^- , the singularities of the projected curve $\pi_{\mathcal{C}}(\mathcal{C})$ are only nodes and cusps. Intuitively, a
 181 node appears when two points of \mathcal{C} project to the same point and a cusp appears when projecting a point with a
 182 tangent line orthogonal to the projection plane (see Figure 3). The idea to encode the nodes is to design a system
 183 whose variables are the coordinates of two different points in \mathbb{R}^n constrained to be on \mathcal{C} and so that they have the

184 same plane projection. To encode a cusp, we design a system whose variables are the coordinates of one point
185 in \mathbb{R}^n constrained to be on \mathfrak{C} with a tangent orthogonal to the projection plane. Furthermore, in order to apply
186 certified numerical methods we need systems that are square and regular (Definition 1). To solve this issue and to
187 gather the two systems into a single one, we first parameterize two different points of \mathfrak{C} with the same projection by
188 $(x_1, x_2, y + r\sqrt{t})$ and $(x_1, x_2, y - r\sqrt{t})$, with $x_1, x_2, t \in \mathbb{R}$, y, r in \mathbb{R}^{n-2} and $\|r\| = 1$. Then, given any function
189 f from \mathbb{R}^n to \mathbb{R} so that $f = 0$ is one of the $n - 1$ hypersurfaces that define \mathfrak{C} , we introduce in Definition 9 two
190 smooth functions $S \cdot f$ and $D \cdot f$. When $t > 0$, they return, roughly speaking, the arithmetic mean and difference
191 of f at the above two points, hence they both vanish if and only if the two points are on the hypersurface $f = 0$.
192 When $t = 0$, the two points coincide and $S \cdot f$ and $D \cdot f$ return, roughly speaking, f evaluated at this point and the
193 gradient of f (at that point) scalar the “vertical” vector $(0, 0, r)$; hence, they both vanish if and only if the point is
194 on the hypersurface $f = 0$ and its tangent hyperplane is normal to the plane of projection. It follows that given a
195 curve defined by $P_1 = \dots = P_{n-1} = 0$, the solutions of the so-called Ball system of all $S \cdot P_i = D \cdot P_i = 0$ is
196 the set of points on the curve that project to nodes and cusps (Theorem 11). Note that we consider \sqrt{t} instead of t
197 in the parameterization $(x_1, x_2, y \pm r\sqrt{t})$ for ensuring the regularity of the Ball system when $t = 0$ (because this
198 ensures that the linear term of the Taylor expansion of $D \cdot f$ does not vanish).

199 **Definition 9.** Let x_1, x_2, t be variables in \mathbb{R} and y, r in \mathbb{R}^{n-2} . For a smooth function $f : \overline{B} \subset \mathbb{R}^n \rightarrow \mathbb{R}$, we define
200 the functions $S \cdot f$ and $D \cdot f$ from \mathbb{R}^{2n-1} to \mathbb{R} .

$$201 \quad S \cdot f(x_1, x_2, y, r, t) = \begin{cases} \frac{1}{2}[f(x_1, x_2, y + r\sqrt{t}) + f(x_1, x_2, y - r\sqrt{t})], & \text{for } t > 0 \\ f(x_1, x_2, y), & \text{for } t = 0 \end{cases}$$

202 and

$$203 \quad D \cdot f(x_1, x_2, y, r, t) = \begin{cases} \frac{1}{2\sqrt{t}}[f(x_1, x_2, y + r\sqrt{t}) - f(x_1, x_2, y - r\sqrt{t})], & \text{for } t > 0 \\ \nabla f(x_1, x_2, y) \cdot (0, 0, r), & \text{for } t = 0. \end{cases}$$

Lemma 10. If f is a smooth function defined on $\overline{B} \subseteq \mathbb{R}^n$, then both $S \cdot f$ and $D \cdot f$ are smooth functions on the
subset

$$\overline{B}_{\text{Ball}} = \{(x_1, x_2, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R} \mid t \geq 0, (x_1, x_2, y \pm r\sqrt{t}) \in \overline{B}, \|r\|^2 = 1\}$$

204 of \mathbb{R}^{2n-1} , where $\|r\|$ denotes the Euclidean norm of r .

205 *Proof.* On the subset $\overline{B}_{\text{Ball}}$ with $t > 0$, both $S \cdot f(x_1, x_2, y, r, t)$ and $D \cdot f(x_1, x_2, y, r, t)$ are the compositions of
206 smooth functions, hence they are smooth functions.

207 For a point $X = (x_1, x_2, y, r, t)$ in B_{Ball} with $t = 0$, we will prove that $S \cdot f$ (resp. $D \cdot f$) is a C^s function for
208 an arbitrarily s which implies that $S \cdot f$ (resp. $D \cdot f$) is smooth. First define the function

$$209 \quad S_0 \cdot f(x_1, x_2, y, r, t) = \begin{cases} \frac{1}{2}[f(x_1, x_2, y + rt) + f(x_1, x_2, y - rt)], & \text{for } t > 0 \\ f(x_1, x_2, y), & \text{for } t = 0. \end{cases}$$

210 Since $S_0 \cdot f(x_1, x_2, y, r, t)$ is an even smooth function with respect to t , the partial derivatives of $S_0 \cdot f$ with respect
211 to t of odd orders, evaluated at X , are zero. For an integer $s > 0$, by the parameterized Taylor formula without
212 remainder [Dem00, Proposition 4.2.2], there exist smooth functions $a_i(x_1, x_2, y, r)$, with integers $0 \leq i < s$ such

213 that $S_0 \cdot f(x_1, x_2, y, r, t) = \sum_{i=0}^{s-1} a_i(x_1, x_2, y, r) t^{2i} + t^{2s} \cdot \phi(x_1, x_2, y, t)$, where $\phi(x_1, x_2, y, t)$ is a smooth function.

214 Notice that $S \cdot f(x_1, x_2, y, r, t) = \sum_{i=0}^{s-1} a_i(x_1, x_2, y, r) t^i + t^s \cdot \phi(x_1, x_2, y, \sqrt{t})$, so that a partial derivative exists
 215 up to order s at $t = 0$. Thus, $S \cdot f(x_1, x_2, y, r, t)$ is a C^{s-1} function. This holds for any arbitrarily large s , hence

216 $S \cdot f(x_1, x_2, y, r, t)$ is a C^∞ function.

Now, we prove that $D \cdot f$ is continuous at $X = (x_1, x_2, y, r, 0)$. Let X_i be a sequence that converges to X . To prove that $D \cdot f(X_i)$ converges to $D \cdot f(X)$, it is enough to show that for a sequence t_i that converges to 0, then we have that $D \cdot f(x_1, x_2, y, r, t_n)$ converges to $D \cdot f(X)$. We can assume that $t_i \neq 0$ for all i , so that

$$\begin{aligned} \lim_{t_i \rightarrow 0} D \cdot f(x_1, x_2, y, r, t_i) &= \lim_{t_i \rightarrow 0} \frac{1}{2\sqrt{t_i}} [f(x_1, x_2, y + r\sqrt{t_i}) - f(x_1, x_2, y - r\sqrt{t_i})] \\ &= \lim_{t_i \rightarrow 0} \frac{1}{2\sqrt{t_i}} [f(x_1, x_2, y + r\sqrt{t_i}) - (f(x_1, x_2, y) - f(x_1, x_2, y)) - f(x_1, x_2, y - r\sqrt{t_i})] \\ &= \lim_{t_i \rightarrow 0} \frac{1}{2\sqrt{t_i}} [f(x_1, x_2, y + r\sqrt{t_i}) - f(x_1, x_2, y)] \\ &\quad + \lim_{t_i \rightarrow 0} \frac{1}{2\sqrt{t_i}} [f(x_1, x_2, y) - f(x_1, x_2, y - r\sqrt{t_i})] \\ &= \frac{1}{2} \nabla f \cdot (0, 0, r) - \frac{1}{2} \nabla f \cdot (0, 0, -r) \\ &= \nabla f \cdot (0, 0, r). \end{aligned}$$

217 We now prove that $D \cdot f$ is smooth at X . Similarly to the proof of the case of $S \cdot f$, since the function $\frac{1}{2}[f(x_1, x_2, y +$
 218 $rt) - f(x_1, x_2, y - rt)]$ is odd with respect to t , there exist smooth functions $b_i(x_1, x_2, y, r)$, for $1 \leq i < s$ and
 219 $\psi(x_1, x_2, y, r, t)$ such that $\frac{1}{2}[f(x_1, x_2, y + rt) - f(x_1, x_2, y - rt)] = \sum_{i=0}^{s-1} b_i(x_1, x_2, y, r) t^{2i+1} + t^{2s+1} \cdot \psi(x_1, x_2, y, t)$.

220 Notice that $D \cdot f(x_1, x_2, y, r, t) = \sum_{i=0}^{s-1} b_i(x_1, x_2, y, r) t^i + t^s \cdot \psi(x_1, x_2, y, \sqrt{t})$, so that a partial derivative exists
 221 up to order s at $t = 0$. Thus, $D \cdot f(x_1, x_2, y, r, t)$ is a C^{s-1} function. This holds for any arbitrarily large s , hence
 222 $D \cdot f(x_1, x_2, y, r, t)$ is a C^∞ function. \square

Theorem 11. Consider $P = (P_1, \dots, P_{n-1}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ that satisfies Assumptions $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and \mathcal{A}_4 .

Then, $X = (x_1, x_2, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R}$ is a solution of the Ball system

$$\text{Ball}(P) = \begin{cases} S \cdot P_1(X) = \dots = S \cdot P_{n-1}(X) = 0 \\ D \cdot P_1(X) = \dots = D \cdot P_{n-1}(X) = 0 \\ \|r\|^2 - 1 = 0 \end{cases} \quad (3.1)$$

223 if and only if (x_1, x_2) is a singular point of $\pi_{\mathcal{C}}(\mathcal{C})$ (see Definition 9 for the notation $S \cdot P_i$ and $D \cdot P_i$).

224 We postpone the proof of Theorem 11 to the end of Section 3.3. As a first step, we study a mapping from the
 225 solutions of the Ball system to pairs of points on the curve \mathcal{C} .

226 **Definition 12.** Let $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$. Define $\widehat{\mathfrak{L}}_n$ to be the set of pairs (q_1, q_2) with $q_1, q_2 \in \mathfrak{C}$, $q_1 \neq q_2$ and
 227 $\pi_{\mathfrak{C}}(q_1) = \pi_{\mathfrak{C}}(q_2)$, also define $\widehat{\mathfrak{L}}_c$ to be the set of pairs (q_1, q_1) with $q_1 \in \mathfrak{L}_c$, and let $\widehat{\mathfrak{L}} = \widehat{\mathfrak{L}}_n \cup \widehat{\mathfrak{L}}_c$.

228 **Lemma 13.** Consider $P = (P_1, \dots, P_{n-1}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ and let $X = (x_1, x_2, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times$
 229 $\mathbb{R}^{n-2} \times \mathbb{R}$, with $\|r\| = 1$. Assume that P satisfies Assumption \mathcal{A}_1 . Then X is a solution of $\text{Ball}(P)$ if and only
 230 if for the points $q_1 = (x_1, x_2, y + r\sqrt{t})$ and $q_2 = (x_1, x_2, y - r\sqrt{t})$, the pair (q_1, q_2) is in $\widehat{\mathfrak{L}}_n$, or in $\widehat{\mathfrak{L}}_c$ with
 231 $(0, 0, r) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$ in $T_{q_1}\mathfrak{C}$.

232 *Proof.* Note that, by Assumption \mathcal{A}_1 , the tangent space to the curve at any of its points is well defined and is a
 233 line. First, assume that X is a solution of $\text{Ball}(P)$. We consider two cases:

- 234 (a) If $t > 0$, then since $r \neq 0 \in \mathbb{R}^{n-2}$ we have that $q_1 \neq q_2$. Moreover, since $S \cdot P_i(X) = D \cdot P_i(X) = 0$
 235 for all $i \in \{1, \dots, n-1\}$, we deduce that $P_i(q_1) = P_i(q_2) = 0$, thus $q_1, q_2 \in \mathfrak{C}$. Moreover, since
 236 $\pi_{\mathfrak{C}}(q_1) = \pi_{\mathfrak{C}}(q_2) = (x_1, x_2)$ we have $q_1, q_2 \in \mathfrak{L}_n$. Thus, $(q_1, q_2) \in \widehat{\mathfrak{L}}_n$.
- 237 (b) If $t = 0$, then $q_1 = q_2$. First, $P_i(q_1) = S \cdot P_i(X) = 0$, for all indices $i \in \{1, \dots, n-1\}$, hence $q_1 \in \mathfrak{C}$.
 238 Moreover, we have that $0 = D \cdot P_i(X) = \nabla P_i(q_1) \cdot (0, 0, r)$, for all $i \in \{1, \dots, n-1\}$, equivalently,
 239 $J_P(q_1) \cdot (0, 0, r)^T = 0 \in \mathbb{R}^{n-1}$, i.e., we have $(0, 0, r) \in T_{q_1}\mathfrak{C}$. Thus, $q_1 \in \mathfrak{L}_c$ and hence, $(q_1, q_1) \in \widehat{\mathfrak{L}}_c$.

240 Now, let us prove the other direction:

- 241 (a) If $(q_1, q_2) \in \widehat{\mathfrak{L}}_n$, then $q_1 \neq q_2$ and $t \neq 0$. Also, since $q_1, q_2 \in \mathfrak{C}$, we can write that $S \cdot P_i(X) =$
 242 $\frac{1}{2}(P_i(q_1) + P_i(q_2)) = 0$, and $D \cdot P_i(X) = \frac{1}{2\sqrt{t}}(P_i(q_1) - P_i(q_2)) = 0$, for all $i \in \{1, \dots, n-1\}$. Thus, X
 243 is a solution of $\text{Ball}(P)$.
- 244 (b) If $(q_1, q_2) \in \widehat{\mathfrak{L}}_c$ and $(0, 0, r) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$ is in $T_{q_1}\mathfrak{C}$, one has $q_1 = q_2 \in \mathfrak{L}_c \subseteq \mathfrak{C}$, and $t = 0$. Moreover,
 245 for all $i \in \{1, \dots, n-1\}$ we have $S \cdot P_i(X) = P_i(q_1) = 0$ and since $(0, 0, r) \in T_{q_1}\mathfrak{C}$, we can equivalently
 246 write $D \cdot P_i(X) = \nabla P_i(q_1) \cdot (0, 0, r) = 0$. Thus, X is a solution of $\text{Ball}(P)$. \square

247 **Definition 14.** Let $\text{Sol}_{\text{Ball}(P)}$ be the solution set of $\text{Ball}(P)$. Define the function Ω_P from $\text{Sol}_{\text{Ball}(P)}$ to $\widehat{\mathfrak{L}}$ that
 248 sends $X = (x_1, x_2, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R}^+$ to the ordered pair $q_1 = (x_1, x_2, y + r\sqrt{t})$ and
 249 $q_2 = (x_1, x_2, y - r\sqrt{t})$. Notice that the function Ω_P is well-defined by Lemma 13.

250 **Lemma 15.** If $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ satisfies Assumption \mathcal{A}_1 , then Ω_P is surjective.

251 *Proof.* For any pair $(q_1, q_2) \in \widehat{\mathfrak{L}}_n$ we have that the point $X = (\frac{1}{2}(q_1 + q_2), \frac{\Pi_{\mathfrak{C}}(q_1 - q_2)}{\|q_1 - q_2\|}, \frac{1}{4}\|q_1 - q_2\|^2) \in \mathbb{R}^n \times$
 252 $\mathbb{R}^{n-1} \times \mathbb{R}^+$ is a solution of $\text{Ball}(P)$, where $\Pi_{\mathfrak{C}}(q_1 - q_2)$ is the vector in \mathbb{R}^{n-2} obtained by omitting the first two
 253 coordinates (which are zeros) from $q_1 - q_2$. Note that $\Omega_P(X) = (q_1, q_2)$. If the pair (q_1, q_1) is in $\widehat{\mathfrak{L}}_c$, we define
 254 r in the following way: we take a unit vector $v \in T_{q_1}\mathfrak{C}$ (the first two coordinates of v are zeros since $q_1 \in \mathfrak{L}_c$).
 255 We set r to be $\Pi_{\mathfrak{C}}(v)$. Again $X = (q_1, r, 0) \in \mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}$ is a solution of $\text{Ball}(P)$, with $\Omega_P(X) = (q_1, q_1)$.
 256 Thus, Ω_P is surjective. \square

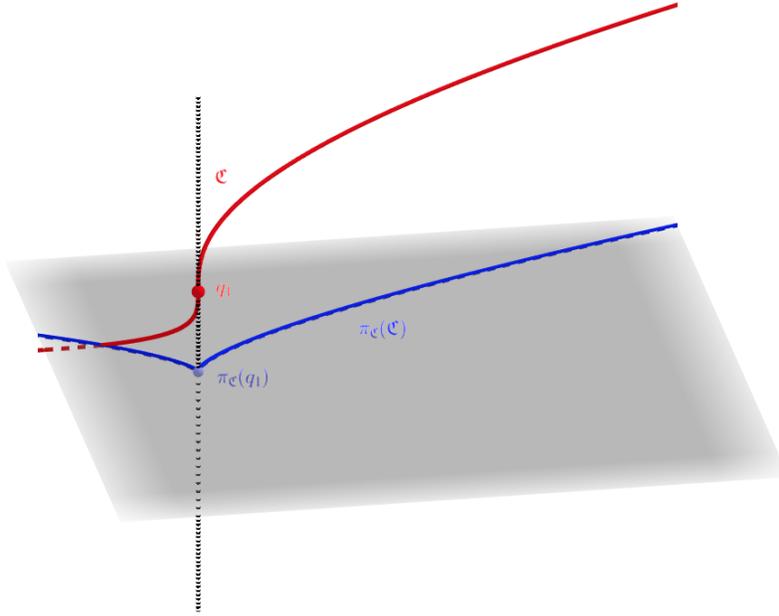


Figure 4: The curve \mathfrak{C} (red) and its plane projection $\pi_{\mathfrak{C}}(\mathfrak{C})$ (blue) of Example 18 displaying a cusp singularity.

257 **Remark 16.** Notice that if $X = (x_1, x_2, y, r, t)$ is in $\text{Sol}_{\text{Ball}(P)}$, then $\Omega_P(X) \in \widehat{\mathfrak{L}}_n$ (resp. $\Omega_P(X) \in \widehat{\mathfrak{L}}_c$) if and
 258 only if $t \neq 0$ (resp. $t = 0$).

259 **Remark 17.** Preserving the notation in Lemma 13, notice that if $X = (x_1, x_2, y, r, t)$ is a solution of $\text{Ball}(P)$,
 260 then $X' = (x_1, x_2, y, -r, t)$ is another solution. Moreover, both solutions characterize the same unordered pair
 261 $\Omega_P(X) = \Omega_P(X') = (q_1, q_2)$. We call X and X' twin solutions. An alternative choice would have been to take r
 262 in a projective space instead of the sphere to identify these twin solutions.

Example 18. Refer to Figure 4. Let $n = 3$ and $B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1, x_2, x_3 \in (-2, 2)\}$. Define
 $P_1(x_1, x_2, x_3) = x_1 - (x_3 - 1)^3$, $P_2(x_1, x_2, x_3) = x_2 - (x_3 - 1)^2$ and $P = (P_1, P_2)$. The Jacobian matrix
of P has full rank over \mathfrak{C} , thus Assumption \mathcal{A}_1 is satisfied. The set \mathfrak{L}_n is empty since $\pi_{\mathfrak{C}}$ is injective over \mathfrak{C} ,
hence Assumption \mathcal{A}_4 is satisfied. The only point of \mathfrak{C} with a tangent line orthogonal to the (x_1, x_2) -plane is
 $q_1 = (0, 0, 1)$, thus $\mathfrak{L}_c = \{q_1\}$ and Assumption \mathcal{A}_2 is satisfied. By Lemma 23, the multiplicity of the system
 $\{P = 0, (x_1, x_2) = \pi_{\mathfrak{C}}(q_1)\}$ at its unique solution q_1 is $\min\{\text{mult}_1((x_3 - 1)^3), \text{mult}_1((x_3 - 1)^2)\} = \min\{3, 2\} =$
2 (mult is defined in Definition 2). Moreover, for any point $q_0 \in \mathfrak{C}$ different from q_1 , the multiplicity of the

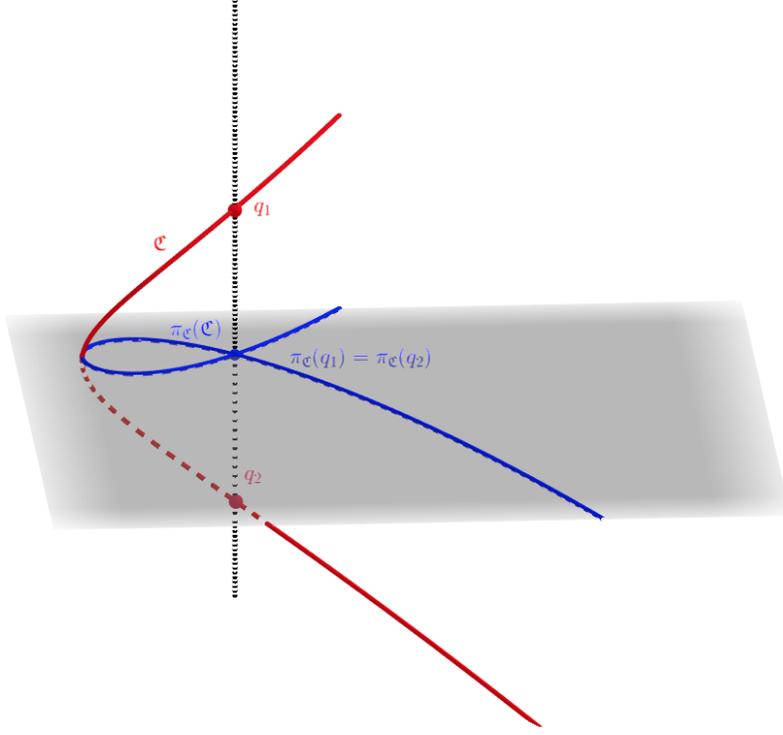


Figure 5: The curve \mathcal{C} (red) and its plane projection $\pi_{\mathcal{C}}(\mathcal{C})$ (blue) of Example 19 displaying a node singularity.

corresponding system at its unique solution q_0 is one, thus P satisfies Assumption \mathcal{A}_3 . The system $\text{Ball}(P)$:

$$\begin{cases} x_1 - 3r^2ty + 3r^2t - y^3 + 3y^2 - 3y + 1 = 0 \\ x_2 - r^2t - y^2 + 2y - 1 = 0 \\ -r^3t - 3ry^2 + 6ry - 3r = 0 \\ -2ry + 2r = 0 \\ r^2 - 1 = 0 \end{cases} \quad (3.2)$$

263 has two twin solutions $X = (0, 0, 1, 1, 0)$ and $X' = (0, 0, 1, -1, 0)$ in $B_{\text{Ball}(P)} \subset \mathbb{R}^{2 \cdot 3-1} = \mathbb{R}^5$ such that
 264 $\Omega_P(X) = \Omega_P(X') = (q_1, q_1) \in \widehat{\mathcal{L}}_{\mathcal{C}}$.

Example 19. Refer to Figure 5. Let B be defined as in Example 18. Define the functions $P_1(x_1, x_2, x_3) = x_1 - (x_3^2 - 1)$, $P_2(x_1, x_2, x_3) = x_2 - (x_3^3 - x_3)$ and $P = (P_1, P_2)$. The Jacobian matrix of P has full rank over \mathcal{C} , thus Assumption \mathcal{A}_1 is satisfied. Moreover, the set $\mathcal{L}_{\mathcal{C}}$ is empty and $\mathcal{L}_{\mathfrak{n}} = \{q_1, q_2\}$, with $q_1 = (0, 0, 1)$, $q_2 = (0, 0, -1)$, i.e., Assumptions \mathcal{A}_2 and \mathcal{A}_4 are satisfied. The multiplicity of the system $\{P = 0 \in \mathbb{R}^{n-1}, x_1 = x_2 = 0\}$ at both

q_1, q_2 is equal to one, thus Assumption \mathcal{A}_3 is also satisfied. The system $\text{Ball}(P)$:

$$\begin{cases} x_1 - r^2t - y^2 + 1 = 0 \\ x_2 - r^2ty - y^3 + y = 0 \\ -2ry = 0 \\ -r^3t - 3ry^2 + r = 0 \\ r^2 - 1 = 0 \end{cases} \quad (3.3)$$

265 has two twin solutions $X = (0, 0, 0, 1, 1)$ and $X' = (0, 0, 0, -1, 1)$ in \mathbb{R}^5 such that $\Omega_P(X) = \Omega_P(X') =$
 266 $(q_1, q_2) \in \widehat{\mathcal{L}}_n$.

267 3.2. Singularities induced by \mathcal{L}_n

268 We now study the types of singularities of the plane curve $\pi_{\mathcal{C}}(\mathcal{C})$ obtained by projecting points in \mathcal{L}_n , that is
 269 when several points of \mathcal{C} project to the same point. We begin by showing that the geometric property that the curve
 270 \mathcal{C} has a tangent orthogonal to the projection plane has an algebraic equivalent in terms of multiplicity.

271 **Lemma 20.** *Let $P = (P_1 \dots, P_{n-1}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ satisfy Assumption \mathcal{A}_1 . Let q be in $\overline{\mathcal{C}}$ such that the*
 272 *multiplicity of the system $S = \{P(x) = 0 \in \mathbb{R}^{n-1}, x_1 - \alpha = x_2 - \beta = 0\}$ at q is finite, where $(\alpha, \beta) = \pi_{\overline{\mathcal{C}}}(q) \in \mathbb{R}^2$.*
 273 *Then, $q \in \overline{\mathcal{L}}_c$ if and only if the multiplicity of the system S at q is at least two.*

274 *Proof.* Without loss of generality assume that $q = 0 \in \mathbb{R}^n$.

275 *Sufficiency:* Assume that $q \in \overline{\mathcal{L}}_c$. Let $v = (v_1, \dots, v_n)$ be a non-trivial vector of the tangent line of $\overline{\mathcal{C}}$ at q .
 276 Thus, $J_P(q) \cdot v^T = 0$. By the definition of $\overline{\mathcal{L}}_c$ we have $v_1 = v_2 = 0$. Define the differential operator $c = \sum_{i=3}^n v_i \frac{\partial}{\partial x_i}$.
 277 Notice that $c \cdot P_j = \sum_{i=3}^n v_i \frac{\partial P_j}{\partial x_i}(q) = 0$ for all integers $1 \leq j \leq n-1$ (see [DLZ11, 2.1] for the definition of $c \cdot P_j$).
 278 Moreover, by the definition of c and since $v_1 = v_2 = 0$, we have $c \cdot (x_1) = c \cdot (x_2) = 0$. Hence, $c \in D_q^1[S] \setminus D_q^0[S]$.
 279 Thus, $\dim(D_q^1) > 1$. Hence, the multiplicity of S at q is at least two.

280 *Necessity:* Assume that the multiplicity of S at q is at least two, then $D_q^0[S] \subsetneq D_q^1[S]$. This implies that there
 281 exists a non-trivial differential operator $c = \sum_{i=1}^n c_i \frac{\partial}{\partial x_i} \in D_q^1[S] \setminus D_q^0[S]$ such that:

282 (a) We have that $c \cdot P_j = 0$ for all integers $1 \leq j \leq n-1$ which implies that if we write $v_i = c_i$, with $1 \leq i \leq n$,
 283 the non-trivial vector v is in the tangent space of $\overline{\mathcal{C}}$ at q .

284 (b) We have that $c \cdot (x_1) = c \cdot (x_2) = 0$, equivalently, $c_1 = c_2 = 0$. Thus, $v_1 = v_2 = 0$.

285 The tangent line to the curve at q is thus orthogonal to the (x_1, x_2) -plane. Thus, $q \in \overline{\mathcal{L}}_c$. □

286 **Lemma 21.** *Under Assumptions $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and \mathcal{A}_4 , if $q \in \mathcal{L}_n$, then $\pi_{\mathcal{C}}(q)$ is a singular point of the plane curve*
 287 *$\pi_{\mathcal{C}}(\mathcal{C})$. More precisely, either $\pi_{\mathcal{C}}(q)$ is of type A_{2k+1}^- with $k \geq 0$, or there exists a non-null smooth function g*
 288 *defined in a neighborhood of $0 \in \mathbb{R}$ with $\text{mult}(g) = \infty$ such that $(\pi_{\mathcal{C}}(q), \pi_{\mathcal{C}}(\mathcal{C}))$ is equivalent (according to*
 289 *Definition 6) to the curve defined by $x^2 - g(y^2) = 0$ at the origin.*

290 *Proof.* Let $p = \pi_{\mathfrak{C}}(q)$, according to \mathcal{A}_3 , $\pi_{\mathfrak{C}}^{-1}(p)$ has at most two points, and since q is in \mathfrak{L}_n , it also has at least two
291 points. Define q' such that $\pi_{\mathfrak{C}}^{-1}(p) = \{q, q'\}$ and denote the plane curve $\pi_{\mathfrak{C}}(\mathfrak{C})$ by C . Without loss of generality,
292 one can assume that $p = (0, 0)$. In addition, \mathcal{A}_3 also implies that the multiplicities of q and q' in the system
293 $\{P(x) = 0 \in \mathbb{R}^{n-1}, x_1 = x_2 = 0\}$ are one. With Assumption \mathcal{A}_1 , Lemma 20 then implies that the tangents to \mathfrak{C}
294 at q and q' are not orthogonal to the (x_1, x_2) -plane. Thus there exists two neighborhoods N_q and $N_{q'}$ of q and q' in
295 \mathbb{R}^n such that π restricted to $\mathfrak{C} \cap N_q$ (resp. $\mathfrak{C} \cap N_{q'}$) is an embedding. Let D_k be a sequence of open disks centered
296 at p and of radius $\frac{1}{k}$. By contradiction, if for all k , there exists points $q_k \in \mathfrak{C}$ such that q_k is not in $N_q \cup N_{q'}$
297 and $\pi_{\mathfrak{C}}(q_k) \in D_k$, then the limit q_∞ is a point of $\overline{\mathfrak{C}}$ distinct from q and q' , and $\pi_{\mathfrak{C}}(q_\infty) = p$. If q_∞ is in B , it
298 contradicts \mathcal{A}_3 and if it is in \overline{B} , it contradicts \mathcal{A}_4 . Thus for a small enough neighborhood of p , the projection of
299 the curve is restricted to the projection of the two branches around q and q' . Finally, if for all D_k , the pre-image of
300 $\pi^{-1}(D_k)$ contains a point in $\overline{\mathfrak{L}_n} \setminus \{q, q'\}$, then this contradicts the discreteness assumption \mathcal{A}_4 . Thus there exists a
301 neighborhood $N \subseteq \mathbb{R}^2$ of p such that $\pi_{\mathfrak{C}}^{-1}(N)$ is a union of two smooth (Assumption \mathcal{A}_1) open subsets of \mathfrak{C} such
302 that q is on one branch and q' on the other, and $\pi_{\mathfrak{C}}$ restricted to $\pi_{\mathfrak{C}}^{-1}(N) \setminus \{p, q'\}$ is an embedding. The projection
303 of these two smooth branches are thus two smooth curves in the plane. Let these two smooth plane branches be
304 defined by the zero sets of the smooth functions f_1 and f_2 in $C^\infty(\mathbb{R}^2, \mathbb{R})$. Let u (resp. u') be a non-zero tangent
305 vector of \mathfrak{C} at q (resp. q') and v (resp. v') be its projection in \mathbb{R}^2 . We distinguish two cases:

306 (a) The vectors v and v' are independent in \mathbb{R}^2 . Thus, v and v' give rise to a local coordinate system (x, y)
307 in a neighborhood of p in \mathbb{R}^2 . The vector v being tangent to the zero set of f_1 , one has $\frac{\partial f_1}{\partial x}(p) = 0$ and
308 $\frac{\partial f_1}{\partial y}(p) \neq 0$. By the implicit function theorem [Dem00, Corollary 2.7.3.], we deduce that there exists a real
309 smooth function h_1 such that $y = x^2 \cdot h_1(x)$ is a local parameterization of the zero set of f_1 . Similarly,
310 there exists a smooth function h_2 such that $x = y^2 \cdot h_2(y)$ is a local parameterization of the zero set of f_2 .
311 Thus $(x, y) \in N$ iff $f(x, y) = f_1(x, y)f_2(x, y) = 0$ iff $(y - x^2 \cdot h_1(x))(x - y^2 \cdot h_2(y)) = 0$, equivalently,
312 $[y - x - x^2 \cdot h_1(x) + y^2 \cdot h_2(y)]^2 - [y + x - x^2 \cdot h_1(x) - y^2 \cdot h_2(y)]^2 = 0$. The change of coordinates
313 $X = y - x + x^2 \cdot h_1(x) + y^2 \cdot h_2(y)$ and $Y = y + x + x^2 \cdot h_1(x) - y^2 \cdot h_2(y)$ is a diffeomorphism since
314 $\det(J_{x,y}(X, Y))_p \neq 0$. Then, the local equation of the curve C at p is of the form $X^2 - Y^2$ with these new
315 coordinates, which means that p is a A_1^- or node singularity.

316 (b) The vectors v and v' are co-linear. Then, choose $v'' \in T_p \mathbb{R}^2$ linearly independent from v , the vectors v, v''
317 give rise to a coordinate system (x, y) at p . In this coordinate system, we thus have $\frac{\partial f_1}{\partial x}(p) = \frac{\partial f_2}{\partial x}(p) = 0$,
318 $\frac{\partial f_1}{\partial y}(p) \neq 0$ and $\frac{\partial f_2}{\partial y}(p) \neq 0$. By the implicit function theorem, there exist smooth functions h_1 and h_2 such
319 that locally $f(x, y) = 0$ if and only if $(y - x^2 \cdot h_1(x))(y - x^2 \cdot h_2(x)) = 0$. The last equality is equivalent
320 to $(2y - x^2(h_1(x) + h_2(x)))^2 - x^4(h_1(x) - h_2(x))^2 = 0$. Assumption \mathcal{A}_4 ensures that the projections of
321 the two branches have only one common point, such that $h_1(x) - h_2(x)$ does not vanish identically. We
322 distinguish two cases:

323 (i) $\text{mult}(h_1(x) - h_2(x)) = k \leq \infty$, then $h_1(x) - h_2(x) = x^k \cdot u$ with $u(p) \neq 0$ and without loss

324 of generality, assume that $u(p) > 0$. The change of coordinates $X = 2y - x^2(h_1(x) + h_2(x))$ and
 325 $Y = x \cdot u^{\frac{1}{2+k}}$ is a diffeomorphism (notice that indeed $u^{\frac{1}{2+k}}$ is a smooth function around p). Then, the
 326 local equation of the curve C at p is of the form $X^2 - Y^{(2k+3)+1}$ with these new coordinates, which
 327 means that p is a singularity of type A_{2k+3}^- .

328 (ii) $\text{mult}(h_1(x) - h_2(x)) = \infty$. Since the function $x^4(h_1(x) - h_2(x))^2$ is even, by Theorem 49, there
 329 exists a smooth function g such that $x^4(h_1(x) - h_2(x))^2 = g(x^2)$. Thus, taking the diffeomorphism
 330 $X = 2y - x^2(h_1(x) + h_2(x))$ and $Y = x$, we get the second case of the claim.

331 □

332 3.3. Singularities induced by \mathfrak{L}_c

333 We now study the types of singularities of the plane curve $\pi_c(\mathfrak{C})$ obtained by projecting points in \mathfrak{L}_c , that is
 334 when the tangent to \mathfrak{C} is orthogonal to the projection plane. We start by locally parametrizing \mathfrak{C} around a point
 335 in \mathfrak{L}_c . This parameterization will ease the computation of $\text{Ball}(P)$ and its Jacobian in Section 4. In the rest
 336 of this section and Section 4, the analysis is simplified by translating relevant points or assuming the curve \mathfrak{C} is
 337 parametrizable by a specific variable. On the other hand, in our algorithmic Section 5, the input is not modifiable
 338 at all but every computation uses interval arithmetic. This implies that the exact coordinates of a point may not be
 339 known, instead we only compute with a box containing it and isolating it from other relevant points. The idea of
 340 our semi-algorithms is to check that some function does not vanish on such a box. This then implies that such a
 341 function does not vanish at the point this box contains. The *theoretical analysis* of this section can then be applied
 342 to the point to deduce the appropriate property without the knowledge of the exact location of that point.

Lemma 22. *Let $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$. Let $q \in \mathfrak{L}_c$ such that Assumption \mathcal{A}_1 is satisfied in a neighborhood of q
 in B . Without loss of generality one can assume $q = 0 \in \mathbb{R}^n$. Then there exist an invertible matrix M of size
 $(n-1) \times (n-1)$ of smooth functions in a neighborhood of q and smooth functions $f_1, f_2, f_3, \dots, f_{n-1}$ defined
 in a neighborhood of $0 \in \mathbb{R}$, such that:*

$$\begin{pmatrix} x_1 - f_1(x_n) \\ x_2 - f_2(x_n) \\ x_3 - f_3(x_n) \\ \dots \\ \dots \\ x_{n-1} - f_{n-1}(x_n) \end{pmatrix} = M \cdot \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ \dots \\ \dots \\ P_{n-1} \end{pmatrix}, \quad (3.4)$$

343 with $\min\{\text{mult}(f_1(x_n)), \text{mult}(f_2(x_n))\} > 1$ (mult is defined in Definition 2).

Proof. Since $\text{rank}(J_P(q)) = n-1$ (Assumption \mathcal{A}_1), there exists $k \in \{1, \dots, n\}$ such that $\det(M_k(q)) \neq 0$,
 where M_k is the minor of J_P obtained by removing the k -th column. Notice that $k \notin \{1, 2\}$, since $q \in \mathfrak{L}_c$ implies
 that $\det(M_1(q)) = \det(M_2(q)) = 0$. Without loss of generality, we assume that $k = n$. Using the implicit

function theorem [Corollary 2.7.3][Dem00], there exist smooth functions f_1, \dots, f_{n-1} of one variable such that we have that

$$P_j(f_1(x_n), \dots, f_{n-1}(x_n), x_n) = 0, j \in \{1, \dots, n-1\}. \quad (3.5)$$

344 Define the function φ that maps x_i to $z_i = x_i - f_i(x_n)$, for all $i \in \{1, \dots, n-1\}$ and x_n to $z_n = x_n$.
 345 We can see that φ is a diffeomorphism and $z = (z_1, \dots, z_n)$ is a local coordinate system around q . Hence, we
 346 can define the function $G_j(z) = P_j \circ \varphi^{-1}(z) = P_j(x)$ for all integers $1 \leq j \leq n-1$. Using Hadamard's
 347 Lemma [Dem00, Proposition 4.2.3] for the first $n-1$ variables of z , we can write $G_j(z) - G_j(0, \dots, 0, z_n) =$
 348 $\sum_{i=1}^{n-1} z_i \cdot h_{ji}(z)$ for some smooth functions h_{ji} . Note that $\varphi^{-1}(z) = (z_1 + f_1(z_n), \dots, z_{n-1} + f_{n-1}(z_n), z_n)$. Hence,
 349 $G_j(0, \dots, 0, z_n) = P_j \circ \varphi^{-1}(0, \dots, 0, z_n) = P_j(f_1(z_n), \dots, f_{n-1}(z_n), z_n) = P_j(f_1(x_n), \dots, f_{n-1}(x_n), x_n)$.
 350 The latter function is equal to zero by (3.5). Thus, $P_j(x) = G_j(z) = \sum_{i=1}^{n-1} z_i \cdot h_{ji}(z) = \sum_{i=1}^{n-1} (x_i - f_i(x_n)) \cdot H_{ji}(x)$,
 351 with $H_{ji}(x) = h_{ji} \circ \varphi(x)$.

352 Defining $M_0 = \left(H_{ji} \right)_{1 \leq j, i \leq n-1}$ we get:

$$\begin{pmatrix} P_1 \\ \dots \\ \dots \\ P_{n-1} \end{pmatrix} = M_0 \cdot \begin{pmatrix} x_1 - f_1(x_n) \\ \dots \\ \dots \\ x_{n-1} - f_{n-1}(x_n) \end{pmatrix}.$$

353 Notice that M_0 evaluated at q is the invertible matrix $M_n(q)$. Hence, by continuity of the determinant function,
 354 there is a neighborhood of q in which M_0 is invertible. Thus, writing M as the inverse of M_0 we get:

$$Q_0 = \begin{pmatrix} x_1 - f_1(x_n) \\ \dots \\ \dots \\ x_{n-1} - f_{n-1}(x_n) \end{pmatrix} = M \cdot \begin{pmatrix} P_1 \\ \dots \\ \dots \\ P_{n-1} \end{pmatrix}. \quad (3.6)$$

355 To prove that $\min\{\text{mult}(f_1(x_n)), \text{mult}(f_2(x_n))\} > 1$, we take the Jacobian matrices of both sides of (3.6) and we
 356 evaluate them at $q = 0$. We get the equation $J_{Q_0}(q) = M(q) \cdot J_P(q)$. By invertibility of $M(q)$ we deduce that the
 357 k -th minors (obtained by removing the k -th column) of $J_{Q_0}(q)$ and $J_P(q)$ have the same rank. Computing $J_{Q_0}(q)$
 358 and considering the fact that $\det(M_1(q)) = \det(M_2(q)) = 0$ implies that $f_1'(0) = f_2'(0) = 0$, we thus have that
 359 $\min\{\text{mult}(f_1(x_n)), \text{mult}(f_2(x_n))\}$ is at least two. \square

360 **Lemma 23.** *Preserving the notation and the assumptions in Lemma 22, the multiplicity m of the system $S =$*
 361 *$\{Q_0(x) = 0 \in \mathbb{R}^{n-1}, x_1 = x_2 = 0\}$ at q is equal to $d = \min\{\text{mult}(f_1(x_n)), \text{mult}(f_2(x_n))\}$.*

362 *Proof.* First, we start with the case $m < \infty$. By Proposition 5, we can assume without loss of generality, that
 363 f_1, \dots, f_{n-1} are polynomials. Following the notation in Definition 3, let $\mathbb{R}[x]$ (resp. $\mathbb{R}[x_n]$) be the ring of poly-
 364 nomials with n variables (resp. one variable) and $\mathbb{R}[x]_q$ (resp. $\mathbb{R}[x_n]_0$) be its localization at q (resp. $0 \in \mathbb{R}$).
 365 Also, define I_S to be the ideal generated by the polynomials of S in $\mathbb{R}[x]_q$ (as I_G is defined in Definition 3),

366 i.e., $I_S = \langle x_1 - f_1(x_n), x_2 - f_2(x_n), \dots, x_{n-1} - f_{n-1}(x_n), x_1, x_2 \rangle = \langle x_1 - f_1(x_n), x_2 - f_2(x_n), \dots, x_{n-1} -$
367 $f_{n-1}(x_n), f_1(x_n), f_2(x_n) \rangle$. If $f_1(x_n) = f_2(x_n) = 0$, then the ideal I_S is of dimension one, hence, S has an
368 infinite number of solutions which contradicts the assumption that $m < \infty$. Thus, $d < \infty$ which means that
369 there exist $h_1, h_2 \in \mathbb{R}[x_n]_0$ such that $h_1(x_n)f_1(x_n) + h_2(x_n)f_2(x_n) = x_n^d$. Thus, $I_S = \langle x_1 - f_1(x_n), x_2 -$
370 $f_2(x_n), \dots, x_{n-1} - f_{n-1}(x_n), x_n^d \rangle$. Note that the set $\{x_1 - f_1(x_n), x_2 - f_2(x_n), \dots, x_{n-1} - f_{n-1}(x_n), x_n^d\}$ is a
371 Gröbner basis of I_S with respect to Local Lexicographical ordering $x_1 > \dots > x_n$. Hence, By [CLO05, Theorem
372 4.4.3] we have $\dim(\frac{\mathbb{R}[x]_q}{I_S}) = \dim(\frac{\mathbb{R}[x]_q}{LT(I_S)}) = \dim(\frac{\mathbb{R}[x]_q}{\langle x_1, x_2, \dots, x_{n-1}, x_n^d \rangle})$, where $LT(I_S)$ is the ideal generated by the
373 leading terms of I_S . Consequently, $m = \dim(\frac{\mathbb{R}[x]_q}{I_S}) = \dim(\frac{\mathbb{R}[x_n]_0}{\langle x_n^d \rangle}) = d$.

374 Second, assume that $m = \infty$. We prove that $d = \infty$, that is, $\frac{\partial^k f_1}{\partial x_n^k}(0) = \frac{\partial^k f_2}{\partial x_n^k}(0) = 0$ for any positive integer
375 k . Preserving the notation in Definition 4, consider the dual space $D_q^k[S]$. We are going to show that for any
376 positive integer k and any element $c \in D_q^k[S] \setminus D_q^{k-1}[S]$ (which always exists since $m = \infty$), the coefficient $c_{x_n^k}$
377 corresponding to $\frac{\partial^k}{\partial x_n^k}$, for c , is non-zero. We consequentially show that $\frac{\partial^k f_1}{\partial x_n^k}(0) = \frac{\partial^k f_2}{\partial x_n^k}(0) = 0$. We prove the
378 previous statements by induction on k .

379 For $k = 1$, since $q \in \mathcal{L}_c$, we already showed in the proof of Lemma 20 that a non-trivial element $c = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}$
380 is in $D_q^1[S] \setminus D_q^0[S]$ if and only if $v = (v_1, \dots, v_n)$ is in $T_q \mathcal{C}$. On the other hand, $T_q \mathcal{C}$ is generated by the vector
381 $(f_1'(0), \dots, f_{n-1}'(0), 1)$, thus $c_{x_n^1} = v_n \neq 0$. The function $f_1(x_n)$ is in the set of functions generated by S thus
382 $0 = c \cdot (f_1(x_n)) = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i} \cdot (f_1(x_n)) = c_{x_n^1} \frac{\partial f_1}{\partial x_n}(0)$, and thus $\frac{\partial f_1}{\partial x_n}(0) = 0$. Thus, the induction hypothesis
383 holds for $k = 1$.

384 Define $c' = \phi_n(c)$ and consider two cases:

385 (a) $c' \in D_q^{k-1}[S] \setminus D_q^{k-2}[S]$: By the induction hypothesis, the coefficient $c'_{x_n^{k-1}}$ corresponding to $\frac{\partial^{k-1}}{\partial x_n^{k-1}}$ for
386 c' is non-zero and $\frac{\partial^{k'} f_1}{\partial x_n^{k'}}(0) = \frac{\partial^{k'} f_2}{\partial x_n^{k'}}(0) = 0$, for all $k' < k$. Notice that by the definition of ϕ_n , we have
387 $c_{x_n^k} = c'_{x_n^{k-1}} \neq 0$. Hence, $0 = c \cdot f_1(x_n) = \sum_{i=1}^k c_{x_n^i} \frac{\partial^i f_1}{\partial x_n^i}(0) = c_{x_n^k} \frac{\partial^k f_1}{\partial x_n^k}(0)$. Hence, $\frac{\partial^k f_1}{\partial x_n^k}(0) = 0$. Similarly,
388 we prove that $\frac{\partial^k f_2}{\partial x_n^k}(0) = 0$. Thus in Case (a), the lemma is proved.

389 (b) $c' \in D_q^{k-2}[S]$: Since $c \in D_q^k[S] \setminus D_q^{k-1}[S]$, there exists $j \in \{1, \dots, n-1\}$ such that the element $c'' = \phi_j(c)$
390 is in $D_q^{k-1}[S] \setminus D_q^{k-2}[S]$. By the induction hypothesis, the coefficient $c''_{x_n^{k-1}}$ corresponding to $\frac{\partial^{k-1}}{\partial x_n^{k-1}}$ for c'' ,
391 is non-zero. On the other hand, $c_{x_j x_n^{k-1}} = c''_{x_n^{k-1}} \neq 0$. Hence, since $\phi_n(c_{x_j x_n^{k-1}} \frac{\partial^{k-1}}{\partial x_j \partial x_n^{k-1}}) \in D_q^{k-1}[S] \setminus$
392 $D_q^{k-2}[S]$, then so is $\phi_n(c) = c'$ which contradicts the assumption. Thus, Case (b) is impossible.

393 □

394 With the additional Assumptions $\mathcal{A}_2, \mathcal{A}_3$ and \mathcal{A}_4 , one can give a more precise form of f_1 and f_2 in Equa-
395 tion (3.4).

Lemma 24. *Let $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$. Let $q \in \mathcal{L}_c$ such that Assumptions $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and \mathcal{A}_4 hold in a neigh-
borhood of q in B , then there exist an invertible matrix \widetilde{M} of size $(n-1) \times (n-1)$ of smooth functions in a*

neighborhood of q , a smooth diffeomorphism φ defined in an open subset of \mathbb{R}^n , with $z = (z_1, \dots, z_n) = \varphi^{-1}(x)$ and smooth functions f_3, \dots, f_{n-1}, g defined in a neighborhood of $0 \in \mathbb{R}$, such that

$$Q = \begin{pmatrix} z_1 - z_n \cdot g(z_n^2) \\ z_2 - z_n^2 \\ z_3 - f_3(z_n) \\ \dots \\ \dots \\ z_{n-1} - f_{n-1}(z_n) \end{pmatrix} = \widetilde{M} \cdot \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ \dots \\ \dots \\ P_{n-1} \end{pmatrix} \circ \varphi, \quad (3.7)$$

396 on a neighborhood of q . Moreover, either $\text{mult}(g(z_n)) = \infty$ or there exists an integer $k > 0$ with $g(z_n) = z_n^k$.

397 *Proof.* Step 1: Equation (3.6) implies that Q_0 and P define the same curve \mathfrak{C} in a neighborhood of q and that the
398 function Q_0 satisfies the same assumptions as P around q . By Lemma 23, $d = \min\{\text{mult}(f_1(x_n)), \text{mult}(f_2(x_n))\}$
399 is the multiplicity of the system $\{Q_0(x) = 0 \in \mathbb{R}^{n-1}, x_1 = 0, x_2 = 0\}$ at q . By Assumption \mathcal{A}_3 , we have that
400 $d = 2$.

401 Without loss of generality, assume that $\text{mult}(f_2(x_n)) = 2$ and $\frac{\partial^2 f_2}{\partial x_n^2}(0) = 2$. Hence, there is a smooth
402 function v such that $f_2(x_n) = x_n^2(1 + x_n \cdot v(x_n))$. Now, consider the diffeomorphism ϕ_n that sends x_n
403 to $z_n = x_n \sqrt{1 + x_n \cdot v(x_n)}$. We have that $x_2 - f_2(x_n) = x_2 - z_n^2$. Define $\tilde{f}_1(z_n) = f_1(\phi_n^{-1}(z_n))$ and
404 $\tilde{f}_2(z_n) = f_2(\phi_n^{-1}(z_n)) = z_n^2$. Since $\text{mult}(\tilde{f}_1(z_n)) = \text{mult}(f_1(x_n)) \geq d = 2$, there exists a smooth function
405 h such that $\tilde{f}_1(z_n) = z_n^2 h(z_n)$. Write $\tilde{f}_1(z_n) = z_n^2 [\frac{h(z_n) + h(-z_n)}{2} + \frac{h(z_n) - h(-z_n)}{2}]$. Since $\frac{h(z_n) + h(-z_n)}{2}$ (resp.
406 $\frac{h(z_n) - h(-z_n)}{2}$) is even (resp. odd), then by Theorem 49 there exists a smooth function ξ_1 (resp. ξ_2) such that
407 $\frac{h(z_n) + h(-z_n)}{2} = \xi_1(z_n^2)$ (resp. $\frac{h(z_n) - h(-z_n)}{2} = z_n \xi_2(z_n^2)$). Thus, $\tilde{f}_1(z_n) = z_n^2 (\xi_1(z_n^2) + z_n \xi_2(z_n^2))$. Notice that
408 $\xi_2(z_n^2)$ cannot be the zero function, otherwise $\tilde{f}_1(\epsilon) = \tilde{f}_1(-\epsilon)$ and $\tilde{f}_2(\epsilon) = \tilde{f}_2(-\epsilon)$ for all small enough $\epsilon > 0$,
409 which contradicts Assumption \mathcal{A}_4 .

410 Step 2: We have two cases:

411 **Case 1:** $\text{mult}(\xi_2(z_n)) = \infty$, then define the diffeomorphism ϕ which sends x_1 to $z_1 = x_1 - x_2 \xi_1(x_2)$, x_i to
412 $z_i = x_i$ for all integers $i \in \{2, \dots, n-1\}$ and x_n to $z_n = x_n \sqrt{1 + x_n \cdot v(x_n)}$. Taking $g(z_n) = z_n \xi_2(z_n)$ and
413 $\varphi = \phi^{-1}$ we prove the claim for the first case.

414 **Case 2:** $\text{mult}(\xi_2(z_n)) = k < \infty$, that is, $\xi_2(z_n) = z_n^k u(z_n)$, for some smooth function u , with $u(0) \neq 0$ and
415 an integer $k \geq 0$. Hence, we can write $x_1 - \tilde{f}_1(z_n) = x_1 - z_n^2 \xi_1(z_n^2) - z_n^{2k+3} u(z_n^2) = x_1 - x_2 \xi_1(x_2) - z_n^{2k+3} u(x_2)$.

416 So, defining the diffeomorphism ϕ which sends x_i to $z_i = x_i$ for all integers $i \in \{2, \dots, n-1\}$, x_n to $z_n =$
417 $x_n \sqrt{1 + x_n \cdot v(x_n)}$ and x_1 to $z_1 = (x_1 - x_2 \xi_1(x_2)) u^{-1}(x_2)$ (which means that $x_1 - f_1(x_n) = u(x_2)[z_1 - z_n^{2k+3}]$),
418 we get that:

$$\begin{pmatrix} x_1 - f_1(x_n) \\ \dots \\ \dots \\ x_{n-1} - f_{n-1}(x_n) \end{pmatrix} = \begin{pmatrix} u(x_2) & 0_{1 \times (n-2)} \\ 0_{(n-2) \times 1} & I_{n-2} \end{pmatrix} \cdot \begin{pmatrix} z_1 - z_n^{2k+3} \\ z_2 - z_n^2 \\ z_3 - f_3(z_n) \\ \dots \\ \dots \\ z_{n-1} - f_{n-1}(z_n) \end{pmatrix} \circ \phi,$$

419 for a small enough neighborhood of q , where I_{n-2} is the identity matrix of size $n - 2$. Comparing with (3.4), we
420 get:

$$M \cdot \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ \dots \\ \dots \\ P_{n-1} \end{pmatrix} = \begin{pmatrix} u(x_2) & 0_{1 \times (n-2)} \\ 0_{(n-2) \times 1} & I_{n-2} \end{pmatrix} \cdot \begin{pmatrix} z_1 - z_n^{2k+3} \\ z_2 - z_n^2 \\ z_3 - f_3(z_n) \\ \dots \\ \dots \\ z_{n-1} - f_{n-1}(z_n) \end{pmatrix} \circ \phi.$$

421 Hence, taking $\widetilde{M} = \begin{pmatrix} u(x_2) & 0_{1 \times (n-2)} \\ 0_{(n-2) \times 1} & I_{n-2} \end{pmatrix}^{-1} \cdot M$ and $\varphi = \phi^{-1}$ we recover (3.7). \square

422 Following the conclusion of Lemma 24, the reader may wonder whether the projection of q in $\pi_{\mathcal{C}}$ is always
423 singular. This is clear when $g(x_n) = x_n^k$ for $0 < k < \infty$ since this implies $z_1^2 - z_2^{k+1} = 0$ and thus $\pi_{\mathcal{C}}(q)$ is a
424 singularity of the type A_{2k} . We next prove that the projection is also singular if $\text{mult}(g(z_n)) = \infty$.

425 **Lemma 25.** *Preserving the notation and the assumptions in Lemma 24, consider the function g defined in (3.7), if*
426 *$\text{mult}(g(z_n)) = \infty$, then $\pi_{\mathcal{C}}(q)$ is singular in $\pi_{\mathcal{C}}(\mathcal{C})$.*

427 *Proof.* Since $\text{mult}(g(z_n)) = \infty$, then **Case 1** in the proof of Lemma 24 holds. Moreover, we saw in the same
428 proof that $\xi_2(z_n^2)$ (restricted to an open neighborhood of $0 \in \mathbb{R}$) cannot be the zero function. This implies that
429 neither is the function $g(z_n^2) = z_n^2 \xi(z_n^2)$, i.e., $g(z_n^2)$, restricted to an open neighborhood of $0 \in \mathbb{R}$, is not the zero
430 function. Assume for the sake of contradiction that $\pi_{\mathcal{C}}(q)$ is smooth in $\pi_{\mathcal{C}}(\mathcal{C})$, then using the implicit function
431 theorem, there exists a C^∞ -function defined in a neighborhood of 0 in \mathbb{R} , with $f(0) = 0$ such that for a small
432 neighborhood of $\pi_{\mathcal{C}}(q)$ in \mathbb{R}^2 , one of the following cases is satisfied:

433 (a) $f(z_1) = z_2 \iff (z_1, z_2) \in \pi_{\mathcal{C}}(\mathcal{C})$. Then, by (3.7), we have $f(z_n g(z_n^2)) = z_n^2$. Taking the second
434 derivative of both sides with respect to z_n and then evaluating at 0 (recall that $\text{mult}(g(z_n)) = \infty$), we get
435 the contradiction $0 = 2$.

436 (b) $f(z_2) = z_1 \iff (z_1, z_2) \in \pi_{\mathcal{C}}(\mathcal{C})$. Then $f(z_n^2) = z_n g(z_n^2)$. The function $z_n g(z_n^2)$ is an odd function but
437 not the zero function, and on the other hand $f(z_2)$ is an even function, which leads to a contradiction.

438 Thus, in both cases we have a contradiction, that is, f does not exist and $\pi_{\mathcal{C}}(q)$ cannot be smooth in $\pi_{\mathcal{C}}(\mathcal{C})$. \square

439 Returning to (3.7), notice that φ is defined in such a way that it preserves the singularity class of $\pi_{\mathcal{C}}(\mathcal{C})$ at the
 440 point $\pi_{\mathcal{C}}(q)$. In other words, if C is the plane projection of the curve defined by Q then $(\pi_{\mathcal{C}}(\mathcal{C}), 0)$ and $(C, 0)$ are
 441 equivalent.

442 3.4. Proof of Theorem 11

443 We first characterize the singularities of the projected curve $\pi_{\mathcal{C}}(\mathcal{C})$ by the points in \mathcal{L}_n and \mathcal{L}_c . The proof of
 444 Theorem 11 is then obtained via the bijection Ω_P (Definition 14) between $\widehat{\mathcal{L}}$ and the solutions of the Ball system.

445 **Lemma 26.** *If P satisfies Assumptions \mathcal{A}_1 , \mathcal{A}_2 , \mathcal{A}_3 and \mathcal{A}_4 , then a point $q \in \mathcal{C}$ projects to a singular point in
 446 $\pi_{\mathcal{C}}(\mathcal{C})$ if and only if $q \in \mathcal{L}_c \cup \mathcal{L}_n$.*

447 *Proof.* If $q \in \mathcal{L}_c \cup \mathcal{L}_n$, then by Lemmas 21, 24, and 25, $\pi_{\mathcal{C}}(q)$ is singular in $\pi_{\mathcal{C}}(\mathcal{C})$. Conversely, if $q \notin \mathcal{L}_c \cup \mathcal{L}_n$,
 448 we prove that $\pi_{\mathcal{C}}(q)$ is smooth in $\pi_{\mathcal{C}}(\mathcal{C})$.

449 Since $q \notin \mathcal{L}_c$, the plane projection of $T_q\mathcal{C}$ is a line, or equivalently, the derivative $T_q\pi_{\mathcal{C}}$ of $\pi_{\mathcal{C}}$ at q is injective.
 450 Thus, $\pi_{\mathcal{C}}$ is an immersion at q ([Dem00, Definition 2.9.3]). Hence, for a small enough neighborhood U_0 of q in
 451 \mathbb{R}^n , we have that $\pi_{\mathcal{C}}$ restricted to $V = U_0 \cap \mathcal{C}$ is an embedding (see [Dem00, Proposition 2.9.6]). We are going to
 452 prove that, assuming that U_0 is small enough, the curve $\pi_{\mathcal{C}}(\mathcal{C})$ has exactly one branch around $\pi_{\mathcal{C}}(q)$ which implies
 453 that $\pi_{\mathcal{C}}(\mathcal{C})$ is smooth at $\pi_{\mathcal{C}}(q)$ since \mathcal{C} is smooth at q by Assumption \mathcal{A}_1 .

454 To prove this claim, assume that there exists an open subset U'_0 in \mathbb{R}^n such that the set $V' = U'_0 \cap \mathcal{C}$ and V are
 455 disjoint, but $\pi_{\mathcal{C}}(q)$ is in the closure of $\pi_{\mathcal{C}}(V')$. Let q_k be a sequence of points in V' such that $\pi_{\mathcal{C}}(q_k)$ converges
 456 to $\pi_{\mathcal{C}}(q)$. Since \overline{B} is compact, there exists a convergent sub-sequence of q_k that has a limit q' in \overline{B} . Notice that
 457 $\pi_{\mathcal{C}}(q') = \pi_{\mathcal{C}}(q)$ by the continuity of $\pi_{\mathcal{C}}$. Hence, q, q' are both in $\overline{\mathcal{L}_n}$. However, since $q \notin \mathcal{L}_n$, we must have that
 458 $q' \notin B$. Hence, q' is in the boundary of B which contradicts Assumption \mathcal{A}_4 . Hence, the curve $\pi_{\mathcal{C}}(\mathcal{C})$ has exactly
 459 one smooth branch around $\pi_{\mathcal{C}}(q)$ which concludes the proof. \square

460 Finally, we prove that the solutions of the Ball system project to the singular points of $\pi_{\mathcal{C}}(\mathcal{C})$.

461 *Proof of Theorem 11:* By Lemma 26, if (x_1, x_2) is singular in $\pi_{\mathcal{C}}(\mathcal{C})$, then there exists a point $q_1 \in \mathcal{L}_c \cup \mathcal{L}_n$, with
 462 $\pi_{\mathcal{C}}(q_1) = (x_1, x_2)$. If $q_1 \in \mathcal{L}_c$, let $q_2 = q_1$ and otherwise let q_2 be the unique (by Assumption \mathcal{A}_3) point in \mathcal{L}_n ,
 463 distinct from q_1 , that projects onto (x_1, x_2) , i.e. $\pi_{\mathcal{C}}(q_1) = \pi_{\mathcal{C}}(q_2) = (x_1, x_2)$. Hence, (q_1, q_2) is in $\widehat{\mathcal{L}}$. Since Ω_P is
 464 surjective (Lemma 15), there exists $X = (x_1, x_2, y, r, t) \in \text{Sol}_{\text{Ball}(P)}$ with $\Omega_P(X) = (q_1, q_2)$.

465 On the other hand, if X is a solution of $\text{Ball}(P)$, then by Lemma 13 the pair $(q_1, q_2) = \Omega_P(X)$ is in $\widehat{\mathcal{L}}$. Hence,
 466 $q_1 = (x_1, x_2, y + r\sqrt{t}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$ is in $\mathcal{L}_c \cup \mathcal{L}_n$. Hence, by Lemma 26 the point (x_1, x_2) is singular in
 467 $\pi_{\mathcal{C}}(\mathcal{C})$. \square

468 **4. Regularity of the Ball system**

469 In this section, our goal is to prove Theorem 27 determining necessary and sufficient conditions for $\text{Ball}(P)$ to
 470 be regular.

471 **Theorem 27.** *Let $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ that satisfies Assumptions $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and \mathcal{A}_4 , then P satisfies Assumption
 472 \mathcal{A}_5^- if and only if $\text{Ball}(P)$ is regular in B_{Ball} .*

473 In order to prove Theorem 27, we are going to show that the Jacobian matrices of $\text{Ball}(P)$ and $\text{Ball}(Q)$
 474 evaluated at X have the same rank, where Q is defined in Equation (3.7). Recall that Equation (3.7) implies that
 475 P and Q define the same curve around q . Notice also that if $X = (q, r, 0) \in \mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}$ is in $\Omega_P^{-1}((q, q))$,
 476 then $X \in \Omega_Q^{-1}((q, q))$.

477 **Lemma 28.** *Let P and Q be as defined in (3.7). Under Assumption \mathcal{A}_1 , let $(q, r, 0) \in \mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}$ be a
 478 solution of the system $\text{Ball}(P)$ in B_{Ball} , then $\text{Ball}(P)$ is regular at $(q, r, 0)$ if and only if $\text{Ball}(Q)$ is regular at the
 479 point $(0, r, 0) \in \mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}$ (recall that for simplicity, we assume in Lemma 24 that $q = 0 \in \mathbb{R}^n$).*

480 *Proof.* Let us write $X = (q, r, 0)$. We are going to prove that the Jacobian matrices of $\text{Ball}(P)$ and $\text{Ball}(Q)$
 481 evaluated at X have the same rank. By Remark 16 we have that $\Omega_P(X) = (q, q) \in \widehat{\mathcal{L}}_c$ (see Definitions 14 and
 482 12), and hence, $q \in \mathcal{L}_c$. By Lemma 13 we have that $(0, 0, r) \in T_q \mathcal{C}$. We prove the claim in three steps:

483 **Step 1:** Let $\widetilde{M} = (f_{ij})_{1 \leq i, j \leq n-1}$ be as defined in the Equality (3.7). We define $S \cdot \widetilde{M}$ (resp. $D \cdot \widetilde{M}$) to be the
 484 matrix $(S \cdot f_{ij})_{1 \leq i, j \leq n-1}$ (resp. $(D \cdot f_{ij})_{1 \leq i, j \leq n-1}$). Using the identity $\frac{1}{2}(ab + cd) = \frac{1}{4}(a + c)(b + d) + \frac{1}{4}(a -$
 485 $c)(b - d)$, one deduces the properties for any $f, g \in C^\infty(\mathbb{R}^n, \mathbb{R})$:

$$S \cdot fg = (S \cdot f)(S \cdot g) + t(D \cdot f)(D \cdot g) \quad (4.1)$$

$$D \cdot fg = (D \cdot f)(S \cdot g) + (S \cdot f)(D \cdot g) \quad (4.2)$$

These identities applied to Equation (3.7) yield

$$\begin{pmatrix} S \cdot Q_1 \\ \cdots \\ \cdots \\ S \cdot Q_{n-1} \end{pmatrix} = \begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} \end{pmatrix} \cdot \begin{pmatrix} S \cdot (P_1 \circ \varphi) \\ \cdots \\ \cdots \\ S \cdot (P_{n-1} \circ \varphi) \\ D \cdot (P_1 \circ \varphi) \\ \cdots \\ \cdots \\ D \cdot (P_{n-1} \circ \varphi) \end{pmatrix}$$

and

$$\begin{pmatrix} D \cdot Q_1 \\ \cdots \\ \cdots \\ D \cdot Q_{n-1} \end{pmatrix} = \begin{pmatrix} D \cdot \widetilde{M} & S \cdot \widetilde{M} \end{pmatrix} \cdot \begin{pmatrix} S \cdot (P_1 \circ \varphi) \\ \cdots \\ \cdots \\ S \cdot (P_{n-1} \circ \varphi) \\ D \cdot (P_1 \circ \varphi) \\ \cdots \\ \cdots \\ D \cdot (P_{n-1} \circ \varphi) \end{pmatrix}$$

Combining the last two equations:

$$\begin{pmatrix} S \cdot Q_1 \\ \cdots \\ \cdots \\ S \cdot Q_{n-1} \\ D \cdot Q_1 \\ \cdots \\ \cdots \\ D \cdot Q_{n-1} \end{pmatrix} = \begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} \\ D \cdot \widetilde{M} & S \cdot \widetilde{M} \end{pmatrix} \cdot \begin{pmatrix} S \cdot (P_1 \circ \varphi) \\ \cdots \\ \cdots \\ S \cdot (P_{n-1} \circ \varphi) \\ D \cdot (P_1 \circ \varphi) \\ \cdots \\ \cdots \\ D \cdot (P_{n-1} \circ \varphi) \end{pmatrix} \quad (4.3)$$

486 Notice that $\begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} \\ D \cdot \widetilde{M} & S \cdot \widetilde{M} \end{pmatrix}_X = \begin{pmatrix} \widetilde{M}(q) & 0 \\ D \cdot \widetilde{M}(X) & \widetilde{M}(q) \end{pmatrix}$ (recall that in our case we have $S \cdot \widetilde{M}(X) = \widetilde{M}(q)$)

487 and that the latter matrix has an inverse (by Lemma 24, $\widetilde{M}(q)$ is an invertible matrix of size $n - 1$), namely,

488 $\begin{pmatrix} \widetilde{M}(q)^{-1} & 0 \\ -\widetilde{M}(q)^{-1} \cdot (D \cdot \widetilde{M})(X) \cdot \widetilde{M}(q)^{-1} & \widetilde{M}(q)^{-1} \end{pmatrix}$ which implies (by continuity of the determinant function)

489 that $\begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} \\ D \cdot \widetilde{M} & S \cdot \widetilde{M} \end{pmatrix}$ is invertible in a neighborhood of X .

Step 2: Writing $y = (y_3, \dots, y_n)$ and $r = (r_3, \dots, r_n)$, consider the diffeomorphism φ defined in Lemma 24 and define the smooth function ψ over an open subset of \mathbb{R}^{2n-1} containing X which maps the point (x_1, x_2, y, r, t) to $(\varphi_1, \varphi_2, S \cdot \varphi_3, \dots, S \cdot \varphi_n, D \cdot \varphi_3, \dots, D \cdot \varphi_n, t)$. Notice that we have:

$$S \cdot (P_j \circ \varphi) = (S \cdot P) \circ \psi \text{ and } D \cdot (P_j \circ \varphi) = (D \cdot P) \circ \psi, \text{ for } 1 \leq j \leq n - 1, \quad (4.4)$$

490 since $\varphi_i(x_1, x_2, y \pm r\sqrt{t}) = \psi_i \pm \psi_{n+i-2} \sqrt{\psi_{2n-1}}$ for all $i \in \{3, \dots, n\}$. In fact, using the last two equations we
491 can also see that ψ^{-1} exists and is smooth. Thus, ψ is a diffeomorphism.

492 **Step 3:** Now, comparing (4.3) with (4.4) we get:

$$SD \cdot Q := \begin{pmatrix} S \cdot Q_1 \\ \dots \\ \dots \\ S \cdot Q_{n-1} \\ D \cdot Q_1 \\ \dots \\ \dots \\ D \cdot Q_{n-1} \end{pmatrix} = \begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} \\ D \cdot \widetilde{M} & S \cdot \widetilde{M} \end{pmatrix} \cdot \begin{pmatrix} S \cdot P_1 \\ \dots \\ \dots \\ S \cdot P_{n-1} \\ D \cdot P_1 \\ \dots \\ \dots \\ D \cdot P_{n-1} \end{pmatrix} \circ \psi.$$

Consider the vector $SD \cdot P = (S \cdot P_1, \dots, S \cdot P_{n-1}, D \cdot P_1, \dots, D \cdot P_{n-1})^T$ and let $J_{SD \cdot P}$, $J_{SD \cdot Q}$ and J_ψ be the Jacobian matrices of $SD \cdot P$, $SD \cdot Q$ and ψ , respectively. Taking the Jacobian matrix of both sides of the last equality:

$$J_{SD \cdot Q} = \begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} \\ D \cdot \widetilde{M} & S \cdot \widetilde{M} \end{pmatrix} \cdot J_{SD \cdot P} \cdot J_\psi + \text{Jacobian} \left(\begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} \\ D \cdot \widetilde{M} & S \cdot \widetilde{M} \end{pmatrix} \right) \cdot \begin{pmatrix} S \cdot P_1 \\ \dots \\ \dots \\ S \cdot P_{n-1} \\ D \cdot P_1 \\ \dots \\ \dots \\ D \cdot P_{n-1} \end{pmatrix} \circ \psi.$$

Evaluating the last equality at $X = (0, r, 0)$ and using the fact that $\psi(X) = \psi(0, r, 0) = (0, r, 0) = X$, we note that the second term of the right-hand side is zero. One has:

$$J_{SD \cdot Q}(X) = \begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} \\ D \cdot \widetilde{M} & S \cdot \widetilde{M} \end{pmatrix}_X \cdot J_{SD \cdot P}(X) \cdot J_\psi(X). \quad (4.5)$$

493 Computing $J_\psi(X)$, we get $J_\psi(X) = \begin{pmatrix} \frac{\partial \varphi_1}{\partial z_1}(0) & \frac{\partial \varphi_1}{\partial z_2}(0) & 0_{1 \times (2n-3)} \\ \dots & \dots & \dots \\ 0_{(2n-2) \times 1} & I_{2n-2} & \dots \end{pmatrix}$, with $\frac{\partial \varphi_1}{\partial z_1}(0) \neq 0$ according to the
494 formula in Lemma 24.

495 Hence by Equation (4.5), it is straightforward to check that:

$$\begin{aligned} J_{\text{Ball}(Q)} &= \begin{pmatrix} J_{SD \cdot Q}(X) \\ 2X \end{pmatrix} = \begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} & 0 \\ D \cdot \widetilde{M} & S \cdot \widetilde{M} & 0 \\ 0_{1 \times (n-1)} & 0_{1 \times (n-1)} & 1 \end{pmatrix}_X \cdot \begin{pmatrix} J_{SD \cdot P}(X) \\ 2X \end{pmatrix} \cdot J_\psi(X) \\ &= \begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} & 0 \\ D \cdot \widetilde{M} & S \cdot \widetilde{M} & 0 \\ 0_{1 \times (n-1)} & 0_{1 \times (n-1)} & 1 \end{pmatrix}_X \cdot J_{\text{Ball}(P)}(X) \cdot J_\psi(X). \end{aligned}$$

496 Recalling that $J_\psi(X)$ and $\begin{pmatrix} S \cdot \widetilde{M} & tD \cdot \widetilde{M} & 0 \\ D \cdot \widetilde{M} & S \cdot \widetilde{M} & 0 \\ 0_{1 \times (n-1)} & 0_{1 \times (n-1)} & 1 \end{pmatrix}_X$ are invertible matrices, the proof of the lemma fol-
 497 lows. □

498 Now, we are ready to prove Theorem 27, which characterizes the regularity of the solutions of $\text{Ball}(P)$ under
 499 generic assumptions. We split the proof in the two Lemmas 32 and 33. Before that, we introduce a new assumption
 500 that helps to simplify the proof.

501 **Definition 29.** Let $(q_1, q_2) \in \widehat{\mathfrak{L}}$. We say that (q_1, q_2) satisfies Assumption $\mathcal{A}_5^{-'}$ if q_1 and q_2 are isolated in $\mathfrak{L}_n \cup \mathfrak{L}_c$
 502 and the following conditions are satisfied:

- 503 (a) If $(q_1, q_2) \in \widehat{\mathfrak{L}}_n$, then the plane projections of the tangent lines of q_1 and q_2 to \mathfrak{C} are linearly independent.
 504 (b) If $(q_1, q_2) \in \widehat{\mathfrak{L}}_c$, then the plane projection of a small enough neighborhood of q_1 in \mathfrak{C} is an ordinary cusp at
 505 $\pi_{\mathfrak{C}}(q_1)$ and the multiplicity of the system $\{P(x) = 0, (x_1, x_2) = \pi_{\mathfrak{C}}(q_1)\}$ at q_1 is two.

506 Assumption $\mathcal{A}_5^{-'}$ can be seen as a "local version" of Assumption \mathcal{A}_5^- . We are going to prove that if Assumptions
 507 $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and \mathcal{A}_4 are satisfied, then Assumption \mathcal{A}_5^- is equivalent to the condition that Assumption $\mathcal{A}_5^{-'}$ is
 508 satisfied for all $\widehat{\mathfrak{L}}$.

509 The main reason behind introducing Assumption $\mathcal{A}_5^{-'}$, is that we are going to prove in Lemma 32 that, under
 510 Assumption \mathcal{A}_1 , a pair $(q_1, q_2) \in \widehat{\mathfrak{L}}$ satisfies Assumption $\mathcal{A}_5^{-'}$ if and only if every X in $\Omega_P^{-1}(q_1, q_2)$ is a regular
 511 solution of $\text{Ball}(P)$, whereas Assumption \mathcal{A}_5^- is, in general, not sufficient for the regularity of the solutions of
 512 $\text{Ball}(P)$. For example, take $n = 3$ and $P = (x_1 - x_3^6, x_2 - x_3^9)$. We can see that P satisfies Assumption \mathcal{A}_1 ,
 513 the set \mathfrak{L}_c consists of a unique point $q = (0, 0, 0)$ and the set \mathfrak{L}_n is empty. The plane projection of \mathfrak{C} is the curve
 514 given by the equation $x_1^3 - x_2^2 = 0$. Hence, Assumption \mathcal{A}_5^- is satisfied. However, the multiplicity of the system
 515 $\{P(x_1, x_2, x_3) = 0 \in \mathbb{R}^2, x_1 = x_2 = 0\}$ at the point q equals 6 (Lemma 23). Hence, Assumption $\mathcal{A}_5^{-'}$ is not
 516 satisfied and one can also check that $\text{Ball}(P)$ is not regular.

517 The next definition and lemma are technical tools to handle the case of nodes in Lemma 32, and later in
 518 Lemma 54.

519 **Definition 30.** Consider $P = (P_1, \dots, P_{n-1}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ satisfying Assumption \mathcal{A}_1 and recall that we
 520 denote the Jacobian matrix of P at the point q by $J_P(q)$. We define the $(n-1) \times (n-2)$ sub-matrix $M_P(q)$ obtained
 521 by removing the first two columns of $J_P(q)$ and the $(n-1) \times 2$ sub-matrix $N_P(q)$ formed by the first two columns
 522 of $J_P(q)$. Let $q_1, q_2 \in \mathfrak{C}$, we define the $2n-2$ square matrix $M(q_1, q_2) = \begin{pmatrix} N_P(q_1) & 0 & M_P(q_1) \\ N_P(q_2) & M_P(q_2) & 0 \end{pmatrix}$.

523 **Lemma 31.** Using the same assumption and notation as in Definition 30, let q_1 and q_2 be distinct points of \mathfrak{C} with
 524 $\pi_{\mathfrak{C}}(q_1) = \pi_{\mathfrak{C}}(q_2)$, then $M(q_1, q_2)$ is invertible if and only if neither q_1 nor q_2 is in \mathfrak{L}_c and the plane projections of
 525 the tangent lines of \mathfrak{C} at q_1 and q_2 do not coincide.

526 *Proof.* We prove the converse statement using

$$\det(M(q_1, q_2)) = 0 \iff \text{There exist } \alpha \in \mathbb{R}^2 \text{ and } \beta, \gamma \in \mathbb{R}^{n-2} \text{ such that the vector}$$

$$x = (\alpha, \beta, \gamma) \text{ is not trivial and } M(q_1, q_2) \cdot x^T = 0.$$

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$$\iff (\alpha, \beta) \text{ and } (\alpha, \gamma) \text{ are in the tangent lines } T_{q_1} \mathcal{C} \text{ and } T_{q_2} \mathcal{C}, \text{ respectively,}$$

$$\text{and at least one of them is not trivial.}$$

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The last statement can be split in two cases:

529

• α is not trivial, which is equivalent to saying that the plane projections of $T_{q_1} \mathcal{C}$ and $T_{q_2} \mathcal{C}$ are both generated by α and coincide.

530

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• $\alpha = (0, 0)$, which is equivalent to β or γ are not trivial, which is equivalent to $T_{q_2} \mathcal{C}$ or $T_{q_1} \mathcal{C}$ projects to a point in the plane, which is equivalent to q_1 or q_2 is in \mathcal{L}_c . \square

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533

Lemma 32. Let $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ satisfy Assumption \mathcal{A}_1 . Let X be a solution of $\text{Ball}(P)$ and $(q_1, q_2) = \Omega_P(X)$ (Definition 14), then X is a regular solution of $\text{Ball}(P)$ if and only if (q_1, q_2) satisfies Assumption \mathcal{A}_5' .

534

535

Proof. Let $X = (x_1, x_2, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R}$ be a solution of $\text{Ball}(P)$. We consider two cases:

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Case (a): $t \neq 0$, i.e., $q_1 \neq q_2$.

537

It is easy to see that $\frac{\partial(S \cdot P_i)}{\partial x_j}, \frac{\partial(D \cdot P_i)}{\partial x_j}, \frac{\partial(S \cdot P_i)}{\partial r_k}, \frac{\partial(D \cdot P_i)}{\partial r_k}, \frac{\partial(S \cdot P_i)}{\partial t}, \frac{\partial(D \cdot P_i)}{\partial t}$ are, respectively, equal to: $S \cdot \frac{\partial(P_i)}{\partial x_j}, D \cdot$

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$\frac{\partial(P_i)}{\partial x_j}, t \cdot D \cdot \frac{\partial(P_i)}{\partial x_k}, S \cdot \frac{\partial(P_i)}{\partial x_k}, \frac{1}{2} \sum_{m=3}^n D \cdot \left(\frac{\partial P_i}{\partial x_m} \right) \cdot r_m, \frac{1}{2t} \left[\sum_{m=3}^n S \cdot \left(\frac{\partial P_i}{\partial x_m} \right) \cdot r_m - D \cdot P_i \right]$. Hence, by computing the

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Jacobian matrix of the $\text{Ball}(P)$ we get the matrix:

$$\begin{pmatrix} S \cdot \frac{\partial(P_1)}{\partial x_1} & \dots & S \cdot \frac{\partial P_1}{\partial x_n} & t \cdot D \cdot \frac{\partial(P_1)}{\partial x_3} \dots & t \cdot D \cdot \frac{\partial(P_1)}{\partial x_n} & \frac{1}{2} \sum_{m=3}^n D \cdot \left(\frac{\partial P_1}{\partial x_m} \right) \cdot r_m \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ S \cdot \frac{\partial(P_{n-1})}{\partial x_1} & \dots & S \cdot \frac{\partial(P_{n-1})}{\partial x_n} & t \cdot D \cdot \frac{\partial(P_{n-1})}{\partial x_3} \dots & t \cdot D \cdot \frac{\partial(P_{n-1})}{\partial x_n} & \frac{1}{2} \sum_{m=3}^n D \cdot \left(\frac{\partial P_{n-1}}{\partial x_m} \right) \cdot r_m \\ D \cdot \frac{\partial(P_1)}{\partial x_1} & \dots & D \cdot \frac{\partial(P_1)}{\partial x_n} & S \cdot \frac{\partial(P_1)}{\partial x_3} \dots & S \cdot \frac{\partial(P_1)}{\partial x_n} & \frac{1}{2t} \left[\sum_{m=3}^n S \cdot \left(\frac{\partial P_1}{\partial x_m} \right) \cdot r_m - D \cdot P_1 \right] \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ D \cdot \frac{\partial(P_{n-1})}{\partial x_1} & \dots & D \cdot \frac{\partial(P_{n-1})}{\partial x_n} & S \cdot \frac{\partial(P_{n-1})}{\partial x_3} \dots & S \cdot \frac{\partial(P_{n-1})}{\partial x_n} & \frac{1}{2t} \left[\sum_{m=3}^n S \cdot \left(\frac{\partial P_{n-1}}{\partial x_m} \right) \cdot r_m - D \cdot P_{n-1} \right] \\ 0 & \dots & 0 & 2r_3 \dots & 2r_n & 0 \end{pmatrix}.$$

We denote by C_i (resp. L_i) the i -th column (resp. line) of the latter matrix. Replace the last column C_{2n-1} with $\sum_{m=1}^{n-2} \frac{r_{m+2}}{2t} C_{n+m} + C_{2n-1}$, also for all integers $1 \leq k \leq n-1$ we replace the line L_k with $L_k + \sqrt{t} \cdot L_{k+n-1}$ and

then the line L_{k+n-1} with $L_k - 2\sqrt{t}L_{k+n-1}$. The resulting matrix is:

$$\begin{pmatrix} \frac{\partial(P_1)}{\partial x_1}(q_1) & \dots & \frac{\partial P_1}{\partial x_n}(q_1) & \sqrt{t} \cdot \frac{\partial(P_1)}{\partial x_3}(q_1) & \dots & \sqrt{t} \frac{\partial(P_1)}{\partial x_n}(q_1) & 0 \\ & & \dots & \dots & \dots & \dots & \dots \\ & & \dots & \dots & \dots & \dots & \dots \\ \frac{\partial(P_{n-1})}{\partial x_1}(q_1) & \dots & \frac{\partial P_{n-1}}{\partial x_n}(q_1) & \sqrt{t} \cdot \frac{\partial(P_{n-1})}{\partial x_3}(q_1) & \dots & \sqrt{t} \frac{\partial(P_{n-1})}{\partial x_n}(q_1) & 0 \\ \frac{\partial(P_1)}{\partial x_1}(q_2) & \dots & \frac{\partial(P_1)}{\partial x_n}(q_2) & -\sqrt{t} \frac{\partial(P_1)}{\partial x_3}(q_2) & \dots & -\sqrt{t} \frac{\partial(P_1)}{\partial x_n}(q_2) & 0 \\ \dots & & \dots & \dots & \dots & \dots & \dots \\ & & \dots & \dots & \dots & \dots & \dots \\ \frac{\partial(P_{n-1})}{\partial x_1}(q_2) & \dots & \frac{\partial(P_{n-1})}{\partial x_n}(q_2) & -\sqrt{t} \frac{\partial(P_{n-1})}{\partial x_3}(q_2) & \dots & -\sqrt{t} \frac{\partial(P_{n-1})}{\partial x_n}(q_2) & 0 \\ 0 \dots & & 0 & 2r_3 & \dots & 2r_n & \frac{1}{2t} \end{pmatrix}.$$

The determinant of the latter matrix is zero if and only if the determinant of the following matrix is zero:

$$M_0 = \begin{pmatrix} N_P(q_1) & M_P(q_1) & M_P(q_1) \\ N_P(q_2) & M_P(q_2) & -M_P(q_2) \end{pmatrix},$$

where $M_P(q_1), M_P(q_2)$ are the minors that are obtained, respectively, by removing the first two columns from $J_P(q_1), J_P(q_2)$ and $N_P(q_1), N_P(q_2)$ are the matrices formed by the first two columns of $J_P(q_1), J_P(q_2)$, respectively. By linear operations on M_0 , we can see that M_0 has same rank as the matrix $M(q_1, q_2)$ (see Definition 30). Thus, X is regular for $\text{Ball}(P)$ if and only if $M(q_1, q_2)$ is invertible. By Lemma 31 we have that $M(q_1, q_2)$ is invertible if and only if none of q_1, q_2 is in \mathfrak{L}_c (and hence none of the plane projections of $T_{q_1}\mathfrak{C}, T_{q_2}\mathfrak{C}$ is trivial) and the plane projection of their tangent spaces are different. Equivalently, the pair (q_1, q_2) is in $\widehat{\mathfrak{L}}_n$ and satisfies Assumption $\mathcal{A}_5^{-'}$.

Case (b): $t = 0$, i.e., $q_1 = q_2$.

Let us write $q = q_1$. We prove the claim in three steps:

Step 1: We first simplify P . Without loss of generality and by Lemma 22 we can assume that $q = 0$ and P_1, \dots, P_{n-1} are, respectively, equal to $x_1 - f_1(x_n), x_2 - f_2(x_n), \dots, x_{n-1} - f_{n-1}(x_n)$ with the property that $\min\{\text{mult}(f_1), \text{mult}(f_2)\} \geq 2$. For all $i \in \{3, \dots, n-1\}$, using Taylor's theorem, we can write $f_i(x_n) = \sum_{j=1}^3 a_{i,j} x_n^j + x_n^4 h_i(x_n)$, for some $a_{i,j} \in \mathbb{R}$ and smooth functions $h_i(x_n)$. Since $\min\{\text{mult}(f_1), \text{mult}(f_2)\} \geq 2$, we can write $f_1(x_n) = \sum_{j=2}^3 \alpha_j x_n^j + x_n^4 h_1(x_n)$ and $f_2(x_n) = \sum_{j=2}^3 \beta_j x_n^j + x_n^4 h_2(x_n)$. Notice that

$$(f_1(x_n), f_2(x_n), f_3(x_n), \dots, f_{n-1}(x_n), x_n)$$

is a local parameterization system of \mathfrak{C} around q . Since $\dim(T_q\mathfrak{C}) = 1$ (Assumption \mathcal{A}_1), there exists $\lambda \in \mathbb{R}^*$ with $(a_{3,1}, \dots, a_{n-1,1}, 1) = \lambda r$ (because the vectors $(0, 0, r) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$ and $(0, 0, a_{1,3}, \dots, a_{1,n-1}, 1)$ are in $T_q\mathfrak{C} \setminus \{0\}$). In particular, $r_n \neq 0$.

Step 2: Now, we compute $J_{\text{Ball}(P)}$ at $X = (x_1, x_2, y, r, 0)$ by first computing it for $X_t = (x_1, x_2, y, r, t)$ with $t \neq 0$, and then taking the limit when t goes to 0. Since the operator S is linear, we write $S(x_i - f_i(x_n)) = S(x_i - \sum_{j=1}^3 a_{i,j} x_n^j) - S(x_n^4 h_i(x_n))$. On the other hand, using the identity (4.1) we deduce that $S(x_n^4 h_i(x_n)) =$

556 $S(x_n^4) \cdot S(h_i(x_n)) + tD(x_n^4) \cdot D(h_i(x_n))$, for all $i \in \{1, \dots, n-1\}$. It is straightforward to see that $S(x_n^4) =$
557 $r_n^4 t^2 + 6r_n^2 t x_n^2 + x_n^4$ and $tD(x_n^4) = 4r_n^3 x_n t^2 + 4r_n x_n^3 t$ with $r = (r_3, \dots, r_n)$. Hence, all of the first-order partial
558 derivatives of $S(x_n^4 h_i(x_n))$, evaluated at X_t , converge to zero when t goes to 0. Hence, the partial derivatives of
559 the functions $S(x_i - f_i(x_n))$ and $S(x_i - \sum_{j=1}^3 a_{i,j} x_n^j)$ evaluated at X are equal. Using an analogous argument, we
560 deduce that the evaluation of the partial derivatives of the functions $D(x_i - f_i(x_n))$ and $D(x_i - \sum_{j=1}^3 a_{i,j} x_n^j)$, at
561 X are also equal. Thus, $J_{\text{Ball}(P)}(X_t)$ and $J_{\text{Ball}(\bar{P})}(X_t)$ converge to the same limit $J_{\text{Ball}(P)}(X)$, where \bar{P} is the
562 function obtained by truncating P beyond degree 3 with respect to the variable x_n .

Computing $J_{\text{Ball}(P)}(X) = \lim_{t \rightarrow 0} J_{\text{Ball}(\bar{P})}(X_t)$, we get:

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -\alpha_2 r_n^2 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & -\beta_2 r_n^2 \\ 0 & 0 & \dots & \dots & -a_{3,1} & 0 & \dots & 0 & -a_{3,2} r_n^2 \\ \dots & \dots \\ \dots & \dots \\ 0 & 0 & \dots & 1 & -a_{n-1,1} & 0 & \dots & 0 & -a_{n-1,2} r_n^2 \\ 0 & 0 & \dots & \dots & -2\alpha_2 r_n & 0 & \dots & 0 & -\alpha_3 r_n^3 \\ 0 & 0 & \dots & \dots & -2\beta_2 r_n & 0 & \dots & 0 & -\beta_3 r_n^3 \\ 0 & 0 & \dots & \dots & -2a_{3,2} r_n & 1 & \dots & -a_{3,1} & -a_{3,3} r_n^3 \\ \dots & \dots \\ \dots & \dots \\ 0 & 0 & \dots & \dots & -2a_{n-1,2} r_n & 0 & \dots & 1 & -a_{n-1,1} & -a_{n-1,3} r_n^3 \\ 0 & 0 & \dots & \dots & 0 & 2r_3 & \dots & 2r_{n-1} & 2r_n & 0 \end{pmatrix}.$$

Hence, observing that the matrix is block diagonal, its determinant is zero if and only if the determinant of the following one is:

$$\begin{pmatrix} -2\alpha_2 r_n & 0 & \dots & 0 & 0 & -\alpha_3 r_n^3 \\ -2\beta_2 r_n & 0 & \dots & 0 & 0 & -\beta_3 r_n^3 \\ -2a_{3,2} r_n & 1 & 0 \dots & 0 & -a_{3,1} & -a_{3,3} r_n^3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -2a_{n-1,2} r_n & 0 & 0 \dots & 1 & -a_{n-1,1} & -a_{n-1,3} r_n^3 \\ 0 & 2r_3 & \dots & 2r_{n-1} & 2r_n & 0 \end{pmatrix}.$$

Shifting the columns of the last matrix we get:

$$\begin{pmatrix} -\alpha_3 r_n^3 & -2\alpha_2 r_n & 0 & \dots & 0 & 0 \\ -\beta_3 r_n^3 & -2\beta_2 r_n & 0 & \dots & 0 & 0 \\ -a_{3,3} r_n^3 & -2a_{3,2} r_n & 1 & 0 \dots & 0 & -a_{3,1} \\ & \dots & \dots & & & \\ -a_{n-1,3} r_n^3 & -2a_{n-1,2} r_n & 0 & 0 \dots & 1 & -a_{n-1,1} \\ 0 & 0 & 2r_3 & \dots & 2r_{n-1} & 2r_n \end{pmatrix}.$$

563 To compute the determinant of the second block, we expand it about the last row. Hence, the determinant of
 564 the last matrix is zero if and only if $r_n(\alpha_2\beta_3 - \alpha_3\beta_2)(r_n + \sum_{i=3}^{n-1} a_{i,1}r_i) = 0$. Notice that, by Step 1, we have that
 565 $r_n \neq 0$ and the third factor $(r_n + \sum_{i=3}^{n-1} a_{i,1}r_i)$ is never zero since it is equal to λ . Thus, $J_{\text{Ball}(P)}(X)$ is invertible iff

566 $\alpha_2\beta_3 - \alpha_3\beta_2 \neq 0$, equivalently, the matrix $A = \begin{pmatrix} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{pmatrix}$ is invertible.

567 Step 3: We now show that the invertibility of A is equivalent to the condition that (q, q) satisfies Assump-
 568 tion $\mathcal{A}_5^{-'}$.

569 First assume that A is invertible. It follows that either $\alpha_2 \neq 0$ or $\beta_2 \neq 0$ and this yields that the minimum
 570 of the multiplicities of f_1 and f_2 is 2. By Lemma 23, the multiplicity of the system $\{P(x_1, x_2, y) = 0 \in$
 571 $\mathbb{R}^{n-1}, (x_1, x_2) = \pi_{\mathcal{E}}(q)\}$ at q is equal to 2, thus Assumption $\mathcal{A}_5^{-'}$ (b) is satisfied. Using the same notation as
 572 in the proof of Lemma 24, one can write $\tilde{f}_1(z_n) = z_n^2(\xi_1(z_n^2) + z_n\xi_2(z_n^2))$. Notice that $\xi_2(z_n^2)$ cannot be the zero
 573 function, otherwise $\tilde{f}_1(\epsilon) = \tilde{f}_1(-\epsilon)$ and $\tilde{f}_2(\epsilon) = \tilde{f}_2(-\epsilon)$ for all small enough $\epsilon > 0$, which means that X would
 574 be the limit of solutions X_ϵ of $\text{Ball}(P)$ with $\Omega_P(X_\epsilon) \in \widehat{\mathcal{L}}_n$. X would then be a non-isolated solution and thus a
 575 non-regular solution of $\text{Ball}(P)$ which contradicts the assumption. We then have two cases as in Lemma 24. The
 576 first one is when $\text{mult}(\xi_2(z_n)) = \infty$, that would imply that $\alpha_2 = \alpha_3 = 0$ and contradicts the invertibility of A .
 577 We then must satisfy the second case $\text{mult}(\xi_2(z_n)) = k < \infty$ and, after a change of variables, the first equation
 578 of the system becomes equivalent to $z_1 - z_n^{2k+3} = 0$. The invertibility of A implies that $k = 0$. The projection
 579 of the curve in the plane is thus locally parameterized by (z_n^3, z_n^2) and is an ordinary cusp, Assumption $\mathcal{A}_5^{-'}$ (a) is
 580 satisfied.

581 Second, assume that Assumption $\mathcal{A}_5^{-'}$ is satisfied. By Lemma 23 and Assumption $\mathcal{A}_5^{-'}$ (b), the minimum of
 582 the multiplicities of f_1 and f_2 is 2. Using the proof of Lemma 24 once again, one can assume that $f_2(z_n) = z_n^2$
 583 and $f_1(z_n) = z_n g(z_n^2)$ or $f_1(z_n) = z_n^{2k+3}$. By Assumption $\mathcal{A}_5^{-'}$ (a), the projection is an ordinary cusp and thus
 584 has a parameterization of the form (z_n^2, z_n^3) , that is $f_1(z_n) = z_n^3$. This implies that A is equivalent to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and
 585 hence is invertible. \square

586 **Lemma 33.** *If Assumptions $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and \mathcal{A}_4 are satisfied, then Assumption \mathcal{A}_5^- is satisfied if and only if*
 587 *Assumption $\mathcal{A}_5^{-'}$ is satisfied for all $(q_1, q_2) \in \widehat{\mathcal{L}} \subset B \times B$.*

588 *Proof.* Assume that Assumption \mathcal{A}_5^- is satisfied and $(q_1, q_2) \in \widehat{\mathfrak{L}}$. If $(q_1, q_2) \in \widehat{\mathfrak{L}}_c$, then by Lemma 24 and
589 Assumption \mathcal{A}_5^- we must have that the plane projection of a small enough neighborhood of q_1 in \mathfrak{C} is an ordinary
590 cusp at $\pi_{\mathfrak{C}}(q_1)$. By Assumption \mathcal{A}_3 and Lemma 20, the multiplicity of the mentioned system at $q_1 = q_2$ is two.
591 Thus, (q_1, q_2) satisfies Assumption $\mathcal{A}_5^{-'}$. If $(q_1, q_2) \in \widehat{\mathfrak{L}}_n$, then by Lemma 21 and Assumption \mathcal{A}_5^- , we have that
592 $\pi_{\mathfrak{C}}(q_1)$ is a node in $\pi_{\mathfrak{C}}(\mathfrak{C})$. Thus, we have that $\pi_{\mathfrak{C}}(q_1)$ is a transverse intersection of two smooth branches of
593 $\pi_{\mathfrak{C}}(\mathfrak{C})$. Those branches are the plane projections of two disjoint branches of \mathfrak{C} each of which contains either q_1 or
594 q_2 . Hence, the plane projections of the tangent spaces of q_1 and q_2 to \mathfrak{C} are linearly independent. Thus, (q_1, q_2)
595 satisfies Assumption $\mathcal{A}_5^{-'}$.

596 Assume conversely that $\mathcal{A}_5^{-'}$ is satisfied for all $(q_1, q_2) \in \widehat{\mathfrak{L}}$. By Lemma 26, any singular point of $\pi_{\mathfrak{C}}(\mathfrak{C})$ is the
597 plane projection of a point $q_1 \in \mathfrak{L}_c \cup \mathfrak{L}_n$. For some $q_2 \in \mathfrak{C}$, the pair (q_1, q_2) is in $\widehat{\mathfrak{L}}$ (which satisfies Assumption
598 $\mathcal{A}_5^{-'}$). Hence, if (q_1, q_2) is in $\widehat{\mathfrak{L}}_n$ (resp. in $\widehat{\mathfrak{L}}_c$) the plane projection of q_1 is a node (resp. an ordinary cusp) by
599 Lemma 21 (resp. Lemma 24). \square

600 Lemmas 32 and 33 then imply Theorem 27.

601 5. Semi-algorithms to check assumptions and isolate singularities

602 In this section we present Semi-algorithm 3 that checks the weak assumptions of Section 2.4 and, if it termi-
603 nates, outputs a superset of isolating boxes of the singularities of the plane projection $\pi_{\mathfrak{C}}(\mathfrak{C})$ of \mathfrak{C} . We also present
604 Semi-algorithm 4 that checks the strong assumptions of Section 2.4 and, if it terminates, outputs a set of isolating
605 boxes of the singularities of $\pi_{\mathfrak{C}}(\mathfrak{C})$.

606 The main idea of these semi-algorithms comes from Theorems 11 and 27: Theorem 11 states that, under
607 Assumptions \mathcal{A}_{1-4} , the singularities of $\pi_{\mathfrak{C}}(\mathfrak{C})$ are the plane projections of the solutions of $\text{Ball}(P)$. Theorem 27
608 states that, under the further Assumption \mathcal{A}_5^- , $\text{Ball}(P)$ is regular, so we can use certified numerical methods such
609 as interval Newton methods [MKC09] to solve $\text{Ball}(P)$. In addition, in order to verify these assumptions, we use
610 subdivision approaches based on interval arithmetic in a semi-algorithm framework.

611 We present in Section 5.1 the basics of interval arithmetic with the notation and definitions by Lin and
612 Yap [LY11] and our semi-algorithms in Section 5.2.

613 5.1. Interval arithmetic

614 Recall that for some positive integer k , by a closed (resp. open) k -box \mathfrak{B} , we mean the Cartesian product of
615 k closed (resp. open) intervals. The width of a box \mathfrak{B} , denoted by $w(\mathfrak{B})$, is the maximal length of the intervals
616 of that product. For a subset $A \subset \mathbb{R}^k$, the set IA is the set of all closed k -boxes that are contained in A . For
617 the positive integer m and a function $f : A \rightarrow \mathbb{R}^m$, the function $\square f : IA \rightarrow I\mathbb{R}^m$ is called an inclusion of f if
618 the set $f(\mathfrak{B}) = \{f(x) \mid x \in \mathfrak{B}\}$ is contained in $\square f(\mathfrak{B})$, for all $\mathfrak{B} \in IA$. An inclusion $\square f$ of f is called a box
619 function, if for any descending sequence of closed k -boxes $\mathfrak{B}_1 \supset \mathfrak{B}_2 \supset \dots$ that converges to a point $q \in \mathbb{R}^k$, the
620 sequence $\square f(\mathfrak{B}_1) \supset \square f(\mathfrak{B}_2) \supset \dots$ converges to $f(q)$. In the rest of this section, we assume that we are given a

621 box function $\square f$ for any function f we consider. The command *subdivide* is applied to a closed k -box $\overline{\mathfrak{B}}$, and it
 622 returns the set of boxes obtained by bisecting $\overline{\mathfrak{B}}$ in all dimensions.

623 An interval matrix $\square M$ is a matrix whose coefficients are intervals. It can also be seen as the set of all
 624 matrices whose (i, j) -th coefficients belong to the (i, j) -th interval. The rank of an interval matrix $\square M$, denoted
 625 by $\text{rank}(\square M)$, is the minimum of the ranks of all the matrices in this set.

626 5.2. Semi-algorithms

627 This section is dedicated to the proof of the following theorem. Recall that the weak and strong assumptions
 628 are defined in Definition 8.

629 **Theorem 34.** *For an open n -box B and a smooth function P from \overline{B} to \mathbb{R}^{n-1} , Semi-algorithm 3 stops if and only
 630 if P satisfies the weak assumptions in \overline{B} and then it returns a set of isolating boxes of all the singularities of $\pi_{\mathfrak{C}}(\mathfrak{C})$,
 631 plus possibly other spurious disjoint boxes. Semi-algorithm 4 stops if and only if P satisfies the strong assumptions
 632 in \overline{B} and then it returns a set of isolating boxes of all the singularities of $\pi_{\mathfrak{C}}(\mathfrak{C})$.*

633 To check whether a given function P satisfies the weak assumptions ($\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ and \mathcal{A}_5^-) in \overline{B} , we use
 634 their relation to the solutions of $\text{Ball}(P)$ studied in the previous sections. Recall that for any subset $A \subseteq \mathbb{R}^n$, we
 635 defined $A_{\text{Ball}} = \{(x_1, x_2, y, r, t) \mid t \geq 0, (x_1, x_2, y + r\sqrt{t}), (x_1, x_2, y - r\sqrt{t}) \in A, \|r\|^2 = 1\}$. Let B be an open
 636 n -box and P be a smooth function from \overline{B} to \mathbb{R}^{n-1} that satisfies Assumption \mathcal{A}_1 in \overline{B} . Consider the following
 637 assumptions:

638 \aleph_1 - All solutions of $\text{Ball}(P)$ in $\overline{B}_{\text{Ball}}$ are regular.

639 \aleph_2 - For every solution X of $\text{Ball}(P)$ in $\overline{B}_{\text{Ball}}$, none of the points of the pair $\Omega_P(X)$ (Definition 14) is in the
 640 boundary of B .

641 \aleph_3 - No two distinct solutions of $\text{Ball}(P)$ in $\overline{B}_{\text{Ball}}$, except the twin solutions (Remark 17), have the same plane
 642 projection.

643 The next lemma shows the relation between these new assumptions and those of Section 2.4. The motivation
 644 of these alternative assumptions is that they are stated in terms of $\text{Ball}(P)$, which makes them easier to verify in
 645 our semi-algorithms.

646 **Lemma 35.** *Let B be an open n -box and P be a smooth function from \overline{B} to \mathbb{R}^{n-1} that satisfies Assumption \mathcal{A}_1
 647 in \overline{B} . Then, Assumptions \aleph_1, \aleph_2 and \aleph_3 are satisfied if and only if Assumptions $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ and \mathcal{A}_5^- are satisfied
 648 in \overline{B} .*

649 *Proof.* If Assumptions $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ and \mathcal{A}_5^- are satisfied in \overline{B} , then by Theorem 27 we have Assumption \aleph_1 is
 650 satisfied. Moreover, by Assumptions \mathcal{A}_2 and \mathcal{A}_4 we have that none of $\overline{\mathfrak{L}}_n, \overline{\mathfrak{L}}_c$ intersects the boundary of B . By
 651 Definition 14, for any solution X of $\text{Ball}(P)$, we have that the points of the pair $\Omega_P(X)$ are in $\overline{\mathfrak{L}}_n \cup \overline{\mathfrak{L}}_c$ and hence

652 are not on the boundary of B , which implies that Assumption \aleph_2 is satisfied. Assume that Assumption \aleph_3 is not
653 satisfied, that is, there exist two distinct non-twin solutions X, X' that have the same plane projection $p \in \mathbb{R}^2$. Let
654 $(q_1, q_2) = \Omega_P(X)$ and $(q'_1, q'_2) = \Omega_P(X')$. By Lemma 13, the pairs $(q_1, q_2), (q'_1, q'_2)$ are distinct and the points
655 q_1, q_2, q'_1, q'_2 have the same plane projection p . By Assumption \mathcal{A}_3 , we cannot have three pairwise distinct points
656 among q_1, q_2, q'_1, q'_2 . Moreover, if the multiplicity at all of the points q_1, q_2, q'_1, q'_2 is one, then $(q_1, q_2), (q'_1, q'_2)$
657 are in $\widehat{\mathfrak{L}}_n$ and not distinct. Hence, at least a point say q_1 has multiplicity larger than one, i.e., $q_1 \in \mathfrak{L}_c$ (Lemma
658 20). Hence, the number of solutions counted with multiplicity is at least three which contradicts Assumption \mathcal{A}_3 .
659 Hence, Assumption \aleph_3 is satisfied.

660 Now, assume that Assumptions \aleph_1, \aleph_2 and \aleph_3 are satisfied. Since, by Assumption \aleph_1 , $\text{Ball}(P)$ is a regular
661 square system, its solution set is a zero-dimensional manifold in the compact set $\overline{B}_{\text{Ball}(P)}$ (regular value theorem).
662 Hence, $\text{Ball}(P)$ has a finite number of solutions in $\overline{B}_{\text{Ball}}$. Since Ω_P (Definition 14) is surjective (Lemma 15),
663 the set $\widehat{\mathfrak{L}}$ (Definition 12) is also finite. Hence, the set $\mathfrak{L}_c \cup \mathfrak{L}_n$ is finite (since $\mathfrak{L}_c \cup \mathfrak{L}_n$ is the image of $\widehat{\mathfrak{L}}$ under
664 the surjective function $(q_1, q_2) \rightarrow q_1$). Moreover, by Assumption \aleph_2 , the set $\overline{\mathfrak{L}}_n \cup \overline{\mathfrak{L}}_c$ does not intersect the
665 boundary of B . Hence, Assumptions \mathcal{A}_2 and \mathcal{A}_4 are satisfied in \overline{B} . To prove that Assumption \mathcal{A}_3 is satisfied, let
666 $p = (\alpha, \beta) \in \pi_{\mathfrak{C}}(\mathfrak{C})$ and $|\pi^{-1}(p)| \geq 3$. For pairwise distinct points $q_1, q_2, q_3 \in \pi^{-1}(p)$, by Lemma 13, we have
667 that there exist two distinct non-twin solutions X, X' of $\text{Ball}(P)$, with $\Omega_P(X) = (q_1, q_2)$ and $\Omega_P(X') = (q_1, q_3)$
668 such that we have the same plane projection p which contradicts Assumption \aleph_3 . Hence, $\pi_{\mathfrak{C}}^{-1}(p)$ consists of at
669 most two distinct points. We consider two cases:

- 670 (a) $\pi_{\mathfrak{C}}^{-1}(p)$ has two distinct elements, say q_1, q_2 . By Lemma 13, the pair (q_1, q_2) is in $\widehat{\mathfrak{L}}_n$, and hence, there exists
671 a solution $X = (\alpha, \beta, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R}$ of $\text{Ball}(P)$, with $t \neq 0$ and $\Omega_P(X) = (q_1, q_2)$.
672 Since X is a regular solution (Assumption \aleph_1), by Lemma 32 we have that none of q_1, q_2 is in \mathfrak{L}_c . Hence,
673 by Lemma 20, the multiplicity of $\{P(x_1, x_2, y) = 0 \in \mathbb{R}^{n-1}, x_1 - \alpha = x_2 - \beta = 0\}$ at q_1 (resp. q_2) is one.
674 Thus, the number of solutions counted with multiplicity is two.
- 675 (b) $\pi_{\mathfrak{C}}^{-1}(p)$ has a unique point q . Let m denote the multiplicity of the system $\{P(x_1, x_2, y) = 0 \in \mathbb{R}^{n-1}, x_1 -$
676 $\alpha = x_2 - \beta = 0\}$ at q . If $m = 1$, then we are done. If $m > 1$, then by Lemma 20 we have that $q \in \mathfrak{L}_c$.
677 Hence, there exists a solution of $\text{Ball}(P)$ of the form $X = (\alpha, \beta, y, r, 0) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R}$ such
678 that $\Omega_P(X) = (q, q)$ (Lemma 15). Since X is regular (Assumption \aleph_1), by Lemma 32 we have that (q, q)
679 satisfies assumption \mathcal{A}'_5 . In particular, the multiplicity m is equal to two.

680 Thus, for all $p \in \pi_{\mathfrak{C}}(\mathfrak{C})$ the sum of the multiplicities of the solutions in the system $\{P(x) = 0 \in \mathbb{R}^{n-1}, x_1 - \alpha =$
681 $x_2 - \beta = 0\}$ is at most two, i.e., Assumption \mathcal{A}_3 is satisfied. Now, since Assumptions $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and \mathcal{A}_4
682 are satisfied and since all solutions of $\text{Ball}(P)$ are regular, by Theorem 27, we have that Assumption \mathcal{A}_5^- is also
683 satisfied. \square

684 Using Lemma 35, we are ready to check Assumptions $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ and \mathcal{A}_5^- using \aleph_1, \aleph_2 and \aleph_3 . Since
685 Lemma 35 requires Assumption \mathcal{A}_1 , we start by checking that assumption with Semi-algorithm 1 that is based on

686 subdivision.

687 **Semi-algorithm 1** Checking Assumption \mathcal{A}_1

688 **Input:** An open n -box B and a function P from \overline{B} to \mathbb{R}^{n-1} .

689 **Termination:** If and only if P satisfies Assumption \mathcal{A}_1 in \overline{B} .

690 **Output:** True.

691 1: $L := \{\overline{B}\}$

692 2: **while** $L \neq \emptyset$ **do**

693 3: $\mathfrak{B} := \text{pop}(L)$

694 4: **if** $0 \in \square P(\mathfrak{B})$ and $\text{rank}(\square J_P(\mathfrak{B})) < n - 1$ **then**

695 5: Subdivide \mathfrak{B} and add its children to L .

696 6: **return** True.

697 **Lemma 36.** *Semi-algorithm 1 stops if and only if P satisfies Assumption \mathcal{A}_1 in \overline{B} .*

699 *Proof.* If Semi-algorithm 1 stops, by the conditions in Step (4), the box \overline{B} is partitioned into two sets of boxes. A
700 set of boxes that are disjoint with $\overline{\mathcal{C}}$ and the other one is a set of boxes that contain parts of $\overline{\mathcal{C}}$ that satisfy Assumption
701 \mathcal{A}_1 . Thus, Assumption \mathcal{A}_1 is satisfied in \overline{B} . On the other hand, assume that P satisfies Assumption \mathcal{A}_1 in \overline{B} and
702 Semi-algorithm 1 does not stop, then, for every positive real ϵ there exists a closed box $\overline{\mathfrak{B}}_\epsilon \subset \overline{B}$, with $w(\overline{\mathfrak{B}}_\epsilon) < \epsilon$
703 such that the conditions in Step (4) are satisfied in $\overline{\mathfrak{B}}_\epsilon$. Consider the infinite chain $\overline{\mathfrak{B}}_{\frac{1}{1}}, \overline{\mathfrak{B}}_{\frac{1}{2}}, \overline{\mathfrak{B}}_{\frac{1}{3}} \dots$ and take
704 $q_k \in \overline{\mathfrak{B}}_{\frac{1}{k}}$, with $q_k \neq q_{k'}$ for $k \neq k'$. Since \overline{B} is compact, then there exists a subsequence of q_k that converges to
705 a point on \overline{B} say q . Since $\square P$ and $\square J_P$ are box function we must have that $P(q) = 0$ and $\text{rank}(J_P(q)) < n - 1$.
706 Thus, q is a point in $\overline{\mathcal{C}}$ that does not satisfy Assumption \mathcal{A}_1 which is a contradiction. Hence, Semi-algorithm 1
707 stops. \square

708 The next step is to check Assumptions \aleph_1 and \aleph_2 . For this goal, we want to find a finite set of pairwise disjoint
709 boxes in $\overline{B}_{\text{Ball}}$ such that every box contains at most one solution of $\text{Ball}(P)$ and the union of these boxes contains
710 all solutions of $\text{Ball}(P)$ in $\overline{B}_{\text{Ball}}$. Notice that, by the definition of box functions, for a closed $(2n - 1)$ -box $\overline{\mathfrak{U}}$, if
711 $0 \notin \square \text{Ball}(P)(\overline{\mathfrak{U}})$, then $\overline{\mathfrak{U}}$ does not contain a solution of $\text{Ball}(P)$, whereas the condition $0 \in \square \text{Ball}(P)(\overline{\mathfrak{U}})$ does
712 not necessarily imply that a solution is in $\overline{\mathfrak{U}}$. This is why the set we are going to find might have unnecessary boxes.
713 However, we will see later that this is enough for our purpose. Before introducing Semi-algorithm 2, we define the
714 following functions.

Definition 37. *Consider the set $\mathbb{R}_{t \geq 0}^{2n-1} = \{(x_1, x_2, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R} \mid t \geq 0\}$ and define*

$$f_{\text{Ball}}^\pm : \mathbb{R}_{t \geq 0}^{2n-1} \rightarrow \mathbb{R}^n$$

$$(x_1, x_2, y, r, t) \mapsto (x_1, x_2, y \pm r\sqrt{t})$$

715 Define the function $f_{\text{Ball}} : \mathbb{R}_{t \geq 0}^{2n-1} \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ that maps X to $(f_{\text{Ball}}^+(X), f_{\text{Ball}}^-(X))$. Notice that f_{Ball} is an
716 extension of Ω_P (Definition 14). By abuse of notation, for a set $S \subset \mathbb{R}^{2n-1}$, we define $f_{\text{Ball}}(S)$ as $f_{\text{Ball}}(S \cap \mathbb{R}_{t \geq 0}^{2n-1})$.

717 **Semi-algorithm 2** Isolating the solutions of $\text{Ball}(P)$ (under Assumption \mathcal{A}_1)

718 **Input:** An open n -box B , a function P from \overline{B} to \mathbb{R}^{n-1} such that P satisfies Assumption \mathcal{A}_1 in \overline{B} and a $(2n-1)$ -
719 open box \mathcal{U}_0 that contains $\overline{B}_{\text{Ball}}$ (see Remark 40).

720 **Termination:** If and only if $\text{Ball}(P)$ satisfies Assumptions \aleph_1 and \aleph_2 in $\overline{B}_{\text{Ball}}$.

721 **Output:** A list of pairwise disjoint isolating boxes of the solutions of $\text{Ball}(P)$ in \mathcal{U}_0 such that their images by
722 f_{Ball} lies in $B \times B$.

723 1: $Solutions = \emptyset$.

724 2: $L := \{\mathcal{U}_0\}$.

725 3: **while** $L \neq \emptyset$ **do**

726 4: $\mathcal{U} := \text{pop}(L)$.

727 5: **if** $0 \notin \square \text{Ball}(P)(\overline{\mathcal{U}})$ or $(\square f_{\text{Ball}}(\overline{\mathcal{U}})) \cap (\overline{B} \times \overline{B}) = \emptyset$ **then**

728 6: Do nothing (\mathcal{U} is simply removed from L).

729 7: **else if** $\text{rank}(\square J_{\text{Ball}(P)}(\overline{\mathcal{U}})) = 2n - 1$ and $\square f_{\text{Ball}}(\epsilon\text{-inflation}(\overline{\mathcal{U}}))^2 \subset B \times B$ **then**

730 8: **if** $\epsilon\text{-inflation}(\mathcal{U})$ contains a solution of $\text{Ball}(P)$ (see Remark 38) **then**

731 9: Add $\epsilon\text{-inflation}(\mathcal{U})$ to $Solutions$.

732 10: **else**

733 11: Subdivide \mathcal{U} and add its children to L .

734 12: Remove duplicates in $Solutions$ (see Remark 38).

735 13: **return** $Solutions$

736 **Remark 38.** Steps (8) and (12) are not detailed because they are standard in subdivision algorithms to handle
737 the issue of solutions on or near box boundaries and ensuring that solution boxes are pairwise disjoint. We only
738 sketch below how these steps are done and refer to Sta95, §5.9.1; Kea97; XY19] for details. In Step (8), an
739 existence test is performed by evaluating an interval Newton operator on an ϵ -inflation of the box \mathcal{U} . The inflated
740 box $\epsilon\text{-inflation}(\mathcal{U})$ is certified to contain a solution if its image by the interval Newton operator is contained in the
741 interior of $\epsilon\text{-inflation}(\mathcal{U})$. When the existence test is positive, the solution may be on the boundary or even outside
742 \mathcal{U} , but still in the interior of $\epsilon\text{-inflation}(\mathcal{U})$. The side effect is that the same solution may be reported several times
743 when it is on or near a boundary of the subdivision. This issue is then solved in Step (12) as follows. When two
744 boxes in the set $Solutions$ intersect, they must report the same solution, and in addition, this solution is in the
745 intersection of the two boxes. In Step (12), we thus compute intersections between boxes and replace intersecting
746 ones by their intersection box. The boxes in the output set $Solutions$ are thus pairwise disjoint.

748 **Lemma 39.** Under Assumption \mathcal{A}_1 in \overline{B} , if Semi-algorithm 2 stops, it returns a list of pairwise disjoint isolating
749 boxes of the solutions of $\text{Ball}(P)$ in \mathcal{U}_0 such that their images by f_{Ball} lies in $B \times B$. Moreover, Semi-algorithm 2

²For a box \mathcal{U} and $\epsilon > 0$, $\epsilon\text{-inflation}(\mathcal{U})$ is the box that has the same center as \mathcal{U} and its width is that of \mathcal{U} multiplied by $(1 + \epsilon)$. The box $\epsilon\text{-inflation}(\mathcal{U})$ thus contains \mathcal{U} , the exact value of ϵ is not important for the algorithm and it is usually set to 0.1 in subdivision algorithms [Rum10].

750 stops if and only if $\text{Ball}(P)$ satisfies Assumptions \aleph_1, \aleph_2 in $\overline{B}_{\text{Ball}}$.

751 *Proof.* We first prove the correctness of the Semi-algorithm 2 assuming that it terminates. Since Step (5) is the
 752 only time the algorithm discards boxes, it never discards a box that contains a solution of $\text{Ball}(P)$ in \mathfrak{U}_0 such that
 753 its image by f_{Ball} lies in $B \times B$. Hence, all such solutions of $\text{Ball}(P)$ lie in output boxes. The rank condition in
 754 Step (7) guarantees that each output box contains at most one solution of $\text{Ball}(P)$ [Sny92b, Theorem A.1]. The
 755 fact that every output box contains at least one solution is ensured by a standard algorithm in Step (8) (see e.g.,
 756 Neu91, Theorem 5.6.2; XY19] and Remark 38). Finally, by Step (12), the output boxes are pairwise disjoint, hence
 757 the algorithm outputs isolating boxes of the solutions of $\text{Ball}(P)$ in \mathfrak{U}_0 such that their images by f_{Ball} lie in $B \times B$.

758 To prove the equivalence for the termination, first assume that Semi-algorithm 2 stops and returns *Solutions*.
 759 According to the correctness proof, every solution X of $\text{Ball}(P)$ in $\overline{B}_{\text{Ball}}$ is regular and satisfies $\Omega_P(X) \in B \times B$.
 760 Thus, Assumptions \aleph_1 and \aleph_2 are satisfied in $\overline{B}_{\text{Ball}}$.

761 On the other hand, assume that \aleph_1 and \aleph_2 hold in $\overline{B}_{\text{Ball}}$. We prove that Semi-algorithm 2 terminates. By As-
 762 sumption \aleph_1 all solutions in $\overline{B}_{\text{Ball}}$ of the square system $\text{Ball}(P)$ are regular. Hence, they form a zero-dimensional
 763 manifold in the compact space $\overline{B}_{\text{Ball}}$. Thus, the solution set is finite. We now prove that for any box $\overline{\mathfrak{U}} \in L$ with
 764 a small enough width, one of the conditions in Step (5) or the conditions in Steps (7-8) are satisfied. Thus, in both
 765 cases $\overline{\mathfrak{U}}$ will be removed from L , and hence, Semi-algorithm 2 stops after a finite number of iterations. Due to
 766 Assumption \aleph_2 , after a finite number of iterations, no box \mathfrak{U} in L intersects the boundary of $B \times B$. Moreover, due
 767 to the convergence of the box evaluations, we can also assume that either $\square f_{\text{Ball}}(\epsilon\text{-inflation}(\overline{\mathfrak{U}})) \subset B \times B$, which
 768 is the second condition of Step (7), or $(\square f_{\text{Ball}}(\overline{\mathfrak{U}})) \cap (\overline{B} \times \overline{B}) = \emptyset$ which is the second condition of Step (5).

769 If \mathfrak{U} does not contain a solution of $\text{Ball}(P)$, then due to convergence of the box function evaluation of $\text{Ball}(P)$,
 770 after a finite number of iterations, every children \mathfrak{U}' of \mathfrak{U} satisfies $0 \notin \square \text{Ball}(P)(\overline{\mathfrak{U}'})$, that is, it is discarded in
 771 Step (5).

772 If \mathfrak{U} contains a solution of $\text{Ball}(P)$ in $\overline{B}_{\text{Ball}}$, according to Assumption \aleph_1 , it is a regular solution. Due to
 773 the convergence of the box evaluation $\det(J_{\text{Ball}(P)}(\overline{\mathfrak{U}}))$ will eventually be non zero and thus $\text{rank}(\square J_{\text{Ball}(P)}(\overline{\mathfrak{U}}))$
 774 will eventually be $2n - 1$ after a finite number of iterations, which is the first condition of Step (7). Due to the
 775 convergence of the interval Newton existence test, the condition of Step (8) will also be eventually satisfied (see
 776 Remark 38). The refined box will then eventually be added in the Solutions list.

777 Thus, for any box in L with a small enough width, one of the conditions of Step (5) is satisfied or all of the
 778 conditions in Step (7-8) are satisfied, thus it is either discarded or added to the output. Hence, Semi-algorithm 2
 779 terminates. \square

780 **Remark 40.** *Semi-algorithm 2 requires a closed $(2n - 1)$ -box $\overline{\mathfrak{U}}_0$ that contains $\overline{B}_{\text{Ball}}$. For instance the following*
 781 *set could be used: $\{(q, r, t) \in \mathbb{R}^{2n-1} \mid q \in \overline{B}, -1 \leq r_i \leq 1 \text{ for } i \in \{3, \dots, n\}, 0 \leq t \leq \frac{\xi^2}{4}\}$ with $\xi =$*
 782 *$\max\{\|q - q'\| \mid q, q' \in \overline{B}\}$.*

783 Finally, using Lemma 35, Semi-algorithm 3 checks whether P satisfies Assumptions $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ and \mathcal{A}_5^-

784 in \overline{B} and outputs a superset of isolating boxes of the singularities of $\pi_{\mathcal{C}}(\mathcal{C})$.

785 **Semi-algorithm 3** Checking the weak assumptions and computing a superset of the singularities of $\pi_{\mathcal{C}}(\mathcal{C})$

786 **Input:** An open n -box B and a smooth function P from \overline{B} to \mathbb{R}^{n-1} .

787 **Termination:** If and only if P satisfies Assumptions $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ and \mathcal{A}_5^- in \overline{B} .

788 **Output:** N , a list of certified node singularities: a list of boxes in \mathbb{R}^{2n-1} whose projections in \mathbb{R}^2 contain each a
789 single node of $\pi_{\mathcal{C}}(\mathcal{C})$.

790 U , a list of uncertified singularities: a list of boxes in \mathbb{R}^{2n-1} whose projections in \mathbb{R}^2 contain each at most one
791 node or one cusp of $\pi_{\mathcal{C}}(\mathcal{C})$.

792 The union of all these projected 2D boxes contains all the singularities of $\pi_{\mathcal{C}}(\mathcal{C})$.

793 1: Check Assumption \mathcal{A}_1 (Semi-algorithm 1).

794 2: Compute a closed $(2n - 1)$ -box $\overline{\mathfrak{M}}_0$ that contains $\overline{B}_{\text{Ball}}$ (Remark 40).

795 3: $L :=$ output of Semi-algorithm 2.

796 4: Keep refining all boxes $\overline{\mathfrak{M}} \in L$ (see Remark 41), until no triplets of boxes overlap in projection. Then remove
797 from L one box from every pair (see Remark 17). This ensures Assumption \mathfrak{N}_3 .

798 5: $N :=$ boxes of L that lie in the halfspace $t > 0$.

799 6: $U :=$ boxes of L that intersect the hyperplane $t = 0$.

800 7: **return** N and U .

801 **Remark 41.** *The refinement of an isolating box of a solution is performed by iterative evaluation of an interval*
802 *Newton operator; we refer to [Neu91, Theorem 5.6.2] for details.*
803

804 To identify the possible cusp points in the set U returned by Semi-algorithm 3, one may wish to solve indepen-
805 dently the Ball system with the additional constraint $t = 0$ (by Remark 16). Unfortunately, in this case we have
806 an over-determined system and thus we cannot certify its solutions numerically. In the special case of a silhouette
807 curve, it is possible to identify cusp points with numerical algorithms in the case $n = 3$ [IMP16a, Lemmas 9 &
808 10], but we leave as a conjecture its generalization for $n > 3$.

809 On the other hand, for curves that satisfy the strong assumptions, \mathcal{A}_5 ensures that there are no cusps in the
810 projection, which is equivalent to $\widehat{\mathcal{L}}_{\mathcal{C}}$ being empty and $\text{Ball}(P)$ having no solutions on the hyperplane $t = 0$ (by
811 Remark 16). Hence, if Assumptions \mathcal{A}_{1-5} hold, we can refine the boxes output by Semi-Algorithm 3 until no box
812 intersects $t = 0$. Boxes in the half-space $t < 0$ correspond to imaginary points in \mathbb{C}^n (Definition 14). Then the
813 boxes satisfying $t > 0$ are all the isolating boxes of the nodes of $\pi_{\mathcal{C}}(\mathcal{C})$ by Lemmas 39 and 13, Remark 16 and
814 Lemma 26.

815 **Semi-algorithm 4** Checking Assumptions \mathcal{A}_{1-5} and isolating the singularities of $\pi_{\mathcal{C}}(\mathcal{C})$

816 **Input:** An open n -box B and a smooth function P from \overline{B} to \mathbb{R}^{n-1} .

817 **Termination:** If and only if P satisfies Assumptions \mathcal{A}_{1-5} in \overline{B} .

818 **Output:** A list of boxes in \mathbb{R}^{2n-1} whose projections in \mathbb{R}^2 are isolating boxes of the singularities $\pi_{\mathcal{C}}(\mathcal{C})$ (all
819 singularities are in some boxes and each box contains a unique singularity).
820 1: $N, U :=$ output of Semi-Algorithm 3.
821 2: **for** all $\bar{U} \in U$ **do**
822 3: Keep refining \bar{U} (see Remark 41) until it does not intersect the hyperplane $t = 0$ and discard it if it lies in
823 the half-space $t < 0$.
824 4: **return** $N \cup U$.

825 6. Implementation and experiments

827 We first describe, in Section 6.1, the algorithms we implemented in our software *Isolating_singularities*³ with,
828 in particular, the refinements we considered for improving the running time of those presented in Section 5. We
829 present in Section 6.2 the third-party libraries we use. In Section 6.3, we present our experiments on several
830 analytic curves in 3 and 4 dimensions and discuss the efficiency of our implementation.

831 6.1. Algorithms

832 Semi-algorithms 3 and 4 of Section 5 take as input an open n -box B and a curve \mathcal{C} defined by a smooth
833 function P from \bar{B} to \mathbb{R}^{n-1} . They terminate if and only if P satisfies our weak, respectively strong, assumptions
834 of Definition 8. Upon termination, they output a superset of the singularities, respectively the singularities, of the
835 curve $\pi_{\mathcal{C}}(\mathcal{C})$.

836 The algorithms we implemented are variants of Semi-algorithms 3 and 4. First, for visualization purposes we
837 modified their output as follows. Instead of returning boxes in \mathbb{R}^{2n-1} isolating solutions of the Ball system, they
838 now return their projections in \mathbb{R}^2 that contain singularities of $\pi_{\mathcal{C}}(\mathcal{C})$. In addition, they also return the projection
839 in \mathbb{R}^2 of the boxes enclosing the curve \mathcal{C} , that thus enclose the curve $\pi_{\mathcal{C}}(\mathcal{C})$.

840 The two main improvements of our implementations are described below. The first one solves a stability issue
841 when singular points of the projection are induced by very close branches of the curve \mathcal{C} or by a close to vertical
842 part of the curve. The second one is a generalization of an idea used in the three-dimensional case in [IMP18], it
843 aims at reducing the domain where the Ball system is to be solved. It is critical for the efficiency since solving in
844 this high-dimensional space is costly.

845 *Evaluating the operators S and D of Definition 9.* To solve the Ball system we need box functions for the operators
846 S and D . We first note that if $P(x_1, x_2, y) \in C^\infty(\mathbb{R}^n, \mathbb{R})$ is a polynomial function, $S \cdot P$ (resp. $D \cdot P$) is also a
847 polynomial function and can be computed from the terms of $P(x_1, x_2, y + rt)$ that have even (resp. odd) exponents
848 in the variable t , see [IMP16b, Lemma 6] for details.

³https://github.com/gkrait/Isolating_singularities

849 If P is a more general C^∞ function for which we have a box function, using Definition 9, computing $S \cdot P$
850 on any $(2n - 1)$ -box $\bar{\mathcal{U}}$ or computing $D \cdot P$ on $\bar{\mathcal{U}}$ such that its t -interval does not contain 0, is implemented from
851 the box function of P and interval arithmetic. On the other hand, when the t -interval contains 0 or is close to 0,
852 the division by \sqrt{t} in the formula for $D \cdot P$ makes the computation undefined or unstable. In such non-polynomial
853 cases, we use a Taylor form at order 3 [Ral83], that is, we compute a Taylor expansion with remainder at $t = 0$ of
854 $D \cdot P$ and evaluate by interval the third order derivative. We define a threshold δ_{Taylor} (which we set to 10^{-2} in
855 our experiments) such that this Taylor form is used when the t -interval has values smaller than δ_{Taylor} .

856 *Improvement of Semi-algorithm 3.* The domain B_{Ball} where the Ball system is solved is refined to reduce costly
857 computations in the high-dimensional space \mathbb{R}^{2n-1} by first enclosing the curve \mathcal{C} in a union of boxes in the smaller
858 space \mathbb{R}^n . We denote this set of boxes by *enclosing_curve*. Our approach follows the observation that every cusp
859 of $\pi_{\mathcal{C}}(\mathcal{C})$ lies in the projection of a box of *enclosing_curve* containing a point p of \mathcal{C} with a tangent orthogonal
860 to the (x_1, x_2) -plane. Such a point p is both x_1 and x_2 -critical for \mathcal{C} , that is, both $\det(M_1(p))$ and $\det(M_2(p))$
861 vanish (M_i denotes the minor of J_P obtained by removing the i -th column). In addition, every node in $\pi_{\mathcal{C}}(\mathcal{C})$ is
862 contained in:

- 863 (a) the projection of a box \mathfrak{B} in *enclosing_curve* such that $0 \in \det(\square M_1)(\mathfrak{B})$ and $0 \in \det(\square M_2)(\mathfrak{B})$, or
- 864 (b) the intersection of the plane projections of two boxes in *enclosing_curve*.

865 To understand this observation, we say that \mathcal{C} is parameterizable by x_i in a box \mathfrak{B} , if for any particular value
866 α of x_i in \mathfrak{B} , the hyperplane $x_i = \alpha$ intersects the curve \mathcal{C} at most once in \mathfrak{B} . The interval implicit function
867 theorem [Sny92a, Thm C5 p.291] states that a sufficient condition for \mathcal{C} to be parameterizable by x_i in \mathfrak{B} , is
868 that $0 \notin \det(\square M_i)(\mathfrak{B})$. In a box \mathfrak{B} such that $0 \notin \det(\square M_i)(\mathfrak{B})$ for $i = 1$ or 2 , \mathcal{C} is parameterizable in x_i ,
869 thus \mathcal{C} cannot contain two points with the same x_i value, which implies that the projection of $\mathcal{C} \cap \mathfrak{B}$ does not
870 contain a node. It follows that such a box can only induce a node in the projection when it overlaps in projection
871 with another box, this case being covered by criterion (b). All nodes or cusps are thus in the projection of boxes
872 satisfying criteria (a) or (b). Using the mapping f_{Ball} of Definition 37 from the Ball system space \mathbb{R}^{2n-1} to pairs
873 of points in \mathbb{R}^n , we only have to solve the Ball system on the pre-image of these particular boxes or pairs of boxes
874 in the set *enclosing_curve*. More precisely, the Ball system domains associated to boxes in *enclosing_curve* are
875 defined as follows:

- 876 (i) for a single box \mathfrak{B} , cross product of boxes $\mathfrak{B}_{(x_1, x_2)}$ in \mathbb{R}^2 and \mathfrak{B}_y in \mathbb{R}^{n-2} , the domain is

$$877 \quad \{(x_1, x_2, y, r, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{R}, (x_1, x_2, y) \in \mathfrak{B}, r \in [-1, 1]^{n-2}, 0 \leq t \leq \max(\frac{\|\mathfrak{B}_y - \mathfrak{B}_y\|^2}{4})\},$$

- 878 (ii) for a pair of boxes $\mathfrak{B} = \mathfrak{B}_{(x_1, x_2)} \times \mathfrak{B}_y$ and $\mathfrak{B}' = \mathfrak{B}'_{(x_1, x_2)} \times \mathfrak{B}'_y$, the domain is

$$879 \quad \{(x_1, x_2, y, r, t), (x_1, x_2) \in \mathfrak{B}_{(x_1, x_2)} \cap \mathfrak{B}'_{(x_1, x_2)}, y \in \frac{1}{2}(\mathfrak{B}_y + \mathfrak{B}'_y), r \in [-1, 1]^{n-2} \text{ if } \mathfrak{B} \cap \mathfrak{B}' \neq \emptyset, \text{ otherwise, } r \in$$

$$880 \quad \frac{\mathfrak{B}_y - \mathfrak{B}'_y}{\|\mathfrak{B}_y - \mathfrak{B}'_y\|}, t \in \frac{\|\mathfrak{B}_y - \mathfrak{B}'_y\|^2}{4}\}.$$

881 To sum up, our improved versions of Semi-algorithm 3 and 4 consists of three steps: (i) computing a set
882 *enclosing_curve* of n -boxes that enclose the curve \mathcal{C} , (ii) finding the boxes in *enclosing_curve* that satisfy the
883 above criteria (a) or (b), and (iii) solving $\text{Ball}(P)$ over the pre-image of these boxes under f_{Ball} . When the semi-
884 algorithms terminate, they return the 2D projections of the boxes in the set *enclosing_curve* that cover $\pi_{\mathcal{C}}(\mathcal{C})$
885 together with the boxes that isolate the singularities of $\pi_{\mathcal{C}}(\mathcal{C})$.

886 6.2. Third-party libraries

887 Our software is based on interval arithmetic, interval evaluations of analytic functions and an interval solver.
888 We use the following libraries, Python-FLINT and Ibexsolve, for these tasks.

889 Python-FLINT is a Python extension module wrapping FLINT (Fast Library for Number Theory) and Arb
890 (arbitrary-precision ball arithmetic), which offers a toolbox for interval arithmetic and evaluation of analytic func-
891 tions.

892 Ibexsolve is a C++ end-user program that solves systems of non-linear equations rigorously, that is, it does not
893 lose any solution and return each solution under the form of a small box enclosing the true value. It implements
894 a classical branch-and-prune algorithm that interleaves contractions and branching (bisections) to enclose the so-
895 lutions of a system at any given desired precision. However, as opposed to Arb, Ibexsolve has a fixed precision,
896 hence when several solutions are closer to each other than this precision, it will correctly return an enclosing box
897 for these solutions but it will fail at isolating them. In our software, we use the default precision which is 10^{-7} .
898 Ibexsolve, and thus also our software *Isolating_singularities*, use a parameter *eps_max* that defines a maximum
899 width for the isolating boxes output by the solver (the box bisections are forced until all output boxes are not
900 larger than *eps_max*). We use Ibexsolve for solving the Ball system (Semi-algorithm 2) and also in a variant of
901 Semi-algorithm 1 to check the smoothness of the curve \mathcal{C} and at the same time enclosing \mathcal{C} in a set of boxes of \mathbb{R}^n .

902 6.3. Experiments

903 In this section, we present four experiments performed with our software *Isolating_singularities*. More pre-
904 cisely, we applied our improved Semi-algorithm 3 on Experiment 1 and Semi-algorithm 4 on all other experiments.
905 The first example is pedagogical and considers a simple analytic curve in \mathbb{R}^3 that induces only one node and one
906 ordinary cusp in \mathbb{R}^2 . The second example considers a smooth analytic curve in \mathbb{R}^4 that induces many nodes in \mathbb{R}^2 .
907 The third one considers sparse but reasonably-high-degree algebraic equations in \mathbb{R}^4 . It should be stressed that, up
908 to our knowledge, the two latter examples are out of reach by other methods: indeed, no other certified algorithm
909 can handle non-algebraic curves in dimension higher than 3 and, for reasonably-high-degree algebraic equations in
910 \mathbb{R}^4 , the bivariate equation defining their 2D projection often has a very high degree (see Section 6.3.3 for details).

911 Finally, in the fourth example, we exhibit the behavior of our software when a node in \mathbb{R}^2 is induced by a pair
912 of points (on the space curve) that are very close. Indeed, when the equations defining the space curve are not
913 algebraic, the Ball system contains a division by \sqrt{t} (due to the formula of $D \cdot P$), which may cause instability

		Enclosing curve \mathfrak{C}			Solving Ball system		Output	
Experiments	Boxes max. width	Tree size	Output boxes	Time	Tree size	Time	Total time	Singularity boxes
Experiment 1	0.1	535	134	0.1	70	3.6	3.7	2
	0.03	1835	456	0.3	90	3.8	4.1	
	0.01	5427	1354	0.7	188	4.4	5.1	
Experiment 2	0.1	2243	520	1.1	6098	52	53.1	43
	0.03	6759	1639	3.4	1078	35.4	38.8	
	0.01	19583	4847	10.2	372	35.8	45.6	
Experiment 3	0.1	1151	203	1.0	655	4.2	5.2	7
	0.03	2503	523	1.8	272	3.5	5.3	
	0.01	6347	1482	4.5	163	5.7	10.2	

Table 1: Running times (in seconds) and numbers of boxes in Experiments 1 to 3.

914 since t tends to zero when the distance between the pair of points tend to zero. For that purpose, we consider two
915 very close skew lines defined analytically (and not algebraically).

916 We report the running times and other relevant parameters in Tables 1 and 2. Running times are in seconds on a
917 sequential Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz machine with Linux. We emphasize that the experiments
918 are done with a prototype implementation that is under ongoing development. The *tree size* columns reports the
919 total number of boxes created during the subdivision algorithm either for enclosing the curve \mathfrak{C} in \mathbb{R}^n or for
920 solving the Ball system in \mathbb{R}^{2n-1} . For the enclosing part, the column *output boxes* is the number of boxes in the
921 set *enclosing_curve*. For each experiment, we provide a visualization of the plane projected curve $\pi_{\mathfrak{C}}(\mathfrak{C})$ with its
922 singularities. On each figure, the green boxes are the plane projections of the boxes in *enclosing_curve* that enclose
923 \mathfrak{C} , hence these green boxes enclose $\pi_{\mathfrak{C}}(\mathfrak{C})$. The black boxes are the projections of the Ball system solution boxes
924 identifying nodes of the plane curve $\pi_{\mathfrak{C}}(\mathfrak{C})$.

925 For each experiment, we consider three values of *eps_max* and it can be observed (see Table 1) that the smaller
926 the value of *eps_max*, the larger the set *enclosing_curve*, and the longer it is to compute. As expected, even with
927 the improvement to reduce the Ball system domains to be solved in, the subdivision in the high-dimensional space
928 \mathbb{R}^{2n-1} is the dominant step of the algorithm.

929 6.3.1. Experiment 1: Analytic curve in \mathbb{R}^3 generating one node and one ordinary cusp

We start with a pedagogical example pictured in Figure 6. Running times are given in Table 1. The curve \mathfrak{C} is defined in the box $B = (-1, 4) \times (-1, 4) \times (-4.8, -1.4)$ by

$$P(x_1, x_2, x_3) = [x_1 - \cos(x_3)(3 + \sin^4(x_3)) + 3, \quad x_2 - \sin^2(x_3)(3 + \sin(2x_3))].$$

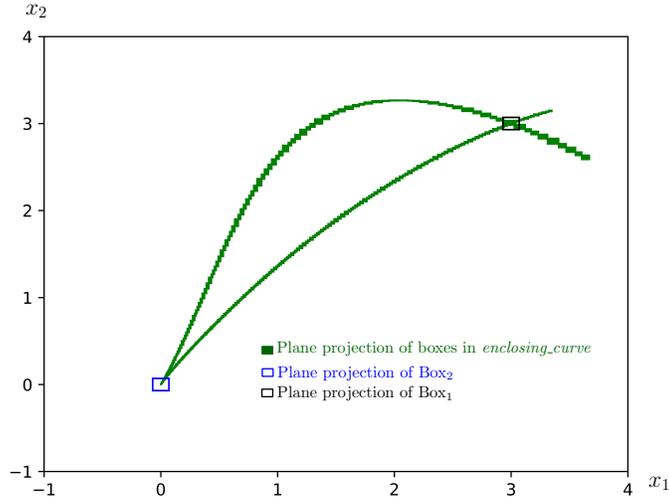


Figure 6: Experiment 1: Plane projection of an analytic curve in \mathbb{R}^3 with one node and one ordinary cusp.

Our improved Semi-algorithm 3 outputs the following solutions for the Ball system in \mathbb{R}^5 :

$$\begin{cases} N = \{\text{Box}_1 = [3, 3] \times [3, 3] \times [-3.15, -3.14] \times [1, 1] \times [2.4673, 2.4675]\} \\ U = \{\text{Box}_2 = [-0.06, 0.04] \times [-0.04, 0.07] \times [-3.15, -3.14] \times [1, 1] \times [-0.01, 0.01]\}. \end{cases}$$

930 Box_1 in the set N thus projects to a node of $\pi_{\mathcal{C}}(\mathcal{C})$. Box_2 being in the set U , one cannot decide whether its
 931 projection in the plane contains a node, a cusp or no singularity at all. On the other hand, one can notice on the
 932 equation $P = 0$ that the curve is parametrizable by the variable x_3 . It is thus an easy computation to check that for
 933 the value $x_3 = -\pi$, the point $q = (0, 0, -\pi)$ is on the curve \mathcal{C} and its tangent line at q is generated by the vector
 934 $(0, 0, 1)$ which is orthogonal to the projection plane. It is then clear that the projection of q generates a cusp that is
 935 witnessed by Box_2 .

936 6.3.2. Experiment 2: Analytic curve in \mathbb{R}^4 with many nodes

Figure 7 illustrates the output of our improved Semi-algorithm 4 for the curve defined by

$$\begin{aligned} P = [x_1 + 2 \sin(x_1) - \cos(x_4) - (3 \cos(x_3) - \cos(2.8571x_3)), \\ x_2 + 0.2 \cos(x_2) + (3 \sin(x_3) - \sin(2.8571x_3)) + \sin(x_4), \\ x_4^2 - \sin(x_2)] \end{aligned}$$

937 over the box $B = (-1, 0) \times (-0.1, 3.5) \times (-20, 20) \times (-10, 10)$. This curve has many nodes, some of them very
 938 close to each other. Running times are given in Table 1.

939 6.3.3. Experiment 3: High degree algebraic curve in \mathbb{R}^4

The goal of this experiment is to emphasize the genericity of the assumptions and the efficiency of our software in the sparse polynomial case. The curve \mathcal{C} is defined in the 4-box $B = (-1, 0.2) \times (-0.2, 1.4) \times (-10, 10)^2$ and

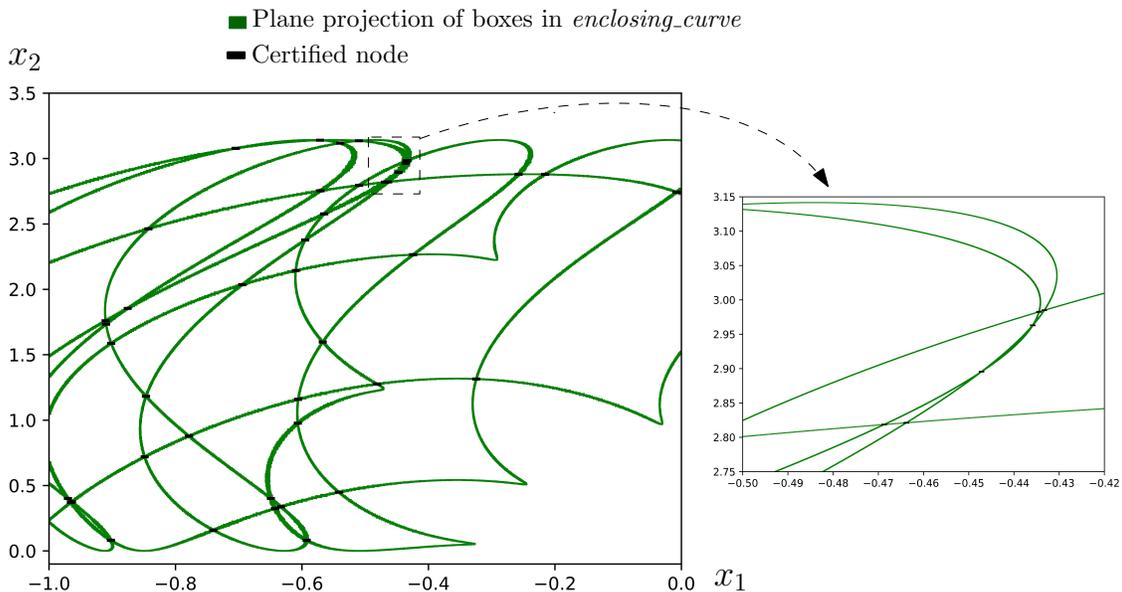


Figure 7: Experiment 2: Plane projection of an analytical curve \mathcal{C} in \mathbb{R}^4 . Each of the 43 black boxes contains a node of $\pi_{\mathcal{C}}(\mathcal{C})$ and is the projection of a box in \mathbb{R}^7 containing one solution of $\text{Ball}(P)$.

is the zero set of three polynomials of degrees 17, 15 and 13, respectively that have a unique monomial of highest degree (which is monic) and 9 other random monomials of degrees at most 2 with integer coefficients in $(-25, 25)$.

$$\begin{aligned}
 P = & [x_1^{17} - 14x_1^2 - 7x_1x_3 - 7x_2^2 - 22x_2x_4 - x_3x_4 - 19x_4^2 + 8x_1 - 14x_3 + 9, \\
 & x_2^{15} + 8x_1x_3 - 14x_1x_4 - 15x_2^2 + 16x_2x_3 + 8x_2x_4 + 2x_3^2 + 13x_4^2 + 11x_1 + 11x_2, \\
 & x_3^{13} + 17x_1^2 - 15x_1x_2 + 4x_1x_3 - 20x_1x_4 + 2x_2^2 - 10x_2x_3 + 4x_2x_4 + 20x_4^2 - 23x_2].
 \end{aligned}$$

940 Figure 8 illustrates the 7 nodes of the projection of \mathcal{C} and running times are given in Table 1.

941 Note that since P is polynomial, the implicit equation of $\pi_{\mathcal{C}}(\mathcal{C})$ can be computed using elimination theory and
 942 its singularities can then be isolated using algebraic solvers. However, the implicit equation we obtained for $\pi_{\mathcal{C}}(\mathcal{C})$
 943 is defined by an irreducible bivariate polynomial of degree 442 with 51074 monomials. Isolating the singularities
 944 of such a high-degree polynomial is then a real challenge.

945 Note also that our class of examples is rather specific and our software does not work that well if our defining
 946 polynomials are dense or even if they have non-monic high-degree monomials. However, it should be stressed
 947 that our software is a prototype and that it would be more efficient to use an interval solver specialized for the
 948 algebraic case than the versatile Ibex solver we used. Such a specialized approach has been proved successful in
 949 the 3-dimensional case by Imbach et al. [IMP18].

950 6.3.4. Experiment 4: Two close lines in \mathbb{R}^3 generating a node

951 As mentioned in the preamble of Section 6.3, the purpose of this experiment { 13 is to study the behavior of our

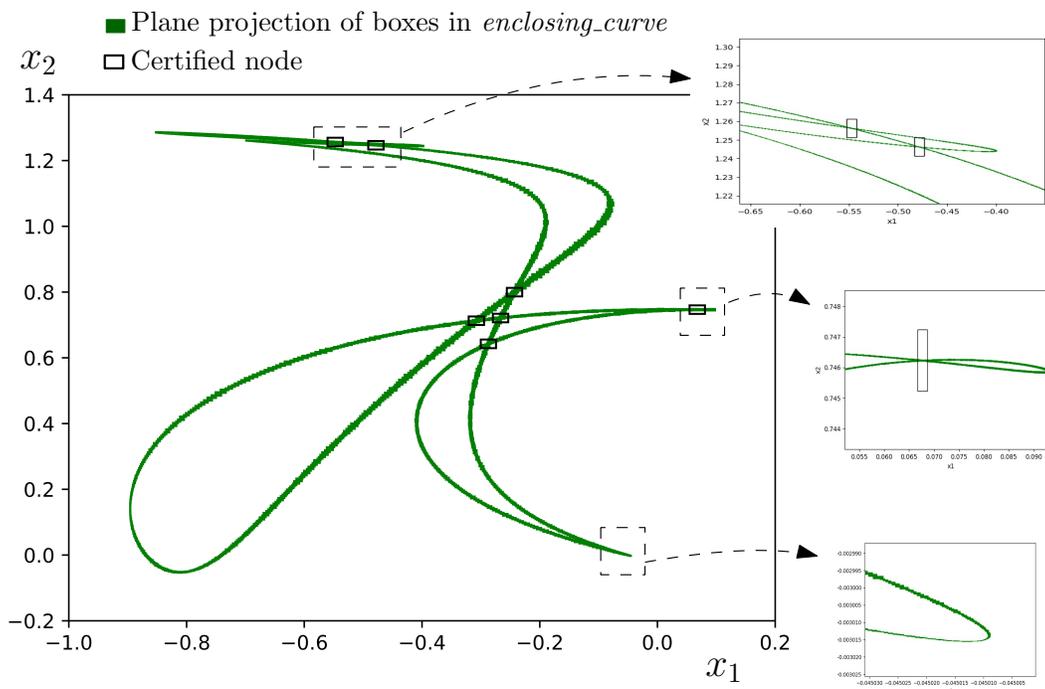


Figure 8: Experiment 3: High degree algebraic curve in \mathbb{R}^3 generating 7 nodes.

952 software when a node in \mathbb{R}^2 is induced by a pair of points (on the space curve) that are very close; namely when
 953 a node $(x_1, x_2) \in \mathbb{R}^2$ is induced by the pair of points $(x_1, x_2, y \pm r\sqrt{t}) \in \mathbb{R}^n$ with t that tends to zero. Indeed,
 954 when the equations defining the space curve are not algebraic, the Ball system contains a division by \sqrt{t} (due to
 955 the formula of $D \cdot P$), which may cause instability since t tends to zero when the distance between the pair of
 956 points tend to zero.

957 The simplest example to consider is the two skew lines $x_2 = x_1$ in the plane $x_3 = \epsilon$ and $x_2 = -x_1$ in the plane
 958 $x_3 = -\epsilon$, whose projection in the (x_1, x_2) -plane has a node at the origin, and to make ϵ vary towards 0. The pair
 959 of lines is thus defined by $[\epsilon x_2 - x_1 x_3, (x_3 - \epsilon)(x_3 + \epsilon)]$ but, in order to have non-algebraic equations, we replace
 960 x_3 by $\sin x_3$ and define our two lines by $P = [\epsilon x_2 - x_1 \sin x_3, \sin^2 x_3 - \epsilon^2]$ in the box $B = (-1, 1)^3$.

961 The goal of this experiment is to illustrate the stability of our software when ϵ varies towards 0. Recall from
 962 Section 6.1 that the D operator is evaluated on a box in two different ways depending on how close to zero is the
 963 t -interval of that box. The Ball system is thus solved either with Equation (3.1) (involving a division by \sqrt{t}) when
 964 the values of the t -interval are larger than a parameter δ_{Taylor} (set to 10^{-2}) or using a Taylor expansion otherwise.

965 To illustrate the stability of our software, we compared in Table 2 its running times when ϵ varies towards 0
 966 with what it would be without using Taylor expansions. It shows that if we do not use Taylor expansions, the
 967 solution is not certified (by the Ibexsolver; see Section 6.2) when $\epsilon \leq 10^{-5}$. On the other hand, our software
 968 is stable, although its running time increases from 0.1 to 0.8 seconds when ϵ gets smaller than or equal to 10^{-2} ,
 969 which is when the D operator starts to be evaluated using a Taylor expansion.

ϵ -values	> 1	1	0.99	0.9	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
Time ($\delta_{\text{Taylor}} = 10^{-2}$)	\mathfrak{C} is empty	\mathfrak{C} is singular	Taylor forms are not triggered			0.8					
Time without Taylor forms			0.1			0.1		0.1		0.3	
Uncertified solutions											

Table 2: Experiment 4: Performances for different values of ϵ .

7. Genericity of the assumptions

The key to prove the genericity of our assumptions is Thom's Transversality Theorem. We thus first recall, in Section 7.1, the basics of transversality theory using the notation of Demazure's book [Dem00]. We then prove, in Section 7.2, that all assumptions of Section 2 are satisfied for a generic curve. Finally, in Section 7.3, we consider the special case where the curve is the silhouette of a surface and prove that Assumptions $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_4$ are generically satisfied in this case.

7.1. Preliminaries

We work with the set of smooth functions $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ with the weak (or compact-open) topology [Dem00, §3.9.2], that is convergence is understood as uniform on compact subsets and for any derivative. A subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ is called residual if it contains the intersection of a countable family of dense open subsets. The space $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ is a Baire space [Dem00, Proposition 3.9.3], that is, every residual subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ is dense. A property is generic in $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ if it is satisfied by a residual subset.

Definition 42 ([Dem00, §3.8.3]). *Let $E \simeq \mathbb{R}^n$ and F be two finite-dimensional real vector spaces and let $r \geq 0$ be an integer. Let $P^r(E, F)$ be the vector space of polynomial functions of degree at most r from E to F . For an open subset U of E (with respect to the usual topology on E), let $J^r(U, F) = U \times P^r(E, F)$ be the space of jets of order r of functions from U to F . Notice that $J^r(U, F)$ can be identified with an open subset of \mathbb{R}^N for some positive integer N . Let $f : U \rightarrow F$ be a smooth function, the jet of order r of f is the function*

$$j^r f : U \subset \mathbb{R}^n \rightarrow J^r(U, F) \subseteq \mathbb{R}^N$$

$$x \mapsto \left(x, f(x), \frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x), \frac{\partial^2 f}{\partial x_1 \partial x_2}(x), \dots, \frac{\partial^r f}{\partial x_n^r}(x) \right).$$

Let W be a sub-manifold of $J^r(U, F)$. We say that $j^r f$ is transverse to W if for all $a \in U$ either $j^r f(a) \notin W$ or every vector of \mathbb{R}^N can be written as a sum of a vector of $T_{j^r f(a)}W$ and a vector in the image of the function $T_a j^r f$, where $T_{j^r f(a)}W$ is the tangent space of W at $j^r f(a)$ and $T_a j^r f$ is the derivative function of $j^r f$ at a .

Theorem 43 (Thom's Transversality Theorem [Dem00, Theorem 3.9.4]). *Let E and F be two finite-dimensional vector spaces with U an open set in E . Let $r \geq 0$ be an integer and W be a sub-manifold of $J^r(U, F)$. Then, the set of functions $f \in C^\infty(U, F)$ such that $j^r f$ is transverse to W is a dense residual subset of $C^\infty(U, F)$.*

988 **Proposition 44** ([Dem00, Corollary 3.7.3]). *Let U be an open subset of \mathbb{R}^n , $N \geq 1$ be an integer and W be a*
989 *sub-manifold of the vector space \mathbb{R}^N of pure co-dimension m . Assume that the smooth function $g : U \rightarrow \mathbb{R}^N$ is*
990 *transverse to W , then $g^{-1}(W)$ is a (possibly empty) sub-manifold of dimension $n - m$.*

991 The idea of the proofs of genericity of our assumptions is to express each assumption as a system of equations
992 in the jet space. When this system defines a manifold W , Proposition 44 directly applies to pull back the manifold
993 from the jet space to the ambient space of the curve. This pull back is a sub-manifold of the same co-dimension
994 as W . A difficulty occurs when the system does not define a manifold. The following corollary overcomes this
995 difficulty in the special case where the system is defined by analytic functions, in other words, the system defines
996 an analytic variety. Such a variety does not need to be a manifold, but, using the Whitney stratification theorem
997 [Whi65], the variety is written as a union of manifolds on which Thom's theorem is then applied.

998 **Corollary 45.** *Let E and F be two finite-dimensional vector spaces with E of dimension n and U an open set in*
999 *E . Let $r \geq 0$ be an integer and W be an analytic variety of $J^r(U, F)$ with co-dimension larger than n , then for a*
1000 *generic $P \in C^\infty(U, F)$, the pre-image of W under $j^r P$ is empty.*

1001 *Proof.* Let $W = \bigcup_{i=1}^m W_i$ be a Whitney stratification of W , where the W_i 's are sub-manifolds. Since $\text{codim}(W) >$
1002 n , we have that $\text{codim}(W_i) > n$ for any integer $1 \leq i \leq m$. Let $\Gamma_i = \{P \in C^\infty(U, F) \mid j^r P \text{ is transverse to } W_i\}$
1003 and $\Gamma = \bigcap_{i=1}^m \Gamma_i$. By Theorem 43, Γ_i is residual and so is Γ . Moreover, by Proposition 44, for $P \in \Gamma_i$ the pre-image
1004 of W_i under $j^r P$ is empty. Hence, $(j^r P)^{-1}(W) = \bigcup_{i=1}^m (j^r P)^{-1}(W_i) = \emptyset$. □

1005 We will also need a refined version of Thom's theorem in a multijet setting, that is for several points in the
1006 source space simultaneously. We give the formal definitions of the multijet space and function but we do not
1007 restate versions of Theorem 43, Proposition 44 and Corollary 45 that also hold for multijets.

Definition 46 ([Dem00, §3.9.6]). *Let U be an open subset of \mathbb{R}^n and $k \geq 1$ be an integer. We denote $\Delta_{(k)}(U)$*
the subset of U^k consisting of sequences (a_1, \dots, a_k) of pairwise distinct points of U . For an integer $r \geq 0$
and a finite-dimensional space F , the k -multijet space of order r , $J_{(k)}^r(U, F)$, is the subset of $J^r(U, F)^k =$
 $(U \times P^r(E, F))^k$ consisting of the k -tuples $((a_1, p_1), \dots, (a_k, p_k))$, with $(a_1, \dots, a_k) \in \Delta_{(k)}(U)$. Let $f : U \rightarrow F$
be a smooth function, the k -multijet of order r of f is the function

$$j_{(k)}^r f : \Delta_{(k)}(U) \rightarrow J_{(k)}^r(U, F)$$

$$(a_1, \dots, a_k) \mapsto (j^r f(a_1), \dots, j^r f(a_k)).$$

1008 Finally, we gather several technical tools from algebra and analysis.

1009 **Proposition 47** ([BV88, Proposition 1.A.1.1]). *Let $M(m, n)$ be the vector space of real matrices of size $m \times n$*
1010 *and r be a positive integer such that $r < \min\{n, m\}$. The determinantal variety, M_r , is the set of matrices in*
1011 *$M(m, n)$ that have rank less than $r + 1$. Then, the following statements hold:*

1012 (a) M_r is an irreducible variety in $M(m, n)$.

1013 (b) M_r is of dimension $r(n + m - r)$.

1014 (c) The singular locus of M_r is M_{r-1} .

1015 **Lemma 48** ([Bôc64, §XIV.61 Theorem 1]). Let $n \geq 2$ be an integer, $\{x_{ij}\}_{1 \leq j, i \leq n}$ be a set of n^2 variables
 1016 and $\mathbb{C}[x_{ij}]_{1 \leq j, i \leq n}$ be the ring of complex polynomials with variables $\{x_{ij}\}$. Then, the determinant of the matrix
 1017 $(x_{ij})_{1 \leq i, j \leq n}$ is an irreducible polynomial in $\mathbb{C}[x_{ij}]_{1 \leq j, i \leq n}$.

1018 **Theorem 49** ([Whi43, Theorem 1 & 2]). Let f be an even (resp. odd) smooth function, then there exists a smooth
 1019 function g such that $f(x) = g(x^2)$ (resp. $f(x) = x \cdot g(x^2)$).

1020 7.2. Genericity of the assumptions for a curve in \mathbb{R}^n

1021 We are going to prove that each assumption in Section 2 is generic. Hence, the combination of these assump-
 1022 tions is also generic since a countable intersection of residual subsets in $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ is residual.

1023 **Lemma 50.** Assumption \mathcal{A}_1 is generic.

Proof. Consider the jet of order 1 of the function $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$:

$$j^1 P : \mathbb{R}^n \rightarrow J^1(\mathbb{R}^n, \mathbb{R}^{n-1}) = \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{(n-1) \times n}$$

$$x \mapsto (x, P(x), J_P(x)) = (x, y, z).$$

1024 We represent the jet space by the variables $x \in \mathbb{R}^n, y \in \mathbb{R}^{n-1}$ and $z \in \mathbb{R}^{(n-1) \times n}$. Abusing notation, we can see
 1025 the variable z as an $(n-1) \times n$ -matrix. Define the variety $W = \{(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{(n-1) \times n} \mid y =$
 1026 $0, \text{rank}(z) \leq n-2\}$. The variety W is a product of a determinantal variety in $\mathbb{R}^{(n-1) \times n}$ of dimension $n^2 - n - 2$
 1027 (by Proposition 47) and a linear space of dimension n in $\mathbb{R}^n \times \mathbb{R}^{n-1}$. Thus, W is a variety of co-dimension $n+1$
 1028 in $\mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{(n-1) \times n}$. Hence, by Corollary 45, there exists a residual subset $\Gamma_1 \subset C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$, such that
 1029 for $P \in \Gamma_1$ the pre-image of W under $j^1 P$ is empty. Consequently, for a generic $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ and any
 1030 $q \in \overline{\mathcal{C}}$, we have that $q \notin (j^1 P)^{-1}(W) = \emptyset$, thus $\text{rank}(J_P(q)) = n-1$, which is Assumption \mathcal{A}_1 . \square

1031 **Lemma 51.** Assumption \mathcal{A}_2 is generic. Moreover, generically, the set $\overline{\mathcal{L}}_c$ is empty.

1032 *Proof.* We consider the jet of order 1 of the function $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ as in the proof of Lemma 50 with the
 1033 same notation. Define the matrix $T_1(z)$ (resp. $T_2(z)$) to be the sub-matrix of z obtained by removing the first
 1034 (resp. second) column. Consider the variety $W \subset J^1(\mathbb{R}^n, \mathbb{R}^{n-1})$ defined by $\{y = 0 \in \mathbb{R}^{n-1}, \det(T_1(z)) =$
 1035 $\det(T_2(z)) = 0\}$. Notice that $\overline{\mathcal{L}}_c$ is included in the pre-image of W under $j^1 P$ since $\overline{\mathcal{L}}_c$ is the set of points of
 1036 the curve $\overline{\mathcal{C}}$ that are both x_1 and x_2 -critical. By Lemma 48, we have that both $\det(T_1(z))$ and $\det(T_2(z))$ are
 1037 irreducible polynomials. By [CLO92, §9.4 Prop 10], a proper sub-variety of an irreducible variety is of lower
 1038 dimension, we deduce that the common zero locus of $\det(T_1(z))$ and $\det(T_2(z))$ is of co-dimension at least two.

1039 We deduce that $\text{codim}(W) > n$. By Corollary 45, there exists a residual subset $\Gamma_2 \subset C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$, such that
 1040 for $P \in \Gamma_2 \cap \Gamma_1$, the pre-image of W under j^1P is empty and hence $\overline{\mathfrak{L}_c}$ is empty, which implies Assumption
 1041 \mathcal{A}_2 . □

1042 **Lemma 52.** *Assumption \mathcal{A}_3 is generic.*

Proof. Let us consider the 3-multijet of order 0:

$$j_{(3)}^0 P : \Delta_{(3)}(\mathbb{R}^n) \rightarrow J_{(3)}^0(\mathbb{R}^n, \mathbb{R}^{n-1}) = (\mathbb{R}^n \times \mathbb{R}^{n-1})^3$$

$$(x, x', x'') \mapsto ((x, P(x)), (x', P(x')), (x'', P(x''))) = ((x, y), (x', y'), (x'', y''))$$

1043 where every element in the jet space $J_{(3)}^0(\mathbb{R}^n, \mathbb{R}^{n-1})$ is of the form $((x, y), (x', y'), (x'', y''))$, where $x =$
 1044 (x_1, \dots, x_n) , $x', x'' \in \mathbb{R}^n$ and $y, y', y'' \in \mathbb{R}^{n-1}$. Consider the linear sub-manifold $W = \{x_1 = x'_1 = x''_1, x_2 =$
 1045 $x'_2 = x''_2, y = y' = y'' = 0\}$, the co-dimension of W is thus $3n + 1$ which is larger than the dimension of the
 1046 source space $\Delta_{(3)}(\mathbb{R}^n)$ which is $3n$. Thus, by Corollary 45, there exists a residual subset $\Gamma_3 \subset C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$,
 1047 such that for $P \in \Gamma_3$, the pre-image of W by $j_{(3)}^0$ is empty, which translates to the fact that there are no pairwise
 1048 distinct points q, q', q'' in $\overline{\mathfrak{C}}$ such that $\pi_{\overline{\mathfrak{C}}}(q) = \pi_{\overline{\mathfrak{C}}}(q') = \pi_{\overline{\mathfrak{C}}}(q'')$. This is also equivalent to saying that the system
 1049 $S = \{P(x) = 0 \in \mathbb{R}^{n-1}, x_1 - \alpha = x_2 - \beta = 0\}$ has at most two distinct solutions (without counting multiplicities)
 1050 for any $(\alpha, \beta) \in \mathbb{R}^2$.

1051 Using Γ_1, Γ_2 as defined in the proofs of Lemmas 50 & 51 and Γ_3 defined above, we define $\Gamma_4 = \Gamma_1 \cap \Gamma_2 \cap \Gamma_3$
 1052 which is thus a residual set and let P be in Γ_4 . Since P is in Γ_3 , the system S has at most two distinct solutions. In
 1053 addition, since P is in $\Gamma_2 \cap \Gamma_1$, one has that $\overline{\mathfrak{L}_c}$ is empty and finally together with Lemma 20, since P is in Γ_1 , this
 1054 implies that these solutions have multiplicity exactly 1 in S . For P in the residual set Γ_4 , the number of solutions
 1055 counted with multiplicities of S is thus at most 2, which is Assumption \mathcal{A}_3 . □

1056 **Lemma 53.** *Assumption \mathcal{A}_4 is generic.*

Proof. Let us consider the 2-multijet of order 0 of P :

$$j_{(2)}^0 P : \Delta_{(2)}(\mathbb{R}^n) \rightarrow J_{(2)}^0(\mathbb{R}^n, \mathbb{R}^{n-1}) = (\mathbb{R}^n \times \mathbb{R}^{n-1})^2$$

$$(x, x') \mapsto ((x, P(x)), (x', P(x'))) = ((x, y), (x', y'))$$

1057 where every element in the jet space $J_{(2)}^0(\mathbb{R}^n, \mathbb{R}^{n-1})$ is of the form $((x, y), (x', y'))$, where $x = (x_1, \dots, x_n)$,
 1058 $x' \in \mathbb{R}^n$ and $y, y' \in \mathbb{R}^{n-1}$. Consider the linear sub-manifold $W = \{x_1 = x'_1, x_2 = x'_2, y = y' = 0\}$ of the jet
 1059 space $J_{(2)}^0(\mathbb{R}^n, \mathbb{R}^{n-1})$. Notice that, $(j_{(2)}^0 P)^{-1}(W)$ contains the set $\widehat{\mathfrak{L}}'_n = \{(q_1, q_2) \in \Delta_{(2)}(\mathbb{R}^n) \cap \overline{\mathfrak{C}} \times \overline{\mathfrak{C}} \mid \pi_{\overline{\mathfrak{C}}}(q_1) =$
 1060 $\pi_{\overline{\mathfrak{C}}}(q_2)\}$ and $\overline{\mathfrak{L}_n}$ is the image of $\widehat{\mathfrak{L}}'_n$ by the projection $(q_1, q_2) \rightarrow q_1$. We have $\dim(\Delta_{(2)}(\mathbb{R}^n)) = 2n$ and, since W
 1061 is linear, its co-dimension is easily computed $\text{codim}(W) = 2(2n - 1) - (2 + 2(n - 1)) = 2n$. Proposition 44
 1062 thus yields that generically $(j_{(2)}^0 P)^{-1}(W)$ is a sub-manifold of dimension zero that is a discrete set in \mathbb{R}^n , and so
 1063 is $\overline{\mathfrak{L}_n}$.

1064 Now, we prove that, generically, $\overline{\mathfrak{L}}_n$ does not intersect the boundary of B . The boundary ∂B of the box B is
1065 included in the union of the supporting hyperplanes H_i of its 2^n faces of dimension $n - 1$, that is $\partial B = \bigcup_{i=1}^{2^n} H_i$.
1066 Define the linear sub-manifold $W_i = \{((x, y), (x', y')) \in W \mid x \in H_i \text{ or } x' \in H_i\}$, notice that this adds one
1067 equation to W and thus increases the co-dimension of W by one, thus $\text{codim}(W_i) = 2n + 1$. By Corollary 45, we
1068 have that generically, the pre-image of W_i under $j_{(2)}^0 P$ is empty, which translates to the fact that there is no point
1069 of $\overline{\mathfrak{L}}_n$ on $\partial B \cap H_i$. This is also true for any i and thus, generically, $\overline{\mathfrak{L}}_n$ does not intersect the boundary of B . \square

1070 **Corollary 54.** *Assumption \mathcal{A}_5 is generic.*

1071 *Proof.* Let B be an open n -box. We prove that for a generic $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$, the singular points of $\pi_{\mathfrak{C}}(\mathfrak{C})$ are
1072 only nodes (recall that by Lemma 26, under the generic assumptions $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and \mathcal{A}_4 , the points in $\mathfrak{C} \setminus (\mathfrak{L}_c \cup \mathfrak{L}_n)$
1073 project to smooth points).

Let Γ_0 be the set of $P \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ such that P satisfies Assumptions $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and \mathcal{A}_4 . The previous lemmas of this section show that Γ_0 is residual in $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$. Let us consider the 2-multijet of order 1 of P :

$$j_{(2)}^1 P : \Delta_{(2)}(\mathbb{R}^n) \rightarrow J_{(2)}^1(\mathbb{R}^n, \mathbb{R}^{n-1}) \subseteq (\mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{(n-1) \times n})^2$$

$$(x, x') \mapsto ((x, P(x), J_P(x)), (x', P(x'), J_P(x'))) = ((x, y, z), (x', y', z'))$$

Let s, s' (resp. r, r') be the sub-matrices of z, z' , respectively, obtained by removing the first two columns (resp. obtained by the first two columns). Define the matrix $M = \begin{pmatrix} r & 0 & s \\ r' & s' & 0 \end{pmatrix}$ and the variety

$$W = \{((x, y, z), (x', y', z')) \in (\mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{(n-1) \times n})^2 \mid y = y' = 0, x_1 = x'_1, x_2 = x'_2, \det(M) = 0\}.$$

1074 The variety W is a product of a determinantal variety and a linear space, thus its co-dimension is $\text{codim}(W) \geq$
1075 $2n + 1 > 2n = \dim(\Delta_{(2)}(\mathbb{R}^n))$. Hence, by Corollary 45, there exists a residual subset Γ'_0 in $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ such
1076 that for all $P \in \Gamma'_0$, the pre-image of W under $j_{(2)}^1 P$ is empty.

1077 Let P be in the residual set $\Gamma_0 \cap \Gamma'_0$. By Lemma 31 and since \mathfrak{L}_c is empty, we deduce that for distinct
1078 $q_1, q_2 \in \mathfrak{C}$ with $\pi_{\mathfrak{C}}(q_1) = \pi_{\mathfrak{C}}(q_2)$, the plane projections of the lines $T_{q_1} \mathfrak{C}$ and $T_{q_2} \mathfrak{C}$ intersect transversely if and
1079 only if $j_{(2)}^1((q_1, q_2)) \notin W$. Finally, by Lemma 21 (Step (a) of the proof), we deduce that $\pi_{\mathfrak{C}}(q_1) = \pi_{\mathfrak{C}}(q_2)$ is a
1080 node in $\pi_{\mathfrak{C}}(\mathfrak{C})$. \square

1081 7.3. Genericity of the assumptions for the silhouette of a surface in \mathbb{R}^n

1082 In this section, we focus on the special case of silhouette curves of surfaces in \mathbb{R}^n . For an open n -box B and \tilde{P}
1083 in $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$ such that $S = \tilde{P}^{-1}(0)$ is a smooth 2-sub-manifold in \mathbb{R}^n , the silhouette of \tilde{P} is the set of points
1084 q of this surface S such that the projection (with respect to a fixed direction) of the tangent plane $T_q S$ to \mathbb{R}^2 is not
1085 surjective. We prove that Assumptions $\mathcal{A}_1, \mathcal{A}_2$ & \mathcal{A}_4 are satisfied for a generic silhouette, and we only conjecture
1086 that Assumptions \mathcal{A}_3 & \mathcal{A}_5^- also hold generically. We start by formalizing the definition of the silhouette curve
1087 algebraically.

1088 **Definition 55.** For an integer $n \geq 3$, let $\tilde{P} = (P_1, \dots, P_{n-2}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$. Define the smooth function
 1089 $P_{n-1} = \det \left(\left(\frac{\partial P_i}{\partial x_j} \right)_{\substack{1 \leq i \leq n-2 \\ 3 \leq j \leq n}} \right)$ and $P = (P_1, \dots, P_{n-1})$. We define the curve \mathfrak{C} (and $\bar{\mathfrak{C}}$) as in Section 2 and call
 1090 it the silhouette of \tilde{P} .

1091 **Proposition 56.** For a generic $\tilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$, the function P satisfies Assumption \mathcal{A}_1 .

Proof. Consider the jet of order 1 of \tilde{P} :

$$j^1 \tilde{P} : \mathbb{R}^n \rightarrow J^1(\mathbb{R}^n, \mathbb{R}^{n-2}) = \mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}^{(n-2) \times n} \simeq \mathbb{R}^{n^2-2} = \mathbb{R}^N$$

$$x \mapsto (x, \tilde{P}(x), J_{\tilde{P}}(x)) = (x, y, z).$$

We represent the jet space by the vectors $x \in \mathbb{R}^n, y \in \mathbb{R}^{n-2}$ and the $((n-2) \times n)$ -matrix $z \in \mathbb{R}^{(n-2) \times n}$. Let $T(z)$ denote the sub-matrix obtained by removing the first two columns of z . Define the variety $W = \{y = 0, \det(T(z)) = 0\} = \{y = 0, \text{rank}(T(z)) \leq n-3\}$ in \mathbb{R}^N . According to Proposition 47, $W = \text{Reg}(W) \cup \text{Sing}(W)$ where $\text{Reg}(W)$ (resp. $\text{Sing}(W)$) is the set of smooth (resp. singular) points in W and

$$\text{Reg}(W) = \{(x, y, z) \in \mathbb{R}^N \mid y = 0, \text{rank}(T(z)) = n-3\}$$

$$\text{Sing}(W) = \{(x, y, z) \in \mathbb{R}^N \mid y = 0, \text{rank}(T(z)) < n-3\}.$$

1092 In addition, Proposition 47 yields that $\text{Reg}(W)$ is a manifold of co-dimension $n-1$ and $\text{Sing}(W)$ is a variety
 1093 of co-dimension $n+2$. Since the co-dimension of $\text{Sing}(W)$ is larger than that of the source space, Corollary 45
 1094 implies that, generically, $(j^1 \tilde{P})^{-1}(\text{Sing}(W)) = \emptyset$. One thus has $(j^1 \tilde{P})^{-1}(W) = (j^1 \tilde{P})^{-1}(\text{Reg}(W))$.

Consider the function

$$\varphi : \mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}^{(n-2) \times n} \rightarrow \mathbb{R}^{n-2} \times \mathbb{R}$$

$$\chi = (x, y, z) \mapsto (y, \det(T(z))),$$

1095 such that $\varphi^{-1}(0) = W$. Its Jacobian matrix is $J_\varphi = \begin{pmatrix} 0_{(n-2) \times n} & I_{(n-2) \times (n-2)} & 0_{(n-2) \times (n-2)n} \\ 0_{1 \times (n)} & 0_{1 \times (n-2)} & v(z) \end{pmatrix}$, where $0_{k_1 \times k_2}$
 1096 (resp. $I_{k_1 \times k_2}$) is the zero (resp. identity) matrix of size $k_1 \times k_2$ and the vector $v(z)$ is the adjugate matrix of $T(z)$
 1097 written as the concatenation of its lines: $v(z) = (\text{Adj}^{ij}(T(z)))_{\substack{1 \leq i \leq n-2 \\ 3 \leq j \leq n}} \in \mathbb{R}^{(n-2)^2}$. Let $\chi = (x, y, z) \in \text{Reg}(W)$,
 1098 then $\text{rank}(T(z)) = n-3$, thus there exists a pair (i, j) such that $\text{Adj}^{ij}(T(z)) \neq 0$. Hence, the vector $v(z)$ is
 1099 non-trivial and $J_\varphi(\chi)$ has full rank $n-1$. The function φ is thus a submersion on $\text{Reg}(W)$.

1100 Theorem 43 yields that, generically, $j^1 \tilde{P}$ is transverse to the manifold $\text{Reg}(W)$. Together with the fact that
 1101 φ is a submersion on $\text{Reg}(W)$, [GG73, Lemma II.4.3 (p.52)] implies that $P = \varphi \circ j^1 \tilde{P}$ is a submersion on
 1102 $(j^1 \tilde{P})^{-1}(\text{Reg}(W)) = (j^1 \tilde{P})^{-1}(W) = (j^1 \tilde{P})^{-1}(\varphi^{-1}(0)) = (\varphi \circ j^1 \tilde{P})^{-1}(0) = P^{-1}(0) = \mathfrak{C}$. In other words,
 1103 J_P has full rank $n-1$ on \mathfrak{C} , which is Assumption \mathcal{A}_1 . \square

1104 **Proposition 57.** For a generic $\tilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$, the function P satisfies Assumption \mathcal{A}_2 .

1105 *Proof.* First we prove that, generically, $\overline{\mathfrak{L}}_c$ is discrete. For any $\tilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$ consider $j^2\tilde{P} : \mathbb{R}^n \rightarrow$
1106 $J^2(\mathbb{R}^n, \mathbb{R}^{n-2}) \subset \mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}^{(n-2) \times n} \times \mathbb{R}^{n^2(n-2)} = \mathbb{R}^N$. Assume that every element in \mathbb{R}^N is represented
1107 as (x, y, z, h) , where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^{n-2}$, $z \in \mathbb{R}^{(n-2) \times n}$ and $h \in \mathbb{R}^{n^2(n-2)}$. Abusing notation, we consider
1108 z as a $((n-2) \times n)$ -matrix. Let $T(z)$ denote the matrix obtained by removing the first two columns of z .
1109 The Jacobian matrix J_P is a function of the derivatives $(\frac{\partial P_i}{\partial x_j}, \frac{\partial^2 P_i}{\partial x_k \partial x_s})_{\substack{1 \leq i, l \leq n-2 \\ 1 \leq j, k, s \leq n}}$, it can thus be seen in the jet
1110 space as a function of z and h , $J_P(z, h)$. Define the matrix $T_1(z, h)$ (resp. $T_2(z, h)$) to be the sub-matrix of
1111 $J_P(z, h)$ obtained by removing the first (resp. second) column. Define the variety $W = \{(x, y, z, h) \mid y = 0 \in$
1112 $\mathbb{R}^{n-2}, \det(T(z)) = \det(T_1(z, h)) = \det(T_2(z, h)) = 0\}$, so that $\overline{\mathfrak{L}}_c$ is included in the pre-image of W under
1113 $j^2\tilde{P}$. Let $W_1 = \{(x, y, z, h) \mid y = 0 \in \mathbb{R}^{n-2}, \det(T(z)) = 0\}$, we already showed in the proof of Proposition 56
1114 that W_1 is an irreducible variety of co-dimension $n - 1$. In addition, $\det(T_1(z, h))$ does not identically vanish
1115 on W_1 , thus W is a proper sub-variety of the irreducible variety W_1 and [CLO92, §9.4 Prop 10] implies that
1116 $\text{codim}(W) > \text{codim}(W_1) = n - 1$.

1117 Now, write $W = \text{Reg}(W) \cup \text{Sing}(W)$, where $\text{Reg}(W)$ (resp. $\text{Sing}(W)$) is the set of smooth (resp. sin-
1118 gular) points in W . Recall that $\text{codim}(\text{Sing}(W)) > n$ since $\text{Sing}(W)$ is a proper closed sub-variety of W
1119 [BCR98, Proposition 3.3.14]. By Corollary 45, there exists a residual set $\Gamma' \subset C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$ such that
1120 if $\tilde{P} \in \Gamma'$, then the pre-image of $\text{Sing}(W)$ under $j^2\tilde{P}$ is empty. Define $\Gamma = \{\tilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2}) \mid$
1121 $j^2\tilde{P}$ is transverse to $\text{Reg}(W)\} \cap \Gamma'$. Notice that if $\tilde{P} \in \Gamma$, then $\overline{\mathfrak{L}}_c$ is included in the pre-image of $\text{Reg}(W)$ under
1122 $j^2\tilde{P}$. Hence, since $\text{codim}(\text{Reg}(W)) = \text{codim}(W) \geq n$, we have by Proposition 44 that $\overline{\mathfrak{L}}_c$ is a sub-manifold of
1123 dimension, at most, zero. Thus, $\overline{\mathfrak{L}}_c$ is discrete for all $\tilde{P} \in \Gamma$. Using Theorem 43 we deduce that Γ is residual.

1124 The proof that $\overline{\mathfrak{L}}_c$ does not intersect the boundary of B can be done analogously as in the proof of Lemma 53.
1125 □

1126 **Proposition 58.** *For a generic $\tilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$, the function P satisfies Assumption \mathcal{A}_4 .*

1127 *Proof.* Consider the 2-multijet $j_{(2)}^1\tilde{P} : \Delta_{(2)}(\mathbb{R}^n) \rightarrow J_{(2)}^1(\mathbb{R}^n, \mathbb{R}^{n-2}) = (\mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}^{(n-2) \times n})^2$ of the function
1128 $\tilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$, where $(\mathbb{R}^n \times \mathbb{R}^{(n-2) \times n} \times \mathbb{R}^{(n-2) \times n})^2$ is described by the coordinates $x, x' \in \mathbb{R}^n, y, y' \in$
1129 \mathbb{R}^{n-2} and $z, z' \in \mathbb{R}^{(n-2) \times n}$. Abusing notation, we consider z and z' as $((n-2) \times n)$ -matrices. Let $T(z)$ (resp.
1130 $T(z')$) denote the matrix obtained by removing the first two columns of z (resp. z'). Define the variety W to be
1131 the solution set of the system $\{y = y' = 0, x_1 - x'_1 = x_2 - x'_2 = \det(T(z)) = \det(T(z')) = 0\}$. Denote the
1132 regular part of W by $\text{Reg}(W)$. By Proposition 47 (a) we deduce that W is of co-dimension $2n$. Using the same
1133 argument in the proof of Proposition 56, we deduce that there exists a residual set $\Gamma \subset C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$ such that
1134 if $\tilde{P} \in \Gamma$, then the image of $\Delta_2(\mathbb{R}^n)$ under $j_{(2)}^1\tilde{P}$ is contained in $\text{Reg}(W)$. Moreover, by Proposition 44, we have
1135 that $M_P = (j_{(2)}^1\tilde{P})^{-1}(\text{Reg}(W)) = (j_{(2)}^1\tilde{P})^{-1}(W)$ is a sub-manifold of dimension zero in $\Delta_2(\mathbb{R}^n)$. Notice that
1136 $\overline{\mathfrak{L}}_n$ is the image of M_P under the projection $(x, x') \rightarrow x$. Since M_P is of dimension zero, then so is $\overline{\mathfrak{L}}_n$. Thus we
1137 have just proven that, if $\tilde{P} \in \Gamma$, then $\overline{\mathfrak{L}}_n$ is a sub-manifold of dimension zero. Hence, $\overline{\mathfrak{L}}_n$ is discrete.

1138 The proof that $\overline{\mathfrak{L}}_n$ does not intersect the boundary of B can be done analogously as in the proof of Lemma 53.
1139 □

1140 Assumption \mathcal{A}_3 can be rephrased by the three following assumptions:

1141 $\mathcal{A}_{3(a)}$ - There are no pairwise distinct $q, q', q'' \in \bar{\mathcal{C}}$ such that $\pi_{\mathcal{C}}(q) = \pi_{\mathcal{C}}(q') = \pi_{\mathcal{C}}(q'')$.

1142 $\mathcal{A}_{3(b)}$ - $\bar{\mathcal{L}}_{\mathcal{C}} \cap \bar{\mathcal{L}}_n = \emptyset$.

1143 $\mathcal{A}_{3(c)}$ - For $q \in \bar{\mathcal{L}}_{\mathcal{C}}$, the multiplicity of the system $\{P(x) = 0 \in \mathbb{R}^{n-1}, (x_1, x_2) = \pi_{\mathcal{C}}(q)\}$ at q is exactly two.

1144 Using this rephrasing, we next show that Assumptions $\mathcal{A}_{3(a)}$ & $\mathcal{A}_{3(b)}$ generically hold and we leave the genericity
1145 of Assumption $\mathcal{A}_{3(c)}$ as a conjecture.

1146 **Proposition 59.** *For a generic function $\tilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$, Assumption $\mathcal{A}_{3(a)}$ holds.*

1147 *Proof.* Consider the 3-multijet $j_{(3)}^1 \tilde{P} : \Delta_{(3)}(\mathbb{R}^n) \rightarrow J_{(3)}^1(\mathbb{R}^n, \mathbb{R}^{n-2}) = (\mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}^{(n-2) \times n})^3$. Assume that
1148 every element in $(\mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}^{(n-2) \times n})^3$ is of the form $((x, y, z), (x', y', z'), (x'', y'', z''))$, where $x, x', x'' \in \mathbb{R}^n$,
1149 $y, y', y'' \in \mathbb{R}^{n-2}$ and $z, z', z'' \in \mathbb{R}^{(n-2) \times n}$. Abusing notation, we consider z, z' and z'' as $((n-2) \times n)$ -matrices.
1150 Let $T(z), T(z'), T(z'')$ denote the matrices obtained by removing the first two columns of z, z', z'' , respectively.
1151 Consider the variety W defined by the equations: $\{x_1 = x'_1 = x''_1, x_2 = x'_2 = x''_2, y = y' = y'' = 0 \in$
1152 $\mathbb{R}^{n-2}, \det(T(z)) = \det(T(z')) = \det(T(z'')) = 0\}$.

1153 Notice that $\dim(\Delta_{(3)}(\mathbb{R}^n)) = 3n < 3n + 1 = \text{codim}(W)$. Hence, by Corollary 45, we have that, generically,
1154 the pre-image of W under $j_{(3)}^1 \tilde{P}$ is empty. Hence, there are no pairwise different $q, q', q'' \in \bar{\mathcal{C}}$ such that $\pi_{\mathcal{C}}(q) =$
1155 $\pi_{\mathcal{C}}(q') = \pi_{\mathcal{C}}(q'')$. \square

1156 **Proposition 60.** *For a generic function $\tilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$, Assumption $\mathcal{A}_{3(b)}$ holds.*

1157 *Proof.* Consider the 2-multijet $j_{(2)}^2 \tilde{P} : \Delta_{(2)}(\mathbb{R}^n) \rightarrow J_{(2)}^2(\mathbb{R}^n, \mathbb{R}^{n-2}) = (\mathbb{R}^n \times \mathbb{R}^{n-2} \times \mathbb{R}^{(n-2) \times n} \times \mathbb{R}^{n^2(n-2)})^2$
1158 of the function $\tilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$, where $(\mathbb{R}^n \times \mathbb{R}^{(n-2) \times n} \times \mathbb{R}^{(n-2) \times n} \times \mathbb{R}^{n^2(n-2)})^2$ is described by the
1159 coordinates $x, x' \in \mathbb{R}^n, y, y' \in \mathbb{R}^{n-2}, z, z' \in \mathbb{R}^{(n-2) \times n}$ and $h, h' \in \mathbb{R}^{n^2(n-2)}$. With abuse of notation we can
1160 consider z and z' as $((n-2) \times n)$ -matrices. Let $T(z)$ (resp. $T(z')$) denote the matrix obtained by removing the
1161 first two columns of z (resp. z'). Define the matrices $T_1(z, h), T_2(z, h)$ as in the proof of Lemma 57 and the variety
1162 W to be the solution set of the system $\{y = y' = 0 \in \mathbb{R}^{n-2}, x_1 - x'_1 = x_2 - x'_2 = \det(T(z)) = \det(T(z')) = 0,$
1163 $\det(T_1(z, h)) = \det(T_2(z, h)) = 0\}$.

1164 Define varieties $W' = \{(x, y, z, h) \mid y = y' = 0, \det(T(z)) = \det(T(z')) = 0, x_1 = x'_1, x_2 = x'_2\}$ and
1165 $W'' = \{(x, y, z, h) \mid y = y' = 0, \det(T_1(z, h)) = \det(T_2(z, h)) = 0\}$. Notice that $W = W' \cap W''$. Moreover,
1166 we can find a smooth silhouette curve C that is not an orthogonal line to the (x_1, x_2) -plane and that contains two
1167 distinct points q, q' , with $\pi_{\mathcal{C}}(q) = \pi_{\mathcal{C}}(q')$ such that the projection of $T_q C$ (resp. $T_{q'} C$) onto \mathbb{R}^2 is injective. Notice
1168 that $j_{(2)}^2 \tilde{P}(q, q') \in W' \setminus W''$. Hence, $W' \not\subseteq W''$. Moreover, since W' is the Cartesian product of determinant
1169 varieties (which are irreducible by Proposition 47(a)) with linear spaces, we have that W' is also irreducible
1170 [BCR98, Theorem 2.8.3 (iii)]. In other words, $W = W' \cap W''$ is a proper sub-variety of the irreducible variety
1171 W' . Hence, $\dim(W) = \dim(W' \cap W'') < \dim(W')$, equivalently, $\text{codim}(W) > \text{codim}(W') = 2n$. Hence,

1172 by Corollary 45 we have that, generically, the pre-image of W under $j_{(2)}^2 \tilde{P}$ is empty. Since, by Proposition 56,
 1173 Assumption \mathcal{A}_1 (which is necessary to guarantee that \mathcal{L}'_c is well-defined) is also generic, we imply that, generically,
 1174 there is no distinct pair $q, q' \in \mathcal{C}$ such that $\pi_{\mathcal{C}}(q) = \pi_{\mathcal{C}}(q')$ and $q \in \overline{\mathcal{L}}_c$, equivalently, $\mathcal{L}'_c \cap \mathcal{L}'_n = \emptyset$ which proves
 1175 the proposition. \square

1176 We thus proved the following proposition that the silhouette of a generic surface in \mathbb{R}^n satisfies all assumptions
 1177 except for Assumptions $\mathcal{A}_{3(c)}$ and \mathcal{A}_{5-} . However, based on previous results with three variables [IMP16b], we
 1178 formulate the following conjecture.

1179 **Proposition 61.** *For a generic function $\tilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$, Assumptions $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_{3(a)}, \mathcal{A}_{3(b)}, \mathcal{A}_4$, hold.*

1180 **Conjecture 62.** *For a generic function $\tilde{P} \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-2})$, Assumptions $\mathcal{A}_{3(c)}$ and \mathcal{A}_{5-} hold.*

1181 8. Conclusion

1182 We proposed a regular square system that encodes the singularities of the plane projection of a curve in \mathbb{R}^n
 1183 under some assumptions, which we proved to be generic via transversality theory. For the case of plane projections
 1184 of silhouette curves, we proved the genericity of only some of the assumptions and we conjecture the genericity
 1185 of the others (Proposition 61 and Conjecture 62). We provided semi-algorithms that check whether a given curve
 1186 satisfies our assumptions and, if they terminate, output isolating boxes of the singularities or the plane projection
 1187 (possibly with spurious boxes under some weak assumptions). The drawback of our approach is that the number
 1188 of variables is doubled, which is an issue for subdivision methods that are exponential in the dimension. We
 1189 partially overcame this issue by applying subdivision schemes in the space of doubled dimension only locally in
 1190 the neighborhood of the points that project onto singularities (Section 6.1).

1191 A natural open question is the complexity of our semi-algorithms. It is worth noticing that our semi-algorithms
 1192 use a subdivision approach with *diameter distance tests* as defined by Burr et al. [BGT20]. As such, it should be
 1193 possible to study our complexities using the method of *continuous amortization*. This should yield explicit bounds
 1194 for the case of polynomial input either in the worst case (as in [BGT20]), or in a smoothed complexity setting (as
 1195 in [CETC19]).

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Symbol	Description (see Section 2 unless specified otherwise)
$C^\infty(\mathbb{R}^n, \mathbb{R}^k)$	Class of functions from \mathbb{R}^n to \mathbb{R}^k that are differentiable infinitely many times
$P = (P_1, \dots, P_{n-1})$	A function in $C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$
\mathfrak{C}	Curve defined by $P = 0$ in an open box B in \mathbb{R}^n
$\pi_{\mathfrak{C}}$	Projection of \mathfrak{C} onto the (x_1, x_2) -plane
\mathfrak{L}_c	Set of points in \mathfrak{C} where the tangent line is orthogonal to the (x_1, x_2) -plane
\mathfrak{L}_n	Set of points q in \mathfrak{C} such that the cardinality of the pre-image of $\pi_{\mathfrak{C}}(q)$ is at least two without counting multiplicities
$\widehat{\mathfrak{L}}_n$	Set of $(q_1, q_2) \in \mathfrak{L}_n^2$ such that $q_1 \neq q_2$ and $\pi_{\mathfrak{C}}(q_1) = \pi_{\mathfrak{C}}(q_2)$ (Def. 12)
$\widehat{\mathfrak{L}}_c$	Set of (q_1, q_1) with $q_1 \in \mathfrak{L}_c$ (Def. 12)
$\widehat{\mathfrak{L}}$	$\widehat{\mathfrak{L}}_n \cup \widehat{\mathfrak{L}}_c$ (Def. 12)
$T_q\mathfrak{C}$	Line tangent to \mathfrak{C} at q
$\overline{\mathfrak{C}}$	Topological closure of \mathfrak{C}
$\pi_{\overline{\mathfrak{C}}}, \overline{\mathfrak{L}}_c, \overline{\mathfrak{L}}_n, T_q\overline{\mathfrak{C}}$	Analogous as $\pi_{\mathfrak{C}}, \mathfrak{L}_c, \mathfrak{L}_n, T_q\mathfrak{C}$ with respect to $\overline{\mathfrak{C}}$ instead of \mathfrak{C}
$S \cdot f(x_1, x_2, y, r, t)$	Operator on f : $\frac{1}{2}[f(x_1, x_2, y + r\sqrt{t}) + f(x_1, x_2, y - r\sqrt{t})]$ for $t \geq 0$ (Def. 9)
$D \cdot f(x_1, x_2, y, r, t)$	Operator on f : $\frac{1}{2\sqrt{t}}[f(x_1, x_2, y + r\sqrt{t}) - f(x_1, x_2, y - r\sqrt{t})]$ for $t > 0$, and $\nabla f(x_1, x_2, y) \cdot (0, 0, r)$, otherwise (Def. 9)
Ball(P)	Ball system (Thm. 11)
Ω_P	Maps a solution (x_1, x_2, y, r, t) of Ball(P) to the pair of points $(x_1, x_2, y \pm r\sqrt{t})$ in $\widehat{\mathfrak{L}}$ (Def. 14 & Fig. 3)
$j^r f, J^r(\mathbb{R}^n, \mathbb{R}^k)$	Jet of order r of f and space of jets of order r of functions from \mathbb{R}^n to \mathbb{R}^k (Def. 42)
$j_{(k)}^r f, J_{(k)}^r(\mathbb{R}^n, \mathbb{R}^k)$	Multijet and space of multijets (Def. 46)

Table 3: Table of the main symbols used throughout this paper.

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