# A Robust Nonlinear Model Reference Adaptive Control for Disturbed Linear Systems: An LMI Approach

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Roberto Franco[†], Héctor Ríos[†,⋆], Alejandra Ferreira de Loza[‡,⋆] and Denis Efimov[§,∗]

***Abstract*—In this paper a robust nonlinear Model Reference Adaptive Control (MRAC) is proposed for disturbed linear systems, *i.e.,* linear systems with parameter uncertainties, and external time-dependent perturbations or nonlinear unmodeled dynamics matched with the control input. The proposed nonlinear control law is composed of two nonlinear adaptive gains. Such adaptive gains allow the control to counteract the effects of some perturbations and nonlinear unmodeled dynamics ensuring asymptotic convergence of the tracking error to zero, and the boundedness of the adaptive gains. The nonlinear controller synthesis is given by a constructive method based on the solution of Linear Matrix Inequalities (LMIs). Besides, the simulation results show that, due to the nonlinearities, the rate of convergence of the proposed algorithm is faster than the provided by a classic MRAC.**

***Index Terms*—Uncertain Linear Systems, Model Reference Adaptive Control, Robust Control, Nonlinear Control.**

## I. Introduction

In the last decades, the adaptive control theory has received a great deal of attention. The MRAC is one of the most popular approaches in such a field. This approach uses a stable model reference, and the objective is to design specific control parameters such that the system dynamics behaves like the model reference (see [1]).

In the context of MRAC several approaches have been proposed. In [2], the Mazenc construction method is used to design strict Lyapunov functions for MRAC. Furthermore, a Lyapunov function for a passivity-based adaptive controller of Lagrangian systems is developed, which provides a uniform asymptotic convergence to zero of the regulation error. In [3], it is shown that under the same standard MRAC conditions, without the knowledge of the plant parameters, the tracking error has a stronger higher-order convergence property. In [4], a combined MRAC for unknown multiple-input-multiple-output linear time-invariant systems with guaranteed parameter convergence is developed. This approach ensures exponential decay of the tracking error, as well as plant and control-parameter estimation errors to zero, with imposing a significantly milder initial excitation condition. In [5], a MRAC approach improves the transient performance, and restrains high-frequency oscillation of the control signal, without modifying the selected model reference guaranteeing asymptotic convergence of the tracking error to zero. In [6], a parameter estimation method is proposed in the framework of composite MRAC for improved parameter convergence. The convergence to zero of both tracking and parameter identification errors can be guaranteed without persistent excitation. In [7], a generalized MRAC is developed to solve the trajectory tracking problem via output feedback for uncertain linear systems. This approach is based on high-order sliding-modes differentiators and dynamic gains, and it guarantees asymptotic stability of the closed-loop system and ultimate robust exact tracking. However, none of the previous works consider the effect of external perturbations.

In the context of MRAC under external perturbations, a robust adaptive tracking controller is proposed in [8]. The external time-dependent perturbations are matched with the control signal. Moreover, it is assumed that such perturbations are bounded as well as their derivatives. This approach ensures the convergence of the tracking error to a region around the origin. To the best of our knowledge, the literature related to the development of MRAC in the presence of external perturbations is limited. Nonetheless, some Input-to-State Stability (ISS) results can be found in [9], [10], and [11].

Note that in addition to the parameter uncertainties, and the possible presence of external perturbations, the nonlinear unmodeled dynamics affect most of the systems. Motivated by these observations, in this paper, following the framework for stability analysis of Persidskii system proposed in [12], a robust nonlinear MRAC is proposed for linear systems with parameter uncertainties, external time-dependent perturbations and nonlinear unmodeled dynamics matched with the control input. The developed nonlinear control law is composed of two nonlinear adaptive gains. Such adaptive gains counteract the effects of some external time-dependent perturbations and nonlinear unmodeled dynamics. The main contribution is threefold:

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- The nonlinear control law ensures asymptotic convergence of the tracking error to zero, and the boundedness of the adaptive gains in the presence of some disturbances.
- In the absence of perturbations, if the reference signal is persistently exciting, then the convergence to zero of the tracking and parameter identification errors is guaranteed.
- The synthesis of the nonlinear controller is constructive, and it is based on the solution of LMIs.

Besides, the simulation results show that, due to the introduced nonlinearities, the rate of convergence of the proposed adaptive control is faster than provided by a classic MRAC.

This paper is organized as follows. The problem statement is given in Section II and the preliminaries are introduced in Section III. The robust Nonlinear-MRAC and the main results are presented in Section IV. Simulation results and conclusions are shown in Sections V-VI, respectively. The proofs are postponed to the Appendix.

**Notation:** The Euclidean norm of a vector $q \in \mathbb{R}^n$ is denoted by $\|q\|$. For a matrix $Q \in \mathbb{R}^{m\times n}$ , denotes its smallest singular value $\sigma_{\min}(Q) = \sqrt{\lambda_{\min}(Q^TQ)}$ and its induced norm as $\|Q\| := \sqrt{\lambda_{\max}(Q^TQ)} = \sigma_{\max}(Q)$, where $\lambda_{\max}$ is the maximum eigenvalue and $\lambda_{\min}$ is the minimum one, $\sigma_{\max}$ is the largest singular value. For a Lebesgue measurable function $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, define the norm $\|u\|_{(t_0,t_1)} := ess\sup_{t\in(t_0,t_1)} \|u(t)\|$, then $\|u\|_\infty := \|u\|_{(0,+\infty)}$ and the set of functions $u$ with the property $\|u\|_\infty < +\infty$ is denoted as $\mathcal{L}_\infty$. For a matrix function $Q : \mathbb{R}_{\geq 0} \to \mathbb{R}^{m\times n}$ , denote $\|Q\|_\infty := \|Q\|_{(0,+\infty)}$. Define the function $\lceil a \rfloor^\gamma := |a|^\gamma \text{sign}(a)$, for any $\gamma \in \mathbb{R}_+$ and any $a \in \mathbb{R}$. For the case $a \in \mathbb{R}^n$, $\lceil a \rfloor^\gamma = [\lceil a_1 \rfloor^\gamma, \lceil a_2 \rfloor^\gamma, ..., \lceil a_n \rfloor^\gamma]^T$. Denote $0_{n\times m}$ as a zero matrix of dimension $n \times m$, $1_n$ as a vector of ones with dimension $n$ and $I_n$ as the identity matrix of dimension $n\times n$. For a matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\delta(\mathcal{A})$ represents the matrix containing only elements in the main diagonal of $\mathcal{A}$, and $\chi(\mathcal{A})$ corresponds to the matrix with zeros on the main diagonal and absolute values for other elements of $\mathcal{A}$.

## II. Problem Statement

Let us consider the following linear system:

$$\dot{x}(t) = Ax(t) + B(u(t) + w(t, x)), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control and $w : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ represents some disturbances comprising time-dependent perturbations and nonlinear unmodeled dynamics. The matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^n$ are unknown but it is assumed that the pair $(A, B)$ is controllable. Let us introduce the following model reference:

$$\dot{x}_m(t) = A_mx_m(t) + B_mr(t), \tag{2}$$

where $A_m \in \mathbb{R}^{n\times n}$ is Hurwitz and known, $B_m \in \mathbb{R}^n$ is known and the reference signal $r \in \mathcal{L}_\infty$.

It is assumed that the model reference is designed such that

$$A_m = A + Bk_x^T, \tag{3a}$$

$$B_m = Bk_r, \tag{3b}$$

where $k_x \in \mathbb{R}^n$ and $k_r \in \mathbb{R} \setminus \{0\}$ are the unknown ideal values that achieve the control objective for $w \equiv 0$.

The manuscript aim is to design a control input $u$ such that the system dynamics (1) behaves like the model reference (2), without the knowledge of the system parameters, and in presence of disturbances $w$.

## III. Preliminaries

Consider a time-varying differential equation:

$$\frac{dx(t)}{dt} = f(t, x(t)),\ t \geq t_0,\ t_0 \in \mathbb{R}, \tag{4}$$

where $x \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n$ ensures forward existence and uniqueness of solutions at least locally in time, $f(t, 0) = 0$. Thus, it is assumed that solution of the system (4) for an initial condition $x_0 \in \mathbb{R}^n$ at time instant $t_0 \in \mathbb{R}$ is denoted as $x(t, t_0, x_0)$ and it is defined on some finite time interval $[t_0, t_0 + T)$ where $0 \leq T < \infty$.

**Theorem 1.** [13]. *Let $V : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that*

$$W_1(x) \leq V(t, x) \leq W_2(x), \tag{5a}$$

$$\dot{V}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x(t)) \leq -W(x), \tag{5b}$$

*for all $t \geq 0$, for all $x \in \Upsilon \subseteq \mathbb{R}^n$, where $W_1$ and $W_2$ are continuous positive definite functions and $W$ is a continuous positive semi-definite function on $\Upsilon$. Choose $r > 0$ such that $B_r \in \Upsilon$ and let $\rho < \min_{\|x\|=r} W_1(x)$. Then, all the solutions of system (4) with $x_0 \in \{x \in B_r | W_2(x) \leq \rho\}$, are bounded and satisfy $W(x(t)) \to 0,\ as\ t \to \infty$.*

*Moreover, if all the assumptions hold globally and $W_1$ is radially unbounded, the statement is true for all $x_0 \in \mathbb{R}^n$.*

## IV. A Robust Nonlinear Model Reference Adaptive Control

First, let us rewrite system (1), using (3), as follows

$$\dot{x} = A_mx + B_mk_r^{-1}\left(u + w(t, x) - k_x^Tx\right).$$

The proposed nonlinear controller and the adaptive laws are given by

$$u = \hat{k}_rr + \hat{k}_x^Tx + \hat{k}_rK^Tf(\tilde{x}), \tag{6a}$$

$$\dot{\hat{k}}_r = -k_1^{-1}\varphi(\tilde{x}, r)(\tilde{x}^TP + f^T(\tilde{x})\Lambda)B_m, \tag{6b}$$

$$\dot{\hat{k}}_x = -K_2^{-1}x(\tilde{x}^TP + f^T(\tilde{x})\Lambda)B_m, \tag{6c}$$

where $k_1 > 0$, $0 < K_2^T = K_2 \in \mathbb{R}^{n\times n}$, $K = [k_3^T, k_4^T, k_5^T]^T \in \mathbb{R}^{3n}$, $0 \leq P^T = P \in \mathbb{R}^{n\times n}$, $\Lambda = [\Lambda_0, \Lambda_1, \Lambda_2]^T$ where $\Lambda_j = \text{diag}\{\lambda_{ji}\}$ with $\lambda_{ji} \in \mathbb{R}_+$ for $i = \overline{1, n}$ and $j = \overline{0, 2}$, $f(\tilde{x}) = [f_0^T(\tilde{x}), f_1^T(\tilde{x}), f_2^T(\tilde{x})]^T$, with

$f_0(\tilde{x}) = \lceil \tilde{x} \rfloor^0$, $f_1(\tilde{x}) = \lceil \tilde{x} \rfloor^\alpha$, $f_2(\tilde{x}) = \lceil \tilde{x} \rfloor^\gamma$ and $\alpha \in (0,1)$, $\gamma > 1$, $\varphi(\tilde{x}, r) = r + K^T f(\tilde{x})$ and $\tilde{x} = x - x_m$. Thus, the closed-loop dynamics is given by

$$\dot{x} = A_m x + B_m[\varphi(\tilde{x}, r)(1 + \tilde{\theta}_1) + \tilde{\theta}_2 x + \bar{w}(t, x)], \quad (7)$$

with the new disturbance term $\bar{w}(t,x) := k_r^{-1} w(t,x)$ and the gain identification errors $\tilde{\theta}_1$ and $\tilde{\theta}_2$ defined as:

$$\tilde{\theta}_1 = k_r^{-1}\hat{k}_r - 1, \quad (8a)$$
$$\tilde{\theta}_2 = k_r^{-1}(\hat{k}_x^T - k_x^T). \quad (8b)$$

Then, taking into account (7) and (8), the error dynamics is given as follows

$$\dot{\tilde{x}} = A_m\tilde{x} + B_m[K^T f(\tilde{x}) + \tilde{\theta}_1\varphi + \tilde{\theta}_2 x + \bar{w}(t,x)], \quad (9a)$$
$$\dot{\tilde{\theta}}_1 = -k_r^{-1}k_1^{-1}\varphi(\tilde{x}^T P + f^T(\tilde{x})\Lambda)B_m, \quad (9b)$$
$$\dot{\tilde{\theta}}_2 = -k_r^{-1}[K_2^{-1}x(\tilde{x}^T P + f^T(\tilde{x})\Lambda)B_m]^T. \quad (9c)$$

Note that fixing $\Lambda = 0_{3n\times n}$ and $K = 0_{3n}$; one recovers the classic MRAC error dynamics. For the case where $w \equiv 0$, one can take $k_3 = k_4 = k_5$ and $\Lambda_0 = \Lambda_1 = \Lambda_2$, then the convergence conditions of the error dynamics (9) are as follows:

**Theorem 2.** *Let the robust nonlinear MRAC* (6) *be applied to the system* (1) *with* $w \equiv 0$. *Suppose that, for* $\alpha \in (0,1)$ *and* $\gamma > 1$, *there exist* $0 < X^T = X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{1\times n}$, $\Phi = \mathrm{diag}\{\phi_i\} \geq 0$; $\Omega_j = \mathrm{diag}\{\omega_{ji}\} \geq 0$ *for* $i = \overline{1,n}$ *and* $j = \overline{0,2}$, *such that the following LMIs*

$$\tilde{Q}_1 \leq 0, \ \Phi + \sum_{j=0}^{2}\Omega_j > 0, \quad (10a)$$

$$1_n^T\left[\delta(B_mY + XA_m^T) + \chi(B_mY + XA_m^T) + \Omega_0\right] \leq 0, \quad (10b)$$

$$1_n^T\left[(1+\alpha)\delta(B_mY + XA_m^T) + \alpha\chi(B_mY + XA_m^T) + \chi^T(B_mY + XA_m^T) + (1+\alpha)\Omega_1\right] \leq 0, \quad (10c)$$

$$1_n^T\left[(1+\gamma)\delta(B_mY + XA_m^T) + \gamma\chi(B_mY + XA_m^T) + \chi^T(B_mY + XA_m^T) + (1+\gamma)\Omega_2\right] \leq 0, \quad (10d)$$

*with*

$$\tilde{Q}_1 = \begin{bmatrix} A_mX + XA_m^T + \Phi & 0 & 0 & 0 \\ \star & B_mY + Y^TB_m^T & B_mY + Y^TB_m^T & B_mY + Y^TB_m^T \\ \star & \star & B_mY + Y^TB_m^T & B_mY + Y^TB_m^T \\ \star & \star & \star & B_mY + Y^TB_m^T \end{bmatrix}, \quad (11)$$

*are feasible for a fixed* $0 < \Lambda_0 = \mathrm{diag}\{\lambda_{0i}\} = \Lambda_1 = \Lambda_2$ *and* $i = \overline{1,n}$. *If the controller parameters are selected as* $k_1 > 0$, $K_2 = K_2^T > 0$, $k_3^T = k_4^T = k_5^T = Y\Lambda_0$ *and* $P = X^{-1}$, *then* $[\tilde{x}, \tilde{\theta}_1, \tilde{\theta}_2] = 0$ *is Globally Uniformly Stable (GUS) and* $\lim_{t\to+\infty}\tilde{x}(t) = 0$.

Note that the persistence of excitation condition over the reference $r$, and thus over $x_m$, implies that the regressor term $[\varphi, x]$ is persistently exciting (PE). This is due to $\lim_{t\to+\infty}\tilde{x}(t) = 0$ and both $\varphi(\tilde{x}, r) = r + K^T f(\tilde{x})$ and $x = \tilde{x} + x_m$ are PE. Hence, it implies that $[\tilde{\theta}_1, \tilde{\theta}_2] = 0$ is Globally Uniformly Asymptotically Stable (GUAS). Therefore, the following result is provided:

**Corollary 1.** *If all the conditions of Theorem 2 are satisfied and* $r$ *is persistently exciting; then,* $[\tilde{x}, \tilde{\theta}_1, \tilde{\theta}_2] = 0$ *is GUAS.*

For the case when $w \equiv 0$, the discontinuous term, *i.e.,* $f_0(\tilde{x})$, could be skipped and the given result is not affected. Now, for the disturbed case, let us introduce the following assumption over the class of disturbances admitted in this work.

**Assumption 1.** *The disturbance term* $\bar{w}$ *satisfies the following inequality:*

$$|\bar{w}|^2 \leq w^+ + L\|\tilde{x}\|^2 + L_1\|\tilde{x}\|^{1+\alpha} + L_2\|\tilde{x}\|^{1+\gamma}, \quad (12)$$

*where* $w^+$, $L$, $L_1$, $L_2 \in \mathbb{R}_+$ *are some known positive constants.*

The disturbances described in Assumption 1 contemplate different kinds of nonlinearities such as linear terms, which are related to $L||\tilde{x}||^2$, the nonlinear terms growing faster than the linear ones given by $L_2||\tilde{x}||^{1+\gamma}$ (for big deviations of the error $\tilde{x}$) and $L_1||\tilde{x}||^{1+\alpha}$ (close to the origin). Note also that this bound considers non-vanishing time-dependent perturbations, which are bounded by $w^+$. In this sense, Assumption 1 is valid for a large class of disturbances. In addition, such perturbations cannot be attenuated by a linear controller (due to presence of the terms with the powers containing $\gamma$ and $\alpha$); thus, a nonlinear control design becomes obligatory.

The convergence properties for the disturbed case, *i.e.,* $w \neq 0$, are described by the following theorem.

**Theorem 3.** *Let the robust nonlinear MRAC* (6) *be applied to the system* (1) *with* $w$ *satisfying Assumption 1. Suppose that, for* $\alpha \in (0,1)$ *and* $\gamma > 1$, *there exist* $0 < X^T = X \in \mathbb{R}^{n\times n}$, $Y_j \in \mathbb{R}^{1\times n}$, $\Phi = \mathrm{diag}\{\phi_i\} > 0$; $\Omega_j = \mathrm{diag}\{\omega_{ji}\} > 0$ *for* $i = \overline{1,n}$ *and* $j = \overline{0,2}$, *such that the following LMIs*

$$\tilde{Q}_2 \leq 0, \quad (13a)$$

$$1_n^T\left[\delta(B_mY_0 + XA_m^T) + \chi(B_mY_0 + XA_m^T) + \Omega_0\right] \leq 0, \quad (13b)$$

$$1_n^T\left[(1+\alpha)\delta(B_mY_1 + XA_m^T) + \alpha\chi(B_mY_1 + XA_m^T) + \chi^T(B_mY_1 + XA_m^T) + (1+\alpha)\Omega_1\right] \leq 0, \quad (13c)$$

$$1_n^T\left[(1+\gamma)\delta(B_mY_2 + XA_m^T) + \gamma\chi(B_mY_2 + XA_m^T) + \chi^T(B_mY_2 + XA_m^T) + (1+\gamma)\Omega_2\right] \leq 0, \quad (13d)$$

$$\Phi - \mu LI_n + \sum_{s=1}^{2} 2\Omega_s - \mu L_s I_n + \Omega_0 > 0, \quad (13e)$$

$$\Phi \geq \mu LI_n, \ 2\Omega_s - \mu L_s I_n \geq 0, \ s = 1, 2, \quad (13f)$$

*with*

$$\tilde{Q}_2 = \begin{bmatrix} A_m X + X A_m^T + \Phi & 0 & 0 & 0 & B_m \\ \star & B_m Y_0 + Y_0^T B_m^T + \frac{\mu w^+}{n}\Lambda_0^{-2} & B_m Y_1 + Y_0^T B_m^T & B_m Y_2 + Y_0^T B_m^T & B_m \\ \star & \star & B_m Y_1 + Y_1^T B_m^T & B_m Y_2 + Y_1^T B_m^T & B_m \\ \star & \star & \star & B_m Y_2 + Y_2^T B_m^T & B_m \\ \star & \star & \star & \star & -\mu I_n \end{bmatrix}. \quad (14)$$

*are feasible for fixed* $0 < \Lambda_j = \mathrm{diag}\{\lambda_{ji}\}$, *with* $j = \overline{0,2}$, *and* $i = \overline{1,n}$, *and* $\mu \in \mathbb{R}_+$. *If the control parameters are selected as* $k_1 > 0$, $K_2 = K_2^T > 0$, $k_{j+3}^T = Y_j \Lambda_j$, *with* $j = \overline{0,2}$, *and* $P = X^{-1}$, *then* $[\tilde{x}, \tilde{\theta}_1, \tilde{\theta}_2] = 0$ *is GUS and* $\lim_{t\to+\infty} \tilde{x}(t) = 0$.

Note that the feasibility of the LMIs (10) is guaranteed due to the controllability of the pair $(A_m, B_m)$, and thus, controllability of the pair $(A, B)$ provided that $A_m$ is diagonally dominant. Indeed, in this case the matrix $X$ can be selected to be diagonal, while the row vector $Y = -\epsilon B_m^\top$ for some sufficiently small $\epsilon > 0$. For the LMIs (13) the conditions are the same for sufficiently small values of constants given in Assumption 1, and in this case $Y_i = -\epsilon_i B_m^\top$ for some sufficiently small $\epsilon_i > 0$ with $i = \overline{0,2}$ is an admissible choice.

*Remark* 1. *The proposed approach cannot completely compensate the unmatched disturbances. However, following the same stability analysis, one could provide ISS properties with respect to some unmatched bounded disturbances.*

Let us illustrate the efficiency of the proposed nonlinear MRAC by numeric simulations with a robotic system.

## V. Simulation Results

The simulations have been done in Matlab with the Euler explicit discretization method and sampling time equal to 0.1[ms], while the solution to the given LMIs were obtained by means of SeDuMi solver among YALMIP in MATLAB.

Let us consider a single-link Flexible-joint robot manipulator, the dynamics in state-space form can be written as follows [14]:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k}{J_1} - \frac{Mglx_1}{J_1} & 0 & \frac{k}{J_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_m} & 0 & \frac{-k}{J_m} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_m} \end{bmatrix} (u + w(t,x)),$$

where $x = [x_1, x_2, x_3, x_4]^T = [q_1, \dot{q}_1, q_2, \dot{q}_2]^T$, $x_1$ and $x_3$ are the angular positions of the link and motor, respectively; $x_2$ and $x_4$ are the velocities of the link and motor, respectively; $J_1 \in \mathbb{R}_+$ and $J_m \in \mathbb{R}_+$ denote the inertia of the link and the motor; $M \in \mathbb{R}_+$ is the mass of the link; $g \in \mathbb{R}_+$ is the gravitational acceleration, $l \in \mathbb{R}_+$ represents the length of the link and $k \in \mathbb{R}_+$ is the stiffness of the spring. The model reference is given as follows:

$$\dot{x}_m = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -32.35 & 0 & 25 & 0 \\ 0 & 0 & 0 & 1 \\ 9.93 & 11.79 & -8.65 & -11 \end{bmatrix} x_m + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r,$$

where the ideal gains are given as $k_r = 1$ and $k_x = [10, -11.7, -11.3, 10]^T$. Consider the initial conditions as $x(0) = [0.4, 1.3, 2.7, 1.5]^T$, $x_m(0) = [0, 0, 0, 0]^T$, $\hat{k}_r(0) = 0$ and $\hat{k}_x(0) = [0, 0, 0, 0]^T$. In order to illustrate the statements of Theorem 3, let us consider an external disturbance $w(t,x) = 0.7\sin(t) + 0.1 + \frac{1}{J_m}x_4 - \frac{1}{J_m}x_4|x_4|$, which represents some non-modeled dynamics and external perturbations. Note that an upper bound for $w$ can be rewritten as $|w(t,x)| \le 0.8 + \frac{1}{J_m}\left(\tilde{x}_4^2 + 2\left(|\tilde{x}_4| + |x_{m_4}|\right) + x_{m_4}^2\right)$.

Then, taking into account that $x_{m_4}$ is bounded, it is clear that $w$ satisfies the condition (12). Thus, the Robust Nonlinear MRAC (6) is implemented, with a reference signal $r = 3$. Then, based on Theorem 3, fixing the following control gains: $k_1 = 1 \times 10^{-3},\ K_2 = 3 \times 10^{-4} I_n,\ w^+ = 0.9,\ \Lambda_0 = \Lambda_1 = \Lambda_2 = 1.5 I_n,\ \mu = 0.17,\ L = L_1 = L_2 = 1.74$, $\alpha = 0.5$ and $\gamma = 1.5$, the following solution for LMIs (13) is obtained:

$$P = \begin{bmatrix} 0.102 & 0.396 & 0.004 & 0.376 \\ 0.396 & 0.002 & -0.092 & 0.310 \\ 0.004 & -0.092 & 0.077 & 0.758 \\ 0.376 & 0.310 & 0.758 & 0.004 \end{bmatrix},$$

$\Phi$ =diag{0.059, 0.045, 0.072, 0.102}, $\Omega_0$ =diag{0.027, 0.018, 0.027, 0.016}, $\Omega_1$ =diag{0.011, 0.011, 0.012, 0.026}, $\Omega_2$ =diag{0.009, 0.008, 0.009 ,0.070}, $k_3$ =[1.321, 1.102, -0.589, -0.957]$^T$, $k_4$ =[1.648, 3.022, -2.819, -1.414]$^T$ and $k_5$ =[3.268, 5.675, -5.526, -2.828]$^T$.

For comparison purposes, the classic MRAC is also implemented fixing $\Lambda_j = 0_{n\times n}$ and $k_{j+3} = 0_n$ with $j = \overline{0,2}$ in (6). The behavior of the system is depicted by Fig. 1 and the norm of the error $\tilde{x}$ is depicted by Fig. 2. Opposite to the classical MRAC, the robust nonlinear MRAC tracks the desired reference despite the presence of disturbances, matched with the control law, depending on the state variables and the time.

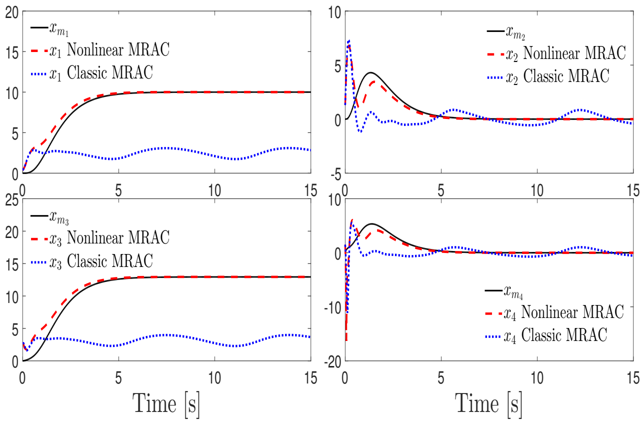


Figure 1. Trajectories of the system and the model reference in presence of disturbances with a constant reference.

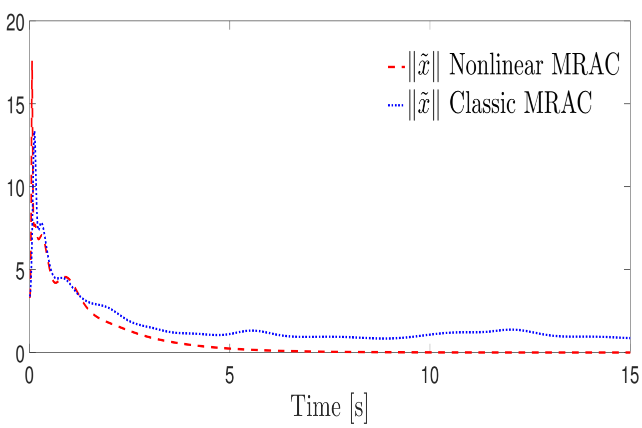


Figure 2. Norm of the error $\tilde{x}$ in presence of disturbances with a constant reference.

The control law $u$ is depicted in Fig. 3; the nonlinear MRAC requires less control effort than the classic MRAC. It has to be remarked that the nonlinear control law tracks the reference signal even in the presence of vanishing and non-vanishing external perturbations. Finally, the evolution of the nonlinear adaptive gains is shown in Figs. 4 and 5. It can be seen that the adaptive gains remain bounded. Based on Theorem 2 and 3, the rate of convergence is asymptotic similarly to the classic MRAC. However, given the comparative simulation results, it can be seen that due to the nonlinear terms, the convergence rate is faster than in the conventional MRAC algorithm.

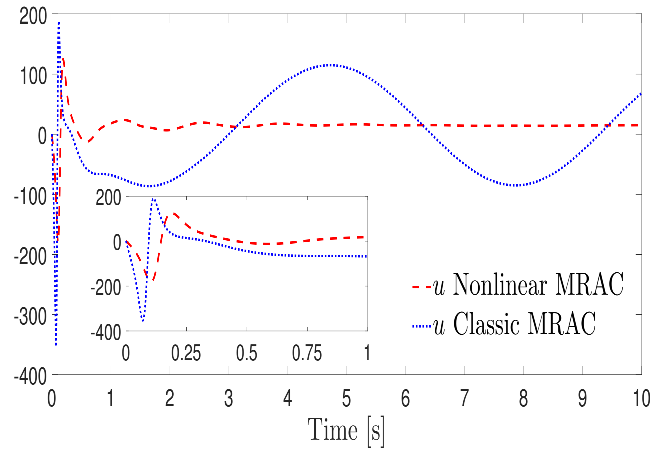


Figure 3. Action law $u$ in presence of disturbances with a constant reference.

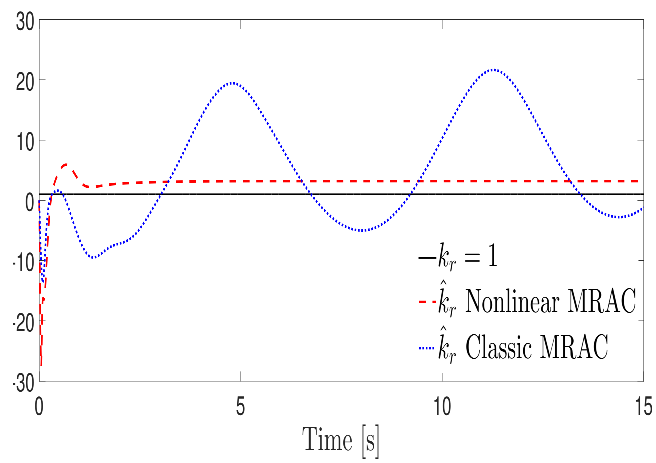


Figure 4. Nonlinear adaptive gains in presence of disturbances with a constant reference.

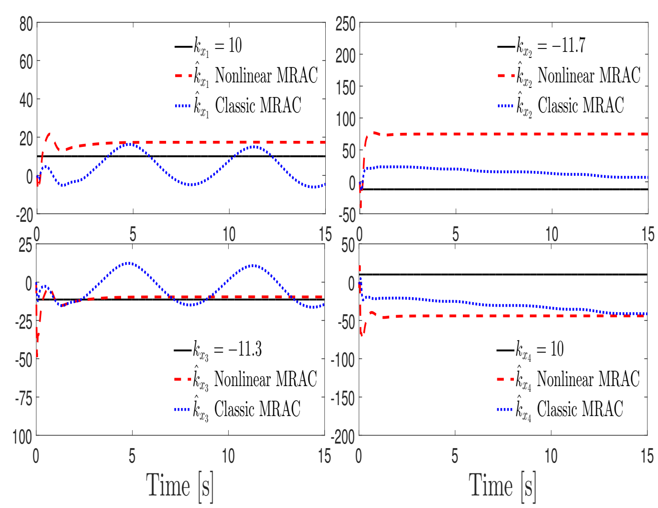


Figure 5. Nonlinear adaptive gains in presence of disturbances with a constant reference.

## VI. Conclusions

In this paper, a robust nonlinear MRAC is proposed for linear systems with parameter uncertainties, external time-dependent perturbations, and nonlinear unmodeled dynamics matched with the control input. The designed nonlinear control law counteracts the effects of such disturbances. The control synthesis is achieved through a constructive method using LMIs. The nonlinear control law guarantees that the tracking error converges asymptotically to zero despite the effect of perturbations and unmodeled dynamics, whereas the adaptive gains remain bounded. The performance of the approach is illustrated by numerical simulations. The results show that the convergence rate is faster than provided by a classic MRAC; the nonlinearities in the control law provoke such effect. The proposed approach inherits common MRAC disadvantages, *i.e.,* the systems under consideration must satisfy the matching and controllability conditions, and the transient quality is sensitive to sampling times.

Future research can be devoted to the case of presence of external disturbances that are not matched with the control signal.


## Acknowledgment

R. Franco, H. Ríos and A. Ferreira de Loza thank the financial support from CONACYT CVU 772057, Cátedras CONACYT CVU 270504 project 922, and Cátedras CONACYT CVU 166403 project 1537, respectively; and from TecNM projects.


## Appendix

**Proof of Theorem 2.** Let us consider the error dynamics (9), with $w \equiv 0$, and the following Lyapunov function

$$V = \tilde{x}^T P \tilde{x} + k_r k_1 \tilde{\theta}_1^2 + k_r \tilde{\theta}_2^T K_2 \tilde{\theta}_2 + 2\sum_{j=0}^{2}\sum_{i=1}^{n} \lambda_{0i} \int_0^{\tilde{x}_i} f_{ji}(s) ds,$$

where $f_j = [f_{j1,\dots,}f_{jn}]^T$ for all $j = \overline{0,2}$. According to [12], $V$ is positive definite and radially unbounded. Due to the shape of the nonlinearities $f_{ji}$ their integrals are positive definite functions, hence there exist $W_1$ and $W_2$ satisfying (5a). Moreover, $W_1(\tilde{x}, \tilde{\theta}_1, \tilde{\theta}_2) = \lambda_{\min}(P)||\tilde{x}||^2 + k_r k_1 \tilde{\theta}_1^2 + k_r \lambda_{\min}(K_2)||\tilde{\theta}_2||^2$. Then, the derivative of $V$ along the trajectories of (9) is given by

$$\dot{V} = \dot{\tilde{x}}^T P \tilde{x} + \tilde{x}^T P \dot{\tilde{x}} + k_r k_1 \tilde{\theta}_1 \dot{\tilde{\theta}}_1 + 2k_r \tilde{\theta}_2^T K_2 \dot{\tilde{\theta}}_2 + 2\sum_{j=0}^{2} \dot{\tilde{x}}^T \Lambda_0 f_j(\tilde{x}),$$

it follows that

$$\begin{aligned}\dot{V} &= \tilde{x}^T \left[PA_m + A_m^T P\right] \tilde{x} + 2\tilde{x}^T P B_m \left(\sum_{j=0}^{2} k_3^T f_j(\tilde{x})\right) \\ &+ 2\tilde{x}^T P B_m \tilde{\theta}_1 \varphi(\tilde{x}, r) + 2\tilde{x}^T P B_m \tilde{\theta}_2 x + 2k_r \tilde{\theta}_1^T K_2 \dot{\tilde{\theta}}_1 \\ &+ 2k_r \varphi^T(\tilde{x}, r) k_1 \dot{\tilde{\theta}}_2 + 2\sum_{j=0}^{2} \Big(\tilde{x}^T A_m^T \Lambda_0 f_j(\tilde{x}) \\ &+ \sum_{s=0}^{2} f_s^T(\tilde{x}) k_3 B_m^T \Lambda_0 f_j(\tilde{x}) + \varphi^T(\tilde{x}, r) \tilde{\theta}_1 B_m^T \Lambda_0 f_j(\tilde{x}) \\ &+ x^T \tilde{\theta}_2^T B_m^T \Lambda_0 f_j(\tilde{x})\Big).\end{aligned}$$

Then, since $\dot{\tilde{\theta}}_1 = k_r^{-1}\dot{\tilde{k}}_r$ and $\dot{\tilde{\theta}}_2 = k_r^{-1}\dot{\tilde{k}}_x^T$, selecting the adaptive gains dynamics as in (6b)-(6c) under substitutions $\Lambda_0 = \Lambda_1 = \Lambda_2$ and $k_3 = k_4 = k_5$, it is obtained

$$\begin{aligned}\dot{V} &= \tilde{x}^T \left[PA_m + A_m^T P\right] \tilde{x} + 2\tilde{x}^T P B_m \left(\sum_{j=0}^{2} k_3^T f_j(\tilde{x})\right) \\ &+ 2\sum_{j=0}^{2}\left(\tilde{x}^T A_m^T \Lambda_0 f_j(\tilde{x}) + \sum_{s=0}^{2} f_s^T(\tilde{x}) k_3 B_m^T \Lambda_0 f_j(\tilde{x})\right).\end{aligned}$$

Thus, the Lyapunov function can be rewritten as

$$\dot{V} = \begin{bmatrix}\tilde{x}\\ f_0\\ f_1\\ f_2\end{bmatrix}^T \begin{bmatrix} PA_m + A_m^T P & PA_0 + A_m^T \Lambda_0 & PA_0 + A_m^T \Lambda_0 & PA_0 + A_m^T \Lambda_0 \\ \star & \Lambda_0 A_0 + A_0^T \Lambda_0 & \Lambda_0 A_0 + A_0^T \Lambda_0 & \Lambda_0 A_0 + A_0^T \Lambda_0 \\ \star & \star & \Lambda_0 A_0 + A_0^T \Lambda_0 & \Lambda_0 A_0 + A_0^T \Lambda_0 \\ \star & \star & \star & \Lambda_0 A_0 + A_0^T \Lambda_0 \end{bmatrix} \begin{bmatrix}\tilde{x}\\ f_0\\ f_1\\ f_2\end{bmatrix} = \xi^T Q_a \xi,$$

where $A_0 = B_m k_3^T$. Pre and post-multiplying $Q_a$ by $R = \text{diag}\{P^{-1}, \Lambda_0^{-1}, \Lambda_0^{-1}, \Lambda_0^{-1}\}$, it follows that

$$\dot{V} = \begin{bmatrix}\tilde{x}\\ f_0\\ f_1\\ f_2\end{bmatrix}^T \begin{bmatrix} A_m X + X A_m^T & B_m Y + X A_m^T & B_m Y + X A_m^T & B_m Y + X A_m^T \\ \star & B_m Y + Y^T B_m^T & B_m Y + Y^T B_m^T & B_m Y + Y^T B_m^T \\ \star & \star & B_m Y + Y^T B_m^T & B_m Y + Y^T B_m^T \\ \star & \star & \star & B_m Y + Y^T B_m^T \end{bmatrix} \begin{bmatrix}\tilde{x}\\ f_0\\ f_1\\ f_2\end{bmatrix} = \xi^T Q \xi,$$

where $X = P^{-1}$ and $Y = k_3^T \Lambda_0^{-1}$. To ensure the non-positive definiteness of $Q$, the blocks $Q_{12}$, $Q_{13}$, $Q_{14}$, *i.e.,* $Q_{12} = Q_{13} = Q_{14} = B_m Y + X A_m^T$, and their symmetric counterparts have to be treated specially [12]. Taking these terms out of $Q$, it is obtained that:

$$\dot{V} = \xi^T \tilde{Q}_1 \xi - \tilde{x}^T \Phi \tilde{x} - 2\sum_{j=0}^{2} \tilde{x}^T \Omega_j f_j(\tilde{x}) + 2\sum_{j=0}^{2} \tilde{x}^T \left(B_m Y + X A_m^T + \Omega_j\right) f_j(\tilde{x}),$$

where $\Phi = \mathrm{diag}(\phi_i)$, with $\phi_i \in \mathbb{R}_+$, $\Omega_j = \mathrm{diag}(\omega_{ji})$, with $\omega_{ji} \in \mathbb{R}_+$, for $i = \overline{1,n}$ and $j = \overline{0,2}$, and

$$\tilde{Q}_1 = \begin{bmatrix} A_mX + XA_m^T + \Phi & 0 & 0 & 0 \\ \star & B_mY + Y^TB_m^T & B_mY + Y^TB_m^T & B_mY + Y^TB_m^T \\ \star & \star & B_mY + Y^TB_m^T & B_mY + Y^TB_m^T \\ \star & \star & \star & B_mY + Y^TB_m^T \end{bmatrix}.$$

Therefore, the elements on the main diagonal of $B_mY + XA_m^T + \Omega_j$, with $j = \overline{0,2}$, are useful if they are negative, meanwhile the cross terms can be treated using Young's inequality, *i.e.,*

$$\tilde{x}_i f_0(\tilde{x}_k) \leq |\tilde{x}_i|, \tag{15a}$$

$$\tilde{x}_i f_1(\tilde{x}_k) \leq \frac{|\tilde{x}_i|^{1+\alpha}}{1+\alpha} + \frac{\alpha|\tilde{x}_k|^{1+\alpha}}{1+\alpha}, \tag{15b}$$

$$\tilde{x}_i f_2(\tilde{x}_k) \leq \frac{|\tilde{x}_i|^{1+\gamma}}{1+\gamma} + \frac{\gamma|\tilde{x}_k|^{1+\gamma}}{1+\gamma}, \tag{15c}$$

for any $i \neq k$, with $i,k = \overline{1,n}$. Let us analyze the block $Q_{13} + \Omega_1$. For such a term it is required that

$$\tilde{x}^T(Q_{13} + \Omega_1)f_1 \leq 0, \tag{16}$$

and by Young's inequality (15b) and the fact that $f_1(\tilde{x}) = \lceil \tilde{x} \rfloor^{\alpha}$, it follows that (16) can be upper bounded by

$$1_n^T \delta(Q_{13})|\tilde{x}|^{\alpha+1} + 1_n^T \frac{\alpha \chi(Q_{13})}{1+\alpha}|\tilde{x}|^{\alpha+1} + 1_n^T \frac{\chi^T(Q_{13})}{1+\alpha}|\tilde{x}|^{\alpha+1} + \Omega_1|\tilde{x}|^{\alpha+1} \leq 0,$$

that by multiplying by $(1+\alpha)$ is equivalent to the LMI (10c). Therefore, following the same procedure, one can obtain the LMIs (10b) and (10d) by analyzing the blocks $Q_{12} + \Omega_0$ and $Q_{14} + \Omega_2$, respectively. Hence, since the LMIs (10) are feasible, it follows that $\tilde{x}^T(B_mY + XA_m^T + \Omega_j)f_j(\tilde{x}) \leq 0$, and, hence an upper bound for the time derivative of $V$ is given as

$$\dot{V} \leq -\tilde{x}^T\Phi\tilde{x} - 2\sum_{j=0}^{2} \tilde{x}^T\Omega_j f_j(\tilde{x}).$$

Moreover, by applying Theorem 1, it is concluded that for any $c > 0$ and for all initial states in the set $\{V(\tilde{x}, \tilde{\theta}_1, \tilde{\theta}_2) \leq c\}$, all state variables are bounded for all $t \geq t_0$ and $\lim_{t\to+\infty} \tilde{x}(t) = 0$, *i.e.,* $[\tilde{x}, \tilde{\theta}_1, \tilde{\theta}_2] = 0$ is GUS and $\lim_{t\to+\infty} \tilde{x}(t) = 0$. This concludes the proof. □

**Proof of Theorem 3.** Let us consider the following Lyapunov function

$$V = \tilde{x}^TP\tilde{x} + k_rk_1\tilde{\theta}_1^2 + k_r\tilde{\theta}_2^TK_2\tilde{\theta}_2 + 2\sum_{j=0}^{2}\sum_{i=1}^{n} \lambda_{ji}\int_0^{\tilde{x}_i} f_{ji}(s)ds,$$

then, the derivative along the trajectories of (9) is given by

$$\begin{aligned}\dot{V} &= \tilde{x}^T\left[PA_m + A_m^TP\right]\tilde{x} + 2\tilde{x}^TPB_m\left(\sum_{j=0}^{2} k_{j+3}^T f_j(\tilde{x})\right) \\ &+ 2\tilde{x}^TPB_m\bar{w} + 2\tilde{x}^TPB_m\tilde{\theta}_1\varphi(\tilde{x}, r) + 2\tilde{x}^TPB_m\tilde{\theta}_2 x \\ &+ 2k_r\tilde{\theta}_1^TK_2\dot{\tilde{\theta}}_1 + 2k_r\varphi^T(\tilde{x}, r)k_1\dot{\tilde{\theta}}_2 + 2\sum_{j=0}^{2}\Big(\tilde{x}^TA_m^T\Lambda_j f_j(\tilde{x}) \\ &+ \sum_{s=0}^{2} f_s^T(\tilde{x})k_{s+3}B_m^T\Lambda_j f_j(\tilde{x}) + \varphi^T(\tilde{x}, r)\tilde{\theta}_1B_m^T\Lambda_j f_j(\tilde{x}) \\ &+ x^T\tilde{\theta}_2^TB_m^T\Lambda_j f_j(\tilde{x}) + \bar{w}^TB_m^T\Lambda_j f_j(\tilde{x})\Big),\end{aligned}$$

Then, since $\dot{\tilde{\theta}}_1 = k_r^{-1}\dot{\tilde{k}}_r$ and $\dot{\tilde{\theta}}_2 = k_r^{-1}\dot{\tilde{k}}_x^T$, selecting the adaptive gains dynamics as (6b)-(6c), it follows that

$$\begin{aligned}\dot{V} &= \tilde{x}^T\left[PA_m + A_m^TP\right]\tilde{x} + 2\tilde{x}^TPB_m\left(\sum_{j=0}^{2} k_{j+3}^T f_j(\tilde{x})\right) \\ &+ 2\sum_{j=0}^{2}\left(\tilde{x}^TA_m^T\Lambda_j f_j(\tilde{x}) + \sum_{s=0}^{2} f_s^T(\tilde{x})k_{s+3}B_m^T\Lambda_j f_j(\tilde{x})\right) \\ &+ 2\tilde{x}^TPB_m\bar{w} + 2\sum_{j=0}^{2}\bar{w}^TB_m^T\Lambda_j f_j(\tilde{x}).\end{aligned}$$

Thus, the Lyapunov function can be rewritten as

$$\dot{V} = \begin{bmatrix}\tilde{x}\\ f_0\\ f_1\\ f_2\\ \bar{w}\end{bmatrix}^T \begin{bmatrix} PA_m + A_m^TP & PA_0 + A_m^T\Lambda_0 & PA_1 + A_m^T\Lambda_1 & PA_2 + A_m^T\Lambda_2 & PB_m \\ \star & \Lambda_0A_0 + A_0^T\Lambda_0 + \frac{\mu w^+}{n}I_n & \Lambda_1A_1 + A_0^T\Lambda_0 & \Lambda_2A_2 + A_0^T\Lambda_0 & B_m^T\Lambda_0 \\ \star & \star & \Lambda_1A_1 + A_1^T\Lambda_1 & \Lambda_2A_2 + A_1^T\Lambda_1 & B_m^T\Lambda_1 \\ \star & \star & \star & \Lambda_2A_2 + A_2^T\Lambda_2 & B_m^T\Lambda_2 \\ \star & \star & \star & \star & -\mu \end{bmatrix} \begin{bmatrix}\tilde{x}\\ f_0\\ f_1\\ f_2\\ \bar{w}\end{bmatrix} - \frac{\mu w^+}{n}f_0^Tf_0 + \mu\bar{w}^T\bar{w} = \xi^TQ_b\xi - \frac{\mu w^+}{n}f_0^Tf_0 + \mu\bar{w}^T\bar{w},$$

where $A_0 = B_mk_3$, $A_1 = B_mk_4$, $A_2 = B_mk_5$ and $w^+ \in \mathbb{R}_+$ is given in Assumption 1. Then, pre and post-multiplying $Q_b$ by $R = \mathrm{diag}\{P^{-1}, \Lambda_0^{-1}, \Lambda_1^{-1}, \Lambda_2^{-1}, 1\}$, it follows that

$$\dot{V} = \begin{bmatrix}\tilde{x}\\ f_0\\ f_1\\ f_2\\ \bar{w}\end{bmatrix}^T \begin{bmatrix} A_mX + XA_m^T & B_mY_0 + XA_m^T & B_mY_1 + XA_m^T & B_mY_2 + XA_m^T & B_m \\ \star & B_mY_0 + Y_0^TB_m^T + \frac{\mu w^+}{n}\Lambda_0^{-2} & B_mY_1 + Y_0^TB_m^T & B_mY_2 + Y_0^TB_m^T & B_m \\ \star & \star & B_mY_1 + Y_1^TB_m^T & B_mY_2 + Y_1^TB_m^T & B_m \\ \star & \star & \star & B_mY_2 + Y_2^TB_m^T & B_m \\ \star & \star & \star & \star & -\mu \end{bmatrix} \begin{bmatrix}\tilde{x}\\ f_0\\ f_1\\ f_2\\ \bar{w}\end{bmatrix} - \frac{\mu w^+}{n}f_0^Tf_0 + \mu\bar{w}^T\bar{w} = \xi^TQ_w\xi - \frac{\mu w^+}{n}f_0^Tf_0 + \mu\bar{w}^T\bar{w},$$

where $X = P^{-1}$, $Y_0 = k_3^T \Lambda_0^{-1}$, $Y_1 = k_4^T \Lambda_1^{-1}$ and $Y_2 = k_5^T \Lambda_2^{-1}$. To ensure the non-positive definiteness of $Q_w$, the blocks $Q_{w_{12}}$, $Q_{w_{13}}$, $Q_{w_{14}}$, *i.e.*, $Q_{w_{12}} = B_m Y_0 + X A_m^T$, $Q_{w_{13}} = B_m Y_1 + X A_m^T$, $Q_{w_{14}} = B_m Y_2 + X A_m^T$ and their symmetric counterparts, have to be treated as in the proof of Theorem 2. Taking these terms out of $Q_w$, it is obtained that:

$$\dot{V} \le \xi^T \tilde{Q}_2 \xi - \tilde{x}^T \Phi \tilde{x} - 2\sum_{j=0}^{2} \tilde{x}^T \Omega_j f_j(\tilde{x}) - \frac{\mu w^+}{n} f_0^T f_0 + 2\sum_{j=0}^{2} \tilde{x}^T \left(B_m Y_j + X A_m^T + \Omega_j\right) f_j(\tilde{x}) + \mu \bar{w}^T \bar{w},$$

where

$$\tilde{Q}_2 = \begin{bmatrix} A_m X + X A_m^T + \Phi & 0 & 0 & 0 & B_m \\ \star & B_m Y_0 + Y_0^T B_m^T + \frac{\mu w^+}{n}\Lambda_0^{-2} & B_m Y_1 + Y_0^T B_m^T & B_m Y_2 + Y_0^T B_m^T & B_m \\ \star & \star & B_m Y_1 + Y_1^T B_m^T & B_m Y_2 + Y_1^T B_m^T & B_m \\ \star & \star & \star & B_m Y_2 + Y_2^T B_m^T & B_m \\ \star & \star & \star & \star & -\mu \end{bmatrix}.$$

Therefore, the elements on the main diagonal of $B_m Y_j + X A_m^T + \Omega_j$, with $j = \overline{0,2}$, are useful if they are negative, meanwhile the cross terms can be also treated using Young's inequality given in (15).

Thus, following the same procedure given in the proof of Theorem 2, one can obtain the LMIs (13b), (13c) and (13d) by analyzing the blocks $Q_{w_{12}} + \Omega_0$, $Q_{w_{13}} + \Omega_1$ and $Q_{w_{14}} + \Omega_2$, respectively. Therefore, since the LMIs (13b)-(13d) are feasible, it follows that $\tilde{x}^T (B_m Y_j + X A_m^T + \Omega_j) f_j(\tilde{x}) \le 0$, and, hence an upper bound for the time derivative of $V$ is given as

$$\dot{V} \le -\tilde{x}^T \Phi \tilde{x} - 2\sum_{j=0}^{2} \tilde{x}^T \Omega_j f_j(\tilde{x}) - \frac{\mu w^+}{n} f_0^T f_0 + \mu \bar{w}^T \bar{w}.$$

Furthermore, due to Assumption 1, it is obtained

$$\dot{V} \le -\tilde{x}^T (\Phi - \mu L)\tilde{x} - \sum_{s=1}^{2} \tilde{x}^T (2\Omega_s - \mu L_s) f_s(\tilde{x}) - 2\tilde{x}^T \Omega_0 f_0(\tilde{x}).$$

Moreover, by applying Theorem 1, it is concluded that for any $c > 0$ and for all initial states in the set $\{V(\tilde{x}, \tilde{\theta}_1, \tilde{\theta}_2) \le c\}$, all state variables are bounded for all $t \ge t_0$ and $\lim_{t\to+\infty} \tilde{x}(t) = 0$, *i.e.*, $[\tilde{x}, \tilde{\theta}_1, \tilde{\theta}_2] = 0$ is GUS and $\lim_{t\to+\infty} \tilde{x}(t) = 0$. This concludes the proof. □

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