Order-theoretic, geometric and combinatorial models of intuitionistic S4 proofs

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Abstract

We propose a few models of proof terms for the intuitionistic modal propositional logic S4. Some of them are based on partial orders, or cpos, or dcpos, some of them on a suitable category of topological spaces and continuous maps. A structure that emerges from these interpretations is that of augmented simplicial sets. This leads to so-called combinatorial models, where simplices play an important role: the point is that the simplicial structure interprets the $\Box$ modality, and that the category of augmented simplicial sets is itself already a model of intuitionistic propositional S4 proof terms. In fact, this category is an elementary topos, and is therefore a prime candidate to interpret all proof terms for intuitionistic S4 set theory. Finally, we suggest that geometric-like realizations functors provide a recipe to build other models of intuitionistic propositional S4 proof terms.

1 Introduction

There are now several different proof term languages for intuitionistic S4 [BdP92, BdP96, PW95, MM96, GL96a, GL96b], with applications in partial evaluation [WLP98], in run-time program generation [DP96], in higher-order abstract syntax [Lel97], etc. These calculi are related, in that we can translate from one to any other. Some of these calculi are even inter-translatable in an untyped setting. Our goal in this paper is to develop a few models of typed proof term languages, much as Scott's cpos provided models for the typed and untyped λ-calculi. Hopefully, this will bring some enlightenment as to what the basic proof term constructions mean.

We develop several such models, starting from the ones that are closest to Scott models of the λ-calculus: these are *order-theoretic models*, based on complete partial orders (cpos), which we present in Section 3. We don't care much about the syntax of proof terms for intuitionistic S4, but we would like to eventually arrive at what we call *combinatorial models*, which are very much related to $\lambda\mathtt{ev}Q_H$ [GL96b], and are related to constructions of simplicial algebra [May67]. We shall explain these combinatorial models last, in Section 5. Until then, since $\lambda\mathtt{ev}Q_H$ is a bit too complex, we shall use the equivalent $\lambda_{\mathrm{S4}\,H}$-calculus [GL96a], a completion of Bierman and De Paiva's calculus [BdP92, BdP96] which we recapitulate in Section 2. The geometric realization of these combinatorial models (in the sense of Milnor's geometric realization for simplicial sets [May67]) yield nice topological spaces that we shall present in Section 4. We conclude in Section 6.

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2 Syntax

Although we don't really wish to delve into the details of syntax, we have to fix a choice: before we find models of intuitionistic S4 proof terms, we have to define what these proof terms are. We shall limit ourselves to a pretty minimal proof term language. We actually consider *minimal* intuitionistic S4, which captures the core of the logic: formulae, a.k.a. types, are defined by the grammar:

$$F ::= A \mid F \supset F \mid \Box F$$

where A ranges over a fixed set of so-called atoms, a.k.a. base types.

We shall use (for convenience) λ_{S4} as a language of proof terms for S4 [BdP92, BdP96, GL96a]. The raw terms of this language are defined by the grammar:

$$s, t ::= x \mid tt \mid \lambda x_F \cdot t \mid \mathsf{unbox}\, t \mid \mathsf{box}\, t\, \mathsf{with}\, \sigma$$

where σ is an *explicit substitution*, that is, a substitution that appears as an explicit component of terms. A substitution σ is any finite mapping from variables x_i to terms t_i, $1 \leq i \leq n$, and is written $\{x_1 := t_1, \ldots, x_n := t_n\}$; its *domain* $\mathrm{dom}\,\sigma$ is the set $\{x_1, \ldots, x_n\}$. The *yield* of σ is defined as $\bigcup_{x \in \mathrm{dom}\,\sigma} \mathrm{fv}(\sigma(x))$, mutually recursively with the set of free variables $\mathrm{fv}(t)$ of the term t, defined by: $\mathrm{fv}(x) \hat{=} \{x\}$, $\mathrm{fv}(st) \hat{=} \mathrm{fv}(s) \cup \mathrm{fv}(t)$, $\mathrm{fv}(\lambda x \cdot t) \hat{=} \mathrm{fv}(t) \setminus \{x\}$, $\mathrm{fv}(\mathsf{unbox}\, t) \hat{=} \mathrm{fv}(t)$, $\mathrm{fv}(\mathsf{box}\, t\, \mathsf{with}\, \sigma) \hat{=} \mathrm{yld}\,\sigma$. Moreover, we assume that, in any term of the form $\mathsf{box}\, t\, \mathsf{with}\, \sigma$, $\mathrm{fv}(t) \subseteq \mathrm{dom}\,\sigma$; we also assume Barendregt's naming convention, namely that no variable occurs both free and bound, or bound at two different places—bound variables are x in $\lambda x \cdot t$ and all variables in $\mathrm{dom}\,\sigma$ in $\mathsf{box}\, t\, \mathsf{with}\, \sigma$.

Substitution application $t\sigma$ is defined by: $x\sigma \hat{=} \sigma(x)$ if $x \in \mathrm{dom}\,\sigma$; $x\sigma \hat{=} x$ if $x \notin \mathrm{dom}\,\sigma$; $(st)\sigma \hat{=} (s\sigma)(t\sigma)$; $(\lambda x \cdot t)\sigma \hat{=} \lambda x \cdot (t\sigma)$ provided $x \notin \mathrm{dom}\,\sigma \cup \mathrm{yld}\,\sigma$; $(\mathsf{unbox}\, t)\sigma \hat{=} \mathsf{unbox}(t\sigma)$; $(\mathsf{box}\, t\, \mathsf{with}\, \sigma')\sigma \hat{=} \mathsf{box}\, t\, \mathsf{with}\, (\sigma' \cdot \sigma)$, where *substitution concatenation* $\sigma' \cdot \sigma$ is defined as $\{x_1 := t_1, \ldots, x_n := t_n\} \cdot \sigma \hat{=} \{x_1 := t_1\sigma, \ldots, x_n := t_n\sigma\}$.

Terms are equated modulo α-conversion, defined as the smallest congruence $\equiv$ such that:

$$\begin{array}{rcl} \lambda x_F \cdot t & \equiv & \lambda y_F \cdot (t\{x := y\}) \\ \mathsf{box}\, t\, \mathsf{with}\, \{x_1 := t_1, \ldots, x_n := t_n\} & \equiv & \mathsf{box}\, t\{x_1 := y_1, \ldots, x_n := y_n\}\, \mathsf{with}\, \{y_1 := t_1, \ldots, y_n := t_n\} \end{array}$$

provided the right-hand side is defined, and $y_1, \ldots, y_n$ are pairwise distinct variables in the second case.

All these definitions are a bit technical. The unbox operator is a kind of "eval", or also of "comma" operator in Lisp. To emphasize the analogy, we shall sometimes write $\mathsf{ev}(t)$ or $\texttt{,}t$ for $\mathsf{unbox}\, t$. The box operator is a bit more complex. Let's first define a special case of box: for any term t such that $\mathrm{fv}(t) = \{x_1, \ldots, x_n\}$, let $\text{‘}t$ be $\mathsf{box}\, t\, \mathsf{with}\, \{x_1 := x_1, \ldots, x_n := x_n\}$—to be formal, we should really write $\mathsf{box}\, t\{x_1 := x'_1, \ldots, x_n := x'_n\}\, \mathsf{with}\, \{x'_1 := x_1, \ldots, x'_n := x_n\}$, but this would be on the verge of being unreadable. Then $\text{‘}t$ behaves like "quote" t in Lisp, or more exactly, "backquote" t. This will become more apparent from the reduction rules and the typing rules below. Then, provided $\mathrm{dom}\,\sigma = \mathrm{fv}(t)$, $\mathsf{box}\, t\, \mathsf{with}\, \sigma$ is exactly $(\text{‘}t)\sigma$: this is a *syntactic closure* in the sense of [BR88], namely a quoted term t together with an environment σ mapping free variables of t to their values.

The typing rules, which encode a natural deduction system for minimal intuitionistic S4 are as follows [BdP92, BdP96], where Γ, Δ, ..., are *typing contexts*, which are lists of *bindings* of the form $x : F$, where x is a variable, F is a type, and no two bindings contain the same variable in any given context:

$$\frac{}{\Gamma, x : F, \Delta \vdash x : F}\ (Ax)$$

$$\frac{\Gamma \vdash s : F \supset G \qquad \Gamma \vdash t : F}{\Gamma \vdash st : G}\ (\supset E) \qquad \frac{\Gamma, x : F \vdash t : G}{\Gamma \vdash \lambda x_F \cdot t : F \supset G}\ (\supset I)$$

$$\frac{\Gamma \vdash t : \Box F}{\Gamma \vdash \mathsf{unbox}\, t : F}\ (\Box E) \qquad \frac{\Gamma \vdash t_i : \Box F_i, 1 \leq i \leq n \quad x_1 : \Box F_1, \ldots, x_n : \Box F_n \vdash t : G}{\Gamma \vdash \mathsf{box}\, t\, \mathsf{with}\, \{x_1 := t_1, \ldots, x_n := t_n\} : \Box G}\ (\Box I)$$

The *exchange rule*:

$$\frac{\Gamma, x : F, y : G, \Delta \vdash t : H}{\Gamma, y : G, x : F, \Delta \vdash t : H}$$

is easily seen to be admissible, so we can consider typing contexts as multisets instead of lists. In particular, this means that there is no choice to be made as to the order of the variables $x_1, \ldots, x_n$ in the context $x_1 : \Box F_1, \ldots, x_n : \Box F_n$ in the right premise of rule $(\Box I)$.

Note that, given Γ and t, there is at most one typing derivation of a judgment $\Gamma \vdash t : F$; in particular, the type F of t is unique when Γ is known.

Define the convertibility relation $=$ on λ_{S4}-terms as the smallest congruence such that [BdP92, BdP96, GL96a]:

$$\begin{array}{ll}
(\beta) & (\lambda x \cdot s)t = s\{x := t\} \qquad (\mathsf{unbox}) \quad \mathsf{unbox}(\mathsf{box}\, t \,\mathsf{with}\, \sigma) = t\sigma \\
(\text{gc}) & \mathsf{box}\, t \,\mathsf{with}\, \{x_1 := t_1, \ldots, x_n := t_n\} = \mathsf{box}\, t \,\mathsf{with}\, \{x_2 := t_2, \ldots, x_n := t_n\} \quad \text{provided } x_1 \notin \text{fv}(t) \\
(\text{ctr}) & \mathsf{box}\, t \,\mathsf{with}\, \{x_1 := t_1, x_2 := t_2, \ldots, x_n := t_n\} = \\
 & \qquad \mathsf{box}\, t\{x_1 := x_2\} \,\mathsf{with}\, \{x_2 := t_2, \ldots, x_n := t_n\} \quad \text{if } t_1 \equiv t_2 \\
(\mathsf{box}) & \mathsf{box}\, t \,\mathsf{with}\, \{x_1 := t_1, x_2 := t_2, \ldots, x_n := t_n\} = \\
 & \qquad \mathsf{box}\, t\{x_1 := {}^{\backprime}s\} \,\mathsf{with}\, \{y_1 := s_1, \ldots, y_m := s_m, x_2 := t_2, \ldots, x_n := t_n\} \\
 & \text{provided } t_1 \equiv \mathsf{box}\, s \,\mathsf{with}\, \{y_1 := s_1, \ldots, y_m := s_m\}
\end{array}$$

Rule (unbox) is much like Lisp's rule for evaluating quoted expressions: observe that it mostly states that evaluating ${}^{\backprime}t$, by $\mathsf{ev}({}^{\backprime}t)$, equals t. Rule (box) can be seen either as an inlining rule, allowing one to inline the definition of x_1 as ${}^{\backprime}s$ inside the body t of the box ${}^{\backprime}t$, or logically as a box-under-box commutation rule. (gc) is a garbage collection rule, while (ctr) is a contraction rule. Note that the choice of x_1 as distinguished variable in these three rules is not essential, and we might have chosen any other x_i, as substitutions are sets, and bindings $x_i := t_i$ permute. We don't care here about orienting these equations, as reduction semantics are not the purpose of this paper.

We shall also introduce the following extensional equalities:

$$\begin{array}{lll}
(\eta) & \lambda x_F \cdot tx = t & \text{provided } x \notin \text{fv}(t) \\
(\eta\, \mathsf{box}) & \mathsf{box}\,(\mathsf{unbox}\, x) \,\mathsf{with}\, \sigma = x\sigma & \text{for every variable } x
\end{array}$$

yielding an enriched notion of conversion that we write $=_\eta$. Note that the $(\eta\, \mathsf{box})$ rule is *not* of the form $\mathsf{box}\,(\mathsf{unbox}\, t) \,\mathsf{with}\, \sigma = t\sigma$ for any term t: t has to be a variable. See [GL96b] for a discussion of this. By the way, it will be easy to check that the more general equation $\mathsf{box}\,(\mathsf{unbox}\, t) \,\mathsf{with}\, \sigma = t\sigma$ won't hold in our models.

The idea of finding models of intuitionistic S4 proof terms is to find categories $\mathcal{C}$ with products, where types F are interpreted as objects $[\![F]\!]$, contexts $\Gamma \hat{=} x_1 : F_1, \ldots, x_n : F_n$ are interpreted as products $[\![\Gamma]\!] \hat{=} [\![F_1]\!] \times \ldots \times [\![F_n]\!]$, and each term t is interpreted as a family $[\![t]\!]$ of morphisms $[\![\Gamma \vdash t : F]\!]$ from $[\![\Gamma]\!]$ to $[\![F]\!]$, for every Γ and F such that $\Gamma \vdash t : F$ is derivable. Moreover, all equations $s = t$ between terms in λ_{S4}, resp. λ_{S4} with the extensional equalities, should induce corresponding equalities $[\![s]\!] = [\![t]\!]$ of interpretations. If an interpretation $[\![\text{-}]\!]$ satisfies all the above, we shall say that it is *sound* with respect to λ_{S4}, resp. λ_{S4} with the extensional equalities.

3 Some Order-Theoretic Models

Recall that a *dcpo* is any partial order $(\mathcal{F}, \leq)$ that is *directed-complete* in the sense that every non-empty directed subset E of $\mathcal{F}$ has a least upper bound $E \uparrow$. A subset E is *directed* provided that every two elements x, y of E have a least upper bound in E. We don't require our dcpos to be *pointed*, namely to have a bottom element. There are some pros and cons to this: we develop a model based on dcpos in Section 3.1, then modify it to use pointed dcpos instead in Section 3.2.

3.1 Using Dcpos, Without Bottom

Let us build our first interpretation, $[\![\text{-}]\!]_s$, in an informal way at first.

A function f from the dcpo $\mathcal{F}$ to the dcpo $\mathcal{G}$ is *continuous* if and only if f is monotonic and $f(E\uparrow) = f(E)\uparrow$ for every non-empty directed subset E of $\mathcal{F}$, where $f(E)\hat{=}\{f(v) \mid v \in E\}$. The set of all continuous functions from $\mathcal{F}$ to $\mathcal{G}$, ordered pointwise, is again a dcpo which we note $\mathcal{F} \to \mathcal{G}$.

Interpreting types as dcpos, we let $[\![F \supset G]\!]_s \hat{=} [\![F]\!]_s \to [\![G]\!]_s$. Application st is interpreted as application, or more formally $[\![\Gamma \vdash st : G]\!]_s$ is defined as the (continuous) function mapping every $g \in [\![\Gamma]\!]_s$ to $[\![\Gamma \vdash s : F \supset G]\!]_s(g)([\![\Gamma \vdash t : F]\!]_s(g))$, where F is the unique type such that $\Gamma \vdash t : F$ is derivable. Abstraction $\lambda x_F \cdot t$ is interpreted by: $[\![\Gamma \vdash \lambda x_F \cdot t : G]\!]_s$ is the (continuous) function mapping every $g \in [\![\Gamma]\!]_s$ to the (continuous) function mapping every $v \in [\![F]\!]_s$ to $[\![\Gamma, x : F \vdash t : G]\!]_s(g, v)$. (Somewhat abusively, we write (g, v) for the tuple $(v_1, \ldots, v_n, v)$ when $g = (v_1, \ldots, v_n)$; this abuse will help us keep the notation simple.) This is currification; indeed, the category of dcpos with continuous functions as morphisms is cartesian closed. (For more about cartesian-closedness and related concepts, see [LS86].)

Before we go on, let us say that variables are interpreted by letting $[\![x_1 : F_1, \ldots, x_n : F_n \vdash x_i : F_i]\!]_s$ map every $(v_1, \ldots, v_n)$ in $[\![F_1]\!]_s \times \ldots \times [\![F_n]\!]_s$ to v_i, which is the only natural choice.

For each dcpo $\mathcal{F}$, define $\mathcal{F}\langle 1\rangle$ as the following partially ordered set: the elements of $\mathcal{F}\langle 1\rangle$ are pairs (x, y) of elements of $\mathcal{F}$ such that $x \leq y$; the ordering on $\mathcal{F}\langle 1\rangle$ is defined by: $(x, y) \leq (x', y')$ if and only if $x = x'$ and $y \leq y'$. Note that the non-empty directed subsets of $\mathcal{F}\langle 1\rangle$ are the sets of pairs (x, y), where x is fixed and y ranges over some non-empty directed subset E of F such that $x \leq z$ for every $z \in E$. Every such directed subset has $(x, E\uparrow)$ as least upper bound, so $\mathcal{F}\langle 1\rangle$ is indeed a dcpo. Note, by the way, that $\mathcal{F}\langle 1\rangle$ has no bottom element in general, since $(\perp, \perp)$, the only candidate for a bottom, is incomparable with most elements of $\mathcal{F}\langle 1\rangle$.

We can think of any element (x, y) in $\mathcal{F}\langle 1\rangle$ as being a given initial value x for some program, which we shall sometimes understand as some syntactical description of this program, plus a *promise* that the program will eventually evaluate to y. As in usual Scott domain theory, y is more precise than x, i.e., $x \leq y$.

Now $\mathsf{unbox}\, t$ evaluates to the final value of the program t, that is, it evaluates the promise of the boxed value t: define $\mathsf{unbox}(x, y)$ as y. Formally, $[\![\Gamma \vdash \mathsf{unbox}\, t : F]\!]_s$ is the function mapping every $g \in [\![\Gamma]\!]_s$ to $\pi_2([\![\Gamma \vdash t : \Box F]\!]_s(g))$, where π_2 is the second projection. This function is continuous, because by assumption $[\![\Gamma \vdash t : \Box F]\!]_s$ is continuous, application is continuous, and π_2 is easily seen to be continuous.

Conversely, define quoting as follows: letting Δ be $x_1 : \Box F_1, \ldots, x_n : \Box F_n$, then $[\![\Delta \vdash `t : \Box G]\!]_s$ is the function mapping every $g\hat{=}((v_1, w_1), \ldots, (v_n, w_n))$ in $[\![\Delta]\!]_s$ to $([\![\Delta \vdash t : G]\!]_s((v_1, v_1), \ldots, (v_n, v_n)), [\![\Delta \vdash t : G]\!]_s(g))$ in $[\![\Box G]\!]_s$. More synthetically, let $\textsc{dup}(v)$ denote (v, v), let π_1 be the first projection, and for every function f, let $\textsc{map}\, f\ (x_1, \ldots, x_n)$ denote the tuple $(f(x_1), \ldots, f(x_n))$. Then $[\![\Delta \vdash `t : \Box G]\!]_s$ maps $g \in [\![\Delta]\!]_s$ to $([\![\Delta \vdash t]\!]_s(\textsc{map}(\textsc{dup} \circ \pi_1)(g)), [\![\Delta \vdash t]\!]_s(g))$. The interpretation of $\mathsf{box}\, t\, \mathsf{with}\, \sigma$ will follow by taking it to be the same as $(`x)\sigma$, namely: let σ be $\{x_1 := t_1, \ldots, x_n := t_n\}$, and assume that $\Gamma \vdash t_i : \Box F_i$ is derivable for all i, $1 \leq i \leq n$. Then we define $[\![\Gamma \vdash \mathsf{box}\, t\, \mathsf{with}\, \sigma : \Box G]\!]_s$ as the function mapping every $g \in [\![\Gamma]\!]_s$ to $([\![\Delta \vdash t : G]\!]_s(\textsc{map}(\textsc{dup} \circ \pi_1)(d)), [\![\Delta \vdash t : G]\!]_s(d))$, where $d\hat{=}([\![\Gamma \vdash t_1 : \Box F_1]\!]_s(g), \ldots, [\![\Gamma \vdash t_n : \Box F_n]\!]_s(g))$.

Now this really needs to be explained and justified. So let's look at some fundamental special cases. Recall that, in Hilbert-style presentation, S4 obeys the necessitation rule:

$$\text{If } \vdash F \text{ is provable, then } \vdash \Box F \text{ is, too.}$$

Given $\vdash t : F$, let $'t$ be a proof term for $\Box F$; in Lisp parlance, this is analogous to *quoting*, a special case of backquoting. A natural choice is to define $'t \hat{=} \mathsf{box}\, t\, \mathsf{with}\, \{\}$. Its interpretation is: $[\![\vdash {'t} : \Box F]\!]_s() = \textsc{dup}([\![\vdash t : F]\!]_s())$. In other words, quoting is duplication: quoting x returns x together with the promise that the final value will be exactly x, and no computation will occur when evaluating x.

It is important to note that $\textsc{dup}$ (quoting) is *not* continuous in general. In fact, $\textsc{dup}$ is not even monotonic in general: when $v \leq w$, $\textsc{dup}(v) \leq \textsc{dup}(w)$ only if $v = w$. This is normal: $F \supset \Box F$ is not provable in general in S4, so there does not need to be any continuous function in the Scott model from $[\![F]\!]_s$ to $[\![\Box F]\!]_s$.

S4 also obeys the axioms:

$$\begin{array}{ll} (K) & \Box(F \supset G) \supset \Box F \supset G \\ (T) & \Box F \supset F \\ (4) & \Box F \supset \Box\Box F \end{array}$$

The λ_{S4}-calculus provides standard proof-terms for these formulae. In the case of (T), this is unbox, or more precisely, $\lambda x_{\Box F} \cdot \mathsf{unbox}\, x$, which we have already discussed. For (K), take $\lambda x_{\Box(F \supset G)} \cdot \lambda y_{\Box F} \cdot$

$\mathsf{box}\,(\mathsf{unbox}\,x_1)(\mathsf{unbox}\,x_2)$ with $\{x_1 := x, x_2 := y\}$, that is, $\lambda x_{\Box(F\supset G)} \cdot \lambda y_{\Box F} \cdot$ `‘(,x,y)` in Lisp-like notation. The interpretation of this term is the function that maps $(f,g) \in [\![\Box(F \supset G)]\!]_s$ to the function that maps $(x,y) \in [\![\Box F]\!]_s$ to $(f(x), g(y))$. This is easily seen to be continuous.

Finally, for (4), we take $\lambda x_{\Box F} \cdot \mathsf{box}\,x'$ with $\{x' := x\}$, or more informally $\lambda x_{\Box F} \cdot {}`x$. This is a kind of analogue to Lisp's `kwote` function, which takes a value and returns a quoted term that evaluates to this very value. Indeed, in the model, this term is the function that maps $(x,y) \in [\![\Box F]\!]_s$ to $((x,x),(x,y))$ in $[\![\Box\Box F]\!]_s$. Informally, given a program x together with a promise that x will evaluate to y, $\mathtt{kwote}(x,y)$ consists of the program $'x = (x,x)$ together with the promise that it will evaluate to a program x whose promise is y.

This interpretation is summed up in Figure 1, together with an extension of the notation $\mathcal{F}\langle 1\rangle$ to $\mathcal{F}\langle n\rangle$ for every $n \geq 0$; this will be discussed later on. It is assumed that all interpretations $[\![A]\!]_s$, where A is a base type, are given.

$$
\begin{array}{rcll}
[\![F \supset G]\!]_s & \hat{=} & [\![F]\!]_s \to [\![G]\!]_s & \\
\mathcal{F} \to \mathcal{G} & \hat{=} & \{\text{continuous functions from } \mathcal{F} \text{ to } \mathcal{G}\} & \text{pointwise ordering} \\
[\![\Box F]\!]_s & \hat{=} & [\![F]\!]_s\langle 1\rangle & \\
\mathcal{F}\langle n\rangle & \hat{=} & \{(v_{n-1},\ldots,v_0,v_{-1}) \mid v_{n-1} \leq \ldots \leq v_0 \leq v_{-1} \in \mathcal{F}\} & \\
 & & (v_{n-1},\ldots,v_0,v_{-1}) \leq (w_{n-1},\ldots,w_0,w_{-1}) \text{ iff } v_{n-1} = w_{n-1},\ldots,v_0 = w_0, v_{-1} \leq w_{-1} &
\end{array}
$$

$$
\begin{array}{rcl}
[\![\Gamma, x : F, \Delta \vdash x : F]\!]_s(g,v,d) & \hat{=} & v \\
[\![\Gamma \vdash st : G]\!]_s(g) & \hat{=} & [\![\Gamma \vdash s : F \supset G]\!]_s(g)([\![\Gamma \vdash t : F]\!]_s(g)) \\
[\![\Gamma \vdash \lambda x_F \cdot t : F \supset G]\!]_s(g) & \hat{=} & \lambda v \in [\![F]\!]_s \cdot [\![\Gamma, x : F \vdash t : G]\!]_s(g,v) \\
[\![\Gamma \vdash \mathsf{unbox}\,t : F]\!]_s(g) & \hat{=} & \pi_2([\![\Gamma \vdash t : \Box F]\!]_s(g)) \\
[\![\Gamma \vdash \mathsf{box}\,t \;\mathsf{with}\; \sigma : \Box G]\!]_s(g) & \hat{=} & ([\![\Delta \vdash t : G]\!]_s(\text{MAP}(\text{DUP} \circ \pi_1)(d)), [\![\Delta \vdash t : G]\!]_s(d)) \\
 & & \text{where } \Delta \hat{=} x_1 : \Box F_1, \ldots, x_n : \Box F_n,\ \sigma \hat{=} \{x_1 := t_1, \ldots, x_n := t_n\} \\
 & & \text{and } d \hat{=} ([\![\Gamma \vdash t_1 : \Box F_1]\!]_s(g), \ldots, [\![\Gamma \vdash t_n : \Box F_n]\!]_s(g))
\end{array}
$$

Figure 1: The Dcpo Interpretation

We now prove that we have got a model of λ_{S4}, as expected:

Lemma 3.1 *For every term t such that $\Gamma \vdash t : F$ is derivable, $[\![\Gamma \vdash t : F]\!]_s$ is a continuous function from $[\![\Gamma]\!]_s$ to $[\![F]\!]_s$.*

Proof: We first claim that: (i) for every continuous function f from $\mathcal{F}_1\langle 1\rangle \times \ldots \times \mathcal{F}_n\langle 1\rangle$ to $\mathcal{F}$, the function f' mapping $((v_1,w_1),\ldots,(v_n,w_n)) \in \mathcal{F}_1\langle 1\rangle \times \ldots \times \mathcal{F}_n\langle 1\rangle$ to $(f((v_1,v_1),\ldots,(v_n,v_n)), f((v_1,w_1),\ldots,(v_n,w_n)))$ in $\mathcal{F}\langle 1\rangle$ is continuous. First, f' is monotonic: if $((v_1,w_1),\ldots,(v_n,w_n)) \leq ((v'_1,w'_1),\ldots,(v'_n,w'_n))$, then $v_1 = v'_1$, $w_1 \leq w'_1$, …, $v_n = v'_n$, $w_n \leq w'_n$. So $f((v_1,v_1),\ldots,(v_n,v_n)) = f((v'_1,v'_1),\ldots,(v'_n,v'_n))$; and on the other hand $f((v_1,w_1),\ldots,(v_n,w_n)) \leq f((v'_1,w'_1),\ldots,(v'_n,w'_n))$ since f is monotonic. So indeed $f'((v_1,w_1),\ldots,(v_n,w_n)) \leq f'((v'_1,w'_1),\ldots,(v'_n,w'_n))$. Second, f' preserves least upper bounds: let E be a non-empty directed subset of $\mathcal{F}_1\langle 1\rangle \times \ldots \times \mathcal{F}_n\langle 1\rangle$, we must show that $f'(E\uparrow) = f'(E)\uparrow$. But E is necessarily a set of values of the form $((v_1,w_1),\ldots,(v_n,w_n))$, with v_1, …, v_n fixed, and $(w_1,\ldots,w_n)$ ranging over some non-empty directed subset E_0 of $\mathcal{F}_1 \times \ldots \times \mathcal{F}_n$. Since v_1, …, v_n are fixed, so is $f((v_1,v_1),\ldots,(v_n,v_n))$. And since f is continuous, $f(E\uparrow) = f(E)\uparrow$, so:

$$
\begin{array}{rcl}
f'(E\uparrow) & = & (f((v_1,v_1),\ldots,(v_n,v_n)), f(E\uparrow)) \\
 & = & (f((v_1,v_1),\ldots,(v_n,v_n)), f(E)\uparrow) \\
 & = & \{(f((v_1,v_1),\ldots,(v_n,v_n)), f((v_1,w_1),\ldots,(v_n,w_n))) \mid (w_1,\ldots,w_n) \in E_0\}\uparrow \\
 & = & f'(E)\uparrow
\end{array}
$$

The Lemma is then proved by structural induction on t. All the cases except when t is a box-term have already been justified. So let t be of the form $\mathsf{box}\,s$ with $\{x_1 := t_1, \ldots, x_n := t_n\}$, where $\Delta \vdash s : F$ is provable and $\Delta \hat{=} x_1 : \Box F_1, \ldots, x_n : \Box F_n$. By induction hypothesis, $[\![\Delta \vdash s : F]\!]_s$ is continuous. Taking this function as the f in (i), it follows that the function f' mapping d to $([\![\Delta \vdash s : F]\!]_s(\text{MAP}(\text{DUP} \circ \pi_1)(d)), [\![\Delta \vdash s : F]\!]_s(d))$ is continuous. Therefore $[\![\Gamma \vdash \mathsf{box}\, s \,\mathsf{with}\, \{x_1 := t_1, \ldots, x_n := t_n\} : \Box F]\!]_s$, which is the function mapping $g \in [\![\Gamma]\!]_s$ to $f'([\![\Gamma \vdash t_1 : \Box F_1]\!]_s(g), \ldots, [\![\Gamma \vdash t_n : \Box F_n]\!]_s(g))$ is continuous, since $[\![\Gamma \vdash t_i : \Box F_i]\!]_s$ is continuous for all i, $1 \leq i \leq n$, by induction hypothesis. □

Lemma 3.2 *For every term t such that $\Gamma, x_1 : F_1, \ldots, x_n : F_n \vdash t : F$ is derivable, for every terms t_1, ..., t_n such that $\Gamma, \Delta \vdash t_i : F_i$ is derivable for every i, $1 \leq i \leq n$, $[\![\Gamma, \Delta \vdash t\{x_1 := t_1, \ldots, x_n := t_n\} : F]\!]_s$ is the function mapping (g, d), where $g \in [\![\Gamma]\!]_s$ and $d \in [\![\Delta]\!]_s$, to :*

$$[\![\Gamma, x_1 : F_1, \ldots, x_n : F_n \vdash t : F]\!]_s(g, [\![\Gamma, \Delta \vdash t_1 : F_1]\!]_s(g, d), \ldots, [\![\Gamma, \Delta \vdash t_n : F_n]\!]_s(g, d))$$

Proof: By structural induction on t. □

Lemma 3.3 *If x is not free in t, then for every $g \in [\![\Gamma]\!]_s$, $v \in [\![F]\!]_s$, $d \in [\![\Delta]\!]_s$, $[\![\Gamma, x : F, \Delta \vdash t : G]\!]_s(g, v, d) = [\![\Gamma, \Delta \vdash t : G]\!]_s(g, d)$.*

Proof: Easy structural induction on t. □

Theorem 3.4 *The dcpo interpretation is sound wrt. λ_{S4} with the extensional equalities: for every terms s and t such that $\Gamma \vdash s : F$ and $\Gamma \vdash t : F$ are both derivable, and such that $s =_\eta t$, we have $[\![\Gamma \vdash s : F]\!]_s = [\![\Gamma \vdash t : F]\!]_s$.*

Proof: We first check each α-equivalence rule:

- $\equiv$, first rule.

$$\begin{aligned}
[\![\Gamma \vdash \lambda y_F \cdot t\{x := y\} : F \supset G]\!]_s(g) &= \lambda v \in [\![F]\!]_s \cdot [\![\Gamma, y : F \vdash t\{x := y\} : G]\!]_s(g, v) \\
&= \lambda v \in [\![F]\!]_s \cdot [\![\Gamma, x : F \vdash t : G]\!]_s(g, [\![\Gamma, y : F \vdash y : F]\!]_s(g, v)) \\
&\qquad \text{(by Lemma 3.2 with } \Delta \hat{=} y : F,\ n \hat{=} 1,\ x_1 \hat{=} x,\ t_1 \hat{=} y) \\
&= \lambda v \in [\![F]\!]_s \cdot [\![\Gamma, x : F \vdash t : G]\!]_s(g, v) = [\![\Gamma \vdash \lambda x_F \cdot t : F \supset G]\!]_s(g)
\end{aligned}$$

- $\equiv$, second rule. $[\![\Gamma \vdash \mathsf{box}\, t\{x_1 := y_1, \ldots, x_n := y_n\} \,\mathsf{with}\, \{y_1 := t_1, \ldots, y_n := t_n\} : \Box G]\!]_s$ is the function mapping every $g \in [\![\Gamma]\!]_s$ to $[\![\Delta \vdash t\{x_1 := y_1, \ldots, x_n := y_n\} : G]\!]_s(\text{MAP}(\text{DUP} \circ \pi_1)(d), d)$, where $d \hat{=} ([\![\Gamma \vdash t_1 : \Box F_1]\!]_s(g), \ldots, [\![\Gamma \vdash t_n : \Box F_n]\!]_s(g))$ and $\Delta \hat{=} y_1 : \Box F_1, \ldots, y_n : \Box F_n$. By Lemma 3.2 with Γ empty, this is also the same as $[\![x_1 : \Box F_1, \ldots, x_n : \Box F_n \vdash t : G]\!]_s(\text{MAP}(\text{DUP} \circ \pi_1)(d), d)$, that is, $[\![\Gamma \vdash \mathsf{box}\, t \,\mathsf{with}\, \{x_1 := t_1, \ldots, x_n := t_n\}]\!]_s(d)$.

An easy induction on the number of α-equivalence rules, then on the structure of terms, now shows that: (i) $s \equiv t$ implies $[\![\Gamma \vdash s : F]\!]_s(g) = [\![\Gamma \vdash t : F]\!]_s(g)$. We have just shown the base cases, the induction cases are straightforward.

- (β):

$$\begin{aligned}
[\![\Gamma \vdash (\lambda x_F \cdot s)t : G]\!]_s(g) &= [\![\Gamma \vdash \lambda x_F \cdot s : F \supset G]\!]_s(g)([\![\Gamma \vdash t : F]\!]_s(g)) \\
&= (\lambda v \in [\![F \supset G]\!]_s \cdot [\![\Gamma, x : F \vdash s : G]\!]_s(g, v))([\![\Gamma \vdash t : F]\!]_s(g)) \\
&= [\![\Gamma, x : F \vdash s : G]\!]_s(g, [\![\Gamma \vdash t : F]\!]_s(g)) = [\![\Gamma \vdash s\{x := t\} : G]\!]_s(g)
\end{aligned}$$

  by Lemma 3.2 with Δ empty, $n \hat{=} 1$, $x_1 \hat{=} x$, $t_1 \hat{=} t$.

- (unbox):

$$\begin{aligned}
&[\![\Gamma \vdash \mathsf{unbox}(\mathsf{box}\, t \,\mathsf{with}\, \sigma) : G]\!]_s(g) \\
=\ & \pi_2([\![x_1 : \Box F_1, \ldots, x_n : \Box F_n \vdash t : G]\!]_s(\text{MAP}(\text{DUP} \circ \pi_1)(d)), [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n \vdash t : G]\!]_s(d)) \\
&\quad \text{where } d \hat{=} ([\![\Gamma \vdash t_1 : \Box F_1]\!]_s(g), \ldots, [\![\Gamma \vdash t_n : \Box F_n]\!]_s(g)) \\
=\ & [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n \vdash t : G]\!]_s(d)
\end{aligned}$$

On the other hand:

$$\begin{array}{rl} & [\![\Gamma \vdash t\sigma : G]\!]_s(g) \\ = & [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n \vdash t : G]\!]_s([\![\Gamma \vdash t_1 : \Box F_1]\!]_s(g), \ldots, [\![\Gamma \vdash t_n : \Box F_n]\!]_s(g)) \quad \text{(by Lemma 3.2)} \\ = & [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n \vdash t : G]\!]_s(d) \end{array}$$

- (gc): assume that x_1 is not free in t, then $[\![\Gamma \vdash \mathsf{box}\, t \;\mathsf{with}\; \{x_1 := t_1, \ldots, x_n := t_n\} : \Box G]\!]_s(g)$ equals $[\![x_1 : \Box F_1, x_2 : \Box F_2, \ldots, x_n : \Box F_n \vdash t : G]\!]_s([\![\Gamma \vdash t_1 : \Box F_1]\!]_s(g), \ldots, [\![\Gamma \vdash t_n : \Box F_n]\!]_s(g))$; this is equal to $[\![x_2 : \Box F_2, \ldots, x_n : \Box F_n \vdash t : G]\!]_s([\![\Gamma \vdash t_2 : \Box F_2]\!]_s(g), \ldots, [\![\Gamma \vdash t_n : \Box F_n]\!]_s(g))$ by Lemma 3.3; but the latter is just $[\![\Gamma \vdash \mathsf{box}\, t \;\mathsf{with}\; \{x_2 := t_2, \ldots, x_n := t_n\}]\!]_s(g)$.

- (ctr): assume $t_1 \equiv t_2$:

$$\begin{array}{rl} & [\![\Gamma \vdash \mathsf{box}\, t\{x_1 := x_2\} \;\mathsf{with}\; \{x_2 := t_2, \ldots, x_n := t_n\} : \Box G]\!]_s(g) \\ = & ([\![x_2 : \Box F_2, \ldots, x_n : \Box F_n \vdash t\{x_1 := x_2\} : G]\!]_s((\textsc{dup}(\pi_1(v_2)), \ldots, \textsc{dup}(\pi_1(v_n))), \\ & \;[\![x_2 : \Box F_2, \ldots, x_n : \Box F_n \vdash t\{x_1 := x_2\} : G]\!]_s(v_2, \ldots, v_n)) \\ & \qquad \text{where } v_2 \hat{=} [\![\Gamma \vdash t_2 : \Box F_2]\!]_s(g), \ldots, v_n \hat{=} [\![\Gamma \vdash t_n : \Box F_n]\!]_s(g)) \\ = & ([\![x_1 : \Box F_1, x_2 : \Box F_2, \ldots, x_n : \Box F_n \vdash t : G]\!]_s((\textsc{dup}(\pi_1(v_2)), \textsc{dup}(\pi_1(v_2)), \ldots, \textsc{dup}(\pi_1(v_n))), \\ & \;[\![x_1 : \Box F_1, x_2 : \Box F_2, \ldots, x_n : \Box F_n \vdash t : G]\!]_s(v_2, v_2, \ldots, v_n)) \qquad \text{(by Lemma 3.2)} \\ = & [\![\Gamma \vdash \mathsf{box}\, t \;\mathsf{with}\; \{x_1 := t_1, x_2 := t_2, \ldots, x_n := t_n\} : \Box G]\!]_s(g) \end{array}$$

since indeed $t_1 \equiv t_2$ implies $[\![\Gamma \vdash t_1 : F_1]\!]_s(g) = [\![\Gamma \vdash t_2 : F_2]\!]_s(g) = v_2$ by (i).

- (box): we shall show that the interpretations of $(`s)\{x := `t\}$ and of $`(s\{x := `t\})$ are equal. This special case of (box) will be enough to deal with the general case: indeed, (box) is deducible from the latter, from the fact that $s = t$ implies $s\sigma = t\sigma$ for every well-typed substitution σ, and from (gc).

  So, assume that the free variables of s are among x, x_1, …, x_n, and that $x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G$, and $y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t : F$. Then:

$$\begin{array}{rl} & [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash (`s)\{x := `t\} : \Box G]\!]_s(v_1, \ldots, v_n, w_1, \ldots, w_m) \\ = & [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash `s : \Box G]\!]_s(v_1, \ldots, v_n, \\ & \qquad [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash `t : \Box F]\!]_s(v_1, \ldots, v_n, w_1, \ldots, w_m)) \\ = & [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash `s : \Box G]\!]_s(v_1, \ldots, v_n, \\ & \qquad [\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash `t : \Box F]\!]_s(w_1, \ldots, w_m)) \qquad \text{(by Lemma 3.3, } n \text{ times)} \\ = & [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash `s : \Box G]\!]_s(v_1, \ldots, v_n, \\ & \qquad ([\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t : F]\!]_s(\textsc{dup}(\pi_1(w_1)), \ldots, \textsc{dup}(\pi_1(w_m))), \\ & \qquad \;[\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t : F]\!]_s(w_1, \ldots, w_m))) \\ = & ([\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_s(\textsc{dup}(\pi_1(v_1)), \ldots, \textsc{dup}(\pi_1(v_n)), \\ & \qquad \textsc{dup}([\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t : F]\!]_s(\textsc{dup}(\pi_1(w_1)), \ldots, \textsc{dup}(\pi_1(w_m))))), \\ & \;[\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_s(v_1, \ldots, v_n, \\ & \qquad ([\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t : F]\!]_s(\textsc{dup}(\pi_1(w_1)), \ldots, \textsc{dup}(\pi_1(w_m))), \\ & \qquad \;[\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t : F]\!]_s(w_1, \ldots, w_m))) \end{array}$$

On the other hand:

$$\begin{array}{rl} & [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash `(s\{x := `t\}) : \Box G]\!]_s(v_1, \ldots, v_n, w_1, \ldots, w_m) \\ = & ([\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash s\{x := `t\} : G]\!]_s \\ & \qquad (\textsc{dup}(\pi_1(v_1)), \ldots, \textsc{dup}(\pi_1(v_n)), \textsc{dup}(\pi_1(w_1)), \ldots, \textsc{dup}(\pi_1(w_m))), \end{array}$$

$$
\begin{aligned}
& [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash s\{x := \text{`}t\} : G]\!]_s(v_1, \ldots, v_n, w_1, \ldots, w_m)) \\
= \; & ([\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_s(\textsc{dup}(\pi_1(v_1)), \ldots, \textsc{dup}(\pi_1(v_n)), \\
& \quad [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash \text{`}t : \Box F]\!]_s \\
& \qquad (\textsc{dup}(\pi_1(v_1)), \ldots, \textsc{dup}(\pi_1(v_n)), \textsc{dup}(\pi_1(w_1)), \ldots, \textsc{dup}(\pi_1(w_m)))), \\
& [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_s(v_1, \ldots, v_n, \\
& \quad [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash \text{`}t : \Box F]\!]_s(v_1, \ldots, v_n, w_1, \ldots, w_m))) \\
= \; & ([\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_s(\textsc{dup}(\pi_1(v_1)), \ldots, \textsc{dup}(\pi_1(v_n)), \\
& \quad [\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash \text{`}t : \Box F]\!]_s(\textsc{dup}(\pi_1(w_1)), \ldots, \textsc{dup}(\pi_1(w_m)))), \\
& [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_s(v_1, \ldots, v_n, \\
& \quad [\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash \text{`}t : \Box F]\!]_s(w_1, \ldots, w_m))) \\
& \quad \text{(by Lemma 3.3 } 2n \text{ times)} \\
= \; & ([\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_s(\textsc{dup}(\pi_1(v_1)), \ldots, \textsc{dup}(\pi_1(v_n)), \\
& \quad ([\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t]\!]_s(\textsc{dup}(\pi_1(\textsc{dup}(\pi_1(w_1)))), \ldots, \textsc{dup}(\pi_1(\textsc{dup}(\pi_1(w_m))))), \\
& \quad [\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t]\!]_s(\textsc{dup}(\pi_1(w_1)), \ldots, \textsc{dup}(\pi_1(w_m))))), \\
& [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_s(v_1, \ldots, v_n, \\
& \quad ([\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t]\!]_s(\textsc{dup}(\pi_1(w_1)), \ldots, \textsc{dup}(\pi_1(w_m))), \\
& \quad [\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t]\!]_s(w_1, \ldots, w_m)))) \\
= \; & ([\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_s(\textsc{dup}(\pi_1(v_1)), \ldots, \textsc{dup}(\pi_1(v_n)), \\
& \quad ([\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t]\!]_s(\textsc{dup}(\pi_1(w_1)), \ldots, \textsc{dup}(\pi_1(w_m))), \\
& \quad [\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t]\!]_s(\textsc{dup}(\pi_1(w_1)), \ldots, \textsc{dup}(\pi_1(w_m))))), \\
& [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_s(v_1, \ldots, v_n, \\
& \quad ([\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t]\!]_s(\textsc{dup}(\pi_1(w_1)), \ldots, \textsc{dup}(\pi_1(w_m))), \\
& \quad [\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t]\!]_s(w_1, \ldots, w_m)))) \\
& \quad \text{(because } \textsc{dup} \circ \pi_1 \circ \textsc{dup} \circ \pi_1 = \textsc{dup} \circ \pi_1 \text{)} \\
= \; & [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash (\text{`}s)\{x := \text{`}t\} : \Box G]\!]_s(v_1, \ldots, v_n, w_1, \ldots, w_m)
\end{aligned}
$$

- (η): assume that $\Gamma \vdash t : F \supset G$ is derivable, and x is not free in t, then $[\![\Gamma \vdash \lambda x_F \cdot tx : F \supset G]\!]_s(g)$ is the function mapping every $v \in [\![F]\!]_s$ to:

$$
\begin{aligned}
[\![\Gamma, x : F \vdash tx : G]\!]_s(g, v) &= [\![\Gamma, x : F \vdash t : F \supset G]\!]_s(g, v)([\![\Gamma, x : F \vdash x : F]\!]_s(g, v)) \\
&= [\![\Gamma, x : F \vdash t : F \supset G]\!]_s(g, v)(v) = [\![\Gamma \vdash t : F \supset G]\!]_s(g)(v)
\end{aligned}
$$

  by Lemma 3.3. So $[\![\Gamma \vdash \lambda x_F \cdot tx : F \supset G]\!]_s(g) = [\![\Gamma \vdash t : F \supset G]\!]_s(g)$, for every g.

- ($\eta\,\mathsf{box}$): let $\sigma \hat{=} \{x_1 := t_1, \ldots, x_n : t_n\}$, and assume that $\Gamma \vdash t_i : \Box F_i$ is derivable for each i, $1 \leq i \leq n$; then:

$$
\begin{aligned}
& [\![\Gamma \vdash \mathsf{box}\,(\mathsf{unbox}\, x_i)\ \mathsf{with}\ \sigma : \Box F]\!]_s(g) \\
= \; & ([\![x_1 : \Box F_1, \ldots, x_n : \Box F_n \vdash \mathsf{unbox}\, x_i : F]\!]_s \\
& \qquad (\textsc{dup}(\pi_1([\![\Gamma \vdash t_1 : \Box F_1]\!]_s(g))), \ldots, \textsc{dup}(\pi_1([\![\Gamma \vdash t_n : \Box F_n]\!]_s(g)))), \\
& [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n \vdash \mathsf{unbox}\, x_i : F]\!]_s([\![\Gamma \vdash t_1 : \Box F_1]\!]_s(g), \ldots, [\![\Gamma \vdash t_n : \Box F_n]\!]_s(g))) \\
= \; & (\pi_2(\textsc{dup}(\pi_1([\![\Gamma \vdash t_i : \Box F_i]\!]_s(g)))), \pi_2([\![\Gamma \vdash t_i : \Box F_i]\!]_s(g))) \\
= \; & (\pi_1([\![\Gamma \vdash t_i : \Box F_i]\!]_s(g)), \pi_2([\![\Gamma \vdash t_i : \Box F_i]\!]_s(g))) \\
= \; & [\![\Gamma \vdash t_i : \Box F_i]\!]_s(g)
\end{aligned}
$$

Then, as for (i), an easy induction, whose base cases we have just examined, shows that $s = t$ implies $[\![\Gamma \vdash s : F]\!]_s(g) = [\![\Gamma \vdash t : F]\!]_s$. $\Box$

Before we go on, notice that the fact that we used dcpos instead of mere cpos does not matter: everything still works with cpos instead of dcpos. In fact, even the continuity assumptions on functions were completely superfluous in this section, and we could have developed the same theory by only considering the category of preorders with monotonic functions, instead of dcpos with continuous functions. Continuity will play a more essential role in the next section.

3.2 Using Pointed Dcpos

For computational relevance, it would be nice to be able to define functions by fixpoints in any type. This is possible in ordinary dcpo theory by having each domain $\mathcal{F}$ contain a bottom element $\perp$, and defining the least fixpoint of $f : \mathcal{F} \to \mathcal{F}$ as $Y(f) \hat{=} \{\perp, f(\perp), f^2(\perp), \ldots, f^n(\perp), \ldots\} \uparrow$. However, our dcpos don't have bottoms in general, in particular no dcpo of the form $\mathcal{F}\langle 1 \rangle$ has a bottom.

A natural fix is to *lift* the latter dcpos, and define $[\![\Box F]\!]_p$ as $[\![F]\!]_p\langle 1 \rangle_\perp$, where $\mathcal{F}_\perp$ denotes the disjoint union $\mathcal{F} \cup \{\perp\}$, with the ordering $\leq_{\mathcal{F}_\perp}$ such that $\perp \leq_{\mathcal{F}_\perp} x$ for all x, and $x \leq_{\mathcal{F}_\perp} y$ if and only if $x \leq y$, for all $x, y \in \mathcal{F}$, where $\leq$ is the ordering on $\mathcal{F}$.

All our dcpos will then be *pointed*, i.e. they will have a bottom. The modified interpretation $[\![\text{-}]\!]_p$ is defined on Figure 2.

$$
\begin{array}{rcl}
[\![F \supset G]\!]_p & \hat{=} & [\![F]\!]_p \to [\![G]\!]_p \\
[\![\Box F]\!]_p & \hat{=} & [\![F]\!]_p\langle 1 \rangle_\perp
\end{array}
$$

$$
\begin{array}{rcl}
[\![\Gamma, x : F, \Delta \vdash x : F]\!]_p(g, v, d) & \hat{=} & v \\
[\![\Gamma \vdash st : G]\!]_p(g) & \hat{=} & [\![\Gamma \vdash s : F \supset G]\!]_p(g)([\![\Gamma \vdash t : F]\!]_p(g)) \\
[\![\Gamma \vdash \lambda x_F \cdot t : F \supset G]\!]_p(g) & \hat{=} & \lambda v \in [\![F]\!]_p \cdot [\![\Gamma, x : F \vdash t : G]\!]_p(g, v) \\
[\![\Gamma \vdash \mathsf{unbox}\, t : F]\!]_p(g) & \hat{=} & \pi_2^\perp([\![\Gamma \vdash t : \Box F]\!]_p(g)) \qquad \text{where } \pi_2^\perp(v, w) \hat{=} w,\ \pi_2^\perp(\perp) \hat{=} \perp \\
[\![\Gamma \vdash \mathsf{box}\, t\ \mathsf{with}\ \sigma : \Box G]\!]_p(g) & \hat{=} & ([\![\Delta \vdash t : G]\!]_p(\textsc{map}(\textsc{dup} \circ \pi_1^\perp)(d)), [\![\Delta \vdash t : G]\!]_p(d)) \\
 & & \text{where } \Delta \hat{=} x_1 : \Box F_1, \ldots, x_n : \Box F_n,\ \sigma \hat{=} \{x_1 := t_1, \ldots, x_n := t_n\} \\
 & & \text{and } d \hat{=} ([\![\Gamma \vdash t_1 : \Box F_1]\!]_p(g), \ldots, [\![\Gamma \vdash t_n : \Box F_n]\!]_p(g)) \\
 & & \text{and } \pi_1^\perp(v, w) \hat{=} w,\ \pi_1^\perp(\perp) \hat{=} \perp
\end{array}
$$

Figure 2: The Pointed Dcpo Interpretation

So quoting still maps v to (v, v), even when $v = \perp$. On the other hand, unbox is a strict function, mapping $\perp$ to $\perp$. The proof term $\lambda x_{\Box(F \supset G)} \cdot \lambda y_{\Box F} \cdot \mathsf{box}\,(\mathsf{unbox}\, x_1)(\mathsf{unbox}\, x_2)\ \mathsf{with}\ \{x_1 := x, x_2 := y\}$ realizing (K) still maps (f, g) and (v, w) to $(f(v), g(w))$; it now also maps $\perp$ and (v, w) to $(\perp, \perp)$, $\perp$ and $\perp$ to $(\perp, \perp)$, and (f, g) and $\perp$ to $(f(\perp), g(\perp))$. In particular, it is not strict, i.e. it is lazy in both its arguments. Similarly, the proof term $\lambda x_{\Box F} \cdot \mathsf{box}\, x'\ \mathsf{with}\ \{x' := x\}$ for (4) maps (v, w) to $((v, v), (v, w))$, and $\perp$ to $(\perp, \perp)$, so again `kwote` is given a lazy interpretation.

We leave it as an exercise to the reader to show that this interpretation is sound wrt. λ_{S4}, and even (η), but not $(\eta\, \mathsf{box})$. Indeed, $[\![\Gamma \vdash \mathsf{box}\,(\mathsf{unbox}\, x)\ \mathsf{with}\ \{x := t\} : \Box F]\!]_p(g)$ equals $(\perp, \perp)$ when $[\![\Gamma \vdash t : \Box F]\!]_p(g) = \perp$, and is therefore different from $[\![\Gamma \vdash t : \Box F]\!]_p(g)$.

Adding $\perp$ actually leaves more choices as to the definition of modal constructions. We may for example decide that (K) will be given an interpretation that is strict in its first argument, or that (4) will be strict, and so on.

3.3 A Note on $\Box^n F$

Let's return for a short while to the unpointed dcpo interpretation. The types $\Box^n F$—meaning F boxed n times—gets interpreted as $[\![F]\!]_s\langle 1 \rangle \ldots \langle 1 \rangle$. But it is easy to see that this is isomorphic to $[\![F]\!]_s\langle n \rangle$:

Lemma 3.5 *$\mathcal{F}\langle 1 \rangle \ldots \langle 1 \rangle$, where there are n occurrences of $\langle 1 \rangle$, is canonically isomorphic to $\mathcal{F}\langle n \rangle$.*

Proof: By induction on n. It is enough to show that $\mathcal{F}\langle n+1\rangle$ is canonically isomorphic to $\mathcal{F}\langle n\rangle\langle 1\rangle$. The former is the set of non-decreasing sequences $(v_n, \ldots, v_1, v_0, v_{-1})$, ordered by $\leq$ on the v_{-1} component, and by equality on all other components. The latter is the space of couples $((v_{n-1}, \ldots, v_0, v_{-1}), (v_{n-1}, \ldots, v_0, v'_{-1}))$, with $v_{n-1} \leq \ldots \leq v_0 \leq v_{-1} \leq v'_{-1}$, ordered by $\leq$ on $(v_{n-1}, \ldots, v_0, v'_{-1})$ and by equality of $(v_{n-1}, \ldots, v_0, v_{-1})$, i.e. by $\leq$ on v'_{-1} and equality on all the v_i's. Map the former sequences $(v_n, \ldots, v_1, v_0, v_{-1})$ to $((v_n, \ldots, v_1, v_0), (v_n, \ldots, v_1, v_{-1}))$, and conversely map $((v_{n-1}, \ldots, v_0, v_{-1}), (v_{n-1}, \ldots, v_0, v'_{-1}))$ to $(v_{n-1}, \ldots, v_0, v_{-1}, v'_{-1})$. It is easy to see that this defines an isomorphism of dcpos. □

Now the point here is that the definition of the spaces $\mathcal{F}\langle n\rangle$, $n \geq 0$, looks very much like that of the *nerve* of a category [GM96].

The notion of nerve applies to all categories, but it will suffice to explain it in the case of preorders. The *nerve* of a preorder $\mathcal{F}$ is a graded set $N_n(\mathcal{F})$, $n \geq 0$, consisting of all non-decreasing sequences $(v_0, \ldots, v_n)$ of elements of $\mathcal{F}$. Such elements v are called *n-simplices* of the nerve, and can be seen as geometric objects of dimension n, having $n+1$ *faces* that are $(n-1)$-simplices: face i is the subsequence where v_i has been removed. Also, every n-simplex can be coerced to a degenerate $(n+1)$-simplex by mapping it to its ith *degeneracy* $(v_0, \ldots, v_{i-1}, v_i, v_i, v_{i+1}, \ldots, v_n)$, for every i, $0 \leq i \leq n$.

This looks very much like the description of $(\mathcal{F}\langle n\rangle)_{n\geq 0}$. However, if we wish to be formal, we need a few additional simple constructions. A *slice* $\mathcal{F}/v_{-1}$ is a preorder consisting of elements of the form (v, v_{-1}) with $v \leq v_{-1}$; their elements are ordered by: $(v, v_{-1}) \leq (w, v_{-1})$ if and only if $v \leq w$. The *opposite* $\mathcal{F}^{\mathrm{op}}$ of a preorder $\mathcal{F}$ is a preorder whose elements are those of $\mathcal{F}$, and whose preorder $\leq^{\mathrm{op}}$ is such that $x \leq^{\mathrm{op}} y$ if and only if $y \leq x$ in $\mathcal{F}$. The dual notion of a slice $\mathcal{F}/v_{-1}$ is called the *coslice* $v_{-1}\backslash\mathcal{F}$: this is the set of all pairs (v_{-1}, v) with $v_{-1} \leq v$, ordered by $(v_{-1}, v) \leq (v_{-1}, w)$ if and only if $v \leq w$. Note that $\mathcal{F}\langle 1\rangle$ is just the direct sum of all coslices of $\mathcal{F}$; a formally similar notion will be used in Section 4.

A few simple computations now show that $\mathcal{F}\langle n\rangle$ is just the same as the union over all v_{-1} of all $N_n(v_{-1}\backslash\mathcal{F}^{\mathrm{op}})$. Taking face 0 does `unbox`, and taking degeneracy 0 does `kwote`. In other words, modulo a few gadgets like slices and taking opposites, $N_n(\mathcal{F})$, $n \geq 0$, is a nerve. Our point here is that the S4 □ modality can be essentially interpreted as a geometric construction. This will be more apparent in Section 4.

4 Geometric Models

It is also possible to interpret types as topological spaces, without any order structure underlying them. Morphisms will be continuous maps, again. But for our category of topological spaces to be cartesian closed, we need an additional restriction. A standard trick is to restrict to *compactly generated spaces* [Str, ML71]:

Definition 4.1 *Say that a subset E of a topological space $\mathcal{F}$ is* k-closed *if and only if $f^{-1}(E)$ is closed in K, for every compact space K and every continuous function f from K to $\mathcal{F}$.*

A topological space $\mathcal{F}$ is compactly generated *if and only if for any subset E of $\mathcal{F}$, E is closed if and only if E is k-closed.*

Note that every closed subset is k-closed. The converse is the part that is not true of every topological space.

Recall that a topological space is *Hausdorff*, or T_2, if and only if, for every two elements x, y such that $x \neq y$, there exist opens O_x and O_y such that $x \in O_x$, $y \in O_y$ and $O_x \cap O_y = \emptyset$; and that $\mathcal{F}$ is *compact* if and only if $\mathcal{F}$ is Hausdorff, and from any open cover of $\mathcal{F}$ we can extract a finite subcover of $\mathcal{F}$.

Say that $\mathcal{F}$ is *weakly Hausdorff* [Str] if and only if, for every compact space K, for every continuous function f from K to $\mathcal{F}$, the image of K by f is closed in $\mathcal{F}$. It is clear that every Hausdorff (T_2) space is weakly Hausdorff, and that every weakly Hausdorff space is T_1, i.e., every singleton $\{x\}$ is closed. We shall deal with weakly Hausdorff spaces as this is slightly more general than Hausdorff spaces, but the results would be the same with Hausdorff spaces: see [ML71], Chapter 7, Section 8, for an introduction to compactly generated Hausdorff spaces.

Every Hausdorff space can be turned into a compactly generated space by adding a few opens (equivalently, a few closed) sets to its topology, this is point (i) below. As this will be more practical, we shall see topologies as given not by their sets of opens, rather by their sets of closed subsets. Recall also that a function f is continuous if and only if $f^{-1}(O)$ is open for every open O, if and only if $f^{-1}(F)$ is closed for every closed subset F. The following Lemma is a summary of some of the results in [Str]:

Lemma 4.1 *Let the* kelleyfication $k(\mathcal{F})$ *of the topological space* $\mathcal{F}$ *be defined as follows: the elements of* $k(\mathcal{F})$ *are those of* $\mathcal{F}$*, the closed sets of* $k(\mathcal{F})$ *are the* k*-closed sets of* $\mathcal{F}$*. Then:*

(*i*) $k(k(\mathcal{F})) = k(\mathcal{F})$*, so* $k(\mathcal{F})$ *is compactly generated;*

(*ii*) *("There are many compactly generated spaces") Every locally compact Hausdorff space is compactly generated;*

(*iii*) *the category* CGWH *of compactly generated weakly Hausdorff spaces with continuous maps has products, and the product* $\prod_{i\in I} \mathcal{F}_i$ *of the family* $(\mathcal{F}_i)_{i\in I}$ *is* $k(\prod^0_{i\in I} \mathcal{F}_i)$*, where* $\prod^0$ *denotes the set-theoretic product with the product topology.*

(*iv*) CGWH *is cartesian-closed; we shall note the exponential object* $\mathcal{G}^{\mathcal{F}}$ *as* $\mathcal{F} \to \mathcal{G}$*.*

Alternatively, let $C_0(\mathcal{F}, \mathcal{G})$ *be the space of all continuous maps from* $\mathcal{F}$ *to* $\mathcal{G}$*, with the* compact-open *topology, namely that generated from the basic opens* $\{f$ *continuous from* $\mathcal{F}$ *to* $\mathcal{G} \mid f(K) \subseteq O\}$*, where* K *ranges over compacts in* $\mathcal{F}$ *and* O *over opens in* $\mathcal{G}$*. Then* $\mathcal{F} \to \mathcal{G} = k(C_0(\mathcal{F}, \mathcal{G}))$*.*

(*v*) *if* $\mathcal{F}$ *is locally compact Hausdorff, and* $\mathcal{G}$ *is* CGWH*, then* $\mathcal{F} \to \mathcal{G} = C_0(\mathcal{F}, \mathcal{G})$*.*

(*vi*) CGWH *has sums: the sum of the family* $(\mathcal{F}_i)_{i\in I}$ *of* CGWH*-spaces is the set-theoretical sum* $\amalg_{i\in I}\mathcal{F}_i$*, with the sum topology (the least making all injections* $\mathcal{F}_i \to \amalg_{i\in I}\mathcal{F}_i$ *continuous).*

The compact-open topology is better known, in the special case of uniform, and even of metric spaces, as the topology of uniform convergence on every compact. This means in particular that, if $\mathcal{F}$ is compact, and $\mathcal{G}$ is a metric space with distance d, then $\mathcal{F} \to \mathcal{G}$ is metrizable as well, by defining the distance between f and g as $\max_{v\in\mathcal{F}} d(f(v), g(v))$. (Note that the max is reached, since $\mathcal{F}$ is compact.) The category of CGWH spaces is a kind of souped-up version of that of compact metric spaces, with enough spaces to be cartesian closed.

We now define the interpretation $[\![\text{-}]\!]_t$ as follows. Let $[\![F \supset G]\!]_t \hat{=} [\![F]\!]_t \to [\![G]\!]_t$. We may define $[\![\Gamma, x : F, \Delta \vdash x : F]\!]_t(g, v, d)$ as v, where $g \in [\![\Gamma]\!]_t$, $v \in [\![F]\!]_t$, $d \in [\![\Delta]\!]_t$; this is indeed continuous from $[\![\Gamma]\!]_t \times [\![F]\!]_t \times [\![\Delta]\!]_t$ to $[\![F]\!]_t$: projections from product objects are morphisms in CGWH, hence continuous. Equally easily, $[\![\Gamma \vdash st : G]\!]_t$ is the (continuous) function mapping every $g \in [\![\Gamma]\!]_t$ to $[\![\Gamma \vdash s : F \supset G]\!]_t(g)([\![\Gamma \vdash t : F]\!]_t(g))$, and $[\![\Gamma \vdash \lambda x_F \cdot t : F \supset G]\!]_t$ is defined as the (continuous) function mapping each $g \in [\![\Gamma]\!]_t$ to the (continuous) function mapping each $v \in [\![F]\!]_t$ to $[\![\Gamma, x : F \vdash t : G]\!]_t(g, v)$. That everything is continuous follows from the fact that CGWH is cartesian closed.

It remains to interpret boxed types:

Definition 4.2 *Given a* CGWH *space* $\mathcal{F}$*, and* $v_0 \in \mathcal{F}$*, let the* coslice $v_0\backslash\mathcal{F}$ *be defined as the space of all continuous functions* α *from* $[0, 1]$ *to* $\mathcal{F}$ *such that* $\alpha(0) = v_0$*, with the compact-open topology.*

Let then $\mathcal{F}\langle 1\rangle \hat{=} \amalg_{v_0\in\mathcal{F}} v_0\backslash\mathcal{F}$*, and* $[\![\Box F]\!]_t \hat{=} [\![F]\!]_t\langle 1\rangle$*.*

Lemma 4.2 *For every* CGWH *space* $\mathcal{F}$*,* $\mathcal{F}\langle 1\rangle$ *is* CGWH*.*

Proof: Since $[0, 1]$ is compact, in particular it is locally compact Hausdorff, so $C_0([0, 1], \mathcal{F})$ is CGWH, by Lemma 4.1 (v). Since $\mathcal{F}$ is weakly Hausdorff, hence T_1, $\{v_0\}$ is closed in $\mathcal{F}$. Because the projection $\alpha \mapsto \alpha(0)$ is continuous, it follows that $v_0\backslash\mathcal{F}$ is closed in $C_0([0, 1], \mathcal{F})$. Therefore: (a) every subset E of $v_0\backslash\mathcal{F}$ is closed in $v_0\backslash\mathcal{F}$ if and only if it is closed in $C_0([0, 1], \mathcal{F})$.

It also follows from (a) that: (b) for every function f from $\mathcal{G}$ to $v_0\backslash\mathcal{F}$, f is continuous from $\mathcal{G}$ to $v_0\backslash\mathcal{F}$ if and only if f is continuous when seen as a function from $\mathcal{G}$ to $C_0([0, 1], \mathcal{F})$.

We can now show that: (c) $v_0\backslash\mathcal{F}$ is compactly generated. Since every closed subset is automatically k-closed, it remains to prove the converse. So let E be k-closed in $v_0\backslash\mathcal{F}$. This means that: (*) for every continuous f from a compact K to $v_0\backslash\mathcal{F}$, $f^{-1}(E)$ is closed. So let f be continuous from an arbitrary compact K to $C_0([0, 1], \mathcal{F})$. Since $v_0\backslash\mathcal{F}$ is closed in $C_0([0, 1], \mathcal{F})$, $K' \hat{=} f^{-1}(v_0\backslash\mathcal{F})$ is closed in K, hence compact. The restriction f' of f to K' is then continuous from K' to $C_0([0, 1], \mathcal{F})$, and its range is included in $v_0\backslash\mathcal{F}$, so by (b) f' is continuous from K' to $v_0\backslash\mathcal{F}$. By (*), ${f'}^{-1}(E)$ is then closed in K', hence in K. But by the definition of K' and f', ${f'}^{-1}(E) = f^{-1}(E)$, so $f^{-1}(E)$ is closed in K. As K and f are arbitrary, E is k-closed in $C_0([0,1],\mathcal{F})$. Since the latter is CGWH, E is closed in $C_0([0,1],\mathcal{F})$, hence in $v_0\backslash\mathcal{F}$.

On the other hand, for every compact space K, every continuous function f from K to $v_0\backslash\mathcal{F}$ is continuous from K to $C_0([0,1],\mathcal{F})$ by (b), so since $C_0([0,1],\mathcal{F})$ is weakly Hausdorff, $f(K)$ is closed in $C_0([0,1],\mathcal{F})$, hence in $v_0\backslash\mathcal{F}$ since $f(K)$ is by assumption included in the latter. Since K and f are arbitrary, it follows that: (d) $v_0\backslash\mathcal{F}$ is weakly Hausdorff.

By (c) and (d), $v_0\backslash\mathcal{F}$ is CGWH, hence the sum space $\mathcal{F}\langle 1\rangle$ is CGWH by Lemma 4.1 (*vi*). $\Box$

Compared to the $[\![\text{-}]\!]_s$ interpretation, we have replaced pairs (v,w) with $v \leq w$ by continuous *paths* leading from v to w. It is interesting to see these paths as specifications of *processes* starting at time 0 and stopping at some later time that we decide to name 1. Then:

- Quoting, which used to map v to (v,v) in the dcpo model, will be the function $\texttt{'} : v \in \mathcal{F} \mapsto (\lambda\tau \in [0,1]\cdot v) \in \mathcal{F}\langle 1\rangle$ mapping v to the constant path that stays at v. Because of the topology we put on $\mathcal{F}\langle 1\rangle$, $\texttt{'}$ is *not* continuous in general. This is a feature: it represents semantically the fact that although we can deduce $\Box F$ from F, there is in general no proof of $F \supset \Box F$.

  More precisely, if $\texttt{'}$ is continuous, then it maps every connected component C of $\mathcal{F}$ to some connected subspace of $\mathcal{F}\langle 1\rangle$, hence to a subspace of some $v\backslash\mathcal{F}$: it follows that C must be the singleton set $\{v\}$, for every C. In other words, $\texttt{'}$ is continuous from $\mathcal{F}$ to $\mathcal{F}\langle 1\rangle$ if and only if $\mathcal{F}$ is *totally disconnected*.

- (K), which used to map (f,g) and (v,w) to $(f(v),g(w))$, will here be the function $\star$ mapping every path α from $\alpha(0)\hat{=}f \in \mathcal{F}\to\mathcal{G}$ to $\alpha(1)\hat{=}g \in \mathcal{F}\to\mathcal{G}$, and every path β from $\beta(0)\hat{=}v\in\mathcal{F}$ to $\beta(1)\hat{=}w\in\mathcal{F}$ to the path $\alpha\star\beta\hat{=}\lambda\tau\in[0,1]\cdot\alpha(\tau)(\beta(\tau))$. In terms of processes, this is *synchronous parallel application*. In general, define $\alpha\star(\beta_1,\ldots,\beta_n)$, where $\alpha\in(\mathcal{F}_1\times\ldots\times\mathcal{F}_n\to\mathcal{G})\langle 1\rangle$ and $\beta_i\in\mathcal{F}_i\langle 1\rangle$ for every i, $1\leq i\leq n$, as $\lambda\tau\in[0,1]\cdot\alpha(\tau)(\beta_1(\tau),\ldots,\beta_n(\tau))$.

- (T) is interpreted by the $\texttt{ev}$ function, defined here as the projection mapping $\alpha\in\mathcal{F}\langle 1\rangle$ to $\alpha(1)$. In terms of processes, this is the function waiting for its argument process α to terminate, and which then returns the final value $\alpha(1)$ of α.

- (4) is a bit more complex. Remember that it used to map (v,w) to $((v,v),(v,w))$ in the dcpo model. The function $\texttt{kwote}$ which realizes it here will map the path $\alpha\in\mathcal{F}\langle 1\rangle$ to the path of paths $\beta\in\mathcal{F}\langle 1\rangle\langle 1\rangle$ such that $\beta(\tau)(\tau')=\alpha(\tau.\tau')$, where $\tau.\tau'$ is the ordinary product of $\tau,\tau'\in[0,1]$. That is, β is a path from the constant path $\beta(0)=\texttt{'}(\alpha(0))$ to the path $\beta(1)=\alpha$. In terms of processes, $\texttt{kwote}(\alpha)$ is a process that starts from the syntactic description $\texttt{'}(\alpha(0))$ of the process α, and eventually returns the actual process α at time 1. This is therefore an interpretation in terms of *higher-order processes*, that may compute other processes.

We then define $[\![\Gamma\vdash \mathsf{box}\,t\,\mathsf{with}\,\{x_1:=t_1,\ldots,x_n:=t_n\}:\Box G]\!]_t$ as the function mapping every $g\in[\![\Gamma]\!]_t$ to $\texttt{'}([\![\Delta\vdash t:G]\!]_t)\star(\texttt{kwote}([\![\Gamma\vdash t_1:\Box F_1]\!]_t(g)),\ldots,\texttt{kwote}([\![\Gamma\vdash t_n:\Box F_n]\!]_t(g)))$, where $\Delta\hat{=}x_1:\Box F_1,\ldots,x_n:\Box F_n$—a similar formula in fact works also for the dcpo interpretation $[\![\text{-}]\!]_s$. More synthetically, define $[\![\text{-}]\!]_t$ on terms as in Figure 3.

$$
\begin{array}{rcl}
[\![\Gamma,x:F,\Delta\vdash x:F]\!]_t(g,v,d) & \hat{=} & v\\
[\![\Gamma\vdash st:G]\!]_t(g) & \hat{=} & [\![\Gamma\vdash s:F\supset G]\!]_t(g)([\![\Gamma\vdash t:F]\!]_t(g))\\
[\![\Gamma\vdash\lambda x_F\cdot t:F\supset G]\!]_t(g) & \hat{=} & \lambda v\in[\![F]\!]_t\cdot[\![\Gamma,x:F\vdash t:G]\!]_t(g,v)\\
[\![\Gamma\vdash\mathsf{unbox}\,t:F]\!]_t(g) & \hat{=} & [\![\Gamma\vdash t:\Box F]\!]_t(g)(1)\\
[\![\Gamma\vdash\mathsf{box}\,t\,\mathsf{with}\,\sigma:\Box G]\!]_t(g) & \hat{=} & \lambda\tau\in[0,1]\cdot[\![\Delta\vdash t:G]\!]_t(\alpha_1(\tau),\ldots,\alpha_n(\tau))\\
 & & \text{where } \alpha_i(\tau)\hat{=}\lambda\tau'\in[0,1]\cdot[\![\Gamma\vdash t_i:\Box F_i]\!]_t(g)(\tau.\tau'),1\leq i\leq n
\end{array}
$$

Figure 3: The Geometric Interpretation

For every function f, let $f_{|A}$ be the restriction of f to the set A.

Lemma 4.3 *For every* CGWH *spaces $\mathcal{F}$ and $\mathcal{G}$, $f : \mathcal{F} \to \mathcal{G}\langle 1\rangle$ is continuous if and only if the following two conditions are satisfied, for every connected component C of $\mathcal{F}$:*

(i) for every $x, y \in C$, $f(x)(0) = f(y)(0)$;

(ii) $f_{|C}$ is continuous from C to $C_0([0,1], \mathcal{G})$.

Proof: Only if: since C is connected and f is continuous, the image $f(C)$ is connected. But every $v_0\backslash\mathcal{G}$, $v_0 \in \mathcal{G}$, is both open and closed in $\mathcal{G}\langle 1\rangle$ by construction, so every connected subset of $\mathcal{G}\langle 1\rangle$ is a connected subset of some $v_0\backslash\mathcal{G}$, $v_0 \in \mathcal{G}$. In particular, for every $x, y \in C$, $f(x)$ and $f(y)$ are both in $v_0\backslash\mathcal{G}$, so $f(x)(0) = v_0 = f(y)(0)$, therefore (i) holds. On the other hand, since f is continuous, $f_{|C}$ is also continuous from C to $\mathcal{G}\langle 1\rangle$, hence also from C to $v_0\backslash\mathcal{G}$. But the topology of $v_0\backslash\mathcal{G}$ is a subset topology of $C_0([0,1], \mathcal{G})$, so (ii) holds.

If: assume that (i) and (ii) hold for every connected component C of $\mathcal{F}$. Let v_0 be $f(x)(0)$ for some $x \in C$. By (i), v_0 does not depend on the choice of x, and therefore $f(C) \subseteq v_0\backslash\mathcal{G}$. By (ii), and since the topology of $v_0\backslash\mathcal{G}$ is a subset topology of $C_0([0,1], \mathcal{G})$, $f_{|C}$ is continuous from C to $v_0\backslash\mathcal{G}$, hence also from C to $\mathcal{G}\langle 1\rangle$. For every open O in $\mathcal{G}\langle 1\rangle$, $f^{-1}(O)$ is the union of ${f_{|C}}^{-1}(O)$, when C ranges over the connected components of $\mathcal{F}$, and is therefore open. So f is continuous from $\mathcal{F}$ to $\mathcal{G}\langle 1\rangle$. $\Box$

Lemma 4.4 *Let C be a connected component of $\mathcal{F}_1 \times \ldots \times \mathcal{F}_n$, where each $\mathcal{F}_i$ is* CGWH. *Then $C \subseteq C_1 \times \ldots \times C_n$, where C_i is a connected component of $\mathcal{F}_i$ for each i, $1 \le i \le n$.*

Proof: Every connected component of $\mathcal{F}_i$ is both open and closed in $\mathcal{F}_i$, so every product $C_1 \times \ldots \times C_n$ is both open and closed in the product $\prod^0_{1\le i\le n} \mathcal{F}_i$ (with the product topology). Since every closed set is k-closed, $C_1 \times \ldots \times C_n$ is also both open and closed in $\prod_{1\le i\le n} \mathcal{F}_i$. Let S be the set of all n-tuples of connected components $(C_1, \ldots, C_n)$ as above such that $C \cap (C_1 \times \ldots \times C_n) \neq \emptyset$. Since $C \cap (C_1 \times \ldots \times C_n)$ is both open and closed in C, and since C is connected, there can be at most one element $(C_1, \ldots, C_n)$ in S. It follows that $C \subseteq C_1 \times \ldots \times C_n$. $\Box$

Lemma 4.5 *For every term t such that $\Gamma \vdash t : F$ is derivable, $[\![\Gamma \vdash t : F]\!]_t$ is a continuous function from $[\![\Gamma]\!]_t$ to $[\![F]\!]_t$.*

Proof: We first establish a few claims. First: (a) $\mathtt{ev} : \alpha \in \mathcal{F}\langle 1\rangle \mapsto \alpha(1)$ is continuous. Let F be any closed subset of $\mathcal{F}$, then $\mathtt{ev}^{-1}(F) = \{\alpha \in \mathcal{F}\langle 1\rangle \mid \alpha(1) \in F\} = \bigcup_{v_0\in\mathcal{F}}\{\alpha \in v_0\backslash\mathcal{F} \mid \alpha(1) \in F\} = \bigcup_{v_0\in\mathcal{F}}\{\alpha \in C_0([0,1], \mathcal{F}) \mid \alpha(0) = v_0, \alpha(1) \in F\} = \bigcup_{v_0\in\mathcal{F}}(\pi_0^{-1}(\{v_0\}) \cap \pi_1^{-1}(F))$, where π_τ is the projection $\alpha \mapsto \alpha(\tau)$, which is continuous from $C_0([0,1], \mathcal{F})$ to $\mathcal{F}$. So $F_{v_0} \hat{=} \pi_0^{-1}(\{v_0\}) \cap \pi_1^{-1}(F)$ is closed in $C_0([0,1], \mathcal{F})$, hence also in $v_0\backslash\mathcal{F}$. Recall that a set is closed in a sum space if and only if it is closed in every summand. Here, $\mathtt{ev}^{-1}(F) \cap (v_0\backslash\mathcal{F}) = F_{v_0}$ is closed in every summand $v_0\backslash\mathcal{F}$ of $\mathcal{F}\langle 1\rangle$, so $\mathtt{ev}^{-1}(F)$ is closed. Since F is arbitrary, $\mathtt{ev}$ is continuous.

We now claim that: (b) $\star$ is continuous from $(\mathcal{F}_1 \times \ldots \times \mathcal{F}_n \to \mathcal{G})\langle 1\rangle \times \mathcal{F}_1\langle 1\rangle \times \ldots \times \mathcal{F}_n\langle 1\rangle$ to $\mathcal{G}\langle 1\rangle$. For convenience, when $x \hat{=} (\beta, \alpha_1, \ldots, \alpha_n)$, we shall also write $\beta \star (\alpha_1, \ldots, \alpha_n)$ as $(\star)(x)$. Let C be any connected component of $(\mathcal{F}_1 \times \ldots \times \mathcal{F}_n \to \mathcal{G})\langle 1\rangle \times \mathcal{F}_1\langle 1\rangle \times \ldots \times \mathcal{F}_n\langle 1\rangle$. By Lemma 4.4, $C \subseteq C_0 \times C_1 \times \ldots \times C_n$, where C_0 is some connected component of $(\mathcal{F}_1 \times \ldots \times \mathcal{F}_n \to \mathcal{G})\langle 1\rangle$, C_1 is some connected component of $\mathcal{F}_1\langle 1\rangle$, …, C_n is some connected component of $\mathcal{F}_n\langle 1\rangle$. In particular, C_0 is included in some $f_0\backslash(\mathcal{F}_1 \times \ldots \times \mathcal{F}_n \to \mathcal{G})$, C_1 is included in some $v_1\backslash\mathcal{F}_1$, …, C_n is included in some $v_n\backslash\mathcal{F}_n$. So for every $(\beta, \alpha_1, \ldots, \alpha_n) \in C$, $\beta(0) = f_0$, $\alpha_1(0) = v_0$, …, $\alpha_n(0) = v_n$. Therefore, for every $x \hat{=} (\beta, \alpha_1, \ldots, \alpha_n)$ and $y \hat{=} (\beta', \alpha'_1, \ldots, \alpha'_n)$ in C, $(\star)(x)(0) = \beta(0)(\alpha_1(0), \ldots, \alpha_n(0)) = f_0(v_0, \ldots, v_n) = \beta'(0)(\alpha'_1(0), \ldots, \alpha'_n(0)) = (\star)(y)(0)$, so condition (i) of Lemma 4.3 is satisfied. On the other hand, $\star$ is continuous from $C_0([0,1], (\mathcal{F}_1 \times \ldots \times \mathcal{F}_n \to \mathcal{G})) \times C_0([0,1], \mathcal{F}_1) \times \ldots \times C_0([0,1], \mathcal{F}_n)$ to $C_0([0,1], \mathcal{G})$ because $\star$ is defined from abstractions and applications only, and CGWH is cartesian closed. So given any closed subset F of $C_0([0,1], \mathcal{G})$, ${(\star)_{|C}}^{-1}(F) = {(\star)_{|C}}^{-1}(F \cap (f_0(v_0, \ldots, v_n)\backslash\mathcal{G}))$, since the range of $(\star)_{|C}$ is included in $f_0(v_0, \ldots, v_n)\backslash\mathcal{G}$. But $f_0(v_0, \ldots, v_n)\backslash\mathcal{G}$ is closed in $C_0([0,1], \mathcal{G})$, so $F \cap (f_0(v_0, \ldots, v_n)\backslash\mathcal{G})$ is closed again, therefore ${(\star)_{|C}}^{-1}(F)$ is closed, hence closed in C. So $\star$ is continuous from C to $C_0([0,1], \mathcal{G})$, therefore condition (ii) of Lemma 4.3 is satisfied. It follows that $\star$ is continuous.

Similarly, we claim that: (c) $\mathtt{kwote}$ is continuous from $\mathcal{F}\langle 1\rangle$ to $\mathcal{F}\langle 1\rangle\langle 1\rangle$. Again, this will be by Lemma 4.3. Let C be any connected component of $\mathcal{F}\langle 1\rangle$, in particular $C \subseteq v_0\backslash\mathcal{F}$ for some $v_0 \in \mathcal{F}$. So for every $\alpha \in C$,

$\alpha(0) = v_0$, and therefore $\texttt{kwote}(\alpha)(0) = (\lambda\tau, \tau' \cdot \alpha(\tau.\tau'))(0) = \lambda\tau' \cdot \alpha(0) = \lambda\tau' \cdot v_0$. In particular, condition (i) is satisfied: for every $\alpha, \alpha' \in C$, $\texttt{kwote}(\alpha)(0) = \lambda\tau' \cdot v_0 = \texttt{kwote}(\alpha')(0)$. We now show condition (ii). Given any closed set F of $\mathcal{F}\langle 1\rangle\langle 1\rangle$, ${\texttt{kwote}_{|C}}^{-1}(F) = {\texttt{kwote}_{|C}}^{-1}(F \cap ((\lambda\tau' \cdot v_0)\backslash\mathcal{F}\langle 1\rangle)) = {\texttt{kwote}_{|C}}^{-1}(F \cap ((\lambda\tau' \cdot v_0)\backslash(v_0\backslash\mathcal{F})))$ is closed in $C_0([0,1], \mathcal{F})$. Indeed, $(\lambda\tau' \cdot v_0)\backslash(v_0\backslash\mathcal{F}))$ is closed in $C_0([0,1], C_0([0,1], \mathcal{F}))$, and $\texttt{kwote}_{|C}$ is continuous from $C_0([0,1], \mathcal{F})$ to $C_0([0,1], C_0([0,1], \mathcal{F}))$, since abstraction, application and multiplication are. So ${\texttt{kwote}_{|C}}^{-1}(F)$ is closed in C, hence $\texttt{kwote}_{|C}$ is continuous.

Finally, we claim that: (d) provided $\mathcal{F}$ is CGWH, for every continuous functions $f_i : \mathcal{F} \to \mathcal{G}_i$, $h \hat{=} \lambda v \in \mathcal{F} \cdot (f_1(v), \ldots, f_n(v))$ is continuous from $\mathcal{F}$ to $\prod_{1\leq i\leq n} \mathcal{G}_i$. To do this, we shall show that, for every closed subset F of $\prod_{1\leq i\leq n} \mathcal{G}_i$, $h^{-1}(F)$ is k-closed in $\mathcal{F}$. So let K be any compact space, and f be any continuous map from K to $\mathcal{F}$. To show that $h^{-1}(F)$ is k-closed, we have to show that $f^{-1}(h^{-1}(F))$ is closed in K. But $f^{-1}(h^{-1}(F)) = (h \circ f)^{-1}(F)$, and $h \circ f$ is trivially continuous from K to $\prod^0_{1\leq i\leq n} \mathcal{G}_i$. Recall that F is closed in $\prod_{1\leq i\leq n} \mathcal{G}_i$, so F is k-closed in $\prod^0_{1\leq i\leq n} \mathcal{G}_i$; by definition of being k-closed, using the compact K and the function $h \circ f$, $f^{-1}(h^{-1}(F))$ is closed in K. Since K and f are arbitrary, $h^{-1}(F)$ is k-closed in $\mathcal{F}$. Since $\mathcal{F}$ is CGWH, $h^{-1}(F)$ is closed in $\mathcal{F}$. Because F is arbitrary, h is continuous.

We now prove the Lemma by structural induction on t. It remains to show this in the cases of terms of the form $\mathsf{unbox}\, t$ or $\mathsf{box}\, t\ \mathsf{with}\ \sigma$. In the first case, this follows from (a). In the second case, recall that we can write $f \hat{=} [\![\Gamma \vdash \mathsf{box}\, t\ \mathsf{with}\ \{x_1 := t_1, \ldots, x_n := t_n\} : \Box G]\!]_t$ as the function mapping every $g \in [\![\Gamma]\!]_t$ to $\texttt{'}([\![\Delta \vdash t : G]\!]_t) \star (\texttt{kwote}([\![\Gamma \vdash t_1 : \Box F_1]\!]_t(g)), \ldots, \texttt{kwote}([\![\Gamma \vdash t_n : \Box F_n]\!]_t(g)))$, where $\Delta \hat{=} x_1 : \Box F_1, \ldots, x_n : \Box F_n$. In particular, observe that $\texttt{'}([\![\Delta \vdash t : G]\!]_t)$ makes sense, because $[\![\Delta \vdash t : G]\!]_t$ is continuous by induction. Then f is continuous by (b), (c) and (d). (Note that the left argument to $\star$ is *constant.*) $\Box$

Lemma 4.6 *For every term t such that $\Gamma, x_1 : F_1, \ldots, x_n : F_n \vdash t : F$ is derivable, for every terms $t_1, \ldots, t_n$ such that $\Gamma, \Delta \vdash t_i : F_i$ is derivable for every i, $1 \leq i \leq n$, $[\![\Gamma, \Delta \vdash t\{x_1 := t_1, \ldots, x_n := t_n\} : F]\!]_t$ is the function mapping (g, d), where $g \in [\![\Gamma]\!]_t$ and $d \in [\![\Delta]\!]_t$, to :*

$$[\![\Gamma, x_1 : F_1, \ldots, x_n : F_n \vdash t : F]\!]_t(g, [\![\Gamma, \Delta \vdash t_1 : F_1]\!]_t(g, d), \ldots, [\![\Gamma, \Delta \vdash t_n : F_n]\!]_t(g, d))$$

Proof: As for Lemma 3.2. $\Box$

Lemma 4.7 *If x is not free in t, then for every $g \in [\![\Gamma]\!]_t$, $v \in [\![F]\!]_t$, $d \in [\![\Delta]\!]_t$, $[\![\Gamma, x : F, \Delta \vdash t : G]\!]_t(g, v, d) = [\![\Gamma, \Delta \vdash t : G]\!]_t(g, d)$.*

Proof: Easy structural induction on t. $\Box$

Theorem 4.8 *The $[\![\text{-}]\!]_t$ interpretation is sound wrt. λ_{S4} with the extensional equalities: for every terms s and t such that $\Gamma \vdash s : F$ and $\Gamma \vdash t : F$ are both derivable, and such that $s =_\eta t$, we have $[\![\Gamma \vdash s : F]\!]_t = [\![\Gamma \vdash t : F]\!]_t$.*

Proof: As for Theorem 3.4. We only deal with the rules (unbox), (box), and $(\eta\,\mathsf{box})$, where the difficulty resides.

We first note that: (a) $\texttt{ev}(\texttt{'}v) = v$ for every $v \in \mathcal{F}$. Indeed, $\texttt{ev}(\texttt{'}v) = (\texttt{'}v)(1) = v$.

Also: (b) $\texttt{ev}(v \star (w_1, \ldots, w_n)) = (\texttt{ev}(v))(\texttt{ev}(w_1), \ldots, \texttt{ev}(w_n))$. Indeed, $\texttt{ev}(v \star (w_1, \ldots, w_n)) = (v \star (w_1, \ldots, w_n))(1) = (\lambda\tau \in [0,1] \cdot v(\tau)(w_1(\tau), \ldots, w_n(\tau)))(1) = v(1)(w_1(1), \ldots, w_n(1)) = (\texttt{ev}(v))(\texttt{ev}(w_1), \ldots, \texttt{ev}(w_n))$.

Let's examine rule (unbox): let σ be $\{x_1 := t_1, \ldots, x_n := t_n\}$, and Δ be $x_1 : \Box F_1, \ldots, x_n : \Box F_n$, then:

$$\begin{aligned}
& [\![\Gamma \vdash \mathsf{unbox}(\mathsf{box}\, t\ \mathsf{with}\ \sigma) : G]\!]_t(g) \\
= \;& \texttt{ev}(\texttt{'}[\![\Delta \vdash t : G]\!]_t \star ([\![\Gamma \vdash t_1 : \Box F_1]\!]_t(g), \ldots, [\![\Gamma \vdash t_n : \Box F_n]\!]_t(g))) \\
= \;& (\texttt{ev}(\texttt{'}[\![\Delta \vdash t : G]\!]_t))(\texttt{ev}([\![\Gamma \vdash t_1 : \Box F_1]\!]_t(g)), \ldots, \texttt{ev}([\![\Gamma \vdash t_n : \Box F_n]\!]_t(g))) \qquad \text{(by (b))} \\
= \;& [\![\Delta \vdash t : G]\!]_t(\texttt{ev}([\![\Gamma \vdash t_1 : \Box F_1]\!]_t(g)), \ldots, \texttt{ev}([\![\Gamma \vdash t_n : \Box F_n]\!]_t(g))) \qquad \text{(by (a))} \\
= \;& [\![\Gamma, \Delta \vdash t : G]\!]_t(g, \texttt{ev}([\![\Gamma \vdash t_1 : \Box F_1]\!]_t(g)), \ldots, \texttt{ev}([\![\Gamma \vdash t_n : \Box F_n]\!]_t(g))) \qquad \text{(by Lemma 4.7)} \\
= \;& [\![\Gamma \vdash t\sigma : G]\!]_t(g) \qquad \text{(by Lemma 4.6)}
\end{aligned}$$

Now for rule (box): as in Theorem 3.4, we show that $(\text{`}s)\{x := \text{`}t\}$ and $\text{`}(s\{x := \text{`}t\})$ have the same interpretations. So, assume that the free variables of s are among $x, x_1, \ldots, x_n$, and that $x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G$, and $y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t : F$. Then:

$$\begin{aligned}
& [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash (\text{`}s)\{x := \text{`}t\} : \Box G]\!]_t(v_1, \ldots, v_n, w_1, \ldots, w_m) \\
= \; & [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash \text{`}s : \Box G]\!]_t(v_1, \ldots, v_n, \\
& \qquad [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash \text{`}t : \Box F]\!]_t(v_1, \ldots, v_n, w_1, \ldots, w_m)) \\
= \; & [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash \text{`}s : \Box G]\!]_t(v_1, \ldots, v_n, \\
& \qquad [\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash \text{`}t : \Box F]\!]_t(w_1, \ldots, w_m)) \qquad \text{(by Lemma 4.7, } n \text{ times)} \\
= \; & \lambda\tau \in [0,1] \cdot [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_t(\lambda\tau' \in [0,1] \cdot v_1(\tau.\tau'), \ldots, \lambda\tau' \in [0,1] \cdot v_n(\tau.\tau'), \\
& \qquad \lambda\tau' \in [0,1] \cdot [\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash \text{`}t : \Box F]\!]_t(w_1, \ldots, w_m)(\tau.\tau')) \\
= \; & \lambda\tau \in [0,1] \cdot [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_t(\lambda\tau' \in [0,1] \cdot v_1(\tau.\tau'), \ldots, \lambda\tau' \in [0,1] \cdot v_n(\tau.\tau'), \\
& \qquad \lambda\tau' \in [0,1] \cdot [\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t : F]\!]_t \\
& \qquad\qquad (\lambda\tau'' \in [0,1] \cdot w_1(\tau.\tau'.\tau''), \ldots, \lambda\tau'' \in [0,1] \cdot w_m(\tau.\tau'.\tau'')))
\end{aligned}$$

On the other hand:

$$\begin{aligned}
& [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash \text{`}(s\{x := \text{`}t\}) : \Box G]\!]_t(v_1, \ldots, v_n, w_1, \ldots, w_m) \\
= \; & \lambda\tau \in [0,1] \cdot [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash s\{x := \text{`}t\} : G]\!]_t \\
& \qquad (\lambda\tau' \in [0,1] \cdot v_1(\tau.\tau'), \ldots, \lambda\tau' \in [0,1] \cdot v_n(\tau.\tau'), \\
& \qquad \lambda\tau' \in [0,1] \cdot w_1(\tau.\tau'), \ldots, \lambda\tau' \in [0,1] \cdot w_m(\tau.\tau')) \\
= \; & \lambda\tau \in [0,1] \cdot [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_t(\lambda\tau' \in [0,1] \cdot v_1(\tau.\tau'), \ldots, \lambda\tau' \in [0,1] \cdot v_n(\tau.\tau'), \\
& \qquad [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash \text{`}t : \Box F]\!]_t \\
& \qquad\qquad (\lambda\tau' \in [0,1] \cdot v_1(\tau.\tau'), \ldots, \lambda\tau' \in [0,1] \cdot v_n(\tau.\tau'), \\
& \qquad\qquad \lambda\tau' \in [0,1] \cdot w_1(\tau.\tau'), \ldots, \lambda\tau' \in [0,1] \cdot w_m(\tau.\tau'))) \\
= \; & \lambda\tau \in [0,1] \cdot [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_t(\lambda\tau' \in [0,1] \cdot v_1(\tau.\tau'), \ldots, \lambda\tau' \in [0,1] \cdot v_n(\tau.\tau'), \\
& \qquad [\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash \text{`}t : \Box F]\!]_t(\lambda\tau' \in [0,1] \cdot w_1(\tau.\tau'), \ldots, \lambda\tau' \in [0,1] \cdot w_m(\tau.\tau'))) \\
& \qquad\quad \text{(by Lemma 4.7, } n \text{ times)} \\
= \; & \lambda\tau \in [0,1] \cdot [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_t(\lambda\tau' \in [0,1] \cdot v_1(\tau.\tau'), \ldots, \lambda\tau' \in [0,1] \cdot v_n(\tau.\tau'), \\
& \qquad \lambda\tau' \in [0,1] \cdot [\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t : F]\!]_t \\
& \qquad\qquad (\lambda\tau'' \in [0,1] \cdot w_1(\tau.\tau'.\tau''), \ldots, \lambda\tau'' \in [0,1] \cdot w_m(\tau.\tau'.\tau'')))
\end{aligned}$$

Notice that we have only used the associativity of the product in $[0, 1]$, and that we did not need commutativity.

Finally, let's examine (η box). Let $\sigma \hat{=} \{x_1 := t_1, \ldots, x_n : t_n\}$, and assume that $\Gamma \vdash t_i : \Box F_i$ is derivable for each i, $1 \leq i \leq n$; then:

$$\begin{aligned}
& [\![\Gamma \vdash \mathsf{box}\,(\mathsf{unbox}\,x_i)\ \mathsf{with}\ \sigma : \Box F]\!]_t(g) \\
= \; & \lambda\tau \in [0,1] \cdot [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n \vdash \mathsf{unbox}\,x_i : \Box F_i]\!]_t \\
& \qquad (\lambda\tau' \in [0,1] \cdot [\![\Gamma \vdash t_1 : \Box F_1]\!]_t(g)(\tau.\tau'), \ldots \lambda\tau' \in [0,1] \cdot [\![\Gamma \vdash t_n : \Box F_n]\!]_t(g)(\tau.\tau')) \\
= \; & \lambda\tau \in [0,1] \cdot (\lambda\tau' \in [0,1] \cdot [\![\Gamma \vdash t_i : \Box F_i]\!]_t(g)(\tau.\tau'))(1) \\
= \; & \lambda\tau \in [0,1] \cdot [\![\Gamma \vdash t_i : \Box F_i]\!]_t(g)(\tau) = [\![\Gamma \vdash t_i : \Box F_i]\!]_t(g)
\end{aligned}$$

□

4.1 A Note on $\Box^n F$, and Simplices

If $\mathcal{F}$ is a space of points, and $\mathcal{F}\langle 1\rangle$ is a space of paths, what are $\mathcal{F}\langle 1\rangle\langle 1\rangle$, $\mathcal{F}\langle 1\rangle\langle 1\rangle\langle 1\rangle$, and so on? Let's examine $\mathcal{F}\langle 1\rangle\langle 1\rangle$ first. This is a space of paths β, such that each $\beta(\tau)$, $\tau \in [0, 1]$ is itself a path, so β is a kind of square, up to deformation. However, β is continuous and $[0, 1]$ is connected, so the range of β is connected as well. But the range of β is a subset of $\mathcal{F}\langle 1\rangle$, which is the direct sum of spaces $v_0\backslash\mathcal{F}$, $v_0 \in \mathcal{F}$. But in any direct sum of topological spaces, every summand is both open and closed, hence every connected subspace is in fact a subspace of some summand. In our case, this means that the range of β is a subset of $v_0\backslash\mathcal{F}$, for some given $v_0 \in \mathcal{F}$. In other words, $\beta(\tau)(0) = v_0$ for every τ, so the range of β assumes the shape of a *triangle*, up to deformation: see Figure 4.

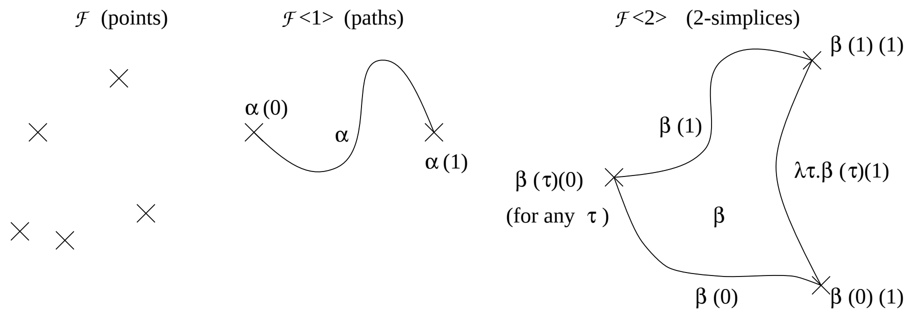

Figure 4: Singular simplices

In general, define $\mathcal{F}\langle n\rangle^\circ$, for $n \geq 0$, as the set of all *extended singular* $(n-1)$*-simplices* in $\mathcal{F}$. For every $q \geq -1$, the *extended singular* q*-simplices* are the continuous maps from $\Delta^+{}_q$ to $\mathcal{F}$, where $\Delta^+{}_q \hat{=} \{(\tau_0, \ldots, \tau_q) \mid \tau_0 \geq 0, \ldots, \tau_q \geq 0, \tau_0 + \ldots + \tau_q \leq 1\}$ is the *standard extended* q*-simplex*; $\Delta^+{}_{-1}$ is the singleton containing only the empty tuple (). Otherwise, $\Delta^+{}_q$ is a polyhedron whose vertices are $(0, \ldots, 0)$ first, and second the points e_0, …, e_q, where $e_i \hat{=} (\tau_0, \ldots, \tau_q)$ with $\tau_i = 1$ and $\tau_j = 0$ for all $j \neq i$. This is analogous to the more usual notion of *standard* q*-simplices* Δ_q, for $q \geq 0$, which are the sub-polyhedra with vertices e_0, …, e_q, namely $\Delta_q \hat{=} \{(\tau_0, \ldots, \tau_q) \mid \tau_0 \geq 0, \ldots, \tau_q \geq 0, \tau_0 + \ldots + \tau_q = 1\}$. The *singular* q*-simplices* of $\mathcal{F}$ are the continuous maps from Δ_q to $\mathcal{F}$. See Figure 5 for an illustration of what the standard simplices, and standard extended simplices, look like.

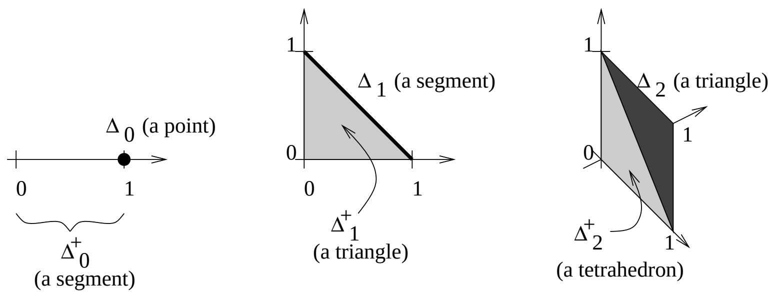

Figure 5: Standard Simplices

The topology on $\mathcal{F}\langle n\rangle^\circ$ is given as follows. When $n = 0$, $\mathcal{F}\langle 0\rangle^\circ$ is isomorphic to $\mathcal{F}$. Otherwise, $\mathcal{F}\langle n\rangle^\circ$ is viewed as the topological sum of all spaces $\gamma\backslash_n\mathcal{F} \hat{=} \{f \in C_0(\Delta^+{}_{n-1}, \mathcal{F}) \mid f_{|\Delta_{n-1}} = \gamma\}$, when γ ranges over all singular $(n-1)$-simplices of $\mathcal{F}$.

Lemma 4.9 *$\mathcal{F}\langle 1\rangle \ldots \langle 1\rangle$, where there are n occurrences of $\langle 1\rangle$, is homeomorphic to $\mathcal{F}\langle n\rangle^\circ$.*

Proof: By induction on n. The base case is by definition, so it remains to show that $\mathcal{F}\langle n\rangle^\circ\langle 1\rangle$ is homeomorphic to $\mathcal{F}\langle n+1\rangle^\circ$.

On the one hand, we map every $\beta \in \mathcal{F}\langle n+1\rangle^\circ$ to $\varphi(\beta) \in \mathcal{F}\langle n\rangle^\circ\langle 1\rangle$, by defining $\varphi(\beta) \hat{=} \lambda\tau \in [0, 1] \cdot \lambda(\tau_0, \ldots, \tau_{n-1}) \in \Delta^+{}_{n-1} \cdot \beta(\tau_0, \ldots, \tau_{n-1}, (1-\tau).(1-\tau_0-\ldots-\tau_{n-1}))$. We must show that indeed $\varphi(\beta) \in \mathcal{F}\langle n\rangle^\circ\langle 1\rangle$, and that φ is continuous.

First, β is in $v\backslash_{n+1}\mathcal{F}$, where $v\hat{=}\beta_{|\Delta_n}$. So $\varphi(\beta)(\tau) = \lambda(\tau_0, \ldots, \tau_{n-1}) \in {\Delta^+}_{n-1} \cdot \beta(\tau_0, \ldots, \tau_{n-1}, (1-\tau).(1-\tau_0 - \ldots - \tau_{n-1}))$ is in $\gamma\backslash_n\mathcal{F}$ for every $\tau \in [0,1]$, where γ is the singular $(n-1)$-simplex $\lambda(\tau_0, \ldots, \tau_{n-1}) \in \Delta_{n-1} \cdot \beta(\tau_0, \ldots, \tau_{n-1}, 0)$. Note that γ is independent of τ, and that $\varphi(\beta)(\tau)$ is continuous in τ (recall that abstraction and application are continuous for the compact-open topology, and that the topology of $\gamma\backslash_n\mathcal{F}$ is exactly the compact-open topology). So $\varphi(\beta)$ is in $w\backslash(\gamma\backslash_n\mathcal{F})$, where $w\hat{=}\varphi(\beta)(0)$, hence it is in $\mathcal{F}\langle n\rangle^\circ\langle 1\rangle$.

Then, observe that φ maps every $v\backslash_{n+1}\mathcal{F}$ to $w\backslash(\gamma\backslash_n\mathcal{F})$, where $v \in C_0(\Delta_n, \mathcal{F})$, $\gamma\hat{=}\lambda(\tau_0, \ldots, \tau_{n-1}) \in \Delta_{n-1} \cdot v(\tau_0, \ldots, \tau_{n-1}, 0)$, and $w\hat{=}\lambda(\tau_0, \ldots, \tau_{n-1}) \in {\Delta^+}_{n-1} \cdot v(\tau_0, \ldots, \tau_{n-1}, 1 - \tau_0 - \ldots - \tau_{n-1})$, and that these spaces have exactly the compact-open topology. Since abstraction and application are continuous, φ is continuous on each summand $v\backslash_{n+1}\mathcal{F}$; so φ is continuous from $\mathcal{F}\langle n+1\rangle^\circ$ to $\mathcal{F}\langle n\rangle^\circ\langle 1\rangle$.

On the other hand, we define a continuous map ψ from $\mathcal{F}\langle n\rangle^\circ\langle 1\rangle$ to $\mathcal{F}\langle n+1\rangle^\circ$ as follows. For every $\alpha \in \mathcal{F}\langle n\rangle^\circ\langle 1\rangle$, α is in $w\backslash\mathcal{F}\langle n\rangle^\circ$, where $w\hat{=}\alpha(0)$. Since α is continuous from $[0,1]$ to $\mathcal{F}\langle n\rangle^\circ$, and since $[0,1]$ is connected, the range of α is, too, so it must be included in some $\gamma\backslash_n\mathcal{F}$. In particular, $\alpha(\tau)_{|\Delta_{n-1}} = \gamma$ for every $\tau \in [0,1]$. Then let:

$$\psi(\alpha)\hat{=}\lambda(\tau_0, \ldots, \tau_n) \in {\Delta^+}_n \cdot \begin{cases} \alpha\left(\frac{1-\tau_0-\ldots-\tau_{n-1}-\tau_n}{1-\tau_0-\ldots-\tau_{n-1}}\right)(\tau_0, \ldots, \tau_{n-1}) & \text{if } \tau_0 + \ldots + \tau_{n-1} \neq 1 \\ \gamma(\tau_0, \ldots, \tau_{n-1}) & \text{if } \tau_0 + \ldots + \tau_{n-1} = 1 \end{cases}$$

$\psi(\alpha)$ is in $v\backslash_{n+1}\mathcal{F}$, where $v\hat{=}\psi(\alpha)_{|\Delta_n}$. Indeed, we just have to check that $\psi(\alpha)$ is a continuous function. Clearly, $\psi(\alpha)$ is continuous over the set of all $(\tau_0, \ldots, \tau_n) \in {\Delta^+}_n$ such that $\tau_0 + \ldots + \tau_{n-1} \neq 1$ (first case of the definition), as well as over the set of those such that $\tau_0 + \ldots + \tau_{n-1} = 1$ (second case). Then, when $(\tau_0, \ldots, \tau_n) \in {\Delta^+}_n$ tends to $(\tau_0^0, \ldots, \tau_{n-1}^0, \tau_n^0)$ in ${\Delta^+}_n$ such that $\tau_0^0 + \ldots + \tau_{n-1}^0 = 1$, then $\alpha((1-\tau_0-\ldots-\tau_{n-1}-\tau_n)/(1-\tau_0-\ldots-\tau_{n-1}))(\tau_0, \ldots, \tau_{n-1})$ tends to $\gamma(\tau_0^0, \ldots, \tau_{n-1}^0)$, although $(1-\tau_0-\ldots-\tau_{n-1}-\tau_n)/(1-\tau_0-\ldots-\tau_{n-1})$ may not tend to any limit. This is because α is continuous, application is continuous and $\alpha(\tau)(\tau_0^0, \ldots, \tau_{n-1}^0) = \gamma(\tau_0^0, \ldots, \tau_{n-1}^0)$ whatever the value of τ is.

Then, ψ is continuous from $w\backslash(\gamma\backslash_n\mathcal{F})$ to $v\backslash_{n+1}\mathcal{F}$, for all $\gamma \in C_0(\Delta_{n-1}, \mathcal{F})$, all $w \in \gamma\backslash_n\mathcal{F}$, and where $v\hat{=}\lambda(\tau_0, \ldots, \tau_n) \in \Delta_n \cdot w(\tau_0, \ldots, \tau_{n-1})$, for similar reasons as φ. So ψ is continuous from $\mathcal{F}\langle n\rangle^\circ\langle 1\rangle$ to $\mathcal{F}\langle n+1\rangle^\circ$.

It remains to show that φ and ψ are mutually inverse:

$$\begin{aligned}
\varphi(\psi(\alpha)) &= \varphi\left(\lambda(\tau_0, \ldots, \tau_n) \in {\Delta^+}_n \cdot \begin{cases} \alpha\left(\frac{1-\tau_0-\ldots-\tau_{n-1}-\tau_n}{1-\tau_0-\ldots-\tau_{n-1}}\right)(\tau_0, \ldots, \tau_{n-1}) & \text{if } \tau_0 + \ldots + \tau_{n-1} \neq 1 \\ \gamma(\tau_0, \ldots, \tau_{n-1}) & \text{if } \tau_0 + \ldots + \tau_{n-1} = 1 \end{cases}\right) \\
&= \lambda\tau \in [0,1] \cdot \lambda(\tau_0, \ldots, \tau_{n-1}) \in {\Delta^+}_{n-1} \cdot \\
&\quad \begin{cases} \alpha\left(\frac{1-\tau_0-\ldots-\tau_{n-1}-(1-\tau).(1-\tau_0-\ldots-\tau_{n-1})}{1-\tau_0-\ldots-\tau_{n-1}}\right)(\tau_0, \ldots, \tau_{n-1}) & \text{if } \tau_0 + \ldots + \tau_{n-1} \neq 1 \\ \gamma(\tau_0, \ldots, \tau_{n-1}) & \text{if } \tau_0 + \ldots + \tau_{n-1} = 1 \end{cases} \\
&= \lambda\tau \in [0,1] \cdot \lambda(\tau_0, \ldots, \tau_{n-1}) \in {\Delta^+}_{n-1} \cdot \alpha(\tau)(\tau_0, \ldots, \tau_{n-1}) = \alpha
\end{aligned}$$

$$\begin{aligned}
\psi(\varphi(\beta)) &= \psi(\lambda\tau \in [0,1] \cdot \lambda(\tau_0, \ldots, \tau_{n-1}) \in {\Delta^+}_{n-1} \cdot \beta(\tau_0, \ldots, \tau_{n-1}, (1-\tau).(1-\tau_0-\ldots-\tau_{n-1}))) \\
&= \lambda(\tau_0, \ldots, \tau_n) \in {\Delta^+}_n \cdot \begin{cases} \beta\left(\tau_0, \ldots, \tau_{n-1}, \left(1 - \frac{1-\tau_0-\ldots-\tau_{n-1}-\tau_n}{1-\tau_0-\ldots-\tau_{n-1}}\right).(1-\tau_0-\ldots-\tau_{n-1}),\right) \\ \quad \text{if } \tau_0 + \ldots + \tau_{n-1} \neq 1 \\ \beta(\tau_0, \ldots, \tau_{n-1}, 0) \quad \text{if } \tau_0 + \ldots + \tau_{n-1} = 1 \text{ (hence } \tau_n = 0) \end{cases} \\
&= \lambda(\tau_0, \ldots, \tau_n) \in {\Delta^+}_n \cdot \beta(\tau_0, \ldots, \tau_{n-1}, \tau_n) = \beta
\end{aligned}$$

□

Note that (extended) simplices over a space of functions $\mathcal{F} \to \mathcal{G}$ also have an elegant geometric interpretation. While $\mathcal{F} \to \mathcal{G}$ is a set of continuous functions, $(\mathcal{F} \to \mathcal{G})\langle 1\rangle$ is a set of continuous paths from functions f to functions g in $\mathcal{F} \to \mathcal{G}$, so $(\mathcal{F} \to \mathcal{G})\langle 1\rangle$ is a set of *homotopies* between continuous functions from $\mathcal{F}$ to $\mathcal{G}$. The elements of $(\mathcal{F} \to \mathcal{G})\langle n\rangle^\circ$, $n \geq 1$, are then known as *higher-order homotopies*: $\mathcal{F} \to \mathcal{G}\langle 2\rangle^\circ$ is the set of homotopies between homotopies, etc. This is a classical construction in algebraic topology [May67].

5 Combinatorial Models

The idea of the combinatorial models is to abstract away from dcpos or CGWH spaces: just take the spaces of extended singular simplices themselves as denotations for the types, and throw away all the topology. What we keep is the information on how all simplices are glued together, i.e., along which faces, and which simplices are degenerate, i.e., which triangles are really flattened and look like lines, and so on. That is, we keep the *simplicial structure* of the spaces [May67]. This looks reasonable, as the simplicial structure is the one feature that emerged from both the dcpo interpretation (as a nerve) and from the geometric interpretation (as extended singular simplices). Our thesis is that the simplicial structure is actually all that we need to interpret S4 proof terms.

5.1 Simplicial Sets, Augmentations, Godement Enriched

First, we recall some classical notions. All the notions we introduce in this section are well-known, but we shall explain them at some length, since they are not standard notions in computer science. The only new result of Section 5.1 is Lemma 5.1, which we use as an illustration.

A *simplicial set* [May67] is a graded set $(K_q)_{q \geq 0}$, that is, an infinite sequence of sets K_q indexed by integers, together with *face* functions $\partial_q^i : K_q \to K_{q-1}$ for every $0 \leq i \leq q$, $q \geq 1$, and *degeneracies* $s_q^i : K_q \to K_{q+1}$ for every $0 \leq i \leq q$, obeying Equations (i)–(vi) below. The elements of K_q are called q-simplices, or simplices of *dimension* q. Every q-simplex u, $q \geq 1$, has $q+1$ faces $\partial_q^0 u, \ldots, \partial_q^q u$: the two endpoints of the segment Δ_1, the three segments that form the sides of Δ_2, for example (see Figure 5). On the other hand, every point (0-simplex) can be seen as a degenerate segment $s_0^0 u$ in exactly one way, every segment u can be seen as a degenerate triangle in two ways $s_1^0 u$ or $s_1^1 u$ (lift the first or the second endpoint by an infinitesimal amount), and so on. Formally, the faces and degeneracies should obey the following equations:

(i)	$\partial_{q-1}^i(\partial_q^j u) = \partial_{q-1}^{j-1}(\partial_q^i u)$ $(0 \leq i < j \leq q, q \geq 2)$	(ii)	$s_{q+1}^i(s_q^{j-1} u) = s_{q+1}^j(s_q^i u)$ $(0 \leq i < j \leq q)$	(iii)	$\partial_{q+1}^i(s_q^j u) = s_{q-1}^{j-1}(\partial_q^i u)$ $(0 \leq i < j \leq q)$
(iv)	$\partial_{q+1}^i(s_q^i u) = u$ $(0 \leq i \leq q)$	(v)	$\partial_{q+1}^{i+1}(s_q^i u) = u$ $(0 \leq i \leq q)$	(vi)	$s_{q-1}^i(\partial_q^j u) = \partial_{q+1}^{j+1}(s_q^i u)$ $(0 \leq i < j \leq q))$

For example, the space of singular q-simplices of a topological space $\mathcal{F}$ is the space of all continuous functions from the standard n-simplex Δ_q to $\mathcal{F}$. The faces of $\alpha : \Delta_q \to \mathcal{F}$ are the functions $\partial_q^i \alpha \hat{=} \lambda(\tau_0, \ldots, \tau_{q-1}) \in \Delta_{q-1} \cdot \alpha(\tau_0, \ldots, \tau_{i-1}, 0, \tau_{i+1}, \ldots, \tau_{q-1})$, and the degeneracies are $s_q^i \alpha \hat{=} \lambda(\tau_0, \ldots, \tau_{q+1}) \in \Delta_{q+1} \cdot \alpha(\tau_0, \ldots, \tau_{i-1}, \tau_i + \tau_{i+1}, \tau_{i+2}, \ldots, \tau_{q+1})$. What we have just defined is a functor **Sing** from the category of topological spaces, or in fact of CGWH spaces, with continuous functions as morphisms, to the category of simplicial sets with simplicial maps as morphisms [May67]. A *simplicial map* f from K to L is a collection of maps $f_q : K_q \to L_q$ which "commute with every face and degeneracy", that is such that $\partial_q^i \circ f_q = f_{q-1} \circ \partial_q^i$ for every $0 \leq i \leq q$, $q \geq 1$, and such that $s_q^i \circ f_q = f_{q+1} \circ s_q^i$ for every $0 \leq i \leq q$. The functor **Sing** maps the continuous function $f : \mathcal{F} \to \mathcal{G}$ to the simplicial map $\mathbf{Sing}(f)$ such that $\mathbf{Sing}(f)_q \hat{=} \lambda\alpha \in C_0(\Delta_q, \mathcal{F}) \cdot f \circ \alpha$.

Another presentation of simplicial sets and simplicial maps is as follows. Let Δ, the *simplicial category*, have as objects all sets $[q] \hat{=} \{0, \ldots, q\}$, $q \geq 0$, and as morphisms all non-decreasing maps $\mu : [m] \to [n]$. As special cases of morphisms in Δ, we find $\delta_q^i : [q-1] \to [q]$, for every $0 \leq i \leq q$, $q \geq 1$, which is the only injective increasing function that does not take the value i; and $\sigma_q^i : [q+1] \to [q]$, $0 \leq i \leq q$, which is the only surjective non-decreasing function that takes the value i twice. It is easy to see that every morphism in Δ can be written as a composition of δ_q^is and σ_q^is. Then the simplicial sets are exactly the functors K from the opposite category Δ^{op} to the category of sets **Set**. Indeed, K maps $[q]$ to the set of q-simplices, the morphisms $\delta_q^i : [q-1] \to [q]$ to $\partial_q^i : K_q \to K_{q-1}$ and $\sigma_q^i : [q+1] \to [q]$ to $s_q^i : K_q \to K_{q+1}$. Simplicial maps are just natural transformations between simplicial sets, viewed as functors. See [ML71], Chapter 7, Section 5, where Δ is called Δ^+ instead. (We use Δ because this is the name most topologists use.)

This categorical way of seeing simplicial sets allows one to define *simplicial objects in a category* $\mathcal{C}$ as functors from Δ^{op} to $\mathcal{C}$. That is, $\mathcal{C}$ replaces **Set**. For example, simplicial topological spaces are just simplicial sets $(K_q)_{q \geq 0}$, where each K_q is a topological space, and the faces and degeneracies are continuous maps, and morphisms between simplicial topological spaces are simplicial continuous functions.

Another interesting category is that of *augmented simplicial sets*. These are simplicial sets $(K_q)_{q\geq 0}$, together with an additional set K_{-1} and an *augmentation* $\epsilon : K_0 \to K_{-1}$ such that $\epsilon(\partial_1^0 u) = \epsilon(\partial_1^1 u)$ for every $u \in K_1$. The main point in augmented simplicial sets is that $f_q \hat{=} \epsilon \circ \partial_1^0 \circ \ldots \circ \partial_q^0$ defines a simplicial map from K to the trivial simplicial set K_{-1}^* whose sets of q-simplices are K_{-1}, independently of q, and whose faces and degeneracies all are the identity function. See [And74] for applications. Another presentation is to say that augmented simplicial sets are graded sets $(K_q)_{q\geq -1}$ together with face maps $\partial_q^i : K_{q-1} \to K_q$ and degeneracies $s_q^i : K_{q+1} \to K_q$, for every i, $0 \leq i \leq q$, obeying equations (i)–(vi), but where the condition $q \geq 2$ is dropped in (i): the augmentation ϵ is just the new face operator ∂_0^0.

Again, augmented simplicial sets can be seen as functors, this time from $\Delta^{0^{\mathrm{op}}}$ to **Set**, where Δ^0 is the category whose objects are $[q]$ for all $q \geq -1$ (instead of $q \geq 0$ for Δ), and whose morphisms are again all non-decreasing maps. That is, we just add the object $[-1] \hat{=} \emptyset$. *Augmented simplicial maps* are natural transformations between these functors, that is, collections of functions $f_q : K_q \to L_q$, $q \geq -1$, such that $\partial_q^i \circ f_q = f_{q-1} \circ \partial_q^i$ for every $0 \leq i \leq q$, and such that $s_q^i \circ f_q = f_{q+1} \circ s_q^i$ for every $0 \leq i \leq q$. See [ML71], Chapter 7, Section 5, where Δ^0 is named Δ.

As for simplicial sets, we may also consider *augmented simplicial objects in a category* $\mathcal{C}$, that is, functors from $\Delta^{0^{\mathrm{op}}}$ to $\mathcal{C}$. For example, the augmented simplicial topological spaces are those such that K_q is a topological space for every $q \geq -1$, and ∂_q^i and s_q^i are continuous.

This is all the more relevant to us as:

Lemma 5.1 *For every* CGWH *space $\mathcal{F}$, $(\mathcal{F}\langle q+1\rangle^\circ)_{q\geq -1}$ defines an augmented simplicial* CGWH *space, with faces and degeneracies given by:*

$$\begin{array}{rcl} \partial_q^i \alpha & \hat{=} & \lambda(\tau_{q-1}, \ldots, \tau_0) \in {\Delta^+}_{q-1} \cdot \alpha(\tau_{q-1}, \ldots, \tau_{i+1}, 0, \tau_{i-1}, \ldots, \tau_0) \\ s_q^i \alpha & \hat{=} & \lambda(\tau_{q+1}, \ldots, \tau_0) \in {\Delta^+}_{q+1} \cdot \alpha(\tau_{q+1}, \ldots, \tau_{i+2}, \tau_{i+1} + \tau_i, \tau_{i-1}, \ldots, \tau_0) \end{array}$$

for all $\alpha \in \mathcal{F}\langle q+1\rangle^\circ$, $0 \leq i \leq q$.

Proof: The equations (i)–(vi) follow by simple computations. It remains to show that ∂_q^i and s_q^i are continuous from $\mathcal{F}\langle q+1\rangle^\circ$ to $\mathcal{F}\langle q\rangle^\circ$, and from $\mathcal{F}\langle q+1\rangle^\circ$ to $\mathcal{F}\langle q+2\rangle^\circ$ respectively. First, they are clearly continuous from $C_0({\Delta^+}_q, \mathcal{F})$ to $C_0({\Delta^+}_{q-1}, \mathcal{F})$ and to $C_0({\Delta^+}_{q+1}, \mathcal{F})$ respectively, since 0, +, abstraction and application are continuous. Moreover, they map $\gamma\backslash_q\mathcal{F}$, for every singular q-simplex γ, to $\partial_q^i\gamma\backslash_{q-1}\mathcal{F}$ and to $s_q^i\gamma\backslash_{q+1}\mathcal{F}$ respectively. So they are continuous from $\mathcal{F}\langle q+1\rangle^\circ$ to $\mathcal{F}\langle q\rangle^\circ$, and to $\mathcal{F}\langle q+2\rangle^\circ$ respectively. The point here is that the notations $\partial_q^i\gamma\backslash_{q-1}\mathcal{F}$ and $s_q^i\gamma\backslash_{q+1}\mathcal{F}$ make sense, because $\partial_q^i\gamma$ and $s_q^i\gamma$ are standard singular simplices: indeed, $\tau_{q-1} + \ldots \tau_{i+1} + 0 + \tau_{i-1} + \ldots + \tau_0 = 1$ whenever $\tau_{q-1} + \ldots + \tau_0 = 1$ in the first case, and $\tau_{q+1} + \ldots \tau_{i+2} + (\tau_{i+1} + \tau_i) + \tau_{i-1} + \ldots + \tau_0 = 1$ whenever $\tau_{q+1} + \ldots + \tau_0 = 1$ in the second case. □

For every $\alpha \in \mathcal{F}\langle n\rangle^\circ\langle 1\rangle$, we may convert α to an element $\psi(\alpha)$ of $\mathcal{F}\langle n+1\rangle^\circ$ (see Lemma 4.9). Then notice that $\partial_n^0(\psi(\alpha)) = \alpha(1) = \mathtt{ev}\alpha$. In other words, `ev` and taking face number 0 are the same thing. Similarly, `kwote` and s^0 correspond, in that for every $\alpha \in \mathcal{F}\langle n\rangle^\circ\langle 1\rangle$, computing $s_n^0(\psi(\alpha)) \in \mathcal{F}\langle n+2\rangle^\circ$ corresponds through $\varphi = \psi^{-1}$ (see Lemma 4.9) to an element in $\mathcal{F}\langle n\rangle^\circ\langle 1\rangle\langle 1\rangle$ that happens to be exactly $\mathtt{kwote}\,\alpha$. Formally, the conversion function φ_2 from $\mathcal{F}\langle n+2\rangle^\circ$ to $\mathcal{F}\langle n\rangle^\circ\langle 1\rangle\langle 1\rangle$ is defined by $\varphi_2(\beta) \hat{=} \lambda\tau \in [0,1] \cdot \varphi(\varphi(\beta)(\tau))$, and we can check that $\varphi_2(s_n^0(\psi(\alpha))) = \mathtt{kwote}\,\alpha$.

However, `’` has no interpretation as a face or a degeneracy in any augmented simplicial CGWH space, because it is not continuous in general. Similar computations as for `ev` and `kwote` above show that `’` corresponds to some additional degeneracy operator s_q^{-1}, $q \geq -1$, defined by $s_q^{-1}\alpha \hat{=} \lambda(\tau_{q+1}, \ldots, \tau_0) \in {\Delta^+}_{q+1} \cdot \alpha(\tau_{q+1}, \ldots, \tau_1)$, which is not continuous in general. This operator obeys the following additional equations:

$$\begin{array}{llll} (ii') & s_{q+1}^{-1}(s_q^{j-1}u) = s_{q+1}^j(s_q^{-1}u) & & \\ & (0 \leq j \leq q) & & \\ (v') & \partial_{q+1}^0(s_q^{-1}u) = u & (vi') & s_{q-1}^{-1}(\partial_q^j u) = \partial_{q+1}^{j+1}(s_q^{-1}u) \\ & (-1 \leq q) & & (0 \leq j \leq q)) \end{array}$$

An augmented simplicial set $(K_q)_{q\geq -1}$ with an additional set of operators $s_q^{-1} : K_q \to K_{q+1}$, $q \geq -1$, obeying equations (ii'), (v') and (vi'), is called a *Godement-enriched simplicial set* (see [Tho95], Section 2.2). Again, it can be described as a functor, as follows. Let Δ^+ be the category whose objects are $[q]^+ \hat{=} \{-1, 0, \ldots, q\}$, $q \geq -1$, and whose morphisms are all non-decreasing functions $\mu : [m]^+ \to [n]^+$ that fix -1, i.e., such that $\mu(-1) = -1$. Then the Godement-enriched simplicial sets are exactly the functors from $\Delta^{+\mathrm{op}}$ to **Set**.

We have just shown that $(\mathcal{F}\langle q+1\rangle^\circ)_{q\geq -1}$ was not only an augmented simplicial CGWH space, but was also a Godement-enriched simplicial set. However, it is not in general a Godement-enriched simplicial CGWH space, because s_q^{-1} is not continuous in general—remember that it is a *feature*, as it explains why $F \supset \Box F$ is not provable in general.

Note that the categories Δ^{op}**Set** of simplicial sets and simplicial maps, $\Delta^{0\mathrm{op}}$**Set** of augmented simplicial sets and augmented simplicial maps, $\Delta^{+\mathrm{op}}$**Set** of Godement-enriched simplicial sets and Godement-enriched simplicial maps, are all cartesian closed. In fact, they are more: as functor categories from some category—Δ^{op}, $\Delta^{0\mathrm{op}}$, or $\Delta^{+\mathrm{op}}$—to **Set**, they are *elementary toposes* [Gol84]. While cartesian closed categories are models of proof terms for intuitionistic propositional logic, elementary toposes are models of proof terms for intuitionistic set theory; the proof terms themselves are known as the *Mitchell-Bénabou language.*

We shall need to use the cartesian closed structure of $\Delta^{0\mathrm{op}}$**Set**, but it is rather complicated, and we shall only need elementary properties of this structure. Let us just recall that being *cartesian* means that we can build the *product* $\mathcal{F}_1 \times \ldots \times \mathcal{F}_n$ of n augmented simplicial sets $\mathcal{F}_1, \ldots, \mathcal{F}_n$, that we can build tuples of augmented simplicial maps f_i from $\mathcal{H}$ to $\mathcal{F}_i$, $1 \leq i \leq n$, as new augmented simplicial maps $\langle f_1, \ldots, f_n\rangle$ from $\mathcal{H}$ to $\mathcal{F}_1 \times \ldots \times \mathcal{F}_n$, and that we have corresponding projections $\bar{i}$, $1 \leq i \leq n$, which are augmented simplicial maps from $\mathcal{F}_1 \times \ldots \times \mathcal{F}_n$ to $\mathcal{F}_i$, obeying all the expected equations. Also, being cartesian *closed* means that for any two augmented simplicial sets $\mathcal{F}$ and $\mathcal{G}$, there is an augmented simplicial set $\mathcal{G}^{\mathcal{F}}$, which will interpret the type of functions from $\mathcal{F}$ to $\mathcal{G}$, that there is a *currification* or *abstraction* operation Λ mapping every augmented simplicial map f from $\mathcal{F} \times \mathcal{G}$ to $\mathcal{H}$ to an augmented simplicial map $\Lambda(f)$ from $\mathcal{G}$ to $\mathcal{H}^{\mathcal{F}}$, and an augmented simplicial map called *application* from $\mathcal{G}^{\mathcal{F}} \times \mathcal{F}$ to $\mathcal{G}$ related to Λ by forms of the β and η rules. More details on the actual definitions of these can be found in Appendix A.

5.2 Defining the Combinatorial Model

Now that we have defined all the required notions, we turn to the definition of our third and last model, the *combinatorial model.* In it, as promised, we forget about all order structures or all topologies, and keep only the simplicial structure.

In view of Lemma 5.1, the right notion is to define the interpretation $[\![F]\!]_c$ of types F as *augmented* simplicial sets. The interpretation of arrow types $F \supset G$, of application and of abstraction, follow from the fact that the category $\Delta^{0\mathrm{op}}$**Set** of augmented simplicial sets is cartesian closed.

There is simple way to interpret box types $\Box F$. For disambiguation purposes, write $\partial^i_{K_q}$ instead of ∂^i_q, $s^i_{K_q}$ instead of s^i_q for the faces and degeneracies of the augmented simplicial set K. Then we may define $[\![\Box F]\!]_c$ as $[\![F]\!]_c\langle 1\rangle$, where the q-simplices of $[\![F]\!]_c\langle 1\rangle$ are the $(q+1)$-simplices of $[\![F]\!]_c$, $q \geq -1$, and $\partial^i_{([\![F]\!]_c\langle 1\rangle)_q} \hat{=} \partial^{i+1}_{[\![F]\!]_{c\,q+1}}$, $s^i_{([\![F]\!]_c\langle 1\rangle)_q} \hat{=} s^{i+1}_{[\![F]\!]_{c\,q+1}}$, $0 \leq i \leq q$. That is, $[\![F]\!]_c\langle 1\rangle$ is $[\![F]\!]_c$ with all dimensions shifted up by 1.

Recall now that we wish to interpret S4 proof terms as morphisms in the category at hand, namely $\Delta^{0\mathrm{op}}$**Set**; in other words, as augmented simplicial maps.

There is a simplicial map $\mathtt{ev} \hat{=} \partial^0$ from $\mathcal{F}\langle 1\rangle$ to $\mathcal{F}$, for every augmented simplicial set $\mathcal{F}$: let $(\partial^0)_q$ map every q-simplex u of $\mathcal{F}\langle 1\rangle$, $q \geq -1$, to $\partial^i_{\mathcal{F}_{q+1}} u \in \mathcal{F}_q$. As suggested in Sections 3.3 and 4.1, ∂^0 will serve as interpretation of `unbox`. This definition is valid, because u is by definition a $(q+1)$-simplex of $\mathcal{F}$ as well. Moreover, ∂^0 is an augmented simplicial map: for every $0 \leq i \leq q$, $\partial^i_{\mathcal{F}_q} \circ (\partial^0)_q = \partial^i_{\mathcal{F}_q} \circ \partial^0_{\mathcal{F}_{q+1}} = \partial^0_{\mathcal{F}_q} \circ \partial^{i+1}_{\mathcal{F}_{q+1}}$ (by Equation (i); recall that in the augmented case, the constraint $q \geq 2$ is dropped) $= (\partial^0)_{q-1} \circ \partial^{i+1}_{\mathcal{F}_{q+1}} = (\partial^0)_{q-1} \circ \partial^i_{(\mathcal{F}\langle 1\rangle)_q}$; and $s^i_{\mathcal{F}_q} \circ (\partial^0)_q = s^i_{\mathcal{F}_q} \circ \partial^0_{\mathcal{F}_{q+1}} = \partial^0_{\mathcal{F}_{q+2}} \circ s^{i+1}_{\mathcal{F}_{q+1}}$ (by Equation (iii)) $= (\partial^0)_{q+1} \circ s^{i+1}_{\mathcal{F}_{q+1}} = (\partial^0)_{q+1} \circ s^i_{(\mathcal{F}\langle 1\rangle)_q}$.

There is also a simplicial map $\mathtt{kwote} \hat{=} s^0$ from $\mathcal{F}\langle 1\rangle$ to $\mathcal{F}\langle 1\rangle\langle 1\rangle$: let $(s^0)_q$ map every q-simplex u of $\mathcal{F}\langle 1\rangle$, i.e., every $(q+1)$-simplex u of $\mathcal{F}$, to $s^0_{q+1} u \in \mathcal{F}_{q+2} = (\mathcal{F}\langle 1\rangle\langle 1\rangle)_q$. s^0 is again an augmented simplicial map: this is by Equations (ii) and (vi).

For any simplicial map f from $\mathcal{F}$ to $\mathcal{G}$, we may also define a simplicial map $\mathtt{'}f$ from $\mathcal{F}\langle 1\rangle$ to $\mathcal{G}\langle 1\rangle$ by letting $(\mathtt{'}f)_q$ map every q-simplex u of $\mathcal{F}\langle 1\rangle$, that is, every $(q+1)$-simplex u of $\mathcal{F}$, to $f_{q+1}(u) \in \mathcal{G}_{q+1} = (\mathcal{G}\langle 1\rangle)_q$. To enforce some analogy with Godement-enriched simplicial sets, we can also write $\mathtt{'}f$ as $s^{-1}f$. But, just like

$'$ was not continuous in the dcpo or in the geometric model, here $s^{-1} = {}'$ is not a simplicial map.

The combinatorial interpretation $[\![\text{-}]\!]_c$ is summed up in Figure 6. As usual, $[\![\Gamma]\!]_c$, where $\Gamma \hat{=} x_1 : F_1, \ldots, x_n : F_n$, is taken to denote the product $[\![F_1]\!]_c \times \ldots \times [\![F_n]\!]_c$.

$$\begin{array}{rcl} [\![F \supset G]\!]_c & \hat{=} & {[\![G]\!]_c}^{[\![F]\!]_c} \\ [\![\Box F]\!]_c & \hat{=} & [\![F]\!]_c\langle 1\rangle \end{array}$$

$$\begin{array}{rcl} [\![\Gamma, x : F, \Delta \vdash x : F]\!]_c & \hat{=} & \bar{i} \\ & & \text{where } \Gamma \hat{=} x_1 : F_1, \ldots, x_{i-1} : F_{i-1} \\ [\![\Gamma \vdash st : G]\!]_c & \hat{=} & App \circ \langle [\![\Gamma \vdash s : F \supset G]\!]_c, [\![\Gamma \vdash t : F]\!]_c \rangle \\ [\![\Gamma \vdash \lambda x_F \cdot t : F \supset G]\!]_c & \hat{=} & \Lambda([\![x : F, \Gamma \vdash t : G]\!]_c) \\ [\![\Gamma \vdash \mathsf{unbox}\, t : F]\!]_c(g) & \hat{=} & \partial^0 \circ [\![\Gamma \vdash t : \Box F]\!]_c \\ [\![\Gamma \vdash \mathsf{box}\, t\ \mathsf{with}\ \sigma : \Box G]\!]_c & \hat{=} & '[\![\Delta \vdash t : G]\!]_c \circ \langle s^0 \circ [\![\Gamma \vdash t_1 : \Box F_1]\!]_c, \ldots, s^0 \circ [\![\Gamma \vdash t_n : \Box F_n]\!]_c \rangle \\ & & \text{where } \Delta \hat{=} x_1 : \Box F_1, \ldots, x_n : \Box F_n,\ \sigma \hat{=} \{x_1 := t_1, \ldots, x_n := t_n\} \end{array}$$

Figure 6: The Combinatorial Interpretation

We can now reproduce the lemmas that we have been proving for the dcpo and the geometric interpretations:

Lemma 5.2 *For every term t such that $\Gamma \vdash t : F$ is derivable, $[\![\Gamma \vdash t : F]\!]_c$ is an augmented simplicial map from $[\![\Gamma]\!]_c$ to $[\![F]\!]_c$.*

Proof: By structural induction on t. Recall that $\bar{i}$, App, ∂^0, s^0 are augmented simplicial maps, and that tupling, Λ, $'$ and composition take augmented simplicial maps to augmented simplicial maps. All cases except possibly that of terms of the form $\mathsf{box}\, t\ \mathsf{with}\ \sigma$ are then trivial. In the latter case, observe that $[\![\Delta \vdash t : G]\!]_c$ is an augmented simplicial map from $[\![\Delta]\!]_c \hat{=} [\![F_1]\!]_c\langle 1\rangle \times \ldots \times [\![F_n]\!]_c\langle 1\rangle$ to $[\![G]\!]_c$, by induction hypothesis, so that $'[\![\Delta \vdash t : G]\!]_c$ is an augmented simplicial map from $([\![F_1]\!]_c\langle 1\rangle \times \ldots \times [\![F_n]\!]_c\langle 1\rangle)\langle 1\rangle$ to $[\![G]\!]_c\langle 1\rangle$. But by construction $([\![F_1]\!]_c\langle 1\rangle \times \ldots \times [\![F_n]\!]_c\langle 1\rangle)\langle 1\rangle = [\![F_1]\!]_c\langle 1\rangle\langle 1\rangle \times \ldots \times [\![F_n]\!]_c\langle 1\rangle\langle 1\rangle$, so: (a) $'[\![\Delta \vdash t : G]\!]_c$ is an augmented simplicial map from $[\![F_1]\!]_c\langle 1\rangle\langle 1\rangle \times \ldots \times [\![F_n]\!]_c\langle 1\rangle\langle 1\rangle$ to $[\![G]\!]_c\langle 1\rangle$. On the other hand, by induction hypothesis $[\![\Gamma \vdash t_i : \Box F_i]\!]_c$ is an augmented simplicial map from $[\![\Gamma]\!]_c$ to $[\![F_i]\!]_c\langle 1\rangle$ for every i, $1 \leq i \leq n$, so $s^0 \circ [\![\Gamma \vdash t_i : \Box F_i]\!]_c$ is an augmented simplicial map from $[\![\Gamma]\!]_c$ to $[\![F_i]\!]_c\langle 1\rangle\langle 1\rangle$, hence: (b) $\langle s^0 \circ [\![\Gamma \vdash t_1 : \Box F_1]\!]_c, \ldots, s^0 \circ [\![\Gamma \vdash t_n : \Box F_n]\!]_c\rangle$ is an augmented simplicial map from $[\![\Gamma]\!]_c$ to $[\![F_1]\!]_c\langle 1\rangle\langle 1\rangle \times \ldots \times [\![F_n]\!]_c\langle 1\rangle\langle 1\rangle$. Composing (a) and (b) then proves the claim. $\Box$

Lemma 5.3 *For every term t such that $\Gamma_1, x_1 : F_1, \ldots, x_n : F_n, \Gamma_2 \vdash t : F$ is derivable, for every terms t_1, $\ldots$, t_n such that $\Gamma_1, \Delta, \Gamma_2 \vdash t_i : F_i$ is derivable for every i, $1 \leq i \leq n$,*

$$\begin{array}{rl} & [\![\Gamma_1, \Delta, \Gamma_2 \vdash t\{x_1 := t_1, \ldots, x_n := t_n\} : F]\!]_c \\ = & [\![\Gamma_1, x_1 : F_1, \ldots, x_n : F_n, \Gamma_2 \vdash t : F]\!]_c \circ \\ & \langle \overline{1}, \ldots, \overline{k_1}, [\![\Gamma_1, \Delta, \Gamma_2 \vdash t_1 : F_1]\!]_c, \ldots, [\![\Gamma_1, \Delta, \Gamma_2 \vdash t_n : F_n]\!]_c, \overline{k_1 + \ell + 1}, \ldots, \overline{k_1 + \ell + k_2} \rangle \end{array}$$

where Γ_1 consists of exactly k_1 bindings, Γ_2 of k_2 bindings, and Δ of ℓ bindings.

Proof: By structural induction on t, using standard equational reasoning in cartesian closed categories [Cur93]. $\Box$

Lemma 5.4 *If x is not free in t, then $[\![\Gamma, x : F, \Delta \vdash t : G]\!]_c = [\![\Gamma, \Delta \vdash t : G]\!]_c \circ \langle \overline{1}, \ldots, \overline{k}, \overline{k+2}, \ldots, \overline{k + \ell + 1}\rangle$, where there are exactly k bindings in Γ and ℓ in Δ.*

Proof: By structural induction on t, as for Lemma 5.3. $\Box$

Theorem 5.5 (Soundness) *The combinatorial interpretation is sound wrt.* λ_{S4} *with the extensional equalities: for every terms* s *and* t *such that* $\Gamma \vdash s : F$ *and* $\Gamma \vdash t : F$ *are both derivable, and such that* $s =_\eta t$*, we have* $[\![\Gamma \vdash s : F]\!]_c = [\![\Gamma \vdash t : F]\!]_c$*.*

Proof: We first check each α-equivalence rule:

- $\equiv$, first rule. Let Γ contain k bindings.

$$\begin{aligned} [\![\Gamma \vdash \lambda y_F \cdot t\{x := y\} : F \supset G]\!]_c &= \Lambda([\![y : F, \Gamma \vdash t\{x := y\} : G]\!]_c) \\ &= \Lambda([\![x : F, \Gamma \vdash t]\!]_c \circ \langle \overline{1}, \ldots, \overline{k+1}\rangle) \quad \text{by Lemma 5.3} \\ &= \Lambda([\![x : F, \Gamma \vdash t]\!]_c) = [\![\Gamma \vdash \lambda x_F \cdot : F \supset G]\!]_c \end{aligned}$$

  Indeed, $\langle \overline{1}, \ldots, \overline{k+1}\rangle$ is the identity, and composing with the identity does nothing.

- $\equiv$, second rule.

$$\begin{aligned} & [\![\Gamma \vdash \mathsf{box}\, t\{x_1 := y_1, \ldots, x_n := y_n\} \,\mathsf{with}\, \{y_1 := t_1, \ldots, y_n := t_n\} : \Box G]\!]_c \\ = \;& {}^{\prime}[\![y_1 : \Box F_1, \ldots, y_n : \Box F_n \vdash t\{x_1 := y_1, \ldots, x_n := y_n\} : G]\!]_c \\ & \quad \circ \langle s^0 \circ [\![\Gamma \vdash t_1 : \Box F_1]\!]_c, \ldots, s^0 \circ [\![\Gamma \vdash t_n : \Box F_n]\!]_c\rangle \\ = \;& {}^{\prime}[\![x_1 : \Box F_1, \ldots, x_n : \Box F_n \vdash t : G]\!]_c \\ & \quad \circ \langle s^0 \circ [\![\Gamma \vdash t_1 : \Box F_1]\!]_c, \ldots, s^0 \circ [\![\Gamma \vdash t_n : \Box F_n]\!]_c\rangle \quad \text{by Lemma 5.3} \\ = \;& [\![\Gamma \vdash \mathsf{box}\, t \,\mathsf{with}\, \{x_1 := t_1, \ldots, x_n := t_n\} : \Box G]\!]_c \end{aligned}$$

- (β). Recall that in any cartesian closed category $\mathit{App} \circ \langle \Lambda(u), v\rangle = u \circ \langle v, \mathrm{id}\rangle$ [Cur93]. Then:

$$\begin{aligned} [\![\Gamma \vdash (\lambda x_F \cdot s)t : G]\!]_c &= \mathit{App} \circ \langle [\![\Gamma \vdash \lambda x_F \cdot s : F \supset G]\!]_c, [\![\Gamma \vdash t : F]\!]_c\rangle \\ &= \mathit{App} \circ \langle \Lambda([\![x : F, \Gamma \vdash s : G]\!]_c), [\![\Gamma \vdash t : F]\!]_c\rangle \\ &= [\![x : F, \Gamma \vdash s : G]\!]_c \circ \langle [\![\Gamma \vdash t : F]\!]_c, \mathrm{id}_{[\![\Gamma]\!]_c}\rangle = [\![\Gamma \vdash s\{x := t\} : G]\!]_c \end{aligned}$$

  by Lemma 5.3.

- (unbox): first, note that: (a) $\partial^0 \circ {}^{\prime} f = f \circ \partial^0$ for every augmented simplicial map f from $\mathcal{F}$ to $\mathcal{G}$, where the equality is between two augmented simplicial maps from $\mathcal{F}\langle 1\rangle$ to $\mathcal{G}$. Indeed, for every q-simplex v of $\mathcal{F}\langle 1\rangle$, $(\partial^0 \circ {}^{\prime} f)_q(v) = \partial^0_{\mathcal{G}_{q+1}}(({}^{\prime} f)_q(v)) = \partial^0_{\mathcal{G}_{q+1}}(f_{q+1}(v)) = f_q(\partial^0_{\mathcal{F}_{q+1}}(v))$ (because f is an augmented simplicial map) $= (f \circ \partial^0)_q(v)$.

  Then, observe also that: (b) $\partial^0 \circ \langle f, g\rangle = \langle \partial^0 \circ f, \partial^0 \circ g\rangle$, for every augmented simplicial maps f from $\mathcal{H}$ to $\mathcal{F}\langle 1\rangle$ and g from $\mathcal{H}$ to $\mathcal{G}\langle 1\rangle$. Indeed, for every q-simplex v in $\mathcal{H}$, $(\partial^0 \circ \langle f, g\rangle)_q(v) = \partial^0_{q+1}(f_q(v), g_q(v)) = (\partial^0_{q+1}(f_q(v)), \partial^0_{q+1}(g_q(v)))$ (since faces operate pointwise) $= (\langle \partial^0 \circ f, \partial^0 \circ g\rangle)_q(v)$.

  Finally: (c) $\partial^0 \circ s^0 = \mathrm{id}_{\mathcal{F}\langle 1\rangle}$. Indeed, for every q-simplex v of $\mathcal{F}\langle 1\rangle$, $(\partial^0 \circ s^0)_q(v) = \partial^0_{q+2}(s^0_{q+1}(v)) = v$ by Equation (iv) defining augmented simplicial sets.

  By (a), (b), (c), then: (d) $\partial^0 \circ {}^{\prime} f \circ \langle s^0 \circ f_1, \ldots, s^0 \circ f_n\rangle = f \circ \langle f_1, \ldots, f_n\rangle$.

  Now let σ be $\{x_1 := t_1, \ldots, x_n := t_n\}$, $\Delta \hat{=} x_1 : \Box F_1, \ldots, x_n : \Box F_n$, and assume that $\Delta \vdash t : G$ has been derived, as well as $\Gamma \vdash t_i : \Box F_i$ for each i, $1 \leq i \leq n$.

$$\begin{aligned} [\![\Gamma \vdash \mathsf{unbox}(\mathsf{box}\, t \,\mathsf{with}\, \sigma) : G]\!]_c &= \partial^0 \circ [\![\Gamma \vdash \mathsf{box}\, t \,\mathsf{with}\, \sigma : \Box G]\!]_c \\ &= \partial^0 \circ {}^{\prime}[\![\Delta \vdash t : G]\!]_c \circ \langle s^0 \circ [\![\Gamma \vdash t_1 : \Box F_1]\!]_c, \ldots, s^0 \circ [\![\Gamma \vdash t_n : \Box F_n]\!]_c\rangle \\ &= [\![\Delta \vdash t : G]\!]_c \circ \langle [\![\Gamma \vdash t_1 : \Box F_1]\!]_c, \ldots, [\![\Gamma \vdash t_n : \Box F_n]\!]_c\rangle \quad \text{by (d)} \\ &= [\![\Gamma \vdash t\sigma : G]\!]_c \quad \text{by Lemma 5.3} \end{aligned}$$

- (gc): First, define π_2 as the augmented simplicial map from $\mathcal{F}_1 \times \ldots \times \mathcal{F}_n$ to $\mathcal{F}_2 \times \ldots \times \mathcal{F}_n$ $(n \geq 1)$, mapping every tuple $(v_1, v_2, \ldots, v_n)$ to $(v_2, \ldots, v_n)$. (This is the same π_2 as in Appendix A.)

Observe that: (e) $\text{'}(f\circ\pi_2) = \text{'}f\circ\pi_2$ for every augmented simplicial map f from $\mathcal{F}$ to $\mathcal{G}$, and where both sides of the equality are viewed as augmented simplicial maps from $(\mathcal{H}\times\mathcal{F})\langle 1\rangle$ to $\mathcal{G}\langle 1\rangle$. Indeed, for every q-simplex v in $(\mathcal{H}\times\mathcal{F})\langle 1\rangle$, we may write v as (v_1, v_2), so $(\text{'}(f\circ\pi_2))_q(v) = (f\circ\pi_2)_{q+1}(v) = f_{q+1}(v_2) = (\text{'}f)_q(v_2) = (\text{'}f\circ\pi_2)_q(v)$.

Now assume that x_1 is not free in t, then:

$$
\begin{aligned}
& [\![\Gamma\vdash \mathsf{box}\, t \,\mathsf{with}\, \{x_1 := t_1,\ldots,x_n := t_n\} : \Box G]\!]_c \\
= \;& \text{'}[\![x_1:\Box F_1, x_2:\Box F_2,\ldots,x_n:\Box F_n\vdash t:G]\!]_c\circ\langle s^0\circ[\![\Gamma\vdash t_1:\Box F_1]\!]_c,\ldots,s^0\circ[\![\Gamma\vdash t_n:\Box F_n]\!]_c\rangle \\
= \;& \text{'}([\![x_2:\Box F_2,\ldots,x_n:\Box F_n\vdash t:G]\!]_c\circ\pi_2) \\
& \qquad\circ\langle s^0\circ[\![\Gamma\vdash t_1:\Box F_1]\!]_c,\ldots,s^0\circ[\![\Gamma\vdash t_n:\Box F_n]\!]_c\rangle \quad \text{by Lemma 5.4} \\
= \;& \text{'}[\![x_2:\Box F_2,\ldots,x_n:\Box F_n\vdash t:G]\!]_c\circ\pi_2 \\
& \qquad\circ\langle s^0\circ[\![\Gamma\vdash t_1:\Box F_1]\!]_c,\ldots,s^0\circ[\![\Gamma\vdash t_n:\Box F_n]\!]_c\rangle \quad \text{by (e)} \\
= \;& \text{'}[\![x_2:\Box F_2,\ldots,x_n:\Box F_n\vdash t:G]\!]_c\circ\langle s^0\circ[\![\Gamma\vdash t_2:\Box F_2]\!]_c,\ldots,s^0\circ[\![\Gamma\vdash t_n:\Box F_n]\!]_c\rangle \\
= \;& [\![\Gamma\vdash \mathsf{box}\, t \,\mathsf{with}\, \{x_2 := t_2,\ldots,x_n := t_n\} : \Box G]\!]_c
\end{aligned}
$$

- (ctr): note that: (f) $\text{'}(f\circ\langle\overline{1},\mathrm{id}_{\mathcal{F}\times\mathcal{G}}\rangle) = \text{'}f\circ\langle\overline{1},\mathrm{id}\rangle$ for every augmented simplicial map f from $\mathcal{F}\times\mathcal{F}\times\mathcal{G}$ to $\mathcal{H}$, and where the equality is between augmented simplicial maps from $(\mathcal{F}\times\mathcal{G})\langle 1\rangle$ to $\mathcal{H}\langle 1\rangle$. Indeed, for every q-simplex (v_1,v_2) of $(\mathcal{F}\times\mathcal{G})\langle 1\rangle$, $(\text{'}(f\circ\langle\overline{1},\mathrm{id}_{\mathcal{F}\times\mathcal{G}}\rangle))_q(v_1,v_2) = (f\circ\langle\overline{1},\mathrm{id}_{\mathcal{F}\times\mathcal{G}}\rangle)_{q+1}(v_1,v_2) = f_{q+1}(v_1,(v_1,v_2)) = (\text{'}f)_q(v_1,(v_1,v_2)) = (\text{'}f\circ\langle\overline{1},\mathrm{id}\rangle)_q(v_1,v_2)$.

  Now assume $t_1\equiv t_2$, then:

$$
\begin{aligned}
& [\![\Gamma\vdash \mathsf{box}\, t\{x_1 := x_2\} \,\mathsf{with}\, \{x_2 := t_2,\ldots,x_n := t_n\} : \Box G]\!]_c \\
= \;& \text{'}[\![x_2:\Box F_2,\ldots,x_n:\Box F_n\vdash t\{x_1:=x_2\}:G]\!]_c\circ\langle s^0\circ[\![\Gamma\vdash t_2:\Box F_2]\!]_c,\ldots,s^0\circ[\![\Gamma\vdash t_n:\Box F_n]\!]_c\rangle \\
= \;& \text{'}([\![x_1:\Box F_1, x_2:\Box F_2,\ldots,x_n:\Box F_n\vdash t:G]\!]_c\circ\langle\overline{1},\mathrm{id}\rangle) \\
& \qquad\circ\langle s^0\circ[\![\Gamma\vdash t_2:\Box F_2]\!]_c,\ldots,s^0\circ[\![\Gamma\vdash t_n:\Box F_n]\!]_c\rangle \quad \text{by Lemma 5.3} \\
= \;& \text{'}[\![x_1:\Box F_1, x_2:\Box F_2,\ldots,x_n:\Box F_n\vdash t:G]\!]_c\circ\langle\overline{1},\mathrm{id}\rangle \\
& \qquad\circ\langle s^0\circ[\![\Gamma\vdash t_2:\Box F_2]\!]_c,\ldots,s^0\circ[\![\Gamma\vdash t_n:\Box F_n]\!]_c\rangle \quad \text{by (f)} \\
= \;& \text{'}[\![x_1:\Box F_1, x_2:\Box F_2,\ldots,x_n:\Box F_n\vdash t:G]\!]_c \\
& \qquad\circ\langle s^0\circ[\![\Gamma\vdash t_2:\Box F_2]\!]_c, s^0\circ[\![\Gamma\vdash t_2:\Box F_2]\!]_c,\ldots,s^0\circ[\![\Gamma\vdash t_n:\Box F_n]\!]_c\rangle \\
= \;& [\![\Gamma\vdash \mathsf{box}\, t \,\mathsf{with}\, \{x_1 := t_1, x_2 := t_2,\ldots,x_n := t_n\} : \Box G]\!]_c
\end{aligned}
$$

  since indeed $t_1\equiv t_2$.

- (box): again, we shall show that the interpretations of $(\text{`}s)\{x := \text{`}t\}$ and of $\text{`}(s\{x := \text{`}t\})$ are equal.

  Notice that: (g) $s^0\circ\langle f,g\rangle = \langle s^0\circ f, s^0\circ g\rangle$ for every augmented simplicial maps f from $\mathcal{H}$ to $\mathcal{F}\langle 1\rangle$ and g from $\mathcal{H}$ to $\mathcal{G}\langle 1\rangle$, where the equality is between augmented simplicial maps from $\mathcal{H}$ to $(\mathcal{F}\times\mathcal{G})\langle 1\rangle\langle 1\rangle$. Indeed, for every q-simplex v of $\mathcal{H}$, $(s^0\circ\langle f,g\rangle)_q(v) = s^0_{q+1}(f_q(v),g_q(v)) = (s^0_{q+1}(f_q(v)), s^0_{q+1}(g_q(v))) = (\langle s^0\circ f, s^0\circ g\rangle)_q(v)$.

  It follows that, for every s for which this makes sense: (h) $[\![\Gamma\vdash \text{`}s:\Box F]\!]_c = \text{'}[\![\Gamma\vdash s:F]\!]_c\circ s^0$. Indeed, $[\![\Gamma\vdash\text{`}s:\Box F]\!]_c = \text{'}[\![\Gamma\vdash s:F]\!]_c\circ\langle s^0\circ\overline{1},\ldots,s^0\circ\overline{k}\rangle$ (where there are k bindings in Γ) $= \text{'}[\![\Gamma\vdash s:F]\!]_c\circ s^0\circ\langle\overline{1},\ldots,\overline{k}\rangle$ (by (g)) $= \text{'}[\![\Gamma\vdash s:F]\!]_c\circ s^0$.

  Then, observe that: (i) $s^0\circ\overline{1} = \overline{1}\circ s^0$ and $s^0\circ\pi_2 = \pi_2\circ s^0$. Let's deal with the first case, as the second is entirely similar. This is an equality between simplicial maps from $\mathcal{F}\langle 1\rangle\times\mathcal{G}$ to $\mathcal{F}\langle 1\rangle\langle 1\rangle$. So for every q-simplex (v_1,v_2) of $\mathcal{F}\langle 1\rangle\times\mathcal{G}$, $(s^0\circ\overline{1})_q(v) = s^0_{q+1}(v_1) = (\overline{1}\circ s^0)_q(v)$.

  Similarly: (j) $s^0\circ\overline{i} = \overline{i}\circ s^0$ for every i.

  Then, note that: (k) $s^0\circ\text{'}f = \text{''}f\circ s^0$ for every augmented simplicial map f from $\mathcal{F}$ to $\mathcal{G}$, where the equality is between two augmented simplicial maps from $\mathcal{F}\langle 1\rangle$ to $\mathcal{G}\langle 1\rangle\langle 1\rangle$. Indeed, for every q-simplex

v of $\mathcal{F}\langle 1\rangle$, $(s^0 \circ \text{'}f)_q(v) = s^0_{q+1}(f_{q+1}(v)) = f_{q+2}(s^0_{q+1}(v))$ (because f is an augmented simplicial map) $= (\text{''}f \circ s^0)_q(v)$.

We also have: (l) $\text{'}(f \circ g) = \text{'}f \circ \text{'}g$, $\text{'}\langle f, g\rangle = \langle \text{'}f, \text{'}g\rangle$, and $\text{'}\overline{i} = \overline{i}$ for every i. This is straightforward.

Finally, let $s^1 \hat{=} \text{'}s^0$. Then: (m) $s^0 \circ s^0 = s^1 \circ s^0$, as an equality between augmented simplicial maps from $\mathcal{F}\langle 1\rangle$ to $\mathcal{F}\langle 1\rangle\langle 1\rangle\langle 1\rangle$. Indeed, let v be any q-simplex in $\mathcal{F}\langle 1\rangle$, then $(s^0 \circ s^0)_q(v) = s^0_{q+2}(s^0_{q+1}(v)) = s^1_{q+2}(s^0_{q+1}(v))$ (by Equation (ii) for augmented simplicial sets) $= (\text{'}s^0)_{q+1}(s^0_{q+1}(v)) = (s^1 \circ s^0)_q(v)$.

So, assume that the free variables of s are among x, x_1, …, x_n, and that $x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G$, and $y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t : F$. Let also π_2^n denote the n-fold composition of π_2. Then:

$$\begin{aligned}
& [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash (\text{`}s)\{x := \text{`}t\} : \Box G]\!]_c \\
= \; & [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash \text{`}s : \Box G]\!]_c \\
& \qquad \circ\langle \overline{1}, \ldots, \overline{n}, [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash \text{`}t : \Box F]\!]_c\rangle \\
& \qquad\qquad \text{by Lemma 5.3} \\
= \; & [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash \text{`}s : \Box G]\!]_c \\
& \qquad \circ\langle \overline{1}, \ldots, \overline{n}, [\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash \text{`}t : \Box F]\!]_c \circ \pi_2^n\rangle \quad \text{by Lemma 5.4} \\
= \; & \text{'}[\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_c \circ s^0 \\
& \qquad \circ\langle \overline{1}, \ldots, \overline{n}, \text{'}[\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t : F]\!]_c \circ s^0 \circ \pi_2^n\rangle \quad \text{by (h)} \\
= \; & \text{'}[\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_c \\
& \qquad \circ\langle s^0 \circ \overline{1}, \ldots, s^0 \circ \overline{n}, s^0 \circ \text{'}[\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t : F]\!]_c \circ s^0 \circ \pi_2^n\rangle \quad \text{by (g)} \\
= \; & \text{'}[\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_c \\
& \qquad \circ\langle \overline{1} \circ s^0, \ldots, \overline{n} \circ s^0, s^0 \circ \text{'}[\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t : F]\!]_c \circ \pi_2^n \circ s^0\rangle \quad \text{by (i) and (j)} \\
= \; & \text{'}[\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_c \\
& \qquad \circ\langle \overline{1} \circ s^0, \ldots, \overline{n} \circ s^0, \text{''}[\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t : F]\!]_c \circ \pi_2^n \circ s^0 \circ s^0\rangle \quad \text{by (k) and (i)} \\
= \; & \text{'}[\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_c \\
& \qquad \circ\langle \overline{1} \circ s^0, \ldots, \overline{n} \circ s^0, \text{''}[\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t : F]\!]_c \circ \pi_2^n \circ s^1 \circ s^0\rangle \quad \text{by (m)}
\end{aligned}$$

On the other hand:

$$\begin{aligned}
& [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash \text{`}(s\{x := \text{`}t\}) : \Box G]\!]_c \\
= \; & \text{'}[\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash s\{x := \text{`}t\} : G]\!]_c \circ s^0 \quad \text{by (h)} \\
= \; & \text{'}([\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_c \\
& \qquad \circ\langle \overline{1}, \ldots, \overline{n}, [\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash \text{`}t : \Box F]\!]_c\rangle) \circ s^0 \\
& \qquad\qquad \text{by Lemma 5.3} \\
= \; & \text{'}([\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_c \\
& \qquad \circ\langle \overline{1}, \ldots, \overline{n}, [\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash \text{`}t : \Box F]\!]_c \circ \pi_2^n\rangle) \circ s^0 \quad \text{by Lemma 5.4} \\
= \; & \text{'}([\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_c \\
& \qquad \circ\langle \overline{1}, \ldots, \overline{n}, \text{'}[\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t : F]\!]_c \circ s^0 \circ \pi_2^n\rangle) \circ s^0 \quad \text{by (h)} \\
= \; & \text{'}([\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_c \\
& \qquad \circ\langle \overline{1}, \ldots, \overline{n}, \text{'}[\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t : F]\!]_c \circ \pi_2^n \circ s^0\rangle) \circ s^0 \quad \text{by (i)} \\
= \; & \text{'}[\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_c \\
& \qquad \circ\langle \overline{1}, \ldots, \overline{n}, \text{''}[\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t : F]\!]_c \circ \pi_2^n \circ s^1\rangle \circ s^0 \quad \text{by (l)} \\
= \; & \text{'}[\![x_1 : \Box F_1, \ldots, x_n : \Box F_n, x : \Box F \vdash s : G]\!]_c \\
& \qquad \circ\langle \overline{1} \circ s^0, \ldots, \overline{n} \circ s^0, \text{''}[\![y_1 : \Box G_1, \ldots, y_m : \Box G_m \vdash t : F]\!]_c \circ \pi_2^n \circ s^1 \circ s^0\rangle
\end{aligned}$$

- (η): assume that $\Gamma \vdash t : F \supset G$ is derivable, and x is not free in t, then:

$$\begin{aligned}
[\![\Gamma \vdash \lambda x_F \cdot tx : F \supset G]\!]_c &= \Lambda(App \circ \langle [\![x : F, \Gamma \vdash t : G]\!]_c, [\![x : F, \Gamma \vdash x : F]\!]_c \rangle) \\
&= \Lambda(App \circ \langle [\![x : F, \Gamma \vdash t : G]\!]_c, \overline{1} \rangle) \\
&= \Lambda(App \circ \langle [\![\Gamma \vdash t : G]\!]_c \circ \pi_2, \overline{1} \rangle) \quad \text{by Lemma 5.4} \\
&= [\![\Gamma \vdash t : G]\!]_c
\end{aligned}$$

since indeed, in any cartesian closed category $\Lambda(App \circ \langle f \circ \pi_2, \overline{1} \rangle) = f$.

- ($\eta\,\mathsf{box}$). Let $\partial^1 \hat{=} \,'\partial^0$. Then: (n) $\partial^1 \circ s^0 = \mathrm{id}_{\mathcal{F}\langle 1 \rangle}$. Indeed, for any q-simplex v in $\mathcal{F}\langle 1 \rangle$, $(\partial^1 \circ s^0)_q(v) = ('\partial^0 \circ s^0)_q(v) = \partial^1_{q+2}(s^0_{q+1}(v)) = v$ by Equation (v).

  Now let $\sigma \hat{=} \{x_1 := t_1, \ldots, x_n := t_n\}$, and assume that $\Gamma \vdash t_i : \Box F_i$ is derivable for each i, $1 \le i \le n$; then:

$$\begin{aligned}
& [\![\Gamma \vdash \mathsf{box}\,(\mathsf{unbox}\, x_i) \;\mathsf{with}\; \sigma : \Box F]\!]_c \\
= \; & '[\![x_1 : \Box F_1, \ldots, x_n : \Box F_n \vdash \mathsf{unbox}\, x_i : F]\!]_c \circ \langle s^0 \circ [\![\Gamma \vdash t_1 : \Box F_1]\!]_c, \ldots, s^0 \circ [\![\Gamma \vdash t_n : \Box F_n]\!]_c \rangle \\
= \; & '(\partial^0 \circ \overline{i}) \circ \langle s^0 \circ [\![\Gamma \vdash t_1 : \Box F_1]\!]_c, \ldots, s^0 \circ [\![\Gamma \vdash t_n : \Box F_n]\!]_c \rangle \\
= \; & '\partial^0 \circ \overline{i} \circ \langle s^0 \circ [\![\Gamma \vdash t_1 : \Box F_1]\!]_c, \ldots, s^0 \circ [\![\Gamma \vdash t_n : \Box F_n]\!]_c \rangle \quad \text{by (l)} \\
= \; & '\partial^0 \circ s^0 \circ [\![\Gamma \vdash t_i : \Box F_i]\!]_c = [\![\Gamma \vdash t_i : \Box F_i]\!]_c \quad \text{by (n)}
\end{aligned}$$

□

5.3 Geometric Interpretation

Just as the functor **Sing** maps every topological space to a simplicial set, and every continuous function to a simplicial map, there is a functor **Geom** running in the opposite direction mapping every simplicial set K to a topological space **Geom**(K) called its *geometric realization*, and mapping simplicial maps to continuous functions [May67]. It can be described as follows: let the *copower* $K_q \cdot \Delta_q$ be the topological sum of K_q copies of the standard q-simplex Δ_q, or equivalently the topological product of K_q and Δ_q, where K_q is equipped with the discrete topology. Then define **Geom**(K) as $(\amalg_{q \ge 0} K_q \cdot \Delta_q)/\sim$, where $\sim$ is the smallest equivalence relation such that:

$$\begin{aligned}
(\partial^i_q x, u) &\sim (x, \delta^i_q u) \quad (0 \le i \le q, q \ge 1, x \in K_q, u \in \Delta_{q-1}) \\
(s^i_q x, u) &\sim (x, \sigma^i_q u) \quad (0 \le i \le q, x \in K_q, u \in \Delta_{q+1})
\end{aligned}$$

where δ^i_q is the standard inclusion of face number i of Δ_q into Δ_q:

$$\delta^i_q(\tau_{q-1}, \ldots, \tau_0) \hat{=} (\tau_{q-1}, \ldots, \tau_i, 0, \tau_{i-1}, \ldots, \tau_0)$$

and where σ^i_q is the standard (flattening) ith projection from Δ_{q+1} to Δ_q:

$$\sigma^i_q(\tau_{q+1}, \ldots, \tau_0) \;\hat{=}\; (\tau_{q+1}, \ldots, \tau_{i+2}, \tau_{i+1} + \tau_i, \tau_{i-1}, \ldots, \tau_0) \quad (0 \le i \le q)$$

The idea is that **Geom**(K) is just a collection of standard extended simplices of all dimensions, glued appropriately along the faces that should be equated, as specified by the equations that hold in K.

Geom(K) is a nice topological space, in that it is a CW-complex, and every CW-complex is compactly generated Hausdorff [May67].

That **Geom** is left adjoint to **Sing** shows how close the notions of topological spaces and of simplicial sets are. In particular, there is a continuous function from **Geom**(**Sing**($\mathcal{F}$)) to $\mathcal{F}$ for every topological space $\mathcal{F}$. This continuous function is even surjective. On the other hand, there is a simplicial map from K to **Sing**(**Geom**(K)) for every simplicial set K, which is in fact an inclusion of simplicial sets (it is injective in every dimension).

Geom and **Sing** preserve all finite limits and colimits, which means that products, sums, truth, falsity, etc., correspond exactly in both the worlds of topological spaces and of simplicial sets.

Similarly, augmented simplicial sets have geometric realizations, which are just the geometric realizations of the underlying simplicial sets (see [ML71], Chapter 9, Section 6 and Chapter 7, Section 5, p.174)

In Section 5.1, Lemma 5.1, however, we did not resort to **Sing** to build the augmented simplicial sets that we considered next from our CGWH spaces $\mathcal{F}$. Rather, we defined a functor **XSing** mapping each CGWH space $\mathcal{F}$ to $\mathbf{XSing}(\mathcal{F})$ whose set of q-simplices, $q \geq -1$, was the set $\mathcal{F}\langle q\rangle^\circ$ of *extended* singular q-simplices of $\mathcal{F}$. This functor maps every continuous function f from $\mathcal{F}$ to $\mathcal{G}$ to an augmented simplicial map g such that g_q sends every extended singular q-simplex α to $f \circ \alpha$.

Just as **Sing** had **Geom** has a left adjoint, we can show:

Theorem 5.6 **XSing** *has a left adjoint* **XGeom***.*

Proof: We use a general category-theoretic argument. Recall (see [Bru]) the following results: let A be a small category, $\mathcal{C}$ be a category, then every functor $F : A \to \mathcal{C}$ determines another functor $F^* : \mathcal{C} \to A^{\mathrm{op}}\mathbf{Set}$, where $A^{\mathrm{op}}\mathbf{Set}$ is the category of contravariant functors from A to **Set**, defined on objects by: $F^*(X)$ is the functor $\mathrm{Hom}_{\mathcal{C}}(F(\_), X)$; that is, $F^*(X)$ maps objects Y of A to the set of morphisms in $\mathcal{C}$ from $F(Y)$ to X, and morphisms μ from Y to Y' in A to the function $\lambda f \in \mathrm{Hom}_{\mathcal{C}}(F(Y), X) \cdot f \circ F(\mu^{\mathrm{op}})$. F^* acts on morphisms in the natural way, which means that it maps any morphism γ from X to X' in $\mathcal{C}$ to the natural transformation $F^*\gamma$ from $F^*(X) = \mathrm{Hom}_{\mathcal{C}}(F(\_), X)$ to $F^*(X') = \mathrm{Hom}_{\mathcal{C}}(F(\_), X')$ that maps every object Y in A to the morphism $\lambda f \in \mathrm{Hom}_{\mathcal{C}}(F(Y), X) \cdot \gamma \circ f$ in **Set**. Then, provided $\mathcal{C}$ has all small inductive limits (a.k.a., small filtered colimits), F^* has a left adjoint $F_! : A^{\mathrm{op}}\mathbf{Set} \to \mathcal{C}$: $F_!$ is the unique functor that commutes with small inductive limits and such that $F_! \circ h^A = F$, where h^A is the Yoneda embedding mapping each object Y in A to $\mathrm{Hom}_A(\_, Y)$ in $A^{\mathrm{op}}\mathbf{Set}$.

In our case, take $A \hat{=} \Delta^0$, $\mathcal{C} \hat{=} \mathrm{CGWH}$, $F : \Delta^0 \to \mathrm{CGWH}$ maps $[q]$ to ${\Delta^+}_q$; we define the action of F on morphisms of Δ^0 by defining $F(\delta^i_q) \hat{=} \partial^i_q$, $F(\sigma^i_q) \hat{=} s^i_q$, $0 \leq i \leq q$ (see Lemma 5.1): this defines F uniquely, since the morphisms of Δ^0 are generated from the δ^i_q and σ^i_q's, under the duals of the simplicial equations. Then F^* is a functor from CGWH to $\Delta^{0^{\mathrm{op}}}\mathbf{Set}$ which is exactly **XSing**, as is easily checked. The Lemma then follows from the fact that CGWH has all small filtered colimits, a consequence of the fact that it has all sums and equalizers (see Lemma 4.1 (*vi*) and Corollary 22 in [Str]). □

Again, we call **XGeom** the (augmented) *geometric realization functor*. We can again give a direct definition of it—a fact which we let the reader check—, and it is similar to that of **Geom**: given an augmented simplicial set K, let the copower $K_q \cdot {\Delta^+}_q$ be the topological sum of K_q copies of ${\Delta^+}_q$, or equivalently the topological product of K_q and ${\Delta^+}_q$, where K_q is equipped with the discrete topology. Then define $\mathbf{XGeom}(K)$ as $(\amalg_{q \geq -1} K_q \cdot {\Delta^+}_q)/\sim$, where $\sim$ is the smallest equivalence relation such that:

$$\begin{array}{rcll} (\partial^i_q x, u) & \sim & (x, \delta^i_q u) & (0 \leq i \leq q, x \in K_q, u \in {\Delta^+}_{q-1}) \\ (s^i_q x, u) & \sim & (x, \sigma^i_q u) & (0 \leq i \leq q, x \in K_q, u \in {\Delta^+}_{q+1}) \end{array}$$

Write $|v, x|$ the class of (v, x) modulo $\sim$. On morphisms, **XGeom** maps every augmented simplicial map f from K to L to the function mapping every $|v, x|$ where $v \in K_q$, $x \in {\Delta^+}_q$, to $|f_q(v), x|$. That this is well-defined, continuous and defines a functor is standard: the arguments of [May67], Chapter III, go through.

Finally, **XGeom** and **XSing** preserve the operators $\partial^0 = \mathtt{ev}$, $s^0 = Q$ and **'** in both directions (up to isomorphism in the category of CGWH spaces, the isomorphism being given by the pair φ, ψ of Lemma 4.9). So they preserve the whole modal structure. This in particular entails that homotopies are preserved, since homotopies are composites of elementary homotopies or their inverses, and elementary homotopies are simplices in the function spaces [May67].

The point that we wish to make is that the combinatorial model gives a *recipe* to build models of intuitionistic S4 proofs. Take a combinatorial model, expressed in terms of augmented simplicial sets, and use any geometric-realization-like functor: this can be done by replacing the sequence of standard extended simplices ${\Delta^+}_q$ by any other sequence of spaces having operators δ^i_q and σ^i_q verifying the dual of Equations (*i*)–(*vi*) (obtained by replacing ∂ by δ, s by σ and reversing the sense of compositions). This general approach to geometric-like realizations is one of the themes of [GZ67]. For completeness, we illustrate it by the analogue of Theorem 5.6 in the case of ordered sets, that is, that of Section 3. We just extract the simplicial structure (the nerve) and find a left adjoint:

Theorem 5.7 *Let* **Ord** *be the category of all posets. Let* $N^{0^{\text{op}}}$ *be the* augmented opposite nerve *functor from* **Ord** *to* $\Delta^{0^{\text{op}}}$**Set***, defined as follows: for every poset* $(\mathcal{F}, \leq)$*,* $N^{0^{\text{op}}}(\mathcal{F})$ *is the simplicial set whose q-simplices are all sequences* $(v_q, \ldots, v_0, v_{-1})$ *of elements of* $\mathcal{F}$ *with* $v_q \leq \ldots \leq v_0 \leq v_{-1}$*, and whose faces and degeneracies are defined by:*

$$\begin{array}{rcl} \partial_q^i(v_q, \ldots, v_0, v_{-1}) & \hat{=} & (v_q, \ldots, v_{i+1}, v_{i-1}, \ldots, v_0, v_{-1}) \\ s_q^i(v_q, \ldots, v_0, v_{-1}) & \hat{=} & (v_q, \ldots, v_{i+1}, v_i, v_i, v_{i-1}, \ldots, v_0, v_{-1}) \end{array}$$

Then $N^{0^{\text{op}}}$ *has a left adjoint* $N^{0^{\text{op}}}_!$*.*

Proof: Let F be the functor from Δ^0 to **Ord** that maps $[q]$ to the the poset $q \leq q-1 \leq \ldots \leq 0 \leq -1$, and the morphism μ from $[m]$ to $[n]$ to the map of posets that sends -1 to -1 and coincides with μ on $q \leq q-1 \leq \ldots \leq 0$. It is easy to see that $F^* = N^{0^{\text{op}}}$, and that **Ord** has all small inductive limits, so that F^* has a left adjoint $F_!$. □

In some sense, $N^{0^{\text{op}}}_!$ is a geometric realization functor: it realizes augmented simplicial sets as posets.

In fact, the order-theoretic model of Section 3 and the geometric model of Section 4 were found precisely by considering such geometric realizations in an informal way, and looking at the way quoting and evaluation operated through the realization. Intuitively, geometric realizations preserve the modal structure: so for any topological space $\mathcal{F}$, $\mathcal{F}\langle 1\rangle$ should be a space of paths in $\mathcal{F}$, evaluation should be taking face number 0 (so we defined $\mathtt{ev}\alpha \hat{=} \partial_1^0 \alpha \hat{=} \alpha(1)$). Quoting some point v should yield some path $\text{'}v$ such that $\partial_1^0(\text{'}v) = v$, i.e., $\alpha(1) = v$: the canonical way is to take for $\text{'}v$ the constant path that stays at v. In fact, the geometric realization mandates that $\text{'}v$ be exactly this path, for already in the augmented simplicial case ' merely shifts dimensions, that is, in a sense, "promotes points to paths in a trivial way". Then ' should be the least continuous that we can, and the simplest way was to impose a topology on paths such that two paths α and β with $\alpha(0) \neq \beta(0)$ did not lie in the same connected component. Similar considerations led us to the model of Section 3.

A last point before we conclude: how come did we get to the idea that the category of augmented simplicial sets would be a model for intuitionistic S4 proofs? It turns out that, if we take the $\lambda\mathtt{ev}Q_H$-calculus of [GL96b]—which is isomorphic to λ_{S4H} in the typed case—, this calculus contains a set of operators that look very much like faces and degeneracies. In particular (see op.cit. for notations), if we define ∂_q^i as the map sending every $\lambda\mathtt{ev}Q$-term t of type $\Box^{q+1}F$, $0 \leq i \leq q$, for any type F, to $\mathtt{ev}^{i+1}t\ \text{id}^i$ of type $\Box^q F$, and where $\text{id}^0 \hat{=} ()$, and if we define s_q^i as the map sending every $\lambda\mathtt{ev}Q$-term t of type $\Box^{q+1}F$, $0 \leq i \leq q$, to $Q^{i+1}t$ of type $\Box^{q+2}F$, then a quick examination of the rewrite rules of $\lambda\mathtt{ev}Q_H$ shows that all Equations (i)–(vi) defining augmented simplicial sets are valid. So the $\lambda\mathtt{ev}Q_H$-terms modulo conversion form an augmented simplicial set. A more careful examination shows that it is even an augmented simplicial CCC, i.e. an augmented simplicial object in the category of (small) cartesian closed categories, and also a Godement-enriched simplicial set provided that we define s_q^{-1} as a quoting operation, noted $t \mapsto t^{\text{`}}[]$ in op.cit.

Another way would have been to observe that the λ_{S4H}-calculus itself modulo conversion has all the syntactic machinery to define a comonad (L, ϵ, δ), where L is quoting ($L(t) \hat{=} \text{'}t = \mathsf{box}\, t\, \mathsf{with}\, \{\}$), the counit ϵ is $\mathtt{ev} = \partial^0 = \mathsf{unbox}$, and comultiplication δ is $Q = s^0 = \lambda t \cdot \mathsf{box}\, x\, \mathsf{with}\, \{x := t\}$; and that comonads and augmented simplicial sets are basically the same thing (see [ML71], Chapter 7, Section 6). The fact that comonads had something to do with S4 was already known to Bierman and De Paiva [BdP92, BdP96], who suggested that models of intuitionistic S4 proofs should be cartesian closed categories with a monoidal comonad, and proved that any such category yields an interpretation of equations (β) and (unbox). (We believe that any cartesian closed category with a monoidal comonad actually interprets the whole of λ_{S4}, including the extensional equalities.) Bierman and De Paiva also considered full propositional intuitionistic logic with false, disjunction, and conjunction. Although we seldom mentioned it, all our models do interpret these other connectives as well, as empty spaces, sums and products respectively. In fact, the combinatorial model even interprets modal intuitionistic set theory, as it is an elementary topos.

It would have been possible, naturally, to rephrase this paper so that, instead of using λ_{S4} as a description of what intuitionistic S4 proofs are, we would use the language of cartesian closed categories with a monoidal comonad. We did not do this for three reasons: first, it has not been formally proved yet that they were two equivalent languages, although this is highly plausible; second, we believe that λ_{S4} carries a clearer computational interpretation; third, this would not make our correctness proofs simpler: doing so would only allow us to trade a few complex computations, like checking that rule (`box`) is interpreted correctly, by lots of slightly less complex ones.

Nevertheless, category-minded people might find it interesting to discuss our models under the angle of comonads. In the combinatorial model, the comonad $(', \partial^0, s^0)$ on the category $\Delta^{0^{\mathrm{op}}}\mathbf{Set}$ is induced by a monad (T, δ^0, σ^0) on Δ^0 as follows. The functor $T : \Delta^0 \to \Delta^0$ maps the object $[q]$ to $[q+1]$, $q \geq -1$, and the morphism $[m]\stackrel{\mu}{\longrightarrow}[n]$ to $[m+1]\stackrel{T(\mu)}{\longrightarrow}[n+1]$ defined by:

$$\begin{array}{rcll} T(\mu)(0) & \hat{=} & 0 & \\ T(\mu)(k) & \hat{=} & \mu(k-1)+1 & \quad (1 \leq k \leq m) \end{array}$$

The unit δ^0 of the monad is the natural transformation from the identity functor on Δ^0 to L that maps each object $[q]$ $(q \geq -1)$ to the arrow $[q]\stackrel{\delta^0_{q+1}}{\longrightarrow}[q+1]$, and the multiplication σ^0 is the natural transformation from L^2 to L that maps each object $[q]$ to the arrow $[q+2]\stackrel{\sigma^0_{q+1}}{\longrightarrow}[q+1]$. That the monad (T, δ^0, σ^0) on Δ^0 *induces* the comonad $(', \partial^0, s^0)$ on $\Delta^{0^{\mathrm{op}}}\mathbf{Set}$ just means that $'$ maps any augmented simplicial set (any functor $K : \Delta^{0^{\mathrm{op}}} \to \mathbf{Set}$) to $K \circ T$, that ∂^0 is the family of augmented simplicial maps from $K\langle 1\rangle$ to K defined as $K \circ (\delta^0)^{\mathrm{op}}$, and that s^0 is the family of all $K \circ (\sigma^0)^{\mathrm{op}}$, both indexed by the augmented simplicial set K. The comonad $(', \partial^0, s^0)$ on the order-theoretic model and on the geometric model follow through composition with the nerve functor (see Section 3.3) and the geometric realization functor respectively.

Finally, the comonad $(', \partial^0, s^0)$ is monoidal in each case, that is, there are functors $m_{F,G} : \Box F \times \Box G \to \Box(F \times G)$ and $m_\top : \top \to \Box\top$ satisfying certain commutation rules (see [BdP96], Appendix A). In the combinatorial model, $m_{K,L}$ is the identity functor from products of augmented simplicial sets $K\langle 1\rangle \times L\langle 1\rangle$ to $(K \times L)\langle 1\rangle$, and $m_\top$ is also the identity, since $\top\langle 1\rangle = \top$: In this case, the comonad $(', \partial^0, s^0)$ is monoidal *on the nose*. In the geometric model, $m_{\mathcal{F},\mathcal{G}}$ maps pairs of paths $\alpha : [0,1] \to \mathcal{F}$ and $\beta : [0,1] \to \mathcal{G}$ to the path of pairs $\lambda\tau \in [0,1] \cdot (\alpha(\tau), \beta(\tau))$, and $m_\top$ (where $\top$ is the one-point topological space $\{*\}$) maps $*$ to the only path $\lambda\tau \in [0,1] \cdot *$. Similarly, in the order-theoretic model, $m_{\mathcal{F},\mathcal{G}}$ maps pairs of pairs (x, y) with $x \leq y$ and (x', y') with $x' \leq y'$ to $((x, x'), (y, y'))$, and $m_\top$ (where $\top$ is the one-point ordered set $\{*\}$) maps $*$ to $(*, *)$. These are all easily seen to be monoidal.

6 Conclusion

It is time to end this journey through models of proof terms for the intuitionistic modal propositional logic S4. We started by presenting a simple model based on dcpos, since it is built with standard tools in theoretical computer science. This model also works with cpos or partial orders. We showed that this model could be enriched to allow one to define functions by fixpoints, i.e., by general recursion. In fact, the standard embedding-projection pairs technique can be used to produce models of untyped S4 proof terms, i.e. of untyped λ_{S4}-terms—this is standard, and did not need to be presented here.

We then shifted to an apparently completely different model based on ordinary topological spaces, with just enough restrictions (the CGWH hypothesis) to get cartesian-closedness. A general theme that emerged from these models was the notion of simplices, and the idea that modalities and modal operators combine forces to produce a whole structure of augmented simplicial topological space.

This led us to so-called *combinatorial models*, where simplices play the most important role: every other notion (products, function spaces, etc.) is *defined* in terms of simplices. This justifies our claim that simplices should be considered as *the* notion that explains the S4 modality semantically.

Combinatorial models arise naturally from the study of $\lambda\mathtt{ev}Q_H$ [GL96b], a calculus for intuitionistic S4 that directly exhibits an augmented simplicial structure.

Finally, combinatorial models are interesting in at least two respects. First, they give us much more than what we expected at first, since they actually interpret all proof terms for intuitionistic S4 *set theory*, not just *propositional logic*. Second, combinatorial models are interesting because, in some way, they are bases for designing other models: we have suggested that a *recipe* to get a model for S4 proof terms was to define some geometric-like realization functor and look at its range. This is what we have done informally to derive our order-theoretic and geometric models.

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A The Cartesian Closed Structure of $\Delta^{0\mathrm{op}}$Set

First, $\Delta^{0\mathrm{op}}$**Set** has products: the product $\mathcal{F} \times \mathcal{G}$ of two augmented simplicial sets $\mathcal{F}$ and $\mathcal{G}$ is defined by $(\mathcal{F} \times \mathcal{G})_q \hat{=} \mathcal{F}_q \times \mathcal{G}_q$, and faces and degeneracies operate componentwise, that is, $\partial^i_q(v,w) \hat{=} (\partial^i_q v, \partial^i_q w)$ and $s^i_q(v,w) \hat{=} (s^i_q v, s^i_q w)$. Given any two augmented simplicial maps f from $\mathcal{H}$ to $\mathcal{F}$ and g from $\mathcal{H}$ to $\mathcal{G}$, the *pairing* $\langle f, g\rangle$ is the augmented simplicial map from $\mathcal{H}$ to $\mathcal{F} \times \mathcal{G}$ mapping every q-simplex v of $\mathcal{H}$ to $(f_q(v), g_q(v))$. Conversely, there are two augmented simplicial maps π_1 and π_2, called the *first* and *second projection* respectively, defined as mapping every q-simplex (v,w) of $\mathcal{F} \times \mathcal{G}$ to v, and to w respectively.

The terminal object (truth, in logical terms) is the augmented simplicial set $\top$ such that $\top_q \hat{=} \{*\}$ for every q, and all faces and degeneracies are the identity. There is a unique augmented simplicial map $*$ from any augmented simplicial set $\mathcal{F}$ to $\top$, and it maps every simplex to the element $*$.

In general, we may define n-ary products, $n \geq 0$, by letting the 0-ary product be $\top$, and the $(n+1)$-ary product $\mathcal{F}_1 \times \ldots \times \mathcal{F}_{n+1}$ be the binary product of $\mathcal{F}_1$ with the n-ary product $\mathcal{F}_2 \times \ldots \times \mathcal{F}_{n+1}$. Given a product $\mathcal{F}_1 \times \ldots \times \mathcal{F}_n$ of augmented simplicial sets, we let $\overline{i}$, $1 \leq i \leq n$, denote the ith projection, that is, the augmented simplicial map mapping every q-simplex $(v_1, \ldots, v_n)$ of $\mathcal{F}_1 \times \ldots \times \mathcal{F}_n$ to v_i. Note that, because of our definition of n-ary products, $(v_1, \ldots, v_n)$ denotes $(v_1, (v_2, \ldots, (v_n, *)\ldots))$. In particular, $\overline{n} \hat{=} \pi_1 \underbrace{\circ \pi_2 \circ \ldots \circ \pi_2}_{n-1 \text{ times}}$.

Conversely, $\langle f_1, \ldots, f_n\rangle$ denotes tupling: this is an augmented simplicial map from $\mathcal{F}$ to $\mathcal{G}_1 \times \ldots \times \mathcal{G}_n$, for all augmented simplicial maps f_i from $\mathcal{F}$ to $\mathcal{G}_i$, defined so that $\langle f_1, \ldots, f_n\rangle$ maps every q-simplex v of $\mathcal{F}$ to $(f_{1q}(v), \ldots, f_{nq}(v))$. Because of our definition of n-ary products, $\langle f_1, f_2, \ldots, f_n\rangle = \langle f_1, \langle f_2, \ldots, \langle f_n, *\rangle \ldots\rangle\rangle$.

Let us describe the exponential objects; this is by far the most complex construction. For every $q \geq -1$, let $\Delta^0[q]$ denote the augmented simplicial set whose n-simplices, $n \geq -1$, are all non-decreasing sequences of elements of $[q]$ (compare the nerves of Section 3.3), with faces and degeneracies given as follows:

$$\begin{aligned}
(\Delta^0[q])_n &\hat{=} \{(j_0, \ldots, j_n) \mid 0 \leq j_0 \leq \ldots \leq j_n \leq q\} \\
\partial^i_n(j_0, \ldots, j_n) &\hat{=} (j_0, \ldots, j_{i-1}, j_{i+1}, \ldots, j_n) \\
s^i_n(j_0, \ldots, j_n) &\hat{=} (j_0, \ldots, j_{i-1}, j_i, j_i, j_{i+1}, \ldots, j_n)
\end{aligned}$$

Note that $\Delta^0[-1]$ is the augmented simplicial set that has one -1-simplex, the empty tuple $()$, and no simplex in any other dimension. There is only one n-simplex in $\Delta^0[0]$, the tuple $(0, \ldots, 0)$, so $\Delta^0[0]$ is isomorphic to $\top$. The n-simplices of $\Delta^0[1]$ are $(0,0,\ldots,0,0)$, $(0,0,\ldots,0,1)$, $(0,0,\ldots,1,1)$, $\ldots$, $(0,1,\ldots,1,1)$, $(1,1,\ldots,1,1)$.

Then, given two augmented simplicial sets $\mathcal{F}$ and $\mathcal{G}$, the exponential object $\mathcal{G}^{\mathcal{F}}$ is the augmented simplicial set whose q-simplices are all augmented simplicial maps from $\mathcal{F} \times \Delta^0[q]$ to $\mathcal{G}$. The -1-simplices in the exponential object are just maps from $\mathcal{F}_{-1}$ to $\mathcal{G}_{-1}$. Since $\Delta^0[0]$ is isomorphic to $\top$, 0-simplices in $\mathcal{G}^{\mathcal{F}}$ are exactly the augmented simplicial maps from $\mathcal{F}$ to $\mathcal{G}$. The 1-simplices are known as *homotopies* between augmented simplicial maps (see e.g. [May67], Proposition I.6.2), and in general the n-simplices are higher homotopies (compare Section 4.1).

The faces $\partial_q^i f$ and degeneracies $s_q^i f$ of higher homotopies f are defined by:

$$\begin{array}{rcl}(\partial_q^i f)_n(v,x) & \hat{=} & f_n(v,\overline{\delta}_q^i(x)) \\ (s_q^i f)_n(v,x) & \hat{=} & f_n(v,\overline{\sigma}_q^i(x))\end{array}$$

for every augmented simplicial map f from $\mathcal{F} \times \Delta^0[q]$ to $\mathcal{G}$, for every $n \geq -1$, for every $v \in \mathcal{F}_n$, and for every $x \in (\Delta^0[q])_n$, and where:

$$\begin{array}{rcl}\overline{\delta}_q^i(j_0,\ldots,j_n) & \hat{=} & (\delta_q^i(j_0),\ldots,\delta_q^i(j_n)) \qquad \delta_q^i(j) \hat{=} \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases} \\ \overline{\sigma}_q^i(j_0,\ldots,j_n) & \hat{=} & (\sigma_q^i(j_0),\ldots,\sigma_q^i(j_n)) \qquad \sigma_q^i(j) \hat{=} \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}\end{array}$$

The δ_q^i and σ_q^i are, by the way, the same as those we introduced earlier in this section.

Application App is an augmented simplicial map from $\mathcal{G}^{\mathcal{F}} \times \mathcal{F}$ to $\mathcal{G}$, defined by:

$$App_q(f,v) \quad \hat{=} \quad f_q(v,(0,1,\ldots,q-1,q))$$

for every $(f,v) \in (\mathcal{G}^{\mathcal{F}})_q \times \mathcal{F}_q$, $q \geq -1$. Note that f is in $(\mathcal{G}^{\mathcal{F}})_q$, so it is itself an augmented simplicial map from $\mathcal{F} \times \Delta^0[q]$ to $\mathcal{G}$, and that $(0,1,\ldots,q-1,q)$ is an element of $(\Delta^0[q])_q$.

Conversely, *abstraction*, i.e. *currification* Λ maps every augmented simplicial map f from $\mathcal{F} \times \mathcal{G}$ to $\mathcal{H}$ to an augmented simplicial map $\Lambda(f)$ from $\mathcal{G}$ to $\mathcal{H}^{\mathcal{F}}$. To define it, we need to observe that every $x \in (\Delta^0[q])_n$ can be written $(j_0,\ldots,j_n)$ with $0 \leq j_0 \leq \ldots \leq j_n \leq q$, and is therefore a non-decreasing function $i \mapsto j_i$ from $[n]$ to $[q]$, that is, a morphism from $[n]$ to $[q]$ in the category Δ^0. Since $\mathcal{G}$, as an augmented simplicial set, is a functor from $\Delta^{0^{\text{op}}}$ to **Set**, it maps x to a map $\mathcal{G}(x)$ from $\mathcal{G}_q$ to $\mathcal{G}_n$. (For those who would like a more concrete description: x, as a non-decreasing function, can be written as a composite of δ^is and σ^is, then $\mathcal{G}(x)$ is the composite of the corresponding operators ∂^i and s^i, in the reverse order.) The definition of Λ is then as follows:

$$((\Lambda(f))_q(v))_n(w,x) \quad \hat{=} \quad f_n(w,\mathcal{G}(x)(v))$$

for every v in $\mathcal{G}_q$, $q \geq -1$, $w \in \mathcal{F}_n$, $n \geq -1$, $x \in (\Delta^0[q])_n$. To help type-check this, notice that $(\Lambda(f))_q(v)$ should be an element of $(\mathcal{H}^{\mathcal{F}})_q$, that is, an augmented simplicial map g from $\mathcal{F} \times \Delta^0[q]$ to $\mathcal{H}$. So g_n should map every $(w,x) \in \mathcal{F}_n \times (\Delta^0[q])_n$ to some element of $\mathcal{H}_n$. But $\mathcal{G}(x)$ is a map from $\mathcal{G}_q$ to $\mathcal{G}_n$, so $\mathcal{G}(x)(v) \in \mathcal{G}_n$, and therefore $f_n(w,\mathcal{G}(x)(v))$ is well-defined and in $\mathcal{H}_n$.