Frequency-explicit approximability estimates for time-harmonic Maxwell's equations

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FREQUENCY-EXPLICIT APPROXIMABILITY ESTIMATES FOR TIME-HARMONIC MAXWELL'S EQUATIONS

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Abstract. We consider time-harmonic Maxwell's equations set in a heterogeneous medium with perfectly conducting boundary conditions. Given a divergence-free right-hand side lying in L^2, we provide a frequency-explicit approximability estimate measuring the difference between the corresponding solution and its best approximation by high-order Nédélec finite elements. Such an approximability estimate is crucial in both the a priori and a posteriori error analysis of finite element discretizations of Maxwell's equations, but the derivation is not trivial. Indeed, it is hard to take advantage of high-order polynomials given that the right-hand side only exhibits L^2 regularity. We proceed in line with previously obtained results for the simpler setting of the scalar Helmholtz equation and propose a regularity splitting of the solution. In turn, this splitting yields sharp approximability estimates generalizing known results for the scalar Helmholtz equation and showing the interest of high-order methods.

Key words. Maxwell's equations, Finite element methods, High-order methods, Pollution effect

1. Introduction

Over the past decades, considerable efforts have been devoted to analyze the stability and convergence of finite element discretizations of high-frequency wave propagation problems. This is in part because the required mathematical analysis is rich and elegant, but also due to the large number of physical and industrial applications for which these problems are relevant.

Except in the low-frequency regime, the bilinear (or sesquilinear) forms associated with time-harmonic wave propagation problems are not coercive. As a consequence, finite element schemes become unstable when the frequency is high and/or close to a resonant frequency, unless heavily refined meshes or high-order elements are employed [15]. For scalar wave propagation problems modeled by the Helmholtz equation, it has become clear that high-order elements are very well-suited to address these stability issues. On the one hand, the interest of high-order elements has been numerically noted in a number of works [3, 21]. On the other hand, thanks to dedicated duality techniques, the stability and convergence theory is now well-understood [6, 15, 17], and is in line with numerical observations. Vectorial problems are less covered in the literature, but the few available results point towards the fact that the analysis techniques employed for the Helmholtz equation as well as the key conclusions can be extended [4, 18]. We also mention [7, 10, 24], where similar duality techniques are used for vectorial wave propagation problems, without focusing on the high-frequency regime though.

Here, we consider time-harmonic Maxwell's equations

$$\left\{ \begin{array}{rcll} -\omega^2 \boldsymbol{\varepsilon} \boldsymbol{e} + \boldsymbol{\nabla} \times \left(\boldsymbol{\mu}^{-1} \boldsymbol{\nabla} \times \boldsymbol{e} \right) & = & \omega \boldsymbol{\varepsilon} \boldsymbol{g} & \text{in } \Omega \\ \boldsymbol{e} \times \boldsymbol{n} & = & \boldsymbol{0} & \text{on } \partial\Omega \end{array} \right. \tag{1}$$

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in a smooth domain Ω with piecewise smooth permittivity and permeability (real-valued, symmetric) tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$. In (1), $\omega > 0$ is the frequency, $\boldsymbol{e} : \Omega \to \mathbb{R}^3$ is the unknown and $\boldsymbol{g} : \Omega \to \mathbb{R}^3$ is a given right-hand side. In practice, the right-hand side takes the form $\boldsymbol{g} = i\boldsymbol{\varepsilon}^{-1}\boldsymbol{J}$ where $\boldsymbol{J} : \Omega \to \mathbb{C}^3$ is a "current density", and $\boldsymbol{e}$ represents the electric fields [2]. Here, we choose to work with $\boldsymbol{g}$ instead of $\boldsymbol{J}$ since it is more relevant mathematically as it naturally appears in convergence analysis by duality.

For a finite element space $\boldsymbol{W}_h$, we define the "approximation factor" as the sharpest constant γ such that the estimate

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{W}_h} \|\boldsymbol{e} - \boldsymbol{v}_h\|_{\mathbf{curl},\omega,\Omega} \leq \gamma \|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega} \tag{2}$$

holds for all $\boldsymbol{g} \in \boldsymbol{L}^2(\Omega)$ with $\boldsymbol{\nabla} \cdot (\boldsymbol{\varepsilon}\boldsymbol{g}) = 0$, where $\|\cdot\|_{\boldsymbol{\varepsilon},\Omega}$ is the $\boldsymbol{\varepsilon}$-weighted $\boldsymbol{L}^2(\Omega)$ norm and $\|\cdot\|_{\mathbf{curl},\omega,\Omega}$ is a suitable "energy" norm (see Section 2.2). The approximation factor γ quantifies the ability of the finite element space $\boldsymbol{W}_h$ to reproduce solutions to (1). Actually, this quantity is central in the stability analysis of the finite element schemes, since it can be shown that the finite element solution is quasi-optimal if and only if γ is "sufficiently small" [4, 6, 17]. The approximation factor also plays a central role in a posteriori error estimation [5, 19].

Since the norm in the right-hand side of (2) is weak, one cannot expect a high regularity for the solution $\boldsymbol{e}$. As a result, taking advantage of high-order polynomials is a subtle task: the solution is only piecewise H^2 in general, so that the optimal approximation rate is of order h, and not h^p. One key idea to overcome this issue is to introduce a regularity splitting as initially done in [17] for scalar wave propagation in homogeneous media and later extended to heterogeneous media [6, 16] and Maxwell's equations in homogeneous media [18].

In this work, we apply the idea of [6] to obtain a regularity splitting for Maxwell's equations in heterogeneous media. Our key result in Corollary 9 is that if $\boldsymbol{W}_h$ is the Nédélec finite element space of order $p \geq 0$ on a shape-regular mesh $\mathcal{T}_h$ with maximal element size h, there exist positive constants c and C independent of ω and h such that if $\omega h/\vartheta_\Omega \leq c$, then

$$\gamma \leq C\left(\frac{\omega h}{\vartheta_\Omega} + \frac{\omega}{\delta}\left(\frac{\omega h}{\vartheta_\Omega}\right)^{p+1}\right) \tag{3}$$

where δ is the distance between ω and the closest resonant frequency (see Section 2.3), and ϑ_Ω is the smallest wavespeed in Ω.

Since $N_\lambda := (\omega h/\vartheta_\Omega)^{-1}$ is a measure of the number of mesh elements per wavelength, one sees from (3) that γ stays small as long as $N_\lambda \geq C(1 + (\omega/\delta)^{1/(p+1)})$ with a constant C independent of ω and h. As a result, while the number of elements per wavelength must be increased to achieved stability when the frequency is high ($\omega d_\Omega/\vartheta_\Omega \gg 1$, d_Ω being the diameter of Ω) or almost resonant ($\delta \ll 1$), the requirement is less demanding for high-order elements.

We close this introduction with two comments. (i) The authors largely expect the upper bound in (3) is sharp, as the same estimate is valid and sharp in the simpler setting of the Helmholtz equation [6]. (ii) The constants c and C in (3) are allowed to depend on p, which is an important limitation. Unfortunately, the authors do not believe that p-explicit results can be obtained in the present setting. Indeed, it appears that p-explicit approximability requires substantially more involved arguments that are not available in heterogeneous media so far [17, 18].

The remaining of this work is organized as follows. Section 2 presents the notation and recalls key preliminary results. In Section 3, we present some initial results concerning the stability of the problem and basic regularity results. We elaborate a regularity splitting in Section 4 that we subsequently apply to derive our approximability result in Section 5, leading to estimate (3).

2. Settings

2.1. **Domain and coefficients.** We consider a simply connected domain $\Omega \subset \mathbb{R}^3$ with an analytic boundary $\partial\Omega$. Ω is partitioned into a set $\mathscr{P}$ of non-overlapping subdomains P with analytic boundaries ∂P such that $\overline{\Omega} = \sup_{P \in \mathscr{P}} \overline{P}$. The notation $d_\Omega := \max_{\boldsymbol{x},\boldsymbol{y} \in \overline{\Omega}} |\boldsymbol{x} - \boldsymbol{y}|$ stands for the diameter of Ω.

$\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are two real symmetric tensor-valued functions defined over Ω. These coefficients are assumed to be piecewise smooth in the sense that for each $P \in \mathscr{P}$ and for $1 \leq j, \ell \leq 3$, $\boldsymbol{\varepsilon}_{j\ell}|_P$ and $\boldsymbol{\mu}_{j\ell}|_P$ are analytic functions. The notations $\boldsymbol{\zeta} := \boldsymbol{\varepsilon}^{-1}$ and $\boldsymbol{\chi} := \boldsymbol{\mu}^{-1}$ will be useful in the sequel.

We denote by $\varepsilon_{\min}, \varepsilon_{\max} : \Omega \to \mathbb{R}$ the (analytic) functions mapping to each $\boldsymbol{x} \in \Omega$ the smallest and largest eigenvalue of $\boldsymbol{\varepsilon}(\boldsymbol{x})$, and we assume that $\boldsymbol{\varepsilon}$ is uniformly bounded and elliptic, which means that

$$0 < \inf_\Omega \varepsilon_{\min}, \qquad \sup_\Omega \varepsilon_{\max} < +\infty.$$

We employ similar notations for $\boldsymbol{\mu}$, $\boldsymbol{\chi}$ and $\boldsymbol{\zeta}$, and assume that $\boldsymbol{\mu}$ is uniformly bounded and elliptic. Finally, we denote by

$$\vartheta_\Omega := \inf_\Omega \frac{1}{\sqrt{\varepsilon_{\max}\mu_{\max}}}$$

the smallest wavespeed in Ω.

2.2. **Functional spaces.** If $D \subset \Omega$ is an open set, $L^2(D)$ is the usual Lebesgue-space of real-valued square-integrable functions over D. In addition, we write $\boldsymbol{L}^2(D) := \left(L^2(D)\right)^3$ for vector-valued functions. The natural inner products and norms of both these spaces are $(\cdot,\cdot)_D$ and $\|\cdot\|_D$, and we drop the subscript when $D = \Omega$. If $\boldsymbol{\varphi}$ is a measurable uniformly bounded and elliptic symmetric tensor-valued function, we also employ the (equivalent) norm

$$\|\boldsymbol{v}\|^2_{\boldsymbol{\varphi},D} := \int_D \boldsymbol{\varphi}\boldsymbol{v} \cdot \boldsymbol{v}$$

for $\boldsymbol{v} \in \boldsymbol{L}^2(D)$.

$\boldsymbol{H}(\mathbf{curl}, \Omega)$ is the Sobolev space of vector-valued functions $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$ such that $\boldsymbol{\nabla} \times \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$. It is equipped with the "energy" norm

$$\|\boldsymbol{v}\|^2_{\mathbf{curl},\omega,\Omega} := \omega^2 \|\boldsymbol{v}\|^2_{\boldsymbol{\varepsilon},\Omega} + \|\boldsymbol{\nabla} \times \boldsymbol{v}\|^2_{\boldsymbol{\chi},\Omega} \qquad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega).$$

$\boldsymbol{H}_0(\mathbf{curl}, \Omega)$ is the closure of smooth compactly supported functions into $\boldsymbol{H}(\mathbf{curl}, \Omega)$ and contains vector-valued functions with vanishing tangential traces.

If $\boldsymbol{\varphi}$ is a measurable, uniformly elliptic and bounded tensor-valued function, $\boldsymbol{H}(\mathrm{div}^0, \boldsymbol{\varphi}, \Omega)$ is the space of functions $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$ such that $\boldsymbol{\nabla} \cdot (\boldsymbol{\varphi}\boldsymbol{v}) = 0$ in Ω. We simply write $\boldsymbol{H}(\mathrm{div}^0, \Omega)$ when $\boldsymbol{\varphi} := \boldsymbol{I}$ is the identity tensor. Besides, the space $\boldsymbol{H}_0(\mathrm{div}^0, \boldsymbol{\varphi}, \Omega)$ is the closure of smooth compactly supported functions in $\boldsymbol{H}(\mathrm{div}^0, \boldsymbol{\varphi}, \Omega)$ and contains functions with vanishing normal traces.

If $m \geq 0$, the space $\boldsymbol{H}^m(\mathscr{P})$ contains those functions $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$ such that for each $P \in \mathscr{P}$, $1 \leq \ell \leq 3$ and all multi-indices $\boldsymbol{\alpha} \in \mathbb{N}^3$ with $|\boldsymbol{\alpha}| \leq m$, we have $\partial^\alpha(\boldsymbol{v}_\ell|_P) \in L^2(P)$. We equip this space with the norms

$$\|\boldsymbol{v}\|^2_{\boldsymbol{\varepsilon},\boldsymbol{H}^m(\mathscr{P})} := \|\boldsymbol{v}\|^2_{\boldsymbol{\varepsilon},\Omega} + \sum_{n=1}^m \sum_{|\boldsymbol{\alpha}|=n} \sum_{P\in\mathscr{P}} \sum_{\ell=1}^3 d_\Omega^{2n} \int_P \varepsilon_{\max} |\partial^{\boldsymbol{\alpha}}(\boldsymbol{v}_\ell|_P)|^2$$

and

$$\|\boldsymbol{v}\|^2_{\boldsymbol{\chi},\boldsymbol{H}^m(\mathscr{P})} := \|\boldsymbol{v}\|^2_{\boldsymbol{\chi},\Omega} + \sum_{n=1}^m \sum_{|\boldsymbol{\alpha}|=n} \sum_{P\in\mathscr{P}} \sum_{\ell=1}^3 d_\Omega^{2n} \int_P \chi_{\max} |\partial^{\boldsymbol{\alpha}}(\boldsymbol{v}_\ell|_P)|^2.$$

We refer the reader to [1] for a detailed exposition concerning Lesbegue and high-order Sobolev spaces, and to [12] and [14] for Sobolev spaces involving the curl and divergence of vector fields.

2.3. **Eigenpairs.** Recalling that Ω is simply connected, it follows from [12, Remark 7.5] that the application

$$\boldsymbol{H}_0(\mathbf{curl},\Omega) \ni \boldsymbol{v} \to \|\boldsymbol{\nabla} \times \boldsymbol{v}\|_{\boldsymbol{\chi},\Omega}$$

is a norm on $\boldsymbol{H}_0(\mathbf{curl},\Omega) \cap \boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$. Besides, the injection $\boldsymbol{H}_0(\mathbf{curl},\Omega) \cap \boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega) \subset \boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$ is compact [22]. As a result (see, e.g. [2, Theorem 4.5.11]), there exists an orthonormal basis $\{\boldsymbol{\phi}_j\}_{j\geq 0}$ of $\boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$ (equipped with the inner-product $(\boldsymbol{\varepsilon}\,\cdot,\cdot)$) and a sequence of strictly positive eigenvalues $\{\lambda_j\}_{j\geq 0}$ such that for all $j \geq 0$,

$$(\boldsymbol{\chi}\boldsymbol{\nabla} \times \boldsymbol{\phi}_j, \boldsymbol{\nabla} \times \boldsymbol{v}) = \lambda_j(\boldsymbol{\varepsilon}\boldsymbol{\phi}_j\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl},\Omega).$$

In addition, if $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl},\Omega) \cap \boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$, we have

$$\|\boldsymbol{v}\|^2_{\boldsymbol{\varepsilon},\Omega} = \sum_{j\geq 0} |v_j|^2 \quad \text{ and } \quad \|\boldsymbol{\nabla} \times \boldsymbol{v}\|^2_{\boldsymbol{\chi},\Omega} = \sum_{j\geq 0} \lambda_j |v_j|^2,$$

where $v_j := (\boldsymbol{\varepsilon}\boldsymbol{v}, \boldsymbol{\phi}_j)$.

In the remaining of this work, we set $\delta := \min_{j\geq 0} |\sqrt{\lambda_j} - \omega|$ and assume $\delta > 0$.

Remark 1 (What can be said about δ?)**.** *In practice, it is complicated to obtain a bound for δ analytically because it requires information about the localization of the spectrum. In the high-frequency regime, δ will, in general, tend to be smaller due to Weyl's law* [9, Theorem 6.8]*. Alternatively, in the low-frequency regime where $0 < \omega < \sqrt{\lambda_0}$, a lower bound for δ may be computed from a lower bound on λ_0, see* [13].

2.4. **Regularity shifts.** Our analysis heavily relies on regularity shift results where, given a divergence-free vector field with a smooth curl, one deduces smoothness results for the field itself. First [23, Theorem 2.2], for all $p \geq 0$, there exists a constant $\mathscr{C}_{\mathrm{shift},p}$ only depending on p, $\mathscr{P}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ such that, for $0 \leq m \leq p$, if $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl},\Omega) \cap \boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$ with $\boldsymbol{\nabla} \times \boldsymbol{v} \in \boldsymbol{H}^m(\mathscr{P})$, we have

$$\|\boldsymbol{v}\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{m+1}(\mathscr{P})} \leq \mathscr{C}_{\mathrm{shift},p} \frac{d_\Omega}{\vartheta_\Omega} \|\boldsymbol{\nabla} \times \boldsymbol{v}\|_{\boldsymbol{\chi},\boldsymbol{H}^m(\mathscr{P})}. \tag{4}$$

Similarly, if $\boldsymbol{w} \in \boldsymbol{H}_0(\mathrm{div}^0,\Omega)$ with $\boldsymbol{\nabla} \times (\boldsymbol{\chi}\boldsymbol{w}) \in \boldsymbol{H}^m(\mathscr{P})$, then we have

$$\|\boldsymbol{w}\|_{\boldsymbol{\chi},\boldsymbol{H}^{m+1}(\mathscr{P})} \leq \mathscr{C}_{\mathrm{shift},p} \frac{d_\Omega}{\vartheta_\Omega} \|\boldsymbol{\zeta}\boldsymbol{\nabla} \times (\boldsymbol{\chi}\boldsymbol{w})\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^m(\mathscr{P})}, \tag{5}$$

as can be seen by applying [23, Theorem 2.2] to $\boldsymbol{\chi}\boldsymbol{w} \in \boldsymbol{H}(\mathbf{curl}, \Omega) \cap \boldsymbol{H}_0(\mathrm{div}^0, \boldsymbol{\mu}, \Omega)$. We also record the following result obtained by combining (4) and (5): if $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega) \cap \boldsymbol{H}(\mathrm{div}^0, \boldsymbol{\varepsilon}, \Omega)$ satisfies $\boldsymbol{\nabla} \times (\boldsymbol{\chi}\boldsymbol{\nabla} \times \boldsymbol{u}) \in \boldsymbol{H}^m(\mathscr{P})$, we have

$$\|\boldsymbol{u}\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{m+2}(\mathscr{P})} \leq \left(\mathscr{C}_{\mathrm{shift},p}\frac{d_\Omega}{\vartheta_\Omega}\right)^2 \|\boldsymbol{\zeta}\boldsymbol{\nabla} \times (\boldsymbol{\chi}\boldsymbol{\nabla} \times \boldsymbol{u})\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^m(\mathscr{P})} \tag{6}$$

for $0 \leq m \leq p-1$.

Remark 2 (Smoothness assumption)**.** *For the sake of simplicity, we assume that the coefficients are piecewise analytic, which allow for regularity shifts for any $m \in \mathbb{N}$. In turn, this allows us to establish our key results for any polynomial degree $p \in \mathbb{N}$. On the other hand, for a fixed polynomial degree $p_\star \in \mathbb{N}$, these smoothness assumptions can be weakened by simply requiring piecewise finite regularity of the coefficients.*

2.5. **Curved tetrahedral mesh.** We consider a partition of Ω into a conforming mesh $\mathcal{T}_h$ of (curved) tetrahedral elements K as in [18, Assumption 3.1]. For $K \in \mathcal{T}_h$ we denote by $\mathscr{F}_K : \widehat{K} \to K$ the (analytic) mapping between the reference tetrahedra $\widehat{K}$ and the element K. We further assume that the mesh $\mathcal{T}_h$ is conforming with the partition $\mathscr{P}$, which means that for each $K \in \mathcal{T}_h$, there exists a unique $P \in \mathscr{P}$ such that $K \subset \overline{P}$. This last assumption means that the coefficients are smooth inside each mesh cell.

2.6. **Nédélec finite element space.** In the remaining of this work, we fix a polynomial degree $p \geq 0$. Then, following [11, Chapter 15], we introduce the Nédélec polynomial space

$$\boldsymbol{N}_p(\widehat{K}) = \boldsymbol{P}_p(\widehat{K}) + \boldsymbol{x} \times \boldsymbol{P}_p(\widehat{K}),$$

where $\boldsymbol{P}_p(\widehat{K}) := \left(P_p(\widehat{K})\right)^3$ and $P_p(\widehat{K})$ stands for the space of polynomials of degree less than or equal to p defined over $\widehat{K}$. Classically, the associate approximation space is obtained by mapping the Nédélec polynomial space to the mesh cells through a Piola mapping, leading to

$$\boldsymbol{W}_h := \left\{\boldsymbol{v}_h \in \boldsymbol{H}_0(\mathbf{curl}, \Omega) \quad \middle| \quad \left(\boldsymbol{D}\mathscr{F}_K^{-1}\right)\left(\boldsymbol{v}_h|_K \circ \mathscr{F}_K^{-1}\right) \in \boldsymbol{N}_p(\widehat{K}) \quad \forall K \in \mathcal{T}_h\right\},$$

where $\boldsymbol{D}\mathscr{F}_K^{-1}$ is the Jacobian matrix of $\mathscr{F}_K^{-1}$.

2.7. **High-order interpolation.** There exists an interpolation operator $\mathcal{J}_h : \boldsymbol{H}^1(\mathscr{P}) \cap \boldsymbol{H}_0(\mathbf{curl}, \Omega) \to \boldsymbol{W}_h$ and a constant $\mathscr{C}_{\mathrm{i},p}$ solely depending on p, the regularity of the mesh and the coefficients $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ such that

$$\|\boldsymbol{v} - \mathcal{J}_h\boldsymbol{v}\|_{\boldsymbol{\varepsilon},\Omega} \leq \mathscr{C}_{\mathrm{i},p}\left(\frac{h}{d_\Omega}\right)^{p+1} \|\boldsymbol{v}\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{p+1}(\mathscr{P})}, \tag{7a}$$

whenever $\boldsymbol{v} \in \boldsymbol{H}^1(\mathscr{P})$ satisfies $\boldsymbol{v} \in \boldsymbol{H}^{p+1}(\mathscr{P})$ and

$$\|\boldsymbol{\nabla} \times (\boldsymbol{w} - \mathcal{J}_h\boldsymbol{w})\|_{\boldsymbol{\chi},\Omega} \leq \mathscr{C}_{\mathrm{i},p}\left(\frac{h}{d_\Omega}\right)^{p+1} \|\boldsymbol{\nabla} \times \boldsymbol{w}\|_{\boldsymbol{\chi},\boldsymbol{H}^{p+1}(\mathscr{P})} \tag{7b}$$

for all $\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega) \cap \boldsymbol{H}^1(\mathscr{P})$ with $\boldsymbol{\nabla} \times \boldsymbol{w} \in \boldsymbol{H}^{p+1}(\mathscr{P})$. The construction of such an interpolation operator is classical, and we refer the reader to [11, Chapters 13 and 17] and [18, Chapter 8] for more details.

Remark 3 (*p*-explicit interpolation estimates)**.** *It is possible to obtain "p-explicit" versions of the estimates in (7), with a constant $\mathscr{C}_{\mathrm{i},p}$ independent of the polynomial degree. In our case, such estimates are not useful because the dependency of $\mathscr{C}_{\mathrm{shift},p}$ on p is unknown (or at least, not practically useful).*

2.8. **Sharp approximability estimates.** We are now ready to rigorously introduce the approximation factor γ. Given $\boldsymbol{g} \in \boldsymbol{L}^2(\Omega)$, we denote by $\boldsymbol{e}^\star(\boldsymbol{g})$ the unique element of $\boldsymbol{H}_0(\mathbf{curl}, \Omega)$ such that

$$-\omega^2(\boldsymbol{\varepsilon}\boldsymbol{v}, \boldsymbol{e}^\star(\boldsymbol{g})) + (\boldsymbol{\chi}\boldsymbol{\nabla} \times \boldsymbol{v}, \boldsymbol{\nabla} \times \boldsymbol{e}^\star(\boldsymbol{g})) = \omega(\boldsymbol{\varepsilon}\boldsymbol{v}, \boldsymbol{g}) \tag{8}$$

for all $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega)$. Note that the existence and uniqueness of $\boldsymbol{e}^\star(\boldsymbol{g})$ follows from the assumption that $\delta > 0$, i.e. ω is not a resonant frequency. Then, we introduce the approximation factor as

$$\gamma := \sup_{\substack{\boldsymbol{g} \in \boldsymbol{H}(\mathrm{div}^0, \boldsymbol{\varepsilon}, \Omega) \\ \|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega} = 1}} \inf_{\boldsymbol{v}_h \in \boldsymbol{W}_h} \|\boldsymbol{e}^\star(\boldsymbol{g}) - \boldsymbol{v}_h\|_{\mathbf{curl},\omega,\Omega}. \tag{9}$$

The constant γ plays a crucial role in showing the stability of finite element discretizations, as detailed in [6, §2.2] for the Helmholtz equation. It appears in the context of a duality technique often called the "Schatz argument", whereby $\boldsymbol{g}$ is taken to be the finite element error, and the function $\boldsymbol{e}^\star(\boldsymbol{g})$ is used to compensate for the negative L^2-term of the bilinear form [20]. A variation of the Schatz argument is also employed in a posteriori error analysis [5, 8].

Observing that we can choose $\boldsymbol{v}_h = \boldsymbol{o}$ in the infimum, a crude estimate for the approximation factor is given by $\gamma \leq c_{\mathrm{s}}$ where

$$c_{\mathrm{s}} := \sup_{\substack{\boldsymbol{g} \in \boldsymbol{H}(\mathrm{div}^0, \boldsymbol{\varepsilon}, \Omega) \\ \|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega} = 1}} \|\boldsymbol{e}^\star(\boldsymbol{g})\|_{\mathbf{curl},\omega,\Omega}. \tag{10}$$

This upper bound is of little use in a priori error estimation where one needs γ to become small as $h \to 0$ in a duality argument [4, 6, 7, 10, 24]. On the other hand, it is of interest in a posteriori error estimation, in particular, to obtain guaranteed estimates [5]. Indeed, the constant c_{s} is often easier to compute than sharper estimates since it only depends on the frequency, the domain and the coefficients, and not on the mesh or the discretization order.

3. Stability

Here, we present a stability result, that follows from standard spectral theory.

Theorem 4 (Stability)**.** *The estimates*

$$\omega\|\boldsymbol{e}^\star(\boldsymbol{g})\|_{\boldsymbol{\varepsilon},\Omega} \leq \frac{\omega}{\delta}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}, \qquad \|\boldsymbol{\nabla} \times \boldsymbol{e}^\star(\boldsymbol{g})\|_{\boldsymbol{\chi},\Omega} \leq \frac{\omega}{\delta}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega} \tag{11}$$

hold true for all $\boldsymbol{g} \in \boldsymbol{H}(\mathrm{div}^0, \boldsymbol{\varepsilon}, \Omega)$. In addition, we have

$$c_{\mathrm{s}} \leq \frac{\omega}{\delta}. \tag{12}$$

Proof. Let $\boldsymbol{g} \in \boldsymbol{H}(\mathrm{div}^0, \boldsymbol{\varepsilon}, \Omega)$ and set $\boldsymbol{e} := \boldsymbol{e}^\star(\boldsymbol{g})$. Since $\boldsymbol{e}, \boldsymbol{g} \in \boldsymbol{H}(\mathrm{div}^0, \boldsymbol{\varepsilon}, \Omega)$, we may expand $\boldsymbol{e}$ and $\boldsymbol{g}$ in the basis $\{\boldsymbol{\phi}_j\}_{j \geq 0}$ by letting $e_j := (\boldsymbol{e}, \boldsymbol{\phi}_j)$ and $g_j := (\boldsymbol{e}, \boldsymbol{\phi}_j)$. Then, picking $\boldsymbol{v} = \boldsymbol{\phi}_j$ in (8), we see that

$$|e_j| = \frac{\omega}{|\lambda_j - \omega^2|}|g_j| \leq \frac{1}{\delta}\frac{\omega}{\sqrt{\lambda_j} + \omega}|g_j| \quad \text{and} \quad \left(\omega + \sqrt{\lambda_j}\right)|e_j| \leq \frac{\omega}{\delta}|g_j|.$$

Then, (11) follows from

$$\|\boldsymbol{e}\|^2_{\mathbf{curl},\omega,\Omega} = \sum_{j\geq 0}(\omega^2+\lambda_j)|e_j|^2 \leq \sum_{j\geq 0}\left((\omega+\sqrt{\lambda_j})|e_j|\right)^2 \leq \left(\frac{\omega}{\delta}\right)^2 \|\boldsymbol{g}\|^2_{\boldsymbol{\varepsilon},\Omega},$$

and (12) follows from (11) recalling the definition of $c_{\rm s}$ in (10). □ □

4. Regularity splitting

In this section, we provide a regularity splitting result that is solely expressed in terms of $c_{\rm s}$ and $\mathscr{C}_{\mathrm{shift},p}$[1]. We start with a basic regularity result, obtained by combining Theorem 4 with the regularity shift results from Section 2.4.

Lemma 5 (Basic regularity)**.** *For all $\boldsymbol{g} \in \boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$, we have*

$$\omega\|\boldsymbol{e}^\star(\boldsymbol{g})\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^1(\mathscr{P})} \leq c_{\rm s}\mathscr{C}_{\mathrm{shift},p}\frac{\omega d_\Omega}{\vartheta_\Omega}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}, \tag{13}$$

and

$$\|\boldsymbol{\nabla}\times\boldsymbol{e}^\star(\boldsymbol{g})\|_{\boldsymbol{\chi},\boldsymbol{H}^1(\mathscr{P})} \leq (1+c_{\rm s})\mathscr{C}_{\mathrm{shift},p}\frac{\omega d_\Omega}{\vartheta_\Omega}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}. \tag{14}$$

Proof. Pick $\boldsymbol{g} \in \boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$ and set $\boldsymbol{e} := \boldsymbol{e}^\star(\boldsymbol{g})$. We first observe that as $\boldsymbol{e} \in \boldsymbol{H}_0(\mathbf{curl},\Omega)\cap\boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$, shift estimate (4) implies that

$$\omega\|\boldsymbol{e}\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^1(\mathscr{P})} \leq \mathscr{C}_{\mathrm{shift},p}\frac{\omega d_\Omega}{\vartheta_\Omega}\|\boldsymbol{\nabla}\times\boldsymbol{e}\|_{\boldsymbol{\chi},\Omega}$$

and the definition of the stability constant in (10) shows that

$$\|\boldsymbol{e}\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^1(\mathscr{P})} \leq \mathscr{C}_{\mathrm{shift},p}\frac{d_\Omega}{\vartheta_\Omega}\|\boldsymbol{\nabla}\times\boldsymbol{e}\|_{\boldsymbol{\chi},\Omega} \leq \mathscr{C}_{\mathrm{shift},p}c_{\rm s}\frac{d_\Omega}{\vartheta_\Omega}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega},$$

so that (13) follows. On the other hand, we establish (14) with (5), since

$$\|\boldsymbol{\nabla}\times\boldsymbol{e}\|_{\boldsymbol{\chi},\boldsymbol{H}^1(\mathscr{P})} \leq \mathscr{C}_{\mathrm{shift},p}\frac{d_\Omega}{\vartheta_\Omega}\|\boldsymbol{\zeta}\boldsymbol{\nabla}\times(\boldsymbol{\chi}\boldsymbol{\nabla}\times\boldsymbol{e})\|_{\boldsymbol{\varepsilon},\Omega} \leq \mathscr{C}_{\mathrm{shift},p}\frac{d_\Omega}{\vartheta_\Omega}\left(\omega\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}+\omega^2\|\boldsymbol{e}\|_{\boldsymbol{\varepsilon},\Omega}\right),$$

using (10) to estimate the last term. □ □

The regularity results presented in Lemma 5 suffice to obtain sharp estimates for the approximation factor when $p=0$. For high-order elements, however, this is not sufficient. As we only have a limited regularity assumption for the right-hand side $\boldsymbol{g}$ in definition (9) of γ, we may not expect more regularity than established in Lemma 5 for the associated solution $\boldsymbol{e}^\star(\boldsymbol{g})$. As shown in [6, 16, 17] for the Helmholtz equation, the key idea is to introduce a "regularity splitting" of the solution. Here, we shall adapt the approach of [6] to Maxwell's equations and consider the formal expansion

$$\boldsymbol{e}^\star(\boldsymbol{g}) = \sum_{j\geq 0}\left(\frac{\omega d_\Omega}{\vartheta_\Omega}\right)^j \boldsymbol{e}^\star_j(\boldsymbol{g}). \tag{15}$$

[1]The authors believe it is of interest to explicitly mention c_s proofs, since at least in principle, the regularity splitting results may apply in cases where c_s is not obtain via Theorem 4.

After identifying the powers of $(\omega d_\Omega/\vartheta_\Omega)$ in (1), one sees that $\boldsymbol{e}_0^\star(\boldsymbol{g}) := \boldsymbol{o}$, and that the other elements $\boldsymbol{e}_j^\star(\boldsymbol{g}) \in \boldsymbol{H}_0(\mathbf{curl},\Omega)\cap\boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$ are iteratively defined through

$$\boldsymbol{\nabla}\times(\boldsymbol{\chi}\boldsymbol{\nabla}\times\boldsymbol{e}_1^\star(\boldsymbol{g})) = \frac{\vartheta_\Omega}{d_\Omega}\boldsymbol{\varepsilon}\boldsymbol{g}, \tag{16a}$$

and

$$\boldsymbol{\nabla}\times\left(\boldsymbol{\chi}\boldsymbol{\nabla}\times\boldsymbol{e}_j^\star(\boldsymbol{g})\right) = \left(\frac{\vartheta_\Omega}{d_\Omega}\right)^2\boldsymbol{\varepsilon}\boldsymbol{e}_{j-2}^\star(\boldsymbol{g}) \tag{16b}$$

for $j\geq 2$. Note that the boundary value problems in (16) are well-posed, since $\|\boldsymbol{\nabla}\times\cdot\|_{\boldsymbol{\chi},\Omega}$ is a norm on $\boldsymbol{H}_0(\mathbf{curl},\Omega)\cap\boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$. We first show that the iterates in the sequence exhibit increasing regularity.

Lemma 6 (Increasing regularity of the expansion)**.** *Let $\boldsymbol{g}\in\boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$. For all $0\leq j\leq p$, we have $\boldsymbol{e}_j^\star(\boldsymbol{g})\in\boldsymbol{H}^{j+1}(\mathscr{P})$ and $\boldsymbol{\nabla}\times\boldsymbol{e}_j^\star(\boldsymbol{g})\in\boldsymbol{H}^j(\mathscr{P})$ with*

$$\omega\|\boldsymbol{e}_j^\star(\boldsymbol{g})\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{j+1}(\mathscr{P})} \leq \mathscr{C}_{\mathrm{shift},p}^{j+1}\frac{\omega d_\Omega}{\vartheta_\Omega}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}, \tag{17}$$

and

$$\|\boldsymbol{\nabla}\times\boldsymbol{e}_j^\star(\boldsymbol{g})\|_{\boldsymbol{\chi},\boldsymbol{H}^j(\mathscr{P})} \leq \mathscr{C}_{\mathrm{shift},p}^{j}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}. \tag{18}$$

Proof. Let $\boldsymbol{g}\in\boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$. To ease the presentation, we set $\boldsymbol{e}_j := \boldsymbol{e}_j^\star(\boldsymbol{g})$ for $j\geq 0$. We start with (18). It obviously holds for $j=0$ as $\boldsymbol{e}_0 := \boldsymbol{o}$. For $j=1$, recalling (16), we have

$$\|\boldsymbol{\nabla}\times\boldsymbol{e}_1\|_{\boldsymbol{\chi},\boldsymbol{H}^1(\mathscr{P})} \leq \mathscr{C}_{\mathrm{shift},p}\frac{d_\Omega}{\vartheta_\Omega}\|\boldsymbol{\nabla}\times(\boldsymbol{\chi}\boldsymbol{\nabla}\times\boldsymbol{e}_1)\|_{\boldsymbol{\zeta},\Omega} = \mathscr{C}_{\mathrm{shift},p}\|\boldsymbol{\varepsilon}\boldsymbol{g}\|_{\boldsymbol{\zeta},\Omega} = \mathscr{C}_{\mathrm{shift},p}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}.$$

Then, assuming that (18) holds up to some j, (5) and (16) reveal that

$$\begin{aligned}\|\boldsymbol{\nabla}\times\boldsymbol{e}_{j+2}\|_{\boldsymbol{\chi},\boldsymbol{H}^{j+2}(\mathscr{P})} &\leq \mathscr{C}_{\mathrm{shift},p}\frac{d_\Omega}{\vartheta_\Omega}\|\boldsymbol{\zeta}\boldsymbol{\nabla}\times(\boldsymbol{\chi}\boldsymbol{\nabla}\times\boldsymbol{e}_{j+2})\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{j+1}(\mathscr{P})} = \mathscr{C}_{\mathrm{shift},p}\frac{\vartheta_\Omega}{d_\Omega}\|\boldsymbol{e}_j\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{j+1}(\mathscr{P})}\\ &\leq \mathscr{C}_{\mathrm{shift},p}^2\|\boldsymbol{\nabla}\times\boldsymbol{e}_j\|_{\boldsymbol{\chi},\boldsymbol{H}^{j+1}(\mathscr{P})} \leq \mathscr{C}_{\mathrm{shift},p}^{j+2}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega},\end{aligned}$$

and (18) follows by induction.

On the other hand, (17) is a direct consequence of (18), since (4) shows that

$$\|\boldsymbol{e}_j\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{j+1}(\mathscr{P})} \leq \mathscr{C}_{\mathrm{shift},p}\frac{d_\Omega}{\vartheta_\Omega}\|\boldsymbol{\nabla}\times\boldsymbol{e}_j\|_{\boldsymbol{\chi},\boldsymbol{H}^j(\mathscr{P})} \leq \frac{d_\Omega}{\vartheta_\Omega}\mathscr{C}_{\mathrm{shift},p}^{j+1}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}. \qquad \square$$

$\square$

So far, expansion (15) is only formal, and we need to truncate the expansion into a finite sum. To do so, we introduce, for $\ell\geq 0$, the "residual" term

$$\boldsymbol{r}_\ell^\star(\boldsymbol{g}) := \boldsymbol{e}^\star(\boldsymbol{g}) - \sum_{j=0}^{\ell}\left(\frac{\omega d_\Omega}{\vartheta_\Omega}\right)^j\boldsymbol{e}_j^\star(\boldsymbol{g}) \in \boldsymbol{H}_0(\mathbf{curl},\Omega)\cap\boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega),$$

so that

$$\boldsymbol{e}^\star(\boldsymbol{g}) = \sum_{j=0}^{\ell}\left(\frac{\omega d_\Omega}{\vartheta_\Omega}\right)^j\boldsymbol{e}_j^\star(\boldsymbol{g}) + \boldsymbol{r}_\ell^\star(\boldsymbol{g}). \tag{19}$$

As we show next, these residuals have increasing regularity.

Lemma 7 (Regularity of residual terms)**.** *For all $\boldsymbol{g} \in \boldsymbol{H}(\mathrm{div}^0, \boldsymbol{\varepsilon}, \Omega)$ and $0 \leq \ell \leq p$, we have $\boldsymbol{r}^\star_\ell(\boldsymbol{g}) \in \boldsymbol{H}^{\ell+1}(\mathscr{P})$ and $\boldsymbol{\nabla} \times \boldsymbol{r}^\star_\ell(\boldsymbol{g}) \in \boldsymbol{H}^{\ell+1}(\mathscr{P})$ with the estimates*

$$\omega \|\boldsymbol{r}^\star_\ell(\boldsymbol{g})\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{\ell+1}(\mathscr{P})} \leq c_{\mathrm{s}} \left(\mathscr{C}_{\mathrm{shift},p} \frac{\omega d_\Omega}{\vartheta_\Omega}\right)^{\ell+1} \|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}, \tag{20}$$

and

$$\|\boldsymbol{\nabla} \times \boldsymbol{r}^\star_\ell(\boldsymbol{g})\|_{\boldsymbol{\chi},\boldsymbol{H}^{\ell+1}(\mathscr{P})} \leq (1+c_{\mathrm{s}}) \left(\mathscr{C}_{\mathrm{shift},p} \frac{\omega d_\Omega}{\vartheta_\Omega}\right)^{\ell+1} \|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}. \tag{21}$$

Proof. For the sake of simplicity, we fix $\boldsymbol{g} \in \boldsymbol{H}(\mathrm{div}^0, \boldsymbol{\varepsilon}, \Omega)$, and set $\boldsymbol{e} := \boldsymbol{e}^\star(\boldsymbol{g})$ and $\boldsymbol{r}_\ell := \boldsymbol{r}^\star_\ell(\boldsymbol{g})$ for $\ell \geq 0$. We have $\boldsymbol{r}_0 := \boldsymbol{e}$, so that (20) and (21) hold for $\ell = 0$ as a direct consequence of (13) and (14).

For the case $\ell = 1$, simple computations show that $\boldsymbol{\nabla} \times (\boldsymbol{\chi}\boldsymbol{\nabla} \times \boldsymbol{r}_1) = \omega^2 \boldsymbol{\varepsilon}\boldsymbol{e}$. Using (5) and (6), it then follows that

$$\omega\|\boldsymbol{r}_1\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^2(\mathscr{P})} \leq \mathscr{C}^2_{\mathrm{shift},p} \frac{\omega d_\Omega^2}{\vartheta_\Omega^2} \|\boldsymbol{\zeta}\boldsymbol{\nabla} \times (\boldsymbol{\chi}\boldsymbol{\nabla} \times \boldsymbol{r}_1)\|_{\boldsymbol{\varepsilon},\Omega} = \mathscr{C}^2_{\mathrm{shift},p} \left(\frac{\omega d_\Omega}{\vartheta_\Omega}\right)^2 \omega \|\boldsymbol{e}\|_{\boldsymbol{\varepsilon},\Omega}$$

and

$$\|\boldsymbol{\nabla} \times \boldsymbol{r}_1\|_{\boldsymbol{\chi},\boldsymbol{H}^2(\mathscr{P})} \leq \mathscr{C}_{\mathrm{shift},p} \frac{d_\Omega}{\vartheta_\Omega} \|\boldsymbol{\zeta}\boldsymbol{\nabla} \times (\boldsymbol{\chi}\boldsymbol{\nabla} \times \boldsymbol{r}_1)\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^1(\mathscr{P})} = \mathscr{C}_{\mathrm{shift},p} \frac{\omega d_\Omega}{\vartheta_\Omega} \omega \|\boldsymbol{e}\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^1(\mathscr{P})}$$

so that (20) and (21) are also valid when $\ell = 1$ recalling (10) and (13).

For the general case, we first observe that $\boldsymbol{\nabla} \times (\boldsymbol{\chi}\boldsymbol{\nabla} \times \boldsymbol{r}_{\ell+2}) = \omega^2 \boldsymbol{\varepsilon}\boldsymbol{r}_\ell$. Therefore, using (5) and (6), we have

$$\omega\|\boldsymbol{r}_{\ell+2}\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{\ell+3}(\mathscr{P})} \leq \left(\mathscr{C}_{\mathrm{shift},p} \frac{\omega d_\Omega}{\vartheta_\Omega}\right)^2 \omega \|\boldsymbol{r}_\ell\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{\ell+1}(\mathscr{P})},$$

and

$$\begin{aligned}\|\boldsymbol{\nabla} \times \boldsymbol{r}_{\ell+2}\|_{\boldsymbol{\chi},\boldsymbol{H}^{\ell+3}(\mathscr{P})} &\leq \mathscr{C}_{\mathrm{shift},p} \frac{d_\Omega}{\vartheta_\Omega} \|\boldsymbol{\zeta}\boldsymbol{\nabla} \times (\boldsymbol{\chi}\boldsymbol{\nabla} \times \boldsymbol{r}_{\ell+2})\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{\ell+2}(\mathscr{P})} \\ &\leq \mathscr{C}_{\mathrm{shift},p} \frac{d_\Omega}{\vartheta_\Omega} \omega^2 \|\boldsymbol{r}_\ell\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{\ell+2}(\mathscr{P})} \leq \left(\mathscr{C}_{\mathrm{shift},p} \frac{\omega d_\Omega}{\vartheta_\Omega}\right)^2 \|\boldsymbol{\nabla} \times \boldsymbol{r}_\ell\|_{\boldsymbol{\chi},\boldsymbol{H}^{\ell+1}(\mathscr{P})},\end{aligned}$$

and the general case follows by induction. □ □

5. Sharp approximability estimates

Equipped with the regularity splitting from Section 4, we are now ready to establish our main result, providing an upper bound for the approximation factor γ.

Theorem 8 (Approximability estimate)**.** *Assume that $\mathscr{C}_{\mathrm{shift},p}(\omega h/\vartheta_\Omega) \leq 1/2$. Then, the following estimate holds true*

$$\gamma \leq \mathscr{C}_{\mathrm{i},p} \left(2\sqrt{2}\mathscr{C}_{\mathrm{shift},p} \frac{\omega h}{\vartheta_\Omega} + \sqrt{1+2c_{\mathrm{s}}^2} \left(\mathscr{C}_{\mathrm{shift},p} \frac{\omega h}{\vartheta_\Omega}\right)^{p+1}\right).$$

Proof. We consider a right-hand side $\boldsymbol{g} \in \boldsymbol{H}(\mathrm{div}^0, \boldsymbol{\varepsilon}, \Omega)$ and employ the notation $\boldsymbol{e} := \boldsymbol{e}^\star(\boldsymbol{g})$, $\boldsymbol{e}_j := \boldsymbol{e}^\star_j(\boldsymbol{g})$ for $j \geq 0$ and $\boldsymbol{r}_p := \boldsymbol{r}^\star_p(\boldsymbol{g})$. Recalling (9) and the finite expansion (19) for $\boldsymbol{e}$, it is sufficient to provide upper bounds for the high-order interpolation error of $\boldsymbol{e}_j$ and $\boldsymbol{r}_\ell$. For $\boldsymbol{e}_j$, (7) and Lemma 6 imply that

$$\omega\left(\frac{\omega d_\Omega}{\vartheta_\Omega}\right)^j \|\boldsymbol{e}_j - \mathcal{J}_h\boldsymbol{e}_j\|_{\boldsymbol{\varepsilon},\Omega} \leq \mathscr{C}_{\mathrm{i},p}\omega\left(\frac{\omega d_\Omega}{\vartheta_\Omega}\right)^j\left(\frac{h}{d_\Omega}\right)^{j+1}\|\boldsymbol{e}_j\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{j+1}(\mathscr{P})}$$
$$= \mathscr{C}_{\mathrm{i},p}\frac{\vartheta_\Omega}{d_\Omega}\left(\frac{\omega h}{\vartheta_\Omega}\right)^{j+1}\|\boldsymbol{e}_j\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{j+1}(\mathscr{P})} \leq \mathscr{C}_{\mathrm{i},p}\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^{j+1}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega},$$

and

$$\left(\frac{\omega d_\Omega}{\vartheta_\Omega}\right)^j\|\boldsymbol{\nabla}\times(\boldsymbol{e}_j-\mathcal{J}_h\boldsymbol{e}_j)\|_{\boldsymbol{\chi},\Omega} \leq \mathscr{C}_{\mathrm{i},p}\left(\frac{\omega d_\Omega}{\vartheta_\Omega}\right)^j\left(\frac{h}{d_\Omega}\right)^j\|\boldsymbol{\nabla}\times\boldsymbol{e}_j\|_{\boldsymbol{\chi},\boldsymbol{H}^j(\mathscr{P})} \leq \mathscr{C}_{\mathrm{i},p}\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^j\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega},$$

and since $\mathscr{C}_{\mathrm{shift},p}(\omega h/\vartheta_\Omega) \leq 1$, we get

$$\left(\frac{\omega d_\Omega}{\vartheta_\Omega}\right)^j|\!|\!|\boldsymbol{e}_j - \mathcal{J}_h\boldsymbol{e}_j|\!|\!|_{\mathbf{curl},\omega,\Omega} \leq \mathscr{C}_{\mathrm{i},p}\sqrt{2}\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^j\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}.$$

Similarly, using to Lemma 7, we have for the residual $\boldsymbol{r}_p$

$$\omega\|\boldsymbol{r}_p - \mathcal{J}_h\boldsymbol{r}_p\|_{\boldsymbol{\varepsilon},\Omega} \leq \mathscr{C}_{\mathrm{i},p}\left(\frac{h}{d_\Omega}\right)^{p+1}\omega\|\boldsymbol{r}_k\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{p+1}(\mathscr{P})} \leq \mathscr{C}_{\mathrm{i},p}c_{\mathrm{s}}\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^{p+1}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}$$

and

$$\|\boldsymbol{\nabla}\times(\boldsymbol{r}_p - \mathcal{J}_h\boldsymbol{r}_p)\|_{\boldsymbol{\chi},\Omega} \leq \mathscr{C}_{\mathrm{i},p}\left(\frac{h}{d_\Omega}\right)^{p+1}\|\boldsymbol{\nabla}\times\boldsymbol{r}_k\|_{\boldsymbol{\chi},\boldsymbol{H}^{p+1}(\mathscr{P})} \leq \mathscr{C}_{\mathrm{i},p}(1+c_{\mathrm{s}})\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^{p+1}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega},$$

and hence

$$|\!|\!|\boldsymbol{r}_p - \mathcal{J}_h\boldsymbol{r}_p|\!|\!|_{\mathbf{curl},\omega,\Omega} \leq \mathscr{C}_{\mathrm{i},p}\sqrt{1+2c_{\mathrm{s}}^2}\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^{p+1}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}.$$

Then, recalling the expansion (19), the above estimates show that

$$|\!|\!|\boldsymbol{e} - \mathcal{J}_h\boldsymbol{e}|\!|\!|_{\mathbf{curl},\omega,\Omega} \leq \mathscr{C}_{\mathrm{i},p}\left(\sqrt{2}\sum_{j=1}^{p}\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^j + \sqrt{1+2c_{\mathrm{s}}^2}\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^{p+1}\right)\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}.$$

Finally, the result follows by

$$\sum_{j=1}^{p}\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^j = \left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)\sum_{j=0}^{p-1}\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^j = \left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)\frac{1-\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^p}{1-\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)} \leq 2\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}.$$

□ □

We conclude our work with a simplified version of Theorem 8 that is easier to read, but not as explicit in how the estimate depends on c_{s}, $\mathscr{C}_{\mathrm{shift},p}$ and $\mathscr{C}_{\mathrm{i},p}$. We skip the proof as it immediately follows from Theorems 4 and 8.

Corollary 9 (Simplified approximability estimate)**.** *There exist positive constants c and C solely depending on c_{s}, $\mathscr{C}_{\mathrm{shift},p}$ and $\mathscr{C}_{\mathrm{i},p}$ such that whenever $\omega h/\vartheta_\Omega \leq c$, we have*

$$\gamma \leq C\left(\frac{\omega h}{\vartheta_\Omega} + \frac{\omega}{\delta}\left(\frac{\omega h}{\vartheta_\Omega}\right)^{p+1}\right).$$

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