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▶ **To cite this version:**

Théophile Chaumont-Frelet, Patrick Vega. Frequency-explicit approximability estimates for time-harmonic Maxwell's equations. 2021. hal-03221188

HAL Id: hal-03221188

https://hal.inria.fr/hal-03221188

Preprint submitted on 7 May 2021

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FREQUENCY-EXPLICIT APPROXIMABILITY ESTIMATES FOR TIME-HARMONIC MAXWELL'S EQUATIONS

T. CHAUMONT-FRELET[⋆,†] AND P. VEGA[⋆,†]

Abstract. We consider time-harmonic Maxwell's equations set in an heterogeneous medium with perfectly conducting boundary conditions. Given a divergence-free right-hand side lying in L^2, we provide a frequency-explicit approximability estimate measuring the difference between the corresponding solution and its best approximation by high-order Nédélec finite elements. Such an approximability estimate is crucial in both the a priori and a posteriori error analysis of finite element discretizations of Maxwell's equations, but the derivation is not trivial. Indeed, it is hard to take advantage of high-order polynomials given that the right-hand side only exhibits L^2 regularity. We proceed in line with previously obtained results for the simpler setting of the scalar Helmholtz equation, and propose a regularity splitting of the solution. In turn, this splitting yields sharp approximability estimates generalizing known results for the scalar Helmoltz equation and showing the interest of high-order methods.

Key words. Maxwell's equations, Finite element methods, High-order methods

1. Introduction

Over the past decades, considerable efforts have been devoted to analyze the stability and convergence of finite element discretizations of high-frequency wave propagation problems. This in part because the required mathematical analysis is rich and elegant, but also due to the large number of physical and industrial applications for which these problems are relevant.

Apart in the low-frequency regime, the bilinear (or sesquilinear) forms associated with time-harmonic wave propagation problems are not coercive. As a consequence, finite element schemes become unstable when the frequency is high and/or close to a resonant frequency, unless heavily refined meshes or high-order elements are employed [12]. For scalar wave propagation problems modeled by the Helmholtz equation, it has become clear that high-order elements are very well-suited to address these stability issues. One the one hand, the interest of high-order elements has been numerically noted in a number of works [3, 17]. On the other hand, thanks to dedicated duality techniques, the stability and convergence theory is now well-understood [6, 12, 14], and is in line with numerical observations. Vectorial problems are less covered in the literature, but the few available results point towards the fact the analysis techniques employed for the Helmholtz equation as well as the key conclusions can be extended [4, 15]. We also mention [7, 8, 20], where similar duality techniques are used for vectorial wave propagation problems, without a focus on the high-frequency regime though.

Here, we consider time-harmonic Maxwell's equations

$$
(1) \qquad \left\{ \begin{array}{rcll} -\omega^2 \varepsilon \boldsymbol{e} + \boldsymbol{\nabla} \times \left(\boldsymbol{\mu}^{-1} \boldsymbol{\nabla} \times \boldsymbol{e} \right) & = & \omega \varepsilon \boldsymbol{g} & \text{in } \Omega \\ \boldsymbol{e} \times \boldsymbol{n} & = & \boldsymbol{0} & \text{on } \partial\Omega \end{array} \right.
$$

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in a smooth domain Ω with piecewise smooth permittivity and permeability (real valued, symmetric) tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$. In (1), $\omega > 0$ is the frequency, $\boldsymbol{e} : \Omega \to \mathbb{R}^3$ is the unknown and $\boldsymbol{g} : \Omega \to \mathbb{R}^3$ is a given right-hand side. In practice, the right-hand side takes the form $\boldsymbol{g} = i\boldsymbol{\varepsilon}^{-1}\boldsymbol{J}$ where $\boldsymbol{J} : \Omega \to \mathbb{C}^3$ is a "current density", and $\boldsymbol{e}$ represents the electric fields [2]. Here, we choose to work with $\boldsymbol{g}$ instead of $\boldsymbol{J}$ since it is more relevant mathematically as it naturally appears in convergence analysis by duality.

For a finite element space $\boldsymbol{W}_h$, we define the "approximation factor" as the sharpest constant γ such that the estimate

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{W}_h} \|\boldsymbol{e} - \boldsymbol{v}_h\|_{\mathbf{curl},\omega,\Omega} \leq \gamma \|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega} \tag{2}$$

holds for all $\boldsymbol{g} \in \boldsymbol{L}^2(\Omega)$ with $\boldsymbol{\nabla} \cdot (\boldsymbol{\varepsilon}\boldsymbol{g}) = 0$, where $\|\cdot\|_{\boldsymbol{\varepsilon},\Omega}$ is the $\boldsymbol{\varepsilon}$-weighted $\boldsymbol{L}^2(\Omega)$ norm and $\|\cdot\|_{\mathbf{curl},\omega,\Omega}$ is a suitable "energy" norm (see Section 2.2). Clearly, γ quantifies the ability of the finite element space $\boldsymbol{W}_h$ to reproduce solutions to (1). Actually, this quantity is central in the stability analysis of the finite schemes, since it can be shown that the finite element solution is quasi-optimal if and only if γ is "sufficiently small" [4, 6, 14]. The approximation factor also plays a central role in a posteriori error estimations [5, 16].

Since the norm in the right-hand side of (2) is weak, one cannot expect a high regularity for the solution $\boldsymbol{e}$. As a result, taking advantage of high-order polynomials is a subtle task. One key idea to overcome this issue is to introduce a regularity splitting as initially done in [14] for scalar wave propagation in homogeneous media and later extended to heterogeneous media [6, 13] and Maxwell's equations in homogeneous media [15].

In this work, we apply the idea of [6] to obtain a regularity splitting for Maxwell's equations in heterogeneous media. Our key result is that if $\boldsymbol{W}_h$ is the Nédélec finite element space of order $p \geq 0$ on a shape shape-regular mesh $\mathcal{T}_h$ with maximal element size h, there exists a constant C independent of ω and h such that if $\omega h/\vartheta_\Omega$ is sufficiently small, then

$$\gamma \leq C\left(\frac{\omega h}{\vartheta_\Omega} + \frac{\omega}{\delta}\left(\frac{\omega h}{\vartheta_\Omega}\right)^{p+1}\right) \tag{3}$$

where δ is the distance between ω and the closest resonant frequency (see Section 2.3), and ϑ_Ω is the smallest wavespeed in Ω.

Since $N_\lambda := (\omega h/\vartheta_\Omega)^{-1}$ is a measure of the number of mesh elements per wavelength, one sees from (3) that γ stays small as long as $N_\lambda \geq C(1 + (\omega/\delta)^{1/(p+1)})$ with a constant C independent of ω and h. As a result, while the number of elements per wavelength must be increased to achieved stability when the frequency is high ($\omega d_\Omega/\vartheta_\Omega \gg 1$, d_Ω being the diameter of Ω) or almost resonant ($\delta \ll 1$), the requirement is less demanding for high-order elements.

The remaining of this work is organized as follows. Section 2 presents the notation and recalls key preliminary results. In Section 3 we present some initial results concerning the stability of the problem and basic regularity results. We elaborate a regularity splitting in Section 4 that we subsequently apply to derive our approximability result in Section 5, leading to estimate (3).

2. Settings

2.1. **Domain and coefficients.** We consider a simply connected domain $\Omega \subset \mathbb{R}^3$ with an analytic boundary $\partial\Omega$. Ω is partitioned into a set $\mathscr{P}$ of non-overlapping subdomains P with analytic boundaries ∂P such that $\overline{\Omega} = \sup_{P \in \mathscr{P}} \overline{P}$. The notation $d_\Omega := \max_{\boldsymbol{x},\boldsymbol{y} \in \overline{\Omega}} |\boldsymbol{x} - \boldsymbol{y}|$ stands for the diameter of Ω.

$\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are two real symmetric tensor-valued functions defined over Ω. These coefficients are assumed to be piecewise smooth in the sense that for each $P \in \mathscr{P}$ and for $1 \leq j, \ell \leq 3$, $\boldsymbol{\varepsilon}_{j\ell}|_P$ and $\boldsymbol{\mu}_{j\ell}|_P$ are analytic functions. The notations $\boldsymbol{\zeta} := \boldsymbol{\varepsilon}^{-1}$ and $\boldsymbol{\chi} := \boldsymbol{\mu}^{-1}$ will be useful in the sequel.

We denote by $\varepsilon_{\min}, \varepsilon_{\max} : \Omega \to \mathbb{R}$ the (analytic) functions mapping to each $\boldsymbol{x} \in \Omega$ the smallest and largest eigenvalue of $\boldsymbol{\varepsilon}(\boldsymbol{x})$, and we assume that $\boldsymbol{\varepsilon}$ is uniformly bounded and elliptic, which means that

$$0 < \inf_\Omega \varepsilon_{\min}, \qquad \sup_\Omega \varepsilon_{\max} < +\infty.$$

We employ similar notations for $\boldsymbol{\mu}$, $\boldsymbol{\chi}$ and $\boldsymbol{\zeta}$, and assume that $\boldsymbol{\mu}$ is uniformly bounded and elliptic. Finally, we denote by

$$\vartheta_\Omega := \inf_\Omega \frac{1}{\sqrt{\varepsilon_{\max}\mu_{\max}}}$$

the smallest wavespeed in Ω.

2.2. **Functional spaces.** If $D \subset \Omega$ is an open set, $L^2(D)$ is the usual Lebesgue-space of real-valued square integrable functions over D. In addition, we write $\boldsymbol{L}^2(D) := \left(L^2(D)\right)^3$ for vector-valued functions. The natural inner products and norms of both these spaces are $(\cdot,\cdot)_D$ and $\|\cdot\|_D$, and we drop the subscript when $D = \Omega$. If $\boldsymbol{\varphi}$ is a measurable uniformly bounded and elliptic symmetric tensor-valued function, we also employ the (equivalent) norm

$$\|\boldsymbol{v}\|_{\boldsymbol{\varphi},D}^2 := \int_D \boldsymbol{\varphi}\boldsymbol{v} \cdot \boldsymbol{v}$$

for $\boldsymbol{v} \in \boldsymbol{L}^2(D)$.

$\boldsymbol{H}(\mathbf{curl}, \Omega)$ is the Sobolev spaces of vector-valued functions $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$ such that $\boldsymbol{\nabla} \times \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$. It is equipped with the "energy" norm

$$|||\boldsymbol{v}|||_{\mathbf{curl},\omega,\Omega}^2 := \omega^2 \|\boldsymbol{v}\|_{\boldsymbol{\varepsilon},\Omega}^2 + \|\boldsymbol{\nabla} \times \boldsymbol{v}\|_{\boldsymbol{\chi},\Omega}^2 \qquad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega).$$

$\boldsymbol{H}_0(\mathbf{curl}, \Omega)$ is the closure of smooth compactly supported functions into $\boldsymbol{H}(\mathbf{curl}, \Omega)$ and contains vector-valued functions with vanishing tangential traces.

If $\boldsymbol{\varphi}$ is a measurable, uniformly elliptic and bounded tensor-valued function, $\boldsymbol{H}(\mathrm{div}^0, \boldsymbol{\varphi}, \Omega)$ is the space of functions $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$ such that $\boldsymbol{\nabla} \cdot (\boldsymbol{\varphi}\boldsymbol{v}) = 0$ in Ω. We simply write $\boldsymbol{H}(\mathrm{div}^0, \Omega)$ when $\boldsymbol{\varphi} := \boldsymbol{I}$ is the identity tensor. Besides, the space $\boldsymbol{H}_0(\mathrm{div}^0, \boldsymbol{\varphi}, \Omega)$ is for the closure of smooth compactly supported functions in $\boldsymbol{H}(\mathrm{div}^0, \boldsymbol{\varphi}, \Omega)$ and contains functions with vanishing normal traces.

If $m \geq 0$, the space $\boldsymbol{H}^m(\mathscr{P})$ contains those functions $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$ such that for each $P \in \mathscr{P}$, $1 \leq \ell \leq 3$ and all multi-indices $\boldsymbol{\alpha} \in \mathbb{N}^3$ with $|\boldsymbol{\alpha}| \leq m$, we have $\partial^\alpha(\boldsymbol{v}_\ell|_P) \in L^2(P)$. We equip this space with the norms

$$\|\boldsymbol{v}\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^m(\mathscr{P})}^2 := \|\boldsymbol{v}\|_{\boldsymbol{\varepsilon},\Omega}^2 + \sum_{n=1}^m \sum_{|\boldsymbol{\alpha}|=n} \sum_{P \in \mathscr{P}} \sum_{\ell=1}^3 d_\Omega^{2n} \int_P \varepsilon_{\max} |\partial^{\boldsymbol{\alpha}}(\boldsymbol{v}_\ell|_P)|^2$$

and

$$\|\boldsymbol{v}\|_{\boldsymbol{\chi},\boldsymbol{H}^m(\mathscr{P})}^2 := \|\boldsymbol{v}\|_{\boldsymbol{\chi},\Omega}^2 + \sum_{n=1}^m \sum_{|\boldsymbol{\alpha}|=n} \sum_{P \in \mathscr{P}} \sum_{\ell=1}^3 d_\Omega^{2n} \int_P \chi_{\max} |\partial^{\boldsymbol{\alpha}}(\boldsymbol{v}_\ell|_P)|^2.$$

We refer the reader to [1] for a detailed exposition concerning Lesbegue and high-order Sobolev spaces, and to [10] and [11] for Sobolev spaces involving the curl and divergence of vector fields.

2.3. **Eigenpairs.** Recalling that Ω is simply connected, it follows from [10, Remark 7.5] that the application

$$\boldsymbol{H}_0(\mathbf{curl},\Omega) \ni \boldsymbol{v} \to \|\boldsymbol{\nabla}\times\boldsymbol{v}\|_{\boldsymbol{\chi},\Omega}$$

is a norm on $\boldsymbol{H}_0(\mathbf{curl},\Omega)\cap\boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$. Besides, the injection $\boldsymbol{H}_0(\mathbf{curl},\Omega)\cap\boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega) \subset \boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$ is compact [18]. As a result (see, e.g. [2, Theorem 4.5.11]), there exists an orthonormal basis $\{\boldsymbol{\phi}_j\}_{j\geq 0}$ of $\boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$ (equipped with the inner-product $(\boldsymbol{\varepsilon}\,\cdot,\cdot)$) and a sequence of strictly positive eigenvalues $\{\lambda_j\}_{j\geq 0}$ such that for all $j \geq 0$,

$$(\boldsymbol{\chi}\boldsymbol{\nabla}\times\boldsymbol{\phi}_j,\boldsymbol{\nabla}\times\boldsymbol{v}) = \lambda_j(\boldsymbol{\varepsilon}\boldsymbol{\phi}_j\boldsymbol{v}) \quad \forall \boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl},\Omega).$$

In addition, if $\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl},\Omega)\cap\boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$, we have

$$\|\boldsymbol{v}\|^2_{\boldsymbol{\varepsilon},\Omega} = \sum_{j\geq 0}|v_j|^2 \quad \text{and} \quad \|\boldsymbol{\nabla}\times\boldsymbol{v}\|^2_{\boldsymbol{\chi},\Omega} = \sum_{j\geq 0}\lambda_j|v_j|^2,$$

where $v_j := (\boldsymbol{\varepsilon}\boldsymbol{v},\boldsymbol{\phi}_j)$.

In the remaining of this work, we set $\delta := \min_{j\geq 0}|\sqrt{\lambda_j}-\omega|$ and assume $\delta > 0$.

2.4. **Regularity shifts.** Our analysis heavily relies on regularity shift results where, given a divergence-free vector field with a smooth curl, one deduces smoothness results for the field itself. First [19, Theorem 2.2], for all $p \geq 0$, there exists a constant $\mathscr{C}_{\mathrm{shift},p}$ only depending on p, $\mathscr{P}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ such that, for $0 \leq m \leq p$, if $\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl},\Omega)\cap\boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$ with $\boldsymbol{\nabla}\times\boldsymbol{v}\in\boldsymbol{H}^m(\mathscr{P})$, we have

$$\|\boldsymbol{v}\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{m+1}(\mathscr{P})} \leq \mathscr{C}_{\mathrm{shift},p}\frac{d_\Omega}{\vartheta_\Omega}\|\boldsymbol{\nabla}\times\boldsymbol{v}\|_{\boldsymbol{\chi},\boldsymbol{H}^m(\mathscr{P})}. \tag{4}$$

Similarly, if $\boldsymbol{w}\in\boldsymbol{H}_0(\mathrm{div}^0,\Omega)$ with $\boldsymbol{\nabla}\times(\boldsymbol{\chi}\boldsymbol{w})\in\boldsymbol{H}^m(\mathscr{P})$, then we have

$$\|\boldsymbol{w}\|_{\boldsymbol{\chi},\boldsymbol{H}^{m+1}(\mathscr{P})} \leq \mathscr{C}_{\mathrm{shift},p}\frac{d_\Omega}{\vartheta_\Omega}\|\boldsymbol{\zeta}\boldsymbol{\nabla}\times(\boldsymbol{\chi}\boldsymbol{w})\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^m(\mathscr{P})}, \tag{5}$$

as can be seen by applying [19, Theorem 2.2] to $\boldsymbol{\chi}\boldsymbol{w}\in\boldsymbol{H}(\mathbf{curl},\Omega)\cap\boldsymbol{H}_0(\mathrm{div}^0,\boldsymbol{\mu},\Omega)$. We also record the following result obtained by combining (4) and (5): if $\boldsymbol{u}\in\boldsymbol{H}_0(\mathbf{curl},\Omega)\cap\boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$ satisfies $\boldsymbol{\nabla}\times(\boldsymbol{\mu}\boldsymbol{\nabla}\times\boldsymbol{u})\in\boldsymbol{H}^m(\mathscr{P})$, we have

$$\|\boldsymbol{u}\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{m+2}(\mathscr{P})} \leq \left(\mathscr{C}_{\mathrm{shift},p}\frac{d_\Omega}{\vartheta_\Omega}\right)^2\|\boldsymbol{\zeta}\boldsymbol{\nabla}\times(\boldsymbol{\chi}\boldsymbol{\nabla}\times\boldsymbol{u})\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^m(\mathscr{P})} \tag{6}$$

for $0 \leq m \leq p-1$.

2.5. **Curved tetrahedral mesh.** We consider a partition of Ω into a conforming mesh $\mathcal{T}_h$ of (curved) tetrahedral elements K as in [15, Assumption 3.1]. For $K\in\mathcal{T}_h$ we denote by $\mathscr{F}_K : \widehat{K}\to K$ the (analytic) mapping between the reference tetrahedra $\widehat{K}$ and the element K. We further assume that the mesh $\mathcal{T}_h$ is conforming with the partition $\mathscr{P}$, which means that for each $K\in\mathcal{T}_h$ there exists a unique $P\in\mathscr{P}$ such that $K\subset\overline{P}$. This last assumption means that the coefficients are smooth inside each mesh cell.

2.6. **Nédélec finite element space.** In the remaining of this work, we fix a polynomial degree $p \geq 0$. Then, following [9, Chapter 15], we introduce the Nédélec polynomial space

$$\boldsymbol{N}_p(\widehat{K}) = \boldsymbol{P}_p(\widehat{K}) + \boldsymbol{x} \times \boldsymbol{P}_p(\widehat{K}),$$

where $\boldsymbol{P}_p(\widehat{K}) := \left(P_p(\widehat{K})\right)^3$ and $P_p(\widehat{K})$ stands for the space of polynomials of degree less than or equal to p defined over $\widehat{K}$. Classically, the associate approximation space is obtained by mapping the Nédélec polynomial space to the mesh cells through a Piola mapping, leading to

$$\boldsymbol{W}_h := \left\{ \boldsymbol{v}_h \in \boldsymbol{H}_0(\mathbf{curl}, \Omega) \quad \middle| \quad (\boldsymbol{D}\mathscr{F}_K^{-1}) \left(\boldsymbol{v}_h|_K \circ \mathscr{F}_K^{-1}\right) \in \boldsymbol{N}_p(\widehat{K}) \quad \forall K \in \mathcal{T}_h \right\},$$

where $\boldsymbol{D}\mathscr{F}_K^{-1}$ is the Jacobian matrix of $\mathscr{F}_K^{-1}$.

2.7. **High-order interpolation.** There exists an interpolation operator $\mathcal{J}_h : \boldsymbol{H}^1(\mathscr{P}) \cap \boldsymbol{H}_0(\mathbf{curl}, \Omega) \to \boldsymbol{W}_h$ and a constant $\mathscr{C}_{\mathrm{i},p}$ solely depending on p, the regularity of the mesh and the coefficients $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ such that

$$\|\boldsymbol{v} - \mathcal{J}_h \boldsymbol{v}\|_{\boldsymbol{\varepsilon},\Omega} \leq \mathscr{C}_{\mathrm{i},p} \left(\frac{h}{d_\Omega}\right)^{p+1} \|\boldsymbol{v}\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{p+1}(\mathscr{P})}, \tag{7a}$$

whenever $\boldsymbol{v} \in \boldsymbol{H}^1(\mathscr{P})$ satisfies $\boldsymbol{v} \in \boldsymbol{H}^{p+1}(\mathscr{P})$ and

$$\|\boldsymbol{\nabla} \times (\boldsymbol{w} - \mathcal{J}_h \boldsymbol{w})\|_{\boldsymbol{\chi},\Omega} \leq \mathscr{C}_{\mathrm{i},p} \left(\frac{h}{d_\Omega}\right)^{p+1} \|\boldsymbol{\nabla} \times \boldsymbol{w}\|_{\boldsymbol{\chi},\boldsymbol{H}^{p+1}(\mathscr{P})} \tag{7b}$$

for all $\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega) \cap \boldsymbol{H}^1(\mathscr{P})$ with $\boldsymbol{\nabla} \times \boldsymbol{w} \in \boldsymbol{H}^{p+1}(\mathscr{P})$. The construction of such an interpolation operator is classical, and we refer the reader to [9, Chapters 13 and 17] and [15, Chapter 8] for more details.

2.8. **Sharp approximability estimates.** We are now ready to rigorously introduce the approximation factor γ. Given $\boldsymbol{g} \in \boldsymbol{L}^2(\Omega)$, we denote by $\boldsymbol{e}^\star(\boldsymbol{g})$ the unique element of $\boldsymbol{H}_0(\mathbf{curl}, \Omega)$ such that

$$-\omega^2(\boldsymbol{\varepsilon}\boldsymbol{v}, \boldsymbol{e}^\star(\boldsymbol{g})) + (\boldsymbol{\chi}\boldsymbol{\nabla} \times \boldsymbol{v}, \boldsymbol{\nabla} \times \boldsymbol{e}^\star(\boldsymbol{g})) = \omega(\boldsymbol{\varepsilon}\boldsymbol{v}, \boldsymbol{g}) \tag{8}$$

for all $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega)$. Note that the existence and uniqueness of $\boldsymbol{e}^\star(\boldsymbol{g})$ follows from the assumption that $\delta > 0$, i.e. ω is not a resonant frequency. Then, we introduce the approximation factor as

$$\gamma := \sup_{\substack{\boldsymbol{g} \in \boldsymbol{H}_0(\mathrm{div}^0, \boldsymbol{\varepsilon}, \Omega) \\ \|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega} = 1}} \inf_{\boldsymbol{v}_h \in \boldsymbol{W}_h} \|\boldsymbol{e}^\star(\boldsymbol{g}) - \boldsymbol{v}_h\|_{\mathbf{curl},\omega,\Omega}. \tag{9}$$

Observing that we can choose $\boldsymbol{v}_h = \boldsymbol{o}$ in the infimum, a crude estimate for the approximation factor is given by $\gamma \leq c_{\mathrm{s}}$ where

$$c_{\mathrm{s}} := \sup_{\substack{\boldsymbol{g} \in \boldsymbol{H}_0(\mathrm{div}^0, \boldsymbol{\varepsilon}, \Omega) \\ \|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega} = 1}} \|\boldsymbol{e}^\star(\boldsymbol{g})\|_{\mathbf{curl},\omega,\Omega}. \tag{10}$$

This upper bound is of little use in a priori error estimation where one needs γ to become small as $h \to 0$ in a duality argument [4, 6, 7, 8, 20]. On the other hand, it is of interest in a posteriori error estimation, in particular to obtain guaranteed estimates [5]. Indeed, the constant c_{s} is often easier to compute than sharper estimates, since it only depends on the domain and the coefficients, and not on the mesh or the discretization order.

3. Stability and basic regularity results

Here, we present basic stability and regularity results. We start with a stability theorem, that follows from standard spectral theory.

Theorem 1 (Stability)**.** *The estimates*

$$\omega\|\boldsymbol{e}^\star(\boldsymbol{g})\|_{\boldsymbol{\varepsilon},\Omega} \leq \frac{\omega}{\delta}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}, \qquad \|\boldsymbol{\nabla}\times\boldsymbol{e}^\star(\boldsymbol{g})\|_{\boldsymbol{\chi},\Omega} \leq \frac{\omega}{\delta}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega} \tag{11}$$

hold true for all $\boldsymbol{g} \in \boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$*. In addition, we have*

$$c_{\mathrm{s}} \leq \frac{\omega}{\delta}. \tag{12}$$

Proof. Let $\boldsymbol{g} \in \boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$ and set $\boldsymbol{e} := \boldsymbol{e}^\star(\boldsymbol{g})$. Since $\boldsymbol{e},\boldsymbol{g} \in \boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$, we may expand $\boldsymbol{e}$ and $\boldsymbol{g}$ in the basis $\{\boldsymbol{\phi}_j\}_{j\geq 0}$ by letting $e_j := (\boldsymbol{e},\boldsymbol{\phi}_j)$ and $g_j := (\boldsymbol{e},\boldsymbol{\phi}_j)$. Then, picking $\boldsymbol{v} = \boldsymbol{\phi}_j$ in (8), we see that

$$|e_j| = \frac{\omega}{|\lambda_j-\omega^2|}|g_j| \leq \frac{1}{\delta}\frac{\omega}{\sqrt{\lambda_j}+\omega}|g_j| \quad \text{and} \quad \left(\omega+\sqrt{\lambda_j}\right)|e_j| \leq \frac{\omega}{\delta}|g_j|.$$

Then, (11) follows from

$$\|\boldsymbol{e}\|^2_{\mathbf{curl},\omega,\Omega} = \sum_{j\geq 0}(\omega^2+\lambda_j)|e_j|^2 \leq \sum_{j\geq 0}\left((\omega+\sqrt{\lambda_j})|e_j|\right)^2 \leq \left(\frac{\omega}{\delta}\right)^2\|\boldsymbol{g}\|^2_{\boldsymbol{\varepsilon},\Omega},$$

and (12) follows from (11) recalling the definition of c_{s} in (10). □

We next combine Theorem 1 with the regularity shift results from Section 2.4 to provide a basic regularity result.

Lemma 2 (Basic regularity)**.** *For all* $\boldsymbol{g} \in \boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$*, we have*

$$\omega\|\boldsymbol{e}^\star(\boldsymbol{g})\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^1(\mathscr{P})} \leq c_{\mathrm{s}}\mathscr{C}_{\mathrm{shift},p}\frac{\omega d_\Omega}{\vartheta_\Omega}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}, \tag{13}$$

and

$$\|\boldsymbol{\nabla}\times\boldsymbol{e}^\star(\boldsymbol{g})\|_{\boldsymbol{\chi},\boldsymbol{H}^1(\mathscr{P})} \leq (1+c_{\mathrm{s}})\mathscr{C}_{\mathrm{shift},p}\frac{\omega d_\Omega}{\vartheta_\Omega}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}. \tag{14}$$

Proof. Pick $\boldsymbol{g} \in \boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$ and set $\boldsymbol{e} := \boldsymbol{e}^\star(\boldsymbol{g})$. We first observe that as $\boldsymbol{e} \in \boldsymbol{H}_0(\mathbf{curl},\Omega)\cap\boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$, shift estimate (4) implies that

$$\omega\|\boldsymbol{e}\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^1(\mathscr{P})} \leq \mathscr{C}_{\mathrm{shift},p}\frac{\omega d_\Omega}{\vartheta_\Omega}\|\boldsymbol{\nabla}\times\boldsymbol{e}\|_{\boldsymbol{\chi},\Omega}$$

and stability estimate (11) shows that

$$\|\boldsymbol{e}\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^1(\mathscr{P})} \leq \mathscr{C}_{\mathrm{shift},p}d_\Omega\|\boldsymbol{\nabla}\times\boldsymbol{e}\|_{\boldsymbol{\varepsilon},\Omega} \leq \mathscr{C}_{\mathrm{shift},p}c_{\mathrm{s}}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega},$$

so that (13) follows. On the other hand, we establish (14) with (5), since

$$\|\boldsymbol{\nabla}\times\boldsymbol{e}\|_{\boldsymbol{\chi},\boldsymbol{H}^1(\mathscr{P})} \leq \mathscr{C}_{\mathrm{shift},p}\frac{d_\Omega}{\vartheta_\Omega}\|\boldsymbol{\zeta}\boldsymbol{\nabla}\times(\boldsymbol{\mu}\boldsymbol{\nabla}\times\boldsymbol{e})\|_{\boldsymbol{\varepsilon},\Omega} \leq \mathscr{C}_{\mathrm{shift},p}\frac{d_\Omega}{\vartheta_\Omega}\left(\omega\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}+\omega^2\|\boldsymbol{e}\|_{\boldsymbol{\varepsilon},\Omega}\right),$$

using (10) to estimate the last term. □

4. Regularity splitting

The regularity results presented in Lemma 2 are sufficient to obtain sharp estimates for the approximation factor when $p=0$. For high-order elements however, this is not sufficient. As we only have a limited regularity assumption for the right-hand side $\boldsymbol{g}$ in definition (9) of γ, we may not expect more regularity than established in Lemma 2 for the associated solution $\boldsymbol{e}^\star(\boldsymbol{g})$. As shown in [6, 13, 14] for the Helmholtz equation, the key idea is to introduce a "regularity splitting" of the solution. Here, we shall adapt the approach of [6] to Maxwell's equations, and consider the formal expansion

$$\boldsymbol{e}^\star(\boldsymbol{g}) = \sum_{j\geq 0}\left(\frac{\omega d_\Omega}{\vartheta_\Omega}\right)^j \boldsymbol{e}^\star_j(\boldsymbol{g}). \tag{15}$$

After identifying the powers of $(\omega d_\Omega/\vartheta_\Omega)$ in (1), one sees that $\boldsymbol{e}^\star_0(\boldsymbol{g}) := \boldsymbol{o}$, and that the other elements $\boldsymbol{e}^\star_j(\boldsymbol{g}) \in \boldsymbol{H}_0(\mathbf{curl},\Omega)\cap\boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$ are iteratively defined through

$$\boldsymbol{\nabla}\times(\boldsymbol{\chi}\boldsymbol{\nabla}\times\boldsymbol{e}^\star_1(\boldsymbol{g})) = \frac{\vartheta_\Omega}{d_\Omega}\boldsymbol{\varepsilon}\boldsymbol{g}, \tag{16a}$$

and

$$\boldsymbol{\nabla}\times\left(\boldsymbol{\chi}\boldsymbol{\nabla}\times\boldsymbol{e}^\star_j(\boldsymbol{g})\right) = \left(\frac{\vartheta_\Omega}{d_\Omega}\right)^2\boldsymbol{\varepsilon}\boldsymbol{e}^\star_{j-2}(\boldsymbol{g}) \tag{16b}$$

for $j\geq 2$. Note that the boundary value problems in (16) are well-posed, since $\|\boldsymbol{\nabla}\times\cdot\|_{\boldsymbol{\chi},\Omega}$ is a norm on $\boldsymbol{H}_0(\mathbf{curl},\Omega)\cap\boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$. We first show that the iterates in the sequence exhibit increasing regularity.

Lemma 3 (Increasing regularity of the expansion)**.** *Let $\boldsymbol{g}\in\boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$. For all $0\leq j\leq p$, we have $\boldsymbol{e}^\star_j(\boldsymbol{g})\in\boldsymbol{H}^{j+1}(\mathscr{P})$ and $\boldsymbol{\nabla}\times\boldsymbol{e}^\star_j(\boldsymbol{g})\in\boldsymbol{H}^j(\mathscr{P})$ with*

$$\omega\|\boldsymbol{e}^\star_j(\boldsymbol{g})\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{j+1}(\mathscr{P})} \leq \mathscr{C}^{j+1}_{\mathrm{shift},p}\frac{\omega d_\Omega}{\vartheta_\Omega}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}, \tag{17}$$

and

$$\|\boldsymbol{\nabla}\times\boldsymbol{e}^\star_j(\boldsymbol{g})\|_{\boldsymbol{\chi},\boldsymbol{H}^j(\mathscr{P})} \leq \mathscr{C}^j_{\mathrm{shift},p}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}. \tag{18}$$

Proof. Let $\boldsymbol{g}\in\boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$. To ease the presentation, we set $\boldsymbol{e}_j := \boldsymbol{e}^\star_j(\boldsymbol{g})$ for $j\geq 0$. We start with (18). It obviously holds for $j=0$ as $\boldsymbol{e}_0 := \boldsymbol{o}$. For $j=1$, recalling (16), we have

$$\|\boldsymbol{\nabla}\times\boldsymbol{e}_1\|_{\boldsymbol{\chi},\boldsymbol{H}^1(\mathscr{P})} \leq \mathscr{C}_{\mathrm{shift},p}\frac{d_\Omega}{\vartheta_\Omega}\|\boldsymbol{\nabla}\times(\boldsymbol{\mu}\boldsymbol{\nabla}\times\boldsymbol{e}_1)\|_{\boldsymbol{\zeta},\Omega} = \mathscr{C}_{\mathrm{shift},p}\|\boldsymbol{\varepsilon}\boldsymbol{g}\|_{\boldsymbol{\zeta},\Omega} = \mathscr{C}_{\mathrm{shift},p}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}.$$

Then, assuming that (18) holds up to some j, (5) and (16) reveal that

$$\begin{aligned}\|\boldsymbol{\nabla}\times\boldsymbol{e}_{j+2}\|_{\boldsymbol{\mu},\boldsymbol{H}^{j+2}(\mathscr{P})} &\leq \mathscr{C}_{\mathrm{shift},p}\frac{d_\Omega}{\vartheta_\Omega}\|\boldsymbol{\zeta}\boldsymbol{\nabla}\times(\boldsymbol{\chi}\boldsymbol{\nabla}\times\boldsymbol{e}_{j+2})\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{j+1}(\mathscr{P})} = \mathscr{C}_{\mathrm{shift},p}\frac{\vartheta_\Omega}{d_\Omega}\|\boldsymbol{e}_j\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{j+1}(\mathscr{P})}\\ &\leq \mathscr{C}^2_{\mathrm{shift},p}\|\boldsymbol{\nabla}\times\boldsymbol{e}_j\|_{\boldsymbol{\chi},\boldsymbol{H}^{j+1}(\mathscr{P})} \leq \mathscr{C}^{j+2}_{\mathrm{shift},p}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega},\end{aligned}$$

and (18) follows by induction.

On the other hand, (17) is a direct consequence of (18), since (4) shows that

$$\|\boldsymbol{e}_j\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{j+1}(\mathscr{P})} \leq \mathscr{C}_{\mathrm{shift},p}\frac{d_\Omega}{\vartheta_\Omega}\|\boldsymbol{\nabla}\times\boldsymbol{e}_j\|_{\boldsymbol{\chi},\boldsymbol{H}^j(\mathscr{P})} \leq \frac{d_\Omega}{\vartheta_\Omega}\mathscr{C}^{j+1}_{\mathrm{shift},p}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}.$$

□

So far, expansion (15) is only formal, and we need to truncate the expansion into a finite sum. To do so, we introduce, for $\ell \geq 0$, the "residual" term

$$\boldsymbol{r}_\ell^\star(\boldsymbol{g}) := \boldsymbol{e}^\star(\boldsymbol{g}) - \sum_{j=0}^{\ell} \left(\frac{\omega d_\Omega}{\vartheta_\Omega}\right)^j \boldsymbol{e}_j^\star(\boldsymbol{g}) \in \boldsymbol{H}_0(\mathbf{curl},\Omega) \cap \boldsymbol{H}(\operatorname{div}^0,\boldsymbol{\varepsilon},\Omega),$$

so that

$$\boldsymbol{e}^\star(\boldsymbol{g}) = \sum_{j=0}^{\ell} \left(\frac{\omega d_\Omega}{\vartheta_\Omega}\right)^j \boldsymbol{e}_j^\star(\boldsymbol{g}) + \boldsymbol{r}_\ell^\star(\boldsymbol{g}). \tag{19}$$

As we show next, these residuals have increasing regularity.

Lemma 4 (Regularity of residual terms)**.** *For all $\boldsymbol{g} \in \boldsymbol{H}(\operatorname{div}^0,\boldsymbol{\varepsilon},\Omega)$ and $0 \leq \ell \leq p$, we have $\boldsymbol{r}_\ell^\star(\boldsymbol{g}) \in \boldsymbol{H}^{\ell+1}(\mathscr{P})$ and $\boldsymbol{\nabla} \times \boldsymbol{r}_\ell^\star(\boldsymbol{g}) \in \boldsymbol{H}^{\ell+1}(\mathscr{P})$ with the estimates*

$$\omega \|\boldsymbol{r}_\ell^\star(\boldsymbol{g})\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{\ell+1}(\mathscr{P})} \leq c_{\rm s} \left(\mathscr{C}_{\text{shift},p}\frac{\omega d_\Omega}{\vartheta_\Omega}\right)^{\ell+1} \|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}, \tag{20}$$

and

$$\|\boldsymbol{\nabla} \times \boldsymbol{r}_\ell^\star(\boldsymbol{g})\|_{\boldsymbol{\chi},\boldsymbol{H}^{\ell+1}(\mathscr{P})} \leq (1+c_{\rm s}) \left(\mathscr{C}_{\text{shift},p}\frac{\omega d_\Omega}{\vartheta_\Omega}\right)^{\ell+1} \|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}. \tag{21}$$

Proof. For the sake of simplicity, we fix $\boldsymbol{g} \in \boldsymbol{H}(\operatorname{div}^0,\boldsymbol{\varepsilon},\Omega)$, and set set $\boldsymbol{e} := \boldsymbol{e}^\star(\boldsymbol{g})$ and $\boldsymbol{r}_\ell := \boldsymbol{r}_\ell^\star(\boldsymbol{g})$ for $\ell \geq 0$. We have $\boldsymbol{r}_0 := \boldsymbol{e}$, so that (20) and (21) hold for $\ell = 0$ as a direct consequence of (13) and (14).

For the case $\ell = 1$, simple computations show that $\boldsymbol{\nabla} \times (\boldsymbol{\chi}\boldsymbol{\nabla} \times \boldsymbol{r}_1) = \omega^2\boldsymbol{\varepsilon}\boldsymbol{e}$. Using (5) and (6), it then follows from (13) that

$$\omega\|\boldsymbol{r}_1\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^2(\mathscr{P})} \leq \mathscr{C}_{\text{shift},p}^2 \frac{\omega d_\Omega^2}{\vartheta_\Omega^2} \|\boldsymbol{\zeta}\boldsymbol{\nabla} \times (\boldsymbol{\chi}\boldsymbol{\nabla} \times \boldsymbol{r}_1)\|_{\boldsymbol{\varepsilon},\Omega} = \mathscr{C}_{\text{shift},p}^2 \left(\frac{\omega d_\Omega}{\vartheta_\Omega}\right)^2 \omega\|\boldsymbol{e}\|_{\boldsymbol{\varepsilon},\Omega}$$

and

$$\|\boldsymbol{\nabla} \times \boldsymbol{r}_1\|_{\boldsymbol{\chi},\boldsymbol{H}^2(\mathscr{P})} \leq \mathscr{C}_{\text{shift},p}\frac{d_\Omega}{\vartheta_\Omega}\|\boldsymbol{\zeta}\boldsymbol{\nabla} \times (\boldsymbol{\mu}\boldsymbol{\nabla} \times \boldsymbol{r}_1)\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^1(\mathscr{P})} = \mathscr{C}_{\text{shift},p}\frac{\omega d_\Omega}{\vartheta_\Omega}\omega\|\boldsymbol{e}\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^1(\mathscr{P})}$$

so that (20) and (21) are also valid when $\ell = 1$ recalling (11).

For the general case, we first observe that $\boldsymbol{\nabla} \times (\boldsymbol{\chi}\boldsymbol{\nabla} \times \boldsymbol{r}_{\ell+2}) = \omega^2\boldsymbol{\varepsilon}\boldsymbol{r}_\ell$. Therefore, using (5) and (6), we have

$$\omega\|\boldsymbol{r}_{\ell+2}\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{\ell+3}(\mathscr{P})} \leq \mathscr{C}_{\text{shift},p}^2\frac{d_\Omega^2}{\vartheta_\Omega}\omega\|\boldsymbol{r}_\ell\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{\ell+1}(\mathscr{P})},$$

and

$$\begin{aligned}\|\boldsymbol{\nabla} \times \boldsymbol{r}_{\ell+2}\|_{\boldsymbol{\chi},\boldsymbol{H}^{\ell+3}(\mathscr{P})} &\leq \mathscr{C}_{\text{shift},p}\frac{d_\Omega}{\vartheta_\Omega}\|\boldsymbol{\zeta}\boldsymbol{\nabla} \times (\boldsymbol{\chi}\boldsymbol{\nabla} \times \boldsymbol{r}_{\ell+2})\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{\ell+2}(\mathscr{P})} \\ &\leq \mathscr{C}_{\text{shift},p}\frac{d_\Omega}{\vartheta_\Omega}\omega^2\|\boldsymbol{r}_\ell\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{\ell+2}(\mathscr{P})} \leq \left(\mathscr{C}_{\text{shift},p}\frac{\omega d_\Omega}{\vartheta_\Omega}\right)^2 \|\boldsymbol{\nabla} \times \boldsymbol{r}_\ell\|_{\boldsymbol{\chi},\boldsymbol{H}^{\ell+1}(\mathscr{P})},\end{aligned}$$

and the general case follows by induction. $\square$

5. Sharp approximability estimates

Equipped with the regularity splitting from Section 4, we are now ready to establish our main result, providing an upper bound for the approximation factor γ.

Theorem 5 (Approximability estimate)**.** *Assume that $\mathscr{C}_{\mathrm{shift},p}(\omega h/\vartheta_\Omega) \leq 1/2$. Then, the following estimate holds true*

$$\gamma \leq \mathscr{C}_{\mathrm{i},p}\left(2\sqrt{2}\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}+\sqrt{1+2c_{\mathrm{s}}^2}\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^{p+1}\right).$$

Proof. We consider a right-hand side $\boldsymbol{g} \in \boldsymbol{H}(\mathrm{div}^0,\boldsymbol{\varepsilon},\Omega)$ and employ the notation $\boldsymbol{e} := \boldsymbol{e}^\star(\boldsymbol{g})$, $\boldsymbol{e}_j := \boldsymbol{e}_j^\star(\boldsymbol{g})$ for $j \geq 0$ and $\boldsymbol{r}_p := \boldsymbol{r}_p^\star(\boldsymbol{g})$. Recalling (9) and the finite expansion (19) for $\boldsymbol{e}$, it is sufficient to provide upper bounds for the high order interpolation error of $\boldsymbol{e}_j$ and $\boldsymbol{r}_\ell$. For $\boldsymbol{e}_j$, the definition (16) imply that

$$\begin{aligned}\omega\left(\frac{\omega d_\Omega}{\vartheta_\Omega}\right)^j\|\boldsymbol{e}_j-\mathcal{J}_h\boldsymbol{e}_j\|_{\boldsymbol{\varepsilon},\Omega} &\leq \mathscr{C}_{\mathrm{i},p}\omega\left(\frac{\omega d_\Omega}{\vartheta_\Omega}\right)^j\left(\frac{h}{d_\Omega}\right)^j\|\boldsymbol{e}_j\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{j+1}(\mathscr{P})}\\ &= \mathscr{C}_{\mathrm{i},p}\frac{\vartheta_\Omega}{d_\Omega}\left(\frac{\omega h}{\vartheta_\Omega}\right)^{j+1}\|\boldsymbol{e}_j\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{j+1}(\mathscr{P})} \leq \mathscr{C}_{\mathrm{i},p}\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^{j+1}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega},\end{aligned}$$

and

$$\left(\frac{\omega d_\Omega}{\vartheta_\Omega}\right)^j\|\boldsymbol{\nabla}\times(\boldsymbol{e}_j-\mathcal{J}_h\boldsymbol{e}_j)\|_{\boldsymbol{\chi},\Omega} \leq \mathscr{C}_{\mathrm{i},p}\left(\frac{\omega d_\Omega}{\vartheta_\Omega}\right)^j\left(\frac{h}{d_\Omega}\right)^j\|\boldsymbol{\nabla}\times\boldsymbol{e}_j\|_{\boldsymbol{\chi},\boldsymbol{H}^j(\mathscr{P})} \leq \mathscr{C}_{\mathrm{i},p}\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^j\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega},$$

and since $\mathscr{C}_{\mathrm{shift},p}(\omega h/\vartheta_\Omega) \leq 1$, we get

$$\left(\frac{\omega d_\Omega}{\vartheta_\Omega}\right)^j|\!|\!|\boldsymbol{e}_j-\mathcal{J}_h\boldsymbol{e}_j|\!|\!|_{\mathbf{curl},\omega,\Omega} \leq \mathscr{C}_{\mathrm{i},p}\sqrt{2}\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^j\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}.$$

Similarly, for the residual $\boldsymbol{r}_p$, we have

$$\omega\|\boldsymbol{r}_p-\mathcal{J}_h\boldsymbol{r}_p\|_{\boldsymbol{\varepsilon},\Omega} \leq \mathscr{C}_{\mathrm{i},p}\left(\frac{h}{d_\Omega}\right)^{p+1}\omega\|\boldsymbol{r}_k\|_{\boldsymbol{\varepsilon},\boldsymbol{H}^{p+1}(\mathscr{P})} \leq \mathscr{C}_{\mathrm{i},p}c_{\mathrm{s}}\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^{p+1}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}$$

and

$$\|\boldsymbol{\nabla}\times(\boldsymbol{r}_p-\mathcal{J}_h\boldsymbol{r}_p)\|_{\boldsymbol{\chi},\Omega} \leq \mathscr{C}_{\mathrm{i},p}\left(\frac{h}{d_\Omega}\right)^{p+1}\|\boldsymbol{\nabla}\times\boldsymbol{r}_k\|_{\boldsymbol{\chi},\boldsymbol{H}^{p+1}(\mathscr{P})} \leq \mathscr{C}_{\mathrm{i},p}(1+c_{\mathrm{s}})\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^{p+1}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega},$$

and hence

$$|\!|\!|\boldsymbol{r}_p-\mathcal{J}_h\boldsymbol{r}_p|\!|\!|_{\mathbf{curl},\omega,\Omega} \leq \mathscr{C}_{\mathrm{i},p}\sqrt{1+2c_{\mathrm{s}}^2}\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^{p+1}\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}.$$

Then, recalling the expansion (19), the above estimates show that

$$|\!|\!|\boldsymbol{e}-\mathcal{J}_h\boldsymbol{e}|\!|\!|_{\mathbf{curl},\omega,\Omega} \leq \mathscr{C}_{\mathrm{i},p}\left(\sqrt{2}\sum_{j=1}^{p}\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^j+\sqrt{1+2c_{\mathrm{s}}^2}\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^{p+1}\right)\|\boldsymbol{g}\|_{\boldsymbol{\varepsilon},\Omega}.$$

Finally, the result follows by

$$\sum_{j=1}^{p}\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^j = \left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)\sum_{j=0}^{p-1}\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^j = \left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)\frac{1-\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)^p}{1-\left(\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}\right)} \leq 2\mathscr{C}_{\mathrm{shift},p}\frac{\omega h}{\vartheta_\Omega}.$$

□

References

1. R. Adams and J. Fournier, *Sobolev spaces*, Academic Press, 2003.
2. F. Assous, P. Ciarlet Jr., and S. Labrunie, *Mathematical foundations of computational electromagnetism*, Applied Mathematical Sciences, vol. 198, Springer, Cham, 2018.
3. H. Bériot, A. Prinn, and G. Gabard, *Efficient implementation of high-order finite elements for Helmholtz problems*, Int. J. Numer. Meth. Engng. **106** (2016), 213–240.
4. T. Chaumont-Frelet, *Mixed finite element discretization of acoustic Helmholtz problems with high wavenumbers*, Calcolo **56** (2019).
5. T. Chaumont-Frelet, A. Ern, and M. Vohralík, *On the derivation of guaranteed and p-robust a posteriori error estimates for the Helmholtz equation*, Numer. Math. (2021).
6. T. Chaumont-Frelet and S. Nicaise, *Wavenumber explicit convergence analysis for finite element discretizations of general wave propagation problems*, IMA J. Numer. Anal. **40** (2020), 1503–1543.
7. T. Chaumont-Frelet, S. Nicaise, and D. Pardo, *Finite element approximation of electromagnetic fields using nonfitting meshes for Geophysics*, SIAM J. Numer. Anal. **56** (2018), 2288–2321.
8. A. Ern and J.-L. Guermond, *Analysis of the edge finite element approximation of the Maxwell equations with low regularity solutions*, Comp. Math. Appl. **75** (2018), 918–932.
9. ———, *Finite elements I. Approximation and interpolation*, Texts in Applied Mathematics, vol. 72, Springer Nature Switzerland, 2021.
10. P. Fernandes and G. Gilardi, *Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions*, Math. Meth. Appl. Sci. **47** (1997), 2872–2896.
11. V. Girault and P. A. Raviart, *Finite element methods for Navier-Stokes equations: theory and algorithms*, Springer-Verlag, 1986.
12. F. Ihlenburg and I. Babuška, *Finite element solution of the Helmholtz equation with high wave number. Part II: The h-p-version of the FEM*, SIAM J. Numer. Anal. **34** (1997), 315–358.
13. D. Lafontaine, E. A. Spence, and J. Wunsch, *Wavenumber-explicit convergence of the hp-FEM for the full-space heterogeneous Helmholtz equation with smooth coefficients*, 2020.
14. J. M. Melenk and S. Sauter, *Wavenumber explicit convergence analysis for Galerkin discretizations of the Helmholtz equation*, SIAM J. Numer. Anal. **49** (2011), 1210–1243.
15. J. M. Melenk and S. A. Sauter, *Wavenumber-explicit hp-FEM analysis for Maxwell's equations with transparent boundary conditions*, Found. Comp. Math., in press (2020).
16. S. Sauter and J. Zech, *A posteriori error estimation of hp-dG finite element methods for highly indefinite Helmholtz problems*, SIAM J. Numer. Anal. **53** (2015), 2414–2440.
17. M. Taus, L. Zepeda-Núñez, R. Hewett, and L. Demanet, *Pollution-free and fast hybridizable discontinuous Galerkin solvers for the high-frequency Helmholtz equation*, Proc. SEG annual meeting (Houston), 2017.
18. C. Weber, *A local compactness theorem for Maxwell's equations*, Math. Meth. Appl. Sci. **2** (1980), 12–25.
19. ———, *Regularity theorems for Maxwell's equations*, Math. Meth. Appl. Sci. **3** (1981), 523–536.
20. L. Zhong, S. Shu, G. Wittum, and J. Xu, *Optimal error estimates for Nédélec edge elements for time-harmonic Maxwell's equations*, J. Comp. Math. **27** (2009), 563–572.