p-robust equilibrated flux reconstruction in H(curl) based on local minimizations. Application to a posteriori analysis of the curl-curl problem

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p-robust equilibrated flux reconstruction in $\boldsymbol{H}(\mathrm{curl})$ based on local minimizations. Application to a posteriori analysis of the curl–curl problem*

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Abstract

We present a local construction of $\boldsymbol{H}(\mathrm{curl})$-conforming piecewise polynomials satisfying a prescribed curl constraint. We start from a piecewise polynomial not contained in the $\boldsymbol{H}(\mathrm{curl})$ space but satisfying a suitable orthogonality property. The procedure employs minimizations in vertex patches and the outcome is, up to a generic constant independent of the underlying polynomial degree, as accurate as the best-approximations over the entire local versions of $\boldsymbol{H}(\mathrm{curl})$. This allows to design guaranteed, fully computable, constant-free, and polynomial-degree-robust a posteriori error estimates of Prager–Synge type for Nédélec finite element approximations of the curl–curl problem. A divergence-free decomposition of a divergence-free $\boldsymbol{H}(\mathrm{div})$-conforming piecewise polynomial, relying on over-constrained minimizations in Raviart–Thomas spaces, is the key ingredient. Numerical results illustrate the theoretical developments.

Key words: Sobolev space $\boldsymbol{H}(\mathrm{curl})$, Sobolev space $\boldsymbol{H}(\mathrm{div})$, equilibrated flux reconstruction, p-robustness, a posteriori error estimate, divergence-free decomposition, broken polynomial extension

1 Introduction

A posteriori error estimation by equilibrated flux reconstruction has achieved a great attention for model elliptic problems like the Poisson problem. For an H^1-conforming discretization whose flux is not in $\boldsymbol{H}(\mathrm{div})$, one has to reconstruct a flux in $\boldsymbol{H}(\mathrm{div})$ satisfying a prescribed divergence constraint. To design high-performance algorithms, the procedure must furthermore be localized and can not involve a solution of any supplementary global problem. Then, a guaranteed, fully computable, and constant-free upper bound on the unknown discretization error follows from the equality of Prager and Synge [33]. There are several techniques of such an equilibrated flux reconstruction. Following Ladevèze and Leguillon [28] and Ainsworth and Oden [2], normal fluxes on mesh faces can first be constructed and then lifted elementwise, dual Voronoï-type grids can be employed for local non-overlapping minimizations in $\boldsymbol{H}(\mathrm{div})$ as in Luce and Wohlmuth [30] or Hannukainen *et al.* [26], or a localization by the partition of unity via the finite element hat basis functions can be used for an overlapping combination of best-possible vertex-patch fluxes as in Destuynder and Métivet [14] or Braess and Schöberl [8]. This last approach is conceptual and, as established in Braess *et al.* [7] and Ern and Vohralík [20], it gives estimates robust with respect to the polynomial degree p (henceforth termed p-robust).

In contrast, there is only a handful of results available for the curl–curl problem, where, for an $\boldsymbol{H}(\mathrm{curl})$-conforming discretization whose curl is not in $\boldsymbol{H}(\mathrm{curl})$, one has to locally reconstruct a flux in $\boldsymbol{H}(\mathrm{curl})$ satisfying a prescribed curl constraint. An approach based on patchwise minimizations for the lowest-order case $p = 0$ has been designed in [8]. Its generalization for arbitrary $p \geq 1$, however, turns surprisingly difficult and, to the best of our knowledge, has not been presented yet. Several workarounds appeared in the literature recently, though. A conceptual discussion appears in Licht [29], whereas a

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construction following in spirit [28, 2] has been proposed and analyzed in Gedicke *et al.* [22]. This last approach has been recently modified in Gedicke *et al.* [23] in order to achieve p-robustness. A broken patchwise equilibration procedure that bypasses the Prager–Synge theorem is proposed and proved p-robust in Chaumont-Frelet *et al.* [10]; it relies on smaller edge patches, but the arising estimates are not constant-free.

The purpose of this contribution is to design an equilibrated flux reconstruction in $\boldsymbol{H}(\mathrm{curl})$ employing best-possible local fluxes. In doing so, we rely on localization by the partition of unity via the hat functions and overlapping flux combinations, in generalization of the concept of [8] to arbitrary $p \geq 0$. Consequently, we identify the equivalent in $\boldsymbol{H}(\mathrm{curl})$ of the concept of equilibrated flux reconstruction in $\boldsymbol{H}(\mathrm{div})$ from [14, 8, 7, 19, 20]. This is then used for a posteriori error estimation when the Nédélec (edge) finite elements of arbitrary degree $p \geq 0$ are used for approximation of the curl–curl problem. It leads to guaranteed, fully computable, and constant-free a posteriori error estimates, including higher-order data oscillation terms, that are locally efficient and robust with respect to the polynomial degree p. Our p-robust efficiency proofs are based on the seminal volume and tangential trace p-robust extensions on a single tetrahedron of Costabel and McIntosh [12, Proposition 4.2] and Demkowicz *et al.* [13, Theorem 7.2]. These results were recently extended in [10, Theorem 3.1] into a stable broken polynomial extension for an edge patch of tetrahedra. We formulate it here as a conjecture for a vertex patch of tetrahedra.

An important step in the construction of our estimators is to decompose the given divergence-free right-hand side into locally supported divergence-free contributions. Starting from the available (lowest-order Galerkin) orthogonality property, we propose a multi-stage procedure relying on two central technical results of independent interest: over-constrained minimization in Raviart–Thomas spaces leading to suitable elementwise orthogonality properties, and a decomposition of a divergence-free piecewise polynomial with the above elementwise orthogonality properties into local divergence-free contributions. These issues are related to the developments on divergence-free decompositions in Scheichl [35], Alonso Rodríguez *et al.* [3, 4], and the references therein.

This contribution is organized as follows. Section 2 fixes the setting and notation. Section 3 formulates our abstract assumptions and the (lowest-order Galerkin) orthogonality property. In Section 4, we motivate our approach at the continuous level. Section 5 develops a divergence-free decomposition of the given target curl. Section 6 then presents our equilibrated flux reconstruction based on local minimization in $\boldsymbol{H}(\mathrm{curl})$, as well as its p-robust stability. These abstract results are consequently applied in Section 7 to the Nédélec finite element discretization of the curl–curl problem. Section 8 is dedicated to a numerical illustration. Finally, in Appendices A and B, we present the two central technical results on over-constrained minimization and divergence-free decomposition.

2 Setting and notation

Let $\omega, \Omega \subset \mathbb{R}^3$ be open, Lipschitz polyhedra; Ω will be used to denote the computational domain, while we reserve the notation $\omega \subseteq \Omega$ for its simply connected subsets. Notice that we do not require Ω to be simply connected.

2.1 Sobolev spaces H^1, $\boldsymbol{H}(\mathrm{curl})$, and $\boldsymbol{H}(\mathrm{div})$

We let $L^2(\omega)$ be the space of scalar-valued square-integrable functions defined on ω; we use the notation $\boldsymbol{L}^2(\omega) := [L^2(\omega)]^3$ for vector-valued functions with each component in $L^2(\omega)$. We denote by $\|\cdot\|_\omega$ the $L^2(\omega)$ or $\boldsymbol{L}^2(\omega)$ norm and by $(\cdot,\cdot)_\omega$ the corresponding scalar product; we drop the index when $\omega = \Omega$. We will extensively work with the following three Sobolev spaces: 1) $H^1(\omega)$, the space of scalar-valued $L^2(\omega)$ functions with weak gradients in $\boldsymbol{L}^2(\omega)$, $H^1(\omega) := \{v \in L^2(\omega);\ \nabla v \in \boldsymbol{L}^2(\omega)\}$; 2) $\boldsymbol{H}(\mathrm{curl}, \omega)$, the space of vector-valued $\boldsymbol{L}^2(\omega)$ functions with weak curls in $\boldsymbol{L}^2(\omega)$, $\boldsymbol{H}(\mathrm{curl}, \omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\omega);\ \nabla\times\boldsymbol{v} \in \boldsymbol{L}^2(\omega)\}$; and 3) $\boldsymbol{H}(\mathrm{div}, \omega)$, the space of vector-valued $\boldsymbol{L}^2(\omega)$ functions with weak divergences in $L^2(\omega)$, $\boldsymbol{H}(\mathrm{div}, \omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\omega);\ \nabla\cdot\boldsymbol{v} \in L^2(\omega)\}$. We refer the reader to Adams [1] and Girault and Raviart [24] for an in-depth description of these spaces. Moreover, component-wise $H^1(\omega)$ functions will be denoted by $\boldsymbol{H}^1(\omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\omega);\ \boldsymbol{v}_i \in H^1(\omega),\ i = 1, \ldots, 3\}$. We will employ the notation $\langle\cdot,\cdot\rangle_S$ for the integral product on boundary (sub)sets $S \subset \partial\omega$.

2.2 Tetrahedral mesh, patches of elements, and the hat functions

Let $\mathcal{T}_h$ be a simplicial mesh of the domain Ω, i.e., $\cup_{K\in\mathcal{T}_h} K = \overline{\Omega}$, where any element $K \in \mathcal{T}_h$ is a closed tetrahedron with nonzero measure, and where the intersection of two different tetrahedra is either empty or their common vertex, edge, or face. The shape-regularity parameter of the mesh $\mathcal{T}_h$ is the positive real number $\kappa_{\mathcal{T}_h} := \max_{K\in\mathcal{T}_h} h_K/\rho_K$, where h_K is the diameter of the tetrahedron K and ρ_K is the diameter of the largest ball contained in K. These assumptions are standard, and allow for strongly graded meshes with local refinements. We will use the notation $a \lesssim b$ when there exists a positive constant C only depending on $\kappa_{\mathcal{T}_h}$ such that $a \le Cb$.

We denote the set of vertices of the mesh $\mathcal{T}_h$ by $\mathcal{V}_h$; it is composed of interior vertices lying in Ω and of vertices lying on the boundary $\partial\Omega$. For an element $K \in \mathcal{T}_h$, $\mathcal{F}_K$ denotes the set of its faces and $\mathcal{V}_K$ the set of its vertices. Conversely, for a vertex $\boldsymbol{a} \in \mathcal{V}_h$, $\mathcal{T}_{\boldsymbol{a}}$ denotes the patch of the elements of $\mathcal{T}_h$ that share $\boldsymbol{a}$, and $\omega_{\boldsymbol{a}}$ is the corresponding open subdomain with diameter $h_{\omega_{\boldsymbol{a}}}$. A particular role below will be played by the continuous, piecewise affine "hat" function $\psi^{\boldsymbol{a}}$ which takes value 1 at the vertex $\boldsymbol{a}$ and zero at the other vertices. We note that $\omega_{\boldsymbol{a}}$ corresponds to the support of $\psi^{\boldsymbol{a}}$ and that the functions $\psi^{\boldsymbol{a}}$ form the partition of unity

$$\sum_{\boldsymbol{a}\in\mathcal{V}_h} \psi^{\boldsymbol{a}} = 1. \tag{2.1}$$

We will also need the one-layer-extended patch $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$ and the associated subdomain $\widetilde{\omega}_{\boldsymbol{a}}$, corresponding to the supports of the hat functions $\psi^{\boldsymbol{b}}$ for all vertices $\boldsymbol{b}$ contained in the patch $\mathcal{T}_{\boldsymbol{a}}$.

2.3 Sobolev spaces with partially vanishing traces on Ω and $\omega_{\boldsymbol{a}}$

Let Γ_{D}, Γ_{N} be two disjoint, relatively open, and possibly empty subsets of the computational domain boundary $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma_{\mathrm{D}}} \cup \overline{\Gamma_{\mathrm{N}}}$. We assume in addition that each boundary face of the mesh $\mathcal{T}_h$ lies entirely either in $\overline{\Gamma_{\mathrm{D}}}$ or in $\overline{\Gamma_{\mathrm{N}}}$. Then $H^1_{0,\mathrm{D}}(\Omega)$ is the subspace of $H^1(\Omega)$ formed by functions vanishing on Γ_{D} in the sense of traces, $H^1_{0,\mathrm{D}}(\Omega) := \{v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma_{\mathrm{D}}\}$. Let $\boldsymbol{n}_\Omega$ be the unit normal vector on $\partial\Omega$, outward to Ω. Let $\mathrm{T} = \mathrm{D}$ or N; then $\boldsymbol{H}_{0,\mathrm{T}}(\mathrm{curl},\Omega)$ is the subspace of $\boldsymbol{H}(\mathrm{curl},\Omega)$ formed by functions with vanishing tangential trace on Γ_{T}, $\boldsymbol{H}_{0,\mathrm{T}}(\mathrm{curl},\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},\Omega);\ \boldsymbol{v}\times\boldsymbol{n}_\Omega = 0 \text{ on } \Gamma_{\mathrm{T}}\}$, where $\boldsymbol{v}\times\boldsymbol{n}_\Omega = 0$ on Γ_{T} means that $(\nabla\times\boldsymbol{v}, \boldsymbol{\varphi}) - (\boldsymbol{v}, \nabla\times\boldsymbol{\varphi}) = 0$ for all functions $\boldsymbol{\varphi} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{\varphi}\times\boldsymbol{n}_\Omega = \boldsymbol{0}$ on $\partial\Omega \setminus \Gamma_{\mathrm{T}}$. Finally, $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ is the subspace of $\boldsymbol{H}(\mathrm{div},\Omega)$ formed by functions with vanishing normal trace on Γ_{N}, $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n}_\Omega = 0 \text{ on } \Gamma_{\mathrm{N}}\}$, where $\boldsymbol{v}\cdot\boldsymbol{n}_\Omega = 0$ on Γ_{N} means that $(\boldsymbol{v}, \nabla\varphi) + (\nabla\cdot\boldsymbol{v}, \varphi) = 0$ for all functions $\varphi \in H^1_{0,\mathrm{D}}(\Omega)$. Fernandes and Gilardi [21] present a thorough characterization of tangential (resp. normal) traces of $\boldsymbol{H}(\mathrm{curl},\Omega)$ (resp. $\boldsymbol{H}(\mathrm{div},\Omega)$) on a part of the boundary $\partial\Omega$.

We will also need local spaces on the patch subdomains $\omega_{\boldsymbol{a}}$. Let first $\boldsymbol{a} \in \mathcal{V}_h$ be an interior vertex. Then we set 1) $H^1_*(\omega_{\boldsymbol{a}}) := \{v \in H^1(\omega_{\boldsymbol{a}});\ (v, 1)_{\omega_{\boldsymbol{a}}} = 0\}$, so that $H^1_*(\omega_{\boldsymbol{a}})$ is the subspace of those $H^1(\omega_{\boldsymbol{a}})$ functions whose mean value vanishes; 2) $\boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}}) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},\omega_{\boldsymbol{a}});\ \boldsymbol{v}\times\boldsymbol{n}_{\omega_{\boldsymbol{a}}} = 0 \text{ on } \partial\omega_{\boldsymbol{a}}\}$, where the tangential trace is understood as above; and, similarly, 3) $\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\omega_{\boldsymbol{a}});\ \boldsymbol{v}\cdot\boldsymbol{n}_{\omega_{\boldsymbol{a}}} = 0 \text{ on } \partial\omega_{\boldsymbol{a}}\}$. We will also need 4) $\boldsymbol{H}^\dagger(\mathrm{curl},\omega_{\boldsymbol{a}}) := \boldsymbol{H}(\mathrm{curl},\omega_{\boldsymbol{a}})$. The situation is more subtle for boundary vertices. As a first possibility, if $\boldsymbol{a} \in \Gamma_{\mathrm{N}}$ (i.e., $\boldsymbol{a} \in \mathcal{V}_h$ is a boundary vertex such that all the faces sharing the vertex $\boldsymbol{a}$ lie in Γ_{N}), then the spaces $H^1_*(\omega_{\boldsymbol{a}})$, $\boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}})$, $\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$, and $\boldsymbol{H}^\dagger(\mathrm{curl},\omega_{\boldsymbol{a}})$ are defined as above. Secondly, when $\boldsymbol{a} \in \overline{\Gamma_{\mathrm{D}}}$, then at least one of the faces sharing the vertex $\boldsymbol{a}$ lies in $\overline{\Gamma_{\mathrm{D}}}$, and we denote by γ_{D} the subset of Γ_{D} corresponding to all such faces. In this situation, we let 1) $H^1_*(\omega_{\boldsymbol{a}}) := \{v \in H^1(\omega_{\boldsymbol{a}});\ v = 0 \text{ on } \gamma_{\mathrm{D}}\}$; 2) $\boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}}) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},\omega_{\boldsymbol{a}});\ \boldsymbol{v}\times\boldsymbol{n}_{\omega_{\boldsymbol{a}}} = 0 \text{ on } \partial\omega_{\boldsymbol{a}} \setminus \gamma_{\mathrm{D}}\}$; 3) $\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\omega_{\boldsymbol{a}});\ \boldsymbol{v}\cdot\boldsymbol{n}_{\omega_{\boldsymbol{a}}} = 0 \text{ on } \partial\omega_{\boldsymbol{a}} \setminus \gamma_{\mathrm{D}}\}$; and 4) $\boldsymbol{H}^\dagger(\mathrm{curl},\omega_{\boldsymbol{a}}) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},\omega_{\boldsymbol{a}});\ \boldsymbol{v}\times\boldsymbol{n}_{\omega_{\boldsymbol{a}}} = 0 \text{ on } \gamma_{\mathrm{D}}\}$.

The Poincaré–Friedrichs–Weber inequality, see [21, Proposition 7.4] and more precisely [10, Theorem A.1] for the form of the constant, will be useful: for all vertices $\boldsymbol{a} \in \mathcal{V}_h$ and all vector-valued functions $\boldsymbol{v} \in \boldsymbol{H}^\dagger(\mathrm{curl},\omega_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$ with $\nabla\cdot\boldsymbol{v} = 0$, we have

$$\|\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}} \lesssim h_{\omega_{\boldsymbol{a}}} \|\nabla\times\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}. \tag{2.2}$$

Strictly speaking, the inequality is established in [10, Theorem A.1] for edge patches, but the proof can be easily extended to vertex patches.

2.4 Cohomology space

The space $\boldsymbol{\mathcal{H}}(\Omega, \Gamma_{\mathrm{D}})$ of functions $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ such that $\nabla\times\boldsymbol{v} = \boldsymbol{0}$ and $\nabla\cdot\boldsymbol{v} = 0$ is the "cohomology" space associated with the domain Ω and the partition of its boundary $\partial\Omega = \overline{\Gamma_{\mathrm{D}}} \cup \overline{\Gamma_{\mathrm{N}}}$. When Ω is simply connected and Γ_{D} is connected, this space is trivial; then the conditions associated with it below can be disregarded. In the general case, $\boldsymbol{\mathcal{H}}(\Omega, \Gamma_{\mathrm{D}})$ is finite-dimensional, and its dimension depends on the topology of Ω and Γ_{D}, see [21, 25].

2.5 Piecewise polynomial spaces

Let $q \geq 0$ be an integer. For a single tetrahedron $K \in \mathcal{T}_h$, denote by $\mathcal{P}_q(K)$ the space of scalar-valued polynomials on K of total degree at most q, and by $[\mathcal{P}_q(K)]^3$ the space of vector-valued polynomials on K with each component in $\mathcal{P}_q(K)$. The Nédélec [6, 32] space of degree q on K is then given by

$$\boldsymbol{\mathcal{N}}_q(K) := [\mathcal{P}_q(K)]^3 + \boldsymbol{x} \times [\mathcal{P}_q(K)]^3. \tag{2.3}$$

Similarly, the Raviart–Thomas [6, 34] space of degree q on K is given by

$$\boldsymbol{\mathcal{RT}}_q(K) := [\mathcal{P}_q(K)]^3 + \mathcal{P}_q(K)\boldsymbol{x}. \tag{2.4}$$

We note that (2.3) and (2.4) are equivalent to the writing with a direct sum and only homogeneous polynomials in the second terms. The second term in (2.3) is also equivalently given by homogeneous $(q+1)$-degree polynomials $\boldsymbol{v}_h$ such that $\boldsymbol{x} \cdot \boldsymbol{v}_h(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in K$.

We will below extensively use the broken, piecewise polynomial spaces formed from the above element spaces

$$\begin{aligned} \mathcal{P}_q(\mathcal{T}_h) &:= \{v_h \in L^2(\Omega);\, v_h|_K \in \mathcal{P}_q(K) \quad \forall K \in \mathcal{T}_h\}, \\ \boldsymbol{\mathcal{N}}_q(\mathcal{T}_h) &:= \{\boldsymbol{v}_h \in \boldsymbol{L}^2(\Omega);\, \boldsymbol{v}_h|_K \in \boldsymbol{\mathcal{N}}_q(K) \quad \forall K \in \mathcal{T}_h\}, \\ \boldsymbol{\mathcal{RT}}_q(\mathcal{T}_h) &:= \{\boldsymbol{v}_h \in \boldsymbol{L}^2(\Omega);\, \boldsymbol{v}_h|_K \in \boldsymbol{\mathcal{RT}}_q(K) \quad \forall K \in \mathcal{T}_h\}. \end{aligned}$$

To form the usual finite-dimensional Sobolev subspaces, we will write $\mathcal{P}_q(\mathcal{T}_h) \cap H^1(\Omega)$ (for $q \geq 1$), $\boldsymbol{\mathcal{N}}_q(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{curl}, \Omega)$, $\boldsymbol{\mathcal{RT}}_q(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$ (both for $q \geq 0$), and similarly for the subspaces reflecting the different boundary conditions. The same notation will also be used on the patches $\mathcal{T}_{\boldsymbol{a}}$.

2.6 L^2-orthogonal projectors and the Raviart–Thomas interpolator

For $q \geq 0$, let Π_q denote the $L^2(K)$-orthogonal projector onto $\mathcal{P}_q(K)$ or the $L^2(\Omega)$-orthogonal projector onto $\mathcal{P}_q(\mathcal{T}_h)$. Then, $\boldsymbol{\Pi}_q$ is given componentwise by Π_q.

Let $K \in \mathcal{T}_h$ be a mesh tetrahedron and $\boldsymbol{v} \in [C^1(K)]^3$ be given. Following [6, 34], the canonical q-degree Raviart–Thomas interpolate $\boldsymbol{I}_q^{\mathcal{RT}}(\boldsymbol{v}) \in \boldsymbol{\mathcal{RT}}_q(K)$, $q \geq 0$, is given by

$$\langle \boldsymbol{I}_q^{\mathcal{RT}}(\boldsymbol{v}) \cdot \boldsymbol{n}_K, r_h \rangle_F = \langle \boldsymbol{v} \cdot \boldsymbol{n}_K, r_h \rangle_F \qquad \forall r_h \in \mathcal{P}_q(F), \quad \forall F \in \mathcal{F}_K, \tag{2.5a}$$

$$(\boldsymbol{I}_q^{\mathcal{RT}}(\boldsymbol{v}), \boldsymbol{r}_h)_K = (\boldsymbol{v}, \boldsymbol{r}_h)_K \qquad \forall \boldsymbol{r}_h \in [\mathcal{P}_{q-1}(K)]^3. \tag{2.5b}$$

Less regular functions can be used in (2.5), but $\boldsymbol{v} \in [C^1(K)]^3$ will be sufficient for our purposes; we will actually only employ polynomial $\boldsymbol{v}$. This interpolator crucially satisfies, on the tetrahedron K, the commuting property

$$\nabla \cdot \boldsymbol{I}_q^{\mathcal{RT}}(\boldsymbol{v}) = \mathcal{P}_q(\nabla \cdot \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in [C^1(K)]^3. \tag{2.6}$$

2.7 Geometry of boundary patches

Following [20, Corollary 3.8], we will need the following technical assumption to work with boundary patches. It is satisfied in most cases of practical interest.

(Boundary patch geometry) *Let $\boldsymbol{a} \in \mathcal{V}_h$ be a boundary vertex. Then there are either 1) at most two tetrahedra in the patch $\mathcal{T}_{\boldsymbol{a}}$; or 2) all the faces lying in $\partial\omega_{\boldsymbol{a}}$ and not sharing the vertex $\boldsymbol{a}$ have at least one vertex lying in the interior of the corresponding subdomain.*

3 Abstract assumptions and a patchwise orthogonality

In this manuscript, we work with vector-valued functions $\boldsymbol{j}$, $\boldsymbol{A}$, and $\boldsymbol{A}_h$ that will later be the datum, the solution, and the numerical approximation of the curl–curl problem, cf. Section 7.1 below. In order to make clear the main ideas of our developments, we now identify some central abstract assumptions. Let henceforth $p \geq 0$ be a fixed polynomial degree.

3.1 Abstract assumptions

Our first central assumption is:

Assumption 3.1 (Current density $\boldsymbol{j}$). *Let $\boldsymbol{j}$ be $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$-conforming, divergence-free, and $L^2(\Omega)$-orthogonal to the cohomology space $\boldsymbol{\mathcal{H}}(\Omega,\Gamma_{\mathrm{D}})$, i.e.,*

$$\boldsymbol{j} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega), \tag{3.1a}$$
$$\nabla\cdot\boldsymbol{j} = 0, \tag{3.1b}$$
$$(\boldsymbol{j},\boldsymbol{\varphi}) = 0 \quad \forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{H}}(\Omega,\Gamma_{\mathrm{D}}). \tag{3.1c}$$

Sometimes, to illustrate the main ideas, we will additionally suppose that $\boldsymbol{j}$ is a piecewise p-degree Raviart–Thomas polynomial, $\boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$. Assumption 3.1 equivalently means that $\boldsymbol{j}$ belongs to the range of the curl operator, i.e., there exists $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$ such that $\nabla\times\boldsymbol{v} = \boldsymbol{j}$. Congruently, whenever $\boldsymbol{j}$ satisfies Assumption 3.1, there exist vector fields

$$\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega), \tag{3.2a}$$
$$(\nabla\times\boldsymbol{A},\nabla\times\boldsymbol{v}) = (\boldsymbol{j},\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega), \tag{3.2b}$$

which we will refer to as "magnetic vector potentials". Note that the vector field $\boldsymbol{A}$ is defined up to a curl-free component that does not interfere with the forthcoming developments. A direct consequence of (3.2) is that $\nabla\times\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$ with $\nabla\times(\nabla\times\boldsymbol{A}) = \boldsymbol{j}$, so that $\nabla\times\boldsymbol{A}$ can be taken as the $\boldsymbol{v}$ above.

Our second central assumption is:

Assumption 3.2 (Discrete magnetic vector potential $\boldsymbol{A}_h$). *Let $\boldsymbol{A}_h$ be a piecewise p-degree Nédélec polynomial satisfying a lowest-order Nédélec orthogonality,*

$$\boldsymbol{A}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega), \tag{3.3a}$$
$$(\nabla\times\boldsymbol{A}_h,\nabla\times\boldsymbol{v}_h) = (\boldsymbol{j},\boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_0(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega). \tag{3.3b}$$

3.2 Patchwise orthogonality

Recall that $\psi^{\boldsymbol{a}}$ is the piecewise affine "hat" function associated with the vertex $\boldsymbol{a} \in \mathcal{V}_h$, as well as the notation $H^1_*(\omega_{\boldsymbol{a}})$ from Section 2.3. The following technical result holds true:

Lemma 3.3 (Equivalence of images by the curl operator). *There holds*

$$\nabla\times\left[\operatorname*{span}_{\boldsymbol{a}\in\mathcal{V}_h}\left(\psi^{\boldsymbol{a}}|_{\omega_{\boldsymbol{a}}}\nabla(\mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_*(\omega_{\boldsymbol{a}}))\right)\right] = \nabla\times[\boldsymbol{\mathcal{N}}_0(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)]. \tag{3.4}$$

Proof. Let $\boldsymbol{a} \in \mathcal{V}_h$. For any $q_h \in \mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_*(\omega_{\boldsymbol{a}})$, clearly $\psi^{\boldsymbol{a}}|_{\omega_{\boldsymbol{a}}}\nabla q_h$, extended by zero outside of the patch subdomain $\omega_{\boldsymbol{a}}$, lies in $\boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$ (though in general not in $\boldsymbol{\mathcal{N}}_0(\mathcal{T}_h)$). Moreover, $\nabla\times(\psi^{\boldsymbol{a}}|_{\omega_{\boldsymbol{a}}}\nabla q_h) = \nabla\psi^{\boldsymbol{a}}|_{\omega_{\boldsymbol{a}}}\times\nabla q_h$, which is a piecewise constant vector-valued polynomial on the patch $\mathcal{T}_{\boldsymbol{a}}$ whose extension by zero outside of the patch subdomain $\omega_{\boldsymbol{a}}$ has a continuous normal trace on interfaces and zero normal trace on Γ_{D}. Thus, this extension belongs to the lowest-order divergence-free Raviart–Thomas space, which implies $\nabla\times(\psi^{\boldsymbol{a}}|_{\omega_{\boldsymbol{a}}}\nabla q_h) = \nabla\times\boldsymbol{w}_h$ on $\omega_{\boldsymbol{a}}$ for $\boldsymbol{w}_h$ which belongs to $\boldsymbol{\mathcal{N}}_0(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$. Thus, in (3.4), there holds the inclusion $\subseteq$.

Conversely, following, e.g., Monk [31, Section 5.5.1] or Ern and Guermond [17, Section 15.1], the space $\boldsymbol{\mathcal{N}}_0(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$ is spanned by the set of the "edge functions" $\{\boldsymbol{\psi}^e\}_{e\in\mathcal{E}_h^{\mathrm{D}}}$, where $\mathcal{E}_h^{\mathrm{D}}$ denotes the mesh edges not lying in $\overline{\Gamma_{\mathrm{D}}}$. If e is the edge between vertices $\boldsymbol{a},\boldsymbol{b} \in \mathcal{V}_h$, then $\boldsymbol{\psi}^e = \psi^{\boldsymbol{a}}\nabla\psi^{\boldsymbol{b}} - \psi^{\boldsymbol{b}}\nabla\psi^{\boldsymbol{a}}$. Moreover, if one of the vertices of e lies in $\overline{\Gamma_{\mathrm{D}}}$, we chose the convention that $\boldsymbol{a} \in \overline{\Gamma_{\mathrm{D}}}$, so that we have $(\psi^{\boldsymbol{b}} - c_{\boldsymbol{b}})|_{\omega_{\boldsymbol{a}}} \in H^1_*(\omega_{\boldsymbol{a}})$ for some constant $c_{\boldsymbol{b}}$ in all cases. Now, since $\nabla\times\boldsymbol{\psi}^e = 2\nabla\psi^{\boldsymbol{a}}\times\nabla\psi^{\boldsymbol{b}} = 2\nabla\times(\psi^{\boldsymbol{a}}\nabla\psi^{\boldsymbol{b}}) = 2\nabla\times(\psi^{\boldsymbol{a}}\nabla(\psi^{\boldsymbol{b}} - c_{\boldsymbol{b}}))$, we have found $q_h := (\psi^{\boldsymbol{b}} - c_{\boldsymbol{b}})|_{\omega_{\boldsymbol{a}}}/2 \in \mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_*(\omega_{\boldsymbol{a}})$ such that, after zero extension, $\nabla\times(\psi^{\boldsymbol{a}}|_{\omega_{\boldsymbol{a}}}\nabla q_h) = \nabla\times\boldsymbol{\psi}^e$, and the inclusion $\supseteq$ in (3.4) holds. □

The following alternative to Assumption 3.2 is crucial:

Theorem 3.4 (Patchwise orthogonality). *Let $\boldsymbol{j}$ satisfy Assumption 3.1. Then $\boldsymbol{A}_h$ satisfies Assumption 3.2 if and only if $\boldsymbol{A}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$ and*

$$(\psi^{\boldsymbol{a}}\boldsymbol{j},\nabla q_h)_{\omega_{\boldsymbol{a}}} + (\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h),\nabla q_h)_{\omega_{\boldsymbol{a}}} = 0 \quad \forall q_h \in \mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_*(\omega_{\boldsymbol{a}}),\ \forall \boldsymbol{a} \in \mathcal{V}_h. \tag{3.5}$$

Proof. Since $\nabla\psi^{\boldsymbol{a}}|_{\omega_{\boldsymbol{a}}}\times\nabla q_h = \nabla\times(\psi^{\boldsymbol{a}}|_{\omega_{\boldsymbol{a}}}\nabla q_h)$,

$$(\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h),\nabla q_h)_{\omega_{\boldsymbol{a}}} = -(\nabla\times\boldsymbol{A}_h,\nabla\psi^{\boldsymbol{a}}\times\nabla q_h)_{\omega_{\boldsymbol{a}}} = -(\nabla\times\boldsymbol{A}_h,\nabla\times(\psi^{\boldsymbol{a}}\nabla q_h))_{\omega_{\boldsymbol{a}}}.$$

For any $\boldsymbol{v}\in\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$ such that $\boldsymbol{j}=\nabla\times\boldsymbol{v}$, the Green theorem in turn gives

$$(\psi^{\boldsymbol{a}}\boldsymbol{j},\nabla q_h)_{\omega_{\boldsymbol{a}}} = (\boldsymbol{j},\psi^{\boldsymbol{a}}\nabla q_h)_{\omega_{\boldsymbol{a}}} = (\boldsymbol{v},\nabla\times(\psi^{\boldsymbol{a}}\nabla q_h))_{\omega_{\boldsymbol{a}}}.$$

Finally, again by the Green theorem, for any $\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_0(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$,

$$(\boldsymbol{j},\boldsymbol{v}_h) = (\nabla\times\boldsymbol{v},\boldsymbol{v}_h) = (\boldsymbol{v},\nabla\times\boldsymbol{v}_h).$$

Applying these identities respectively in (3.5) and (3.3b), the assertion follows from Lemma 3.4. □

4 Motivation

Let $\boldsymbol{j}$ satisfy Assumption 3.1. We motivate here our approach by showing how an equilibrated flux $\boldsymbol{h}$ may be constructed locally from any $\boldsymbol{A}$ satisfying (3.2) at the continuous level. These observations are the basis of the actual flux equilibration procedure involving $\boldsymbol{A}_h$ satisfying Assumption 3.2 at the discrete level that we develop in Sections 5 and 6 below. We would in particular like to identify a patchwise construction such that

$$\boldsymbol{h}^{\boldsymbol{a}} \in \boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}}), \tag{4.1a}$$

$$\boldsymbol{h} := \sum_{\boldsymbol{a}\in\mathcal{V}_h}\boldsymbol{h}^{\boldsymbol{a}} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega), \tag{4.1b}$$

$$\nabla\times\boldsymbol{h} = \boldsymbol{j}. \tag{4.1c}$$

At the continuous level, the solution is trivially

$$\boldsymbol{h}^{\boldsymbol{a}} = \psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}).$$

We now rewrite the above definition implicitly. The idea is to introduce

$$\boldsymbol{h}^{\boldsymbol{a}} := \arg\min_{\substack{\boldsymbol{v}\in\boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}})\\ \nabla\times\boldsymbol{v}=\boldsymbol{j}^{\boldsymbol{a}}}} \|\boldsymbol{v}-\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A})\|^2_{\omega_{\boldsymbol{a}}} \qquad \forall\boldsymbol{a}\in\mathcal{V}_h \tag{4.2}$$

with a suitable curl constraint $\boldsymbol{j}^{\boldsymbol{a}}$. Since

$$\nabla\times(\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A})) = \psi^{\boldsymbol{a}}\underbrace{(\nabla\times(\nabla\times\boldsymbol{A}))}_{\boldsymbol{j}} + \underbrace{\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A})}_{\boldsymbol{\theta}^{\boldsymbol{a}}} \tag{4.3}$$

we have

$$\boldsymbol{j}^{\boldsymbol{a}} := \psi^{\boldsymbol{a}}\boldsymbol{j}+\boldsymbol{\theta}^{\boldsymbol{a}}, \qquad \boldsymbol{\theta}^{\boldsymbol{a}} := \nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}). \tag{4.4}$$

Importantly, it holds that

$$\boldsymbol{\theta}^{\boldsymbol{a}} \in \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}), \tag{4.5a}$$

$$\nabla\cdot\boldsymbol{\theta}^{\boldsymbol{a}} = \underbrace{\nabla\times\nabla\psi^{\boldsymbol{a}}}_{0}\cdot(\nabla\times\boldsymbol{A}) - \nabla\psi^{\boldsymbol{a}}\cdot\underbrace{\nabla\times(\nabla\times\boldsymbol{A})}_{\boldsymbol{j}} = -\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}, \tag{4.5b}$$

$$\sum_{\boldsymbol{a}\in\mathcal{V}_h}\boldsymbol{\theta}^{\boldsymbol{a}} = \sum_{\boldsymbol{a}\in\mathcal{V}_h}\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}) = \boldsymbol{0}, \tag{4.5c}$$

where the last property follows by the partition of unity (2.1). Consequently,

$$\boldsymbol{j}^{\boldsymbol{a}} = \psi^{\boldsymbol{a}}\boldsymbol{j}+\boldsymbol{\theta}^{\boldsymbol{a}} \in \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}), \tag{4.6a}$$

$$\nabla\cdot\boldsymbol{j}^{\boldsymbol{a}} = \nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j} + \psi^{\boldsymbol{a}}\underbrace{\nabla\cdot\boldsymbol{j}}_{0} + \nabla\cdot\boldsymbol{\theta}^{\boldsymbol{a}} = 0, \tag{4.6b}$$

$$\sum_{\boldsymbol{a}\in\mathcal{V}_h}\boldsymbol{j}^{\boldsymbol{a}} = \boldsymbol{j}. \tag{4.6c}$$

These auxiliary fields $\boldsymbol{\theta}^{\boldsymbol{a}}$ can also be defined implicitly as the solution to the minimization problems:

$$\boldsymbol{\theta}^{\boldsymbol{a}} := \arg\min_{\substack{\boldsymbol{v}\in\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})\\ \nabla\cdot\boldsymbol{v}=-\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}}} \|\boldsymbol{v}-\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A})\|^2_{\omega_{\boldsymbol{a}}} \qquad \forall\boldsymbol{a}\in\mathcal{V}_h. \tag{4.7}$$

We shall now mimic (4.2), (4.6), and (4.7) at the discrete level.

5 Stable divergence-free patchwise decomposition of the given target curl

The central issue for the construction of an equilibrated flux $\boldsymbol{h}_h$ (which we will detail in Section 6 below) is a divergence-free patchwise decomposition of $\boldsymbol{j}$ in the spirit of (4.6). We address this issue here, upon designing an appropriate discrete variant of (4.7). We will crucially rely on the patchwise orthogonality property (3.5) stemming from Assumption 3.2. This will initially request us to work with the increased polynomial degree

$$p' := \min\{p, 1\}, \tag{5.1}$$

recalling that $p \geq 0$ is fixed in Section 3.

5.1 Patchwise contributions $\boldsymbol{j}_h^{\boldsymbol{a}}$

Definition 5.1 (Patchwise contributions $\boldsymbol{j}_h^{\boldsymbol{a}}$)**.** *Let $\boldsymbol{j}$ and $\boldsymbol{A}_h$ satisfy respectively Assumptions 3.1 and 3.2. Carry out the three following steps:*

1. For all vertices $\boldsymbol{a} \in \mathcal{V}_h$, consider the p'-degree Raviart–Thomas patchwise minimizations

$$\boldsymbol{\theta}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_{p'}(-\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}) \\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_K = (\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h),\boldsymbol{r}_h)_K \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^3,\, \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}_h - \nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}}^2, \tag{5.2}$$

where the constraints concern normal trace and divergence but additionally also the elementwise product with piecewise vector-valued constants.

2. Extending $\boldsymbol{\theta}_h^{\boldsymbol{a}}$ by zero outside of the patch subdomain $\omega_{\boldsymbol{a}}$, set

$$\boldsymbol{\delta}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\theta}_h^{\boldsymbol{a}}. \tag{5.3}$$

For all tetrahedra $K \in \mathcal{T}_h$, consider the $(p+1)$-degree Raviart–Thomas elementwise minimizations:

$$\boldsymbol{\delta}_h^{\boldsymbol{a}}|_K := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_1(K) \\ \nabla\cdot\boldsymbol{v}_h = 0 \\ \boldsymbol{v}_h\cdot\boldsymbol{n}_K = \boldsymbol{I}_1^{\boldsymbol{\mathcal{RT}}}(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)\cdot\boldsymbol{n}_K \text{ on } \partial K}} \|\boldsymbol{v}_h - \boldsymbol{I}_1^{\boldsymbol{\mathcal{RT}}}(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)\|_K^2 \quad \forall \boldsymbol{a} \in \mathcal{V}_K \text{ when } p = 0, \tag{5.4a}$$

$$\boldsymbol{\delta}_h^{\boldsymbol{a}}|_K := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_{p+1}(K) \\ \nabla\cdot\boldsymbol{v}_h = 0 \\ \boldsymbol{v}_h\cdot\boldsymbol{n}_K = \psi^{\boldsymbol{a}}\boldsymbol{\delta}_h\cdot\boldsymbol{n}_K \text{ on } \partial K}} \|\boldsymbol{v}_h - \psi^{\boldsymbol{a}}\boldsymbol{\delta}_h\|_K^2 \quad \forall \boldsymbol{a} \in \mathcal{V}_K \text{ when } p \geq 1. \tag{5.4b}$$

3. For all vertices $\boldsymbol{a} \in \mathcal{V}_h$, define

$$\boldsymbol{j}_h^{\boldsymbol{a}} := \psi^{\boldsymbol{a}}\boldsymbol{j} + \boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}}. \tag{5.5}$$

5.2 Properties of the auxiliary fields $\boldsymbol{\theta}_h^{\boldsymbol{a}}$, $\boldsymbol{\delta}_h$, and $\boldsymbol{\delta}_h^{\boldsymbol{a}}$

We collect here some important results on $\boldsymbol{\theta}_h^{\boldsymbol{a}}$, $\boldsymbol{\delta}_h$, and $\boldsymbol{\delta}_h^{\boldsymbol{a}}$ from (5.2)–(5.4). We start with the following application of the self-standing result on over-constrained minimization in the Raviart–Thomas spaces that we present in Appendix A below. Let

$$\eta_{\mathrm{osc},\boldsymbol{j}}^{\boldsymbol{a}} := \left\{ \sum_{K\in\mathcal{T}_{\boldsymbol{a}}} \left(\frac{h_K}{\pi} \|\boldsymbol{j} - \boldsymbol{\Pi}_{p'}(\boldsymbol{j})\|_K \right)^2 \right\}^{1/2}. \tag{5.6}$$

Lemma 5.2 (Existence, uniqueness, and stability of $\boldsymbol{\theta}_h^{\boldsymbol{a}}$ from (5.2))**.** *There exists a unique solution $\boldsymbol{\theta}_h^{\boldsymbol{a}}$ to problem (5.2) for all $\boldsymbol{a} \in \mathcal{V}_h$. Moreover, it satisfies the stability estimate*

$$\|\boldsymbol{\theta}_h^{\boldsymbol{a}} - \nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{v}\in\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v} = -\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}}} \|\boldsymbol{v} - \nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}} + h_{\omega_{\boldsymbol{a}}}^{-1}\eta_{\mathrm{osc},\boldsymbol{j}}^{\boldsymbol{a}}.$$

Proof. We choose $g^{\boldsymbol{a}} := (-\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j})|_{\omega_{\boldsymbol{a}}}$, $\boldsymbol{\tau}_h^{\boldsymbol{a}} := (\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h))|_{\omega_{\boldsymbol{a}}}$, $q := p$ and verify the assumptions of Theorem A.2 in three steps. Note that $\Pi_{p'}(\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}) = \nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\Pi}_{p'}(\boldsymbol{j})$ and that $\|\nabla\psi^{\boldsymbol{a}}\cdot(\boldsymbol{j} - \boldsymbol{\Pi}_{p'}(\boldsymbol{j}))\|_K \leq \|\nabla\psi^{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}}\|\boldsymbol{j} - \boldsymbol{\Pi}_{p'}(\boldsymbol{j})\|_K \lesssim h_{\omega_{\boldsymbol{a}}}^{-1}\|\boldsymbol{j} - \boldsymbol{\Pi}_{p'}(\boldsymbol{j})\|_K$, where $\|\nabla\psi^{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}} \lesssim h_{\omega_{\boldsymbol{a}}}^{-1}$ follows from the shape regularity of the mesh, which gives rise to $h_{\omega_{\boldsymbol{a}}}^{-1}\eta_{\mathrm{osc},\boldsymbol{j}}^{\boldsymbol{a}}$ from the data oscillation term in Theorem A.2.

Step 1. Assumption (A.1a). From (3.1a), $g^{\boldsymbol{a}} \in L^2(\omega_{\boldsymbol{a}})$, so that the first condition in (A.1a) is satisfied. From (3.3a), in turn, on $\omega_{\boldsymbol{a}}$, it follows that $\nabla\times\boldsymbol{A}_h \in [\mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})]^3$, see, e.g., [6, Corollary 2.3.2], so that $\boldsymbol{\tau}_h^{\boldsymbol{a}} = \nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h) \in [\mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})]^3 \subset \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \subset \boldsymbol{\mathcal{RT}}_{p'}(\mathcal{T}_{\boldsymbol{a}})$. Thus the second (polynomial) condition in (A.1a) is also satisfied.

Step 2. Assumption (A.1b). For vertices $\boldsymbol{a} \in \mathcal{V}_h$ such that $\boldsymbol{a} \notin \overline{\Gamma_{\mathrm{D}}}$, the Green theorem and $\boldsymbol{j} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ from (3.1a) together with $\nabla\cdot\boldsymbol{j} = 0$ from (3.1b) imply

$$-(\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}, 1)_{\omega_{\boldsymbol{a}}} = -(\nabla\psi^{\boldsymbol{a}}, \boldsymbol{j})_{\omega_{\boldsymbol{a}}} = (\psi^{\boldsymbol{a}}, \nabla\cdot\boldsymbol{j})_{\omega_{\boldsymbol{a}}} = 0.$$

Step 3. Assumption (A.1c). For any $q_h \in \mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_*(\omega_{\boldsymbol{a}})$, again the Green theorem yields

$$-(\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}, q_h)_{\omega_{\boldsymbol{a}}} \overset{(3.1b)}{=} -(\nabla\cdot(\psi^{\boldsymbol{a}}\boldsymbol{j}), q_h)_{\omega_{\boldsymbol{a}}} = (\psi^{\boldsymbol{a}}\boldsymbol{j}, \nabla q_h)_{\omega_{\boldsymbol{a}}},$$

so that patchwise orthogonality property (3.5) implies

$$(-\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}, q_h)_{\omega_{\boldsymbol{a}}} + (\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h), \nabla q_h)_{\omega_{\boldsymbol{a}}} = 0. \tag{5.7}$$

□

Similarly, an important part of the results of the following lemma are consequences of Appendix B below:

Lemma 5.3 (Auxiliary correction fields $\boldsymbol{\delta}_h$ and $\boldsymbol{\delta}_h^{\boldsymbol{a}}$)**.** *For $\boldsymbol{\delta}_h$ given by* (5.3)*, there holds*

$$\boldsymbol{\delta}_h \in \boldsymbol{\mathcal{RT}}_{p'}(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \quad \text{and} \quad \nabla\cdot\boldsymbol{\delta}_h = 0. \tag{5.8}$$

In addition, there exists a unique solution $\boldsymbol{\delta}_h^{\boldsymbol{a}}|_K$ to problems (5.4) *for all tetrahedra $K \in \mathcal{T}_h$ and all vertices $\boldsymbol{a} \in \mathcal{V}_K$, yielding the local divergence-free decomposition*

$$\boldsymbol{\delta}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \quad \text{and} \quad \nabla\cdot\boldsymbol{\delta}_h^{\boldsymbol{a}} = 0 \qquad \forall \boldsymbol{a} \in \mathcal{V}_h, \tag{5.9a}$$

$$\boldsymbol{\delta}_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\delta}_h^{\boldsymbol{a}}. \tag{5.9b}$$

Moreover, for all tetrahedra $K \in \mathcal{T}_h$ and all vertices $\boldsymbol{a} \in \mathcal{V}_K$, there holds the local stability estimate

$$\|\boldsymbol{\delta}_h^{\boldsymbol{a}}\|_K \lesssim \|\boldsymbol{\delta}_h\|_K. \tag{5.10}$$

Proof. The patchwise contributions $\boldsymbol{\theta}_h^{\boldsymbol{a}}$ extended by zero outside of the patch subdomains $\omega_{\boldsymbol{a}}$ belong to $\boldsymbol{\mathcal{RT}}_{p'}(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$, so that the first property in (5.8) is immediate. The second property in (5.8) then follows by the divergence constraint in (5.2), the linearity of the projector $\Pi_{p'}$, and the partition of unity (2.1)

$$\nabla\cdot\boldsymbol{\delta}_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \nabla\cdot\boldsymbol{\theta}_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \Pi_{p'}(-\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}) = \Pi_{p'}\left[\sum_{\boldsymbol{a}\in\mathcal{V}_h} -\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}\right] = \Pi_{p'}(0) = 0.$$

Let $K \in \mathcal{T}_h$ and $\boldsymbol{r}_h \in [\mathcal{P}_0(K)]^3$ be fixed. Then definition (5.3), which gives $\boldsymbol{\delta}_h|_K = \sum_{\boldsymbol{b}\in\mathcal{V}_K} \boldsymbol{\theta}_h^{\boldsymbol{b}}$, the partition of unity (2.1), which implies $\sum_{\boldsymbol{b}\in\mathcal{V}_K}(\nabla\psi^{\boldsymbol{b}}\times(\nabla\times\boldsymbol{A}_h))|_K = \boldsymbol{0}$, and the elementwise orthogonality constraint in (5.2) lead to

$$(\boldsymbol{\delta}_h, \boldsymbol{r}_h)_K = \sum_{\boldsymbol{b}\in\mathcal{V}_K} (\boldsymbol{\theta}_h^{\boldsymbol{b}} - \nabla\psi^{\boldsymbol{b}}\times(\nabla\times\boldsymbol{A}_h), \boldsymbol{r}_h)_K = 0.$$

This is condition (B.2). Thus, Theorem B.1 can be employed, where we choose $q := p'$ together with $q' := p'$ for $p = 0$ and $q' := p' + 1$ for $p \geq 1$. This implies the existence and uniqueness of solutions $\boldsymbol{\delta}_h^{\boldsymbol{a}}|_K$ to problems (5.4), the properties (5.9a), the decomposition (5.9b), and the stability bound (5.10). Note in particular that we only employ (B.6b) with $q' = q$ in the lowest-order case with $q = 1$, so there is indeed no polynomial degree dependence in (5.10). □

5.3 Decomposition of the current density $\boldsymbol{j}$ and its stability

Combining the above developments brings us to the first main result of this section:

Theorem 5.4 (Divergence-free patchwise decomposition of $\boldsymbol{j}$). *Let $\boldsymbol{j}$ and $\boldsymbol{A}_h$ satisfy respectively Assumptions 3.1 and 3.2. Let $\boldsymbol{j}_h^{\boldsymbol{a}}$ be given by Definition 5.1 for all vertices $\boldsymbol{a} \in \mathcal{V}_h$. Then*

$$\boldsymbol{j}_h^{\boldsymbol{a}} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}), \tag{5.11a}$$

$$\nabla\cdot \boldsymbol{j}_h^{\boldsymbol{a}} = \nabla\psi^{\boldsymbol{a}}\cdot(\boldsymbol{j} - \boldsymbol{\Pi}_{p'}(\boldsymbol{j})), \tag{5.11b}$$

$$\sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{j}_h^{\boldsymbol{a}} = \boldsymbol{j}, \tag{5.11c}$$

where the extension of $\boldsymbol{j}_h^{\boldsymbol{a}}$ by zero outside of the patch subdomain $\omega_{\boldsymbol{a}}$ is understood in the last two properties. Moreover, when $\boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ is piecewise polynomial,

$$\boldsymbol{j}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}), \tag{5.12a}$$

$$\nabla\cdot \boldsymbol{j}_h^{\boldsymbol{a}} = 0. \tag{5.12b}$$

Proof. Property (5.11a) is immediate since $\psi^{\boldsymbol{a}}\boldsymbol{j} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ in view of assumption (3.1a), from (5.2) which gives $\boldsymbol{\theta}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$, and from the first property in (5.9a). Property (5.11b) follows since $\nabla\cdot(\psi^{\boldsymbol{a}}\boldsymbol{j}) = \nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}$ in view of assumption (3.1b) and using $\nabla\cdot\boldsymbol{\theta}_h^{\boldsymbol{a}} = \Pi_{p'}(-\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}) = -\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\Pi}_{p'}(\boldsymbol{j})$ from (5.2) and $\nabla\cdot\boldsymbol{\delta}_h^{\boldsymbol{a}} = 0$, which is the second property in (5.9a). Finally, (5.11c) follows from the partition of unity (2.1) which gives $\sum_{\boldsymbol{a}\in\mathcal{V}_h} \psi^{\boldsymbol{a}}\boldsymbol{j} = \boldsymbol{j}$ together with (5.3) and (5.9b). When $\boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$, (5.12) immediately follows from (5.11) and the fact that $\psi^{\boldsymbol{a}}\boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$. □

Recall from Section 2.2 that $\widetilde{\omega}_{\boldsymbol{a}}$ is the extended vertex patch. The second main result of this section concerns the appropriate stability of the above decomposition. Let $\widetilde{\eta}_{\mathrm{osc},\boldsymbol{j}}^{\boldsymbol{a}}$ be defined as $\eta_{\mathrm{osc},\boldsymbol{j}}^{\boldsymbol{a}}$ in (5.6) but on the extended patch $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$.

Theorem 5.5 (Stability of the contributions $\boldsymbol{j}_h^{\boldsymbol{a}}$). *Let $\boldsymbol{j}$ and $\boldsymbol{A}_h$ satisfy respectively Assumptions 3.1 and 3.2, and consider any $\boldsymbol{A}$ satisfying (3.2). For all vertices $\boldsymbol{a} \in \mathcal{V}_h$, let $\boldsymbol{j}_h^{\boldsymbol{a}}$ be given by Definition 5.1 and let, as in (4.4),*

$$\boldsymbol{j}^{\boldsymbol{a}} := \psi^{\boldsymbol{a}}\boldsymbol{j} + \nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}).$$

Then

$$\|\boldsymbol{j}^{\boldsymbol{a}} - \boldsymbol{j}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \lesssim h_{\omega_{\boldsymbol{a}}}^{-1}\left[\|\nabla\times(\boldsymbol{A} - \boldsymbol{A}_h)\|_{\widetilde{\omega}_{\boldsymbol{a}}} + \widetilde{\eta}_{\mathrm{osc},\boldsymbol{j}}^{\boldsymbol{a}}\right]. \tag{5.13}$$

Proof. We develop

$$\boldsymbol{j}^{\boldsymbol{a}} - \boldsymbol{j}_h^{\boldsymbol{a}} = \nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}) - \boldsymbol{\theta}_h^{\boldsymbol{a}} + \boldsymbol{\delta}_h^{\boldsymbol{a}} = \nabla\psi^{\boldsymbol{a}}\times(\nabla\times(\boldsymbol{A} - \boldsymbol{A}_h)) - (\boldsymbol{\theta}_h^{\boldsymbol{a}} - \nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h)) + \boldsymbol{\delta}_h^{\boldsymbol{a}}.$$

For the first term above, we immediately see

$$\|\nabla\psi^{\boldsymbol{a}}\times(\nabla\times(\boldsymbol{A} - \boldsymbol{A}_h))\|_{\omega_{\boldsymbol{a}}} \leq \|\nabla\psi^{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}}\|\nabla\times(\boldsymbol{A} - \boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}}.$$

For the second term above, we employ Lemma 5.2 with $\boldsymbol{v} = \nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A})$, which lies in $\boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ with divergence equal to $-\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}$ by virtue of (4.5), which leads to

$$\|\boldsymbol{\theta}_h^{\boldsymbol{a}} - \nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}} \lesssim \|\nabla\psi^{\boldsymbol{a}}\times(\nabla\times(\boldsymbol{A} - \boldsymbol{A}_h))\|_{\omega_{\boldsymbol{a}}} + h_{\omega_{\boldsymbol{a}}}^{-1}\eta_{\mathrm{osc},\boldsymbol{j}}^{\boldsymbol{a}}.$$

For the last term, we first recall (5.10), i.e., $\|\boldsymbol{\delta}_h^{\boldsymbol{a}}\|_K \lesssim \|\boldsymbol{\delta}_h\|_K$ for every $K \in \mathcal{T}_{\boldsymbol{a}}$. Now definition (5.3), the partition of unity (2.1), and the triangle inequality imply

$$\|\boldsymbol{\delta}_h\|_K = \left\|\sum_{\boldsymbol{b}\in\mathcal{V}_K} (\boldsymbol{\theta}_h^{\boldsymbol{b}} - \nabla\psi^{\boldsymbol{b}}\times(\nabla\times\boldsymbol{A}_h))\right\|_K \leq \sum_{\boldsymbol{b}\in\mathcal{V}_K} \|\boldsymbol{\theta}_h^{\boldsymbol{b}} - \nabla\psi^{\boldsymbol{b}}\times(\nabla\times\boldsymbol{A}_h)\|_{\omega_{\boldsymbol{b}}},$$

which extends by one layer beyond the patch $\omega_{\boldsymbol{a}}$. The shape regularity of the mesh ensures that $\|\nabla\psi^{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}} \lesssim h_{\omega_{\boldsymbol{a}}}^{-1}$ and $\|\nabla\psi^{\boldsymbol{b}}\|_{\infty,\omega_{\boldsymbol{b}}} \simeq \|\nabla\psi^{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}}$ for all vertices $\boldsymbol{b}$ in the patch $\mathcal{T}_{\boldsymbol{a}}$. Hence, (5.13) follows upon combining the above developments. □

5.4 Remarks

Several remarks are in order:

1. Step 1 of Definition 5.1 mimics (4.7) at the discrete level. The auxiliary field $\boldsymbol{\theta}_h^{\boldsymbol{a}}$ from (5.2) in particular satisfies $\boldsymbol{\theta}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ and $\nabla\cdot\boldsymbol{\theta}_h^{\boldsymbol{a}} = \Pi_{p'}(-\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j})$, which fully mimics (4.5a) and (4.5b). Unfortunately, $\sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\theta}_h^{\boldsymbol{a}} \neq \boldsymbol{0}$, which would mimic (4.5c).
2. Step 2 of Definition 5.1 decomposes $\boldsymbol{\delta}_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\theta}_h^{\boldsymbol{a}}$, which should ideally be zero, such that a discrete equivalent of (4.5c) is also satisfied. Indeed, it follows from (5.3) together with (5.9a) that

$$\begin{aligned} \boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}} &\in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}), \\ \nabla\cdot(\boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}}) &= \Pi_{p'}(-\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}), \\ \sum_{\boldsymbol{a}\in\mathcal{V}_h} (\boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}}) &= \boldsymbol{\delta}_h - \boldsymbol{\delta}_h = \boldsymbol{0}. \end{aligned}$$

   Thus, $\boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}}$ (rather than just $\boldsymbol{\theta}_h^{\boldsymbol{a}}$) is the discrete counterpart of $\boldsymbol{\theta}^{\boldsymbol{a}}$ from (4.5).
3. Step 3 of Definition 5.1 finally materializes (4.6a) at the discrete level. Thus, the divergence-free current density $\boldsymbol{j}$ is decomposed into patchwise (up to data oscillation) divergence-free contributions via (5.11) of Theorem 5.4, which exhibits all properties of decomposition (4.6).
4. Property (5.13) from Theorem 5.5 shows that the local decomposition from Theorem 5.4 compares in a p-robust way to the continuous-level decomposition (4.6), up to data oscillation.
5. In comparison with (4.7), (5.2) also contains a constraint on the elementwise product with piecewise vector-valued constants. This is crucial for the existence and uniqueness of $\boldsymbol{\delta}_h^{\boldsymbol{a}}$ from (5.4), which in turn enables the decomposition. As for the additional constraint in (5.2), it is only possible to add it thanks to the patchwise orthogonality property (3.5), which we recall, is equivalent to Assumption 3.2.
6. The divergence constraint in (5.2) together with (5.7) and the Green theorem imply that $(\boldsymbol{\theta}_h^{\boldsymbol{a}} - \nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h), \nabla q_h)_{\omega_{\boldsymbol{a}}} = 0$ for all $q_h \in \mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_*(\omega_{\boldsymbol{a}})$, giving an insight for the constraint on the elementwise product with piecewise constants in (5.2).
7. Definition (5.3) could be equivalently written as $\boldsymbol{\delta}_h|_K := \sum_{\boldsymbol{b}\in\mathcal{V}_K} \boldsymbol{\theta}_h^{\boldsymbol{b}}|_K$ for the given tetrahedron $K \in \mathcal{T}_h$ of (5.4). Thus, the entire procedure of Definition 5.1 is local.

6 Equilibrated flux reconstruction based on local patchwise minimizations in $\boldsymbol{H}(\mathrm{curl})$ and its p-robust stability

In this section, we identify an appropriate discrete variant of (4.1)–(4.2). We will for this purpose rely on the patchwise contributions $\boldsymbol{j}_h^{\boldsymbol{a}}$ of Definition 5.1 that lead to the decomposition of the current density $\boldsymbol{j}$ following Theorem 5.4. Let $\boldsymbol{V}_{p+1}^{\boldsymbol{a}} := \{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}});\ \nabla\cdot\boldsymbol{v}_h = 0\}$ and let $\boldsymbol{\Pi}_{\boldsymbol{V}_{p+1}^{\boldsymbol{a}}}$ be the $\boldsymbol{L}^2(\omega_{\boldsymbol{a}})$-orthogonal projection onto $\boldsymbol{V}_{p+1}^{\boldsymbol{a}}$; this projector is actually never needed when $\boldsymbol{j}$ is piecewise polynomial, $\boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$, where (5.12) holds. We start immediately with:

Definition 6.1 (Equilibrated flux reconstruction based on local minimization in $\boldsymbol{H}(\mathrm{curl})$)**.** *Let $\boldsymbol{j}$ and $\boldsymbol{A}_h$ satisfy respectively Assumptions 3.1 and 3.2 and let, for all vertices $\boldsymbol{a} \in \mathcal{V}_h$, $\boldsymbol{j}_h^{\boldsymbol{a}}$ be given by Definition 5.1. Consider the patchwise minimizations*

$$\boldsymbol{h}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{curl}, \omega_{\boldsymbol{a}}) \\ \nabla\times\boldsymbol{v}_h = \boldsymbol{\Pi}_{\boldsymbol{V}_{p+1}^{\boldsymbol{a}}}(\boldsymbol{j}_h^{\boldsymbol{a}})}} \|\boldsymbol{v}_h - \psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}_h)\|^2_{\omega_{\boldsymbol{a}}}. \tag{6.1a}$$

Extending $\boldsymbol{h}_h^{\boldsymbol{a}}$ by zero outside of $\omega_{\boldsymbol{a}}$, define

$$\boldsymbol{h}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{h}_h^{\boldsymbol{a}}. \tag{6.1b}$$

Since either $\boldsymbol{j}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ with $\nabla\cdot\boldsymbol{j}_h^{\boldsymbol{a}} = 0$ from (5.12), or the projector $\boldsymbol{\Pi}_{\boldsymbol{V}_{p+1}^{\boldsymbol{a}}}$ enforces this property for non-polynomial $\boldsymbol{j}$, there exists a unique solution $\boldsymbol{h}_h^{\boldsymbol{a}}$ to (6.1a) by standard arguments, see, e.g., [6]. Moreover, the following equilibration property follows immediately from (5.11c):

Theorem 6.2 (Equilibrated flux reconstruction)**.** *The equilibrated flux reconstruction $\boldsymbol{h}_h$ from Definition 6.1 satisfies*

$$\begin{aligned} \boldsymbol{h}_h &\in \boldsymbol{\mathcal{N}}_{p+1}(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega), && (6.2a) \\ \nabla\times\boldsymbol{h}_h &= \boldsymbol{j} \quad \textit{when } \boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega). && (6.2b) \end{aligned}$$

We now present the following conjecture:

Conjecture 6.3 (p-robust stability)**.** *For a vertex* $\boldsymbol{a} \in \mathcal{V}_h$*, let* $\boldsymbol{A}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}(\mathrm{curl}, \omega_{\boldsymbol{a}})$ *and* $\boldsymbol{j}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ *with* $\nabla \cdot \boldsymbol{j}_h^{\boldsymbol{a}} = 0$ *be given. Then*

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{curl}, \omega_{\boldsymbol{a}}) \\ \nabla \times \boldsymbol{v}_h = \boldsymbol{j}_h^{\boldsymbol{a}}}} \|\boldsymbol{v}_h - \psi^{\boldsymbol{a}}(\nabla \times \boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl}, \omega_{\boldsymbol{a}}) \\ \nabla \times \boldsymbol{v} = \boldsymbol{j}_h^{\boldsymbol{a}}}} \|\boldsymbol{v} - \psi^{\boldsymbol{a}}(\nabla \times \boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}}. \tag{6.3}$$

On a single tetrahedron K in place of the vertex patch $\mathcal{T}_{\boldsymbol{a}}$, Conjecture 6.3 follows by the seminal contributions of Costabel and McIntosh [12, Proposition 4.2] and Demkowicz *et al.* [13, Theorem 7.2], see [9, Theorem 2]. Moreover, on an edge patch, such a result has been recently established in [10, Theorem 3.1]. The further extension to a vertex patch can be achieved along the lines of [20]. Since it is rather technical and lengthy, we prefer to state (6.3) as a conjecture and report the proof of Conjecture 6.3 elsewhere. We mention that a non p-robust version of (6.3), where the hidden constant is additionally allowed to depend on the polynomial degree p, trivially holds by usual scaling arguments.

Let $\widetilde{\eta}^{\boldsymbol{a}}_{\mathrm{osc},\boldsymbol{j}}$ be defined as $\eta^{\boldsymbol{a}}_{\mathrm{osc},\boldsymbol{j}}$ in (5.6) but on the extended patch $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$ and define

$$\eta^{\boldsymbol{a}}_{\mathrm{osc},\boldsymbol{j}_h^{\boldsymbol{a}}} := h_{\omega_{\boldsymbol{a}}} \|\boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}(\boldsymbol{j}_h^{\boldsymbol{a}}) - \boldsymbol{j}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}. \tag{6.4}$$

Under Conjecture 6.3, the construction of Definition 6.1 is stable as follows:

Theorem 6.4 (p-robust stability of $\boldsymbol{h}_h^{\boldsymbol{a}}$)**.** *Let* $\boldsymbol{j}$ *and* $\boldsymbol{A}_h$ *satisfy respectively Assumptions 3.1 and 3.2, and consider any* $\boldsymbol{A}$ *satisfying* (3.2)*. For all vertices* $\boldsymbol{a} \in \mathcal{V}_h$*, let* $\boldsymbol{j}_h^{\boldsymbol{a}}$ *be given by Definition 5.1 and* $\boldsymbol{h}_h^{\boldsymbol{a}}$ *by Definition 6.1. Admit Conjecture 6.3. Then*

$$\|\boldsymbol{h}_h^{\boldsymbol{a}} - \psi^{\boldsymbol{a}}(\nabla \times \boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}} \lesssim \|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\|_{\widetilde{\omega}_{\boldsymbol{a}}} + \eta^{\boldsymbol{a}}_{\mathrm{osc},\boldsymbol{j}_h^{\boldsymbol{a}}} + \widetilde{\eta}^{\boldsymbol{a}}_{\mathrm{osc},\boldsymbol{j}}.$$

Proof. Fix a vertex $\boldsymbol{a} \in \mathcal{V}_h$ and use $\boldsymbol{j}^{\boldsymbol{a}} = \psi^{\boldsymbol{a}}\boldsymbol{j} + \nabla\psi^{\boldsymbol{a}} \times (\nabla \times \boldsymbol{A}) = \nabla \times (\psi^{\boldsymbol{a}}(\nabla \times \boldsymbol{A}))$ as in Theorem 5.5 and (4.3), which implies $(\boldsymbol{j}^{\boldsymbol{a}}, \boldsymbol{v})_{\omega_{\boldsymbol{a}}} = (\psi^{\boldsymbol{a}}(\nabla \times \boldsymbol{A}), \nabla \times \boldsymbol{v})_{\omega_{\boldsymbol{a}}}$ for any $\boldsymbol{v} \in \boldsymbol{H}^{\dagger}(\mathrm{curl}, \omega_{\boldsymbol{a}})$. Then Conjecture 6.3 and a primal–dual equivalence as in, e.g., [10, Lemma 5.5] imply

$$\begin{aligned}
\|\boldsymbol{h}_h^{\boldsymbol{a}} - \psi^{\boldsymbol{a}}(\nabla \times \boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}} &\lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl}, \omega_{\boldsymbol{a}}) \\ \nabla \times \boldsymbol{v} = \boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}(\boldsymbol{j}_h^{\boldsymbol{a}})}} \|\boldsymbol{v} - \psi^{\boldsymbol{a}}(\nabla \times \boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}} \\
&= \sup_{\substack{\boldsymbol{v} \in \boldsymbol{H}^{\dagger}(\mathrm{curl}, \omega_{\boldsymbol{a}}) \\ \|\nabla \times \boldsymbol{v}\|_{\omega_{\boldsymbol{a}}} = 1}} \left\{ (\boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}(\boldsymbol{j}_h^{\boldsymbol{a}}), \boldsymbol{v})_{\omega_{\boldsymbol{a}}} - (\psi^{\boldsymbol{a}}(\nabla \times \boldsymbol{A}_h), \nabla \times \boldsymbol{v})_{\omega_{\boldsymbol{a}}} \right\} \\
&\leq \sup_{\substack{\boldsymbol{v} \in \boldsymbol{H}^{\dagger}(\mathrm{curl}, \omega_{\boldsymbol{a}}) \\ \|\nabla \times \boldsymbol{v}\|_{\omega_{\boldsymbol{a}}} = 1}} (\boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}(\boldsymbol{j}_h^{\boldsymbol{a}}) - \boldsymbol{j}^{\boldsymbol{a}}, \boldsymbol{v})_{\omega_{\boldsymbol{a}}} + \|\psi^{\boldsymbol{a}}(\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h))\|_{\omega_{\boldsymbol{a}}} \\
&\leq \sup_{\substack{\boldsymbol{v} \in \boldsymbol{H}^{\dagger}(\mathrm{curl}, \omega_{\boldsymbol{a}}) \\ \|\nabla \times \boldsymbol{v}\|_{\omega_{\boldsymbol{a}}} = 1}} (\boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}(\boldsymbol{j}_h^{\boldsymbol{a}} - \boldsymbol{j}^{\boldsymbol{a}}, \boldsymbol{v})_{\omega_{\boldsymbol{a}}} + \|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}}.
\end{aligned}$$

We are thus left to treat the first term above.

Fix $\boldsymbol{v} \in \boldsymbol{H}^{\dagger}(\mathrm{curl}, \omega_{\boldsymbol{a}})$ with $\|\nabla \times \boldsymbol{v}\|_{\omega_{\boldsymbol{a}}} = 1$. Consider $q \in H^1_*(\omega_{\boldsymbol{a}})$ such that

$$(\nabla q, \nabla w)_{\omega_{\boldsymbol{a}}} = (\boldsymbol{v}, \nabla w)_{\omega_{\boldsymbol{a}}} \qquad \forall w \in H^1_*(\omega_{\boldsymbol{a}}).$$

Then $\tilde{\boldsymbol{v}} := \boldsymbol{v} - \nabla q$ lies in both $\boldsymbol{H}^{\dagger}(\mathrm{curl}, \omega_{\boldsymbol{a}})$ and $\boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ and is divergence-free, $\nabla \cdot \tilde{\boldsymbol{v}} = 0$. Thus, the Poincaré–Friedrichs–Weber inequality (2.2), implies

$$\|\tilde{\boldsymbol{v}}\|_{\omega_{\boldsymbol{a}}} \lesssim h_{\omega_{\boldsymbol{a}}} \|\nabla \times \tilde{\boldsymbol{v}}\|_{\omega_{\boldsymbol{a}}} = h_{\omega_{\boldsymbol{a}}} \|\nabla \times \boldsymbol{v}\|_{\omega_{\boldsymbol{a}}} = h_{\omega_{\boldsymbol{a}}}. \tag{6.5}$$

Note that $\boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}(\boldsymbol{j}_h^{\boldsymbol{a}}) - \boldsymbol{j}^{\boldsymbol{a}} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ with $\nabla \cdot (\boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}(\boldsymbol{j}_h^{\boldsymbol{a}}) - \boldsymbol{j}^{\boldsymbol{a}}) = 0$; indeed, this follows from (4.6a)–(4.6b) together with either (5.12) if $\boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ or the use of the projector $\boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}$ in general. Thus, the Green theorem gives

$$(\boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}(\boldsymbol{j}_h^{\boldsymbol{a}}) - \boldsymbol{j}^{\boldsymbol{a}}, \nabla q)_{\omega_{\boldsymbol{a}}} = 0. \tag{6.6}$$

Thus, by virtue of the Cauchy–Schwarz inequality, (6.5), and the triangle inequality,

$$\begin{aligned}
(\boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}(\boldsymbol{j}_h^{\boldsymbol{a}}) - \boldsymbol{j}^{\boldsymbol{a}}, \boldsymbol{v})_{\omega_{\boldsymbol{a}}} &= (\boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}(\boldsymbol{j}_h^{\boldsymbol{a}}) - \boldsymbol{j}^{\boldsymbol{a}}, \tilde{\boldsymbol{v}})_{\omega_{\boldsymbol{a}}} \leq \|\boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}(\boldsymbol{j}_h^{\boldsymbol{a}}) - \boldsymbol{j}^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \|\tilde{\boldsymbol{v}}\|_{\omega_{\boldsymbol{a}}} \\
&\leq h_{\omega_{\boldsymbol{a}}} \left[\|\boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}(\boldsymbol{j}_h^{\boldsymbol{a}}) - \boldsymbol{j}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} + \|\boldsymbol{j}_h^{\boldsymbol{a}} - \boldsymbol{j}^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \right],
\end{aligned}$$

and we conclude by (5.13) from Theorem 5.5. □

7 Guaranteed, fully computable, constant-free, and p-robust a posteriori error estimates for the curl–curl problem

We apply in this section the previous results to a posteriori error analysis of the curl–curl problem.

7.1 The curl–curl problem

Let the current density $\boldsymbol{j}$ satisfy Assumption 3.1. In the curl–curl problem, one looks for the magnetic vector potential $\boldsymbol{A} : \Omega \to \mathbb{R}^3$ such that

$$\nabla\times(\nabla\times\boldsymbol{A}) = \boldsymbol{j}, \quad \nabla\cdot\boldsymbol{A} = 0 \quad \text{in } \Omega, \tag{7.1a}$$

$$\boldsymbol{A}\times\boldsymbol{n}_\Omega = \boldsymbol{0}, \quad \text{on } \Gamma_\mathrm{D}, \tag{7.1b}$$

$$(\nabla\times\boldsymbol{A})\times\boldsymbol{n}_\Omega = \boldsymbol{0}, \quad \boldsymbol{A}\cdot\boldsymbol{n}_\Omega = 0, \quad \text{on } \Gamma_\mathrm{N}, \tag{7.1c}$$

with the additional requirement that $(\boldsymbol{A}, \boldsymbol{\varphi}) = 0$ for all $\boldsymbol{\varphi}$ from the cohomology space $\boldsymbol{\mathcal{H}}(\Omega, \Gamma_\mathrm{D})$ invoked in Section 2.4 to ensure uniqueness. Introducing $\boldsymbol{K}(\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega);\ \nabla\times\boldsymbol{v} = \boldsymbol{0}\}$, the weak formulation of problem (7.1), cf., e.g., [6], consists in finding a pair $(\boldsymbol{A}, \boldsymbol{q}) \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega) \times \boldsymbol{K}(\Omega)$ such that

$$(\boldsymbol{A}, \boldsymbol{\varphi}) = 0 \qquad \forall \boldsymbol{\varphi} \in \boldsymbol{K}(\Omega) \tag{7.2a}$$

$$(\nabla\times\boldsymbol{A}, \nabla\times\boldsymbol{v}) + (\boldsymbol{q}, \boldsymbol{v}) = (\boldsymbol{j}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega). \tag{7.2b}$$

Picking the test function $\boldsymbol{v} = \boldsymbol{q}$ in (7.2b), we see that $\boldsymbol{q} = \boldsymbol{0}$, so that $\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$ is such that

$$(\nabla\times\boldsymbol{A}, \nabla\times\boldsymbol{v}) = (\boldsymbol{j}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega). \tag{7.3}$$

From (7.3), $\boldsymbol{A}$ satisfies (3.2).

7.2 Nédélec finite element approximation

For the integer $p \geq 0$, let the Nédélec finite element space be given by $\boldsymbol{V}_h := \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$. The subspace $\boldsymbol{K}_h := \{\boldsymbol{v}_h \in \boldsymbol{V}_h;\ \nabla\times\boldsymbol{v}_h = \boldsymbol{0}\}$ is simply $\nabla(\mathcal{P}_{p+1}(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega))$ when Ω is simply connected and Γ_D is connected, and can be readily identified by introducing "cuts" in the mesh mimicking the construction of the cohomology space $\boldsymbol{\mathcal{H}}(\Omega, \Gamma_\mathrm{D})$, see [25, Chapter 6]. The finite element approximation of (7.2) is a pair $(\boldsymbol{A}_h, \boldsymbol{q}_h) \in \boldsymbol{V}_h \times \boldsymbol{K}_h$ such that

$$(\boldsymbol{A}_h, \boldsymbol{\varphi}_h) = 0 \qquad \forall \boldsymbol{\varphi}_h \in \boldsymbol{K}_h, \tag{7.4a}$$

$$(\nabla\times\boldsymbol{A}_h, \nabla\times\boldsymbol{v}_h) + (\boldsymbol{q}_h, \boldsymbol{v}_h) = (\boldsymbol{j}, \boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{7.4b}$$

Observing that $\boldsymbol{K}_h \subset \boldsymbol{K}$, this actually leads to $\boldsymbol{A}_h \in \boldsymbol{V}_h$ such that

$$(\nabla\times\boldsymbol{A}_h, \nabla\times\boldsymbol{v}_h) = (\boldsymbol{j}, \boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{7.5}$$

so that in particular $\boldsymbol{A}_h$ satisfies Assumption 3.2.

7.3 Guaranteed upper bound

Let us recall from [11, Theorems 3.4 and 3.5], [27, Theorem 2.1] and the discussion in [10, Section 3.2.1] that there exists a constant C_L such that for all $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$, there exists $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{w}$ vanishes on Γ_D in the sense of traces (so that $\boldsymbol{w} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$), $\nabla\times\boldsymbol{w} = \nabla\times\boldsymbol{v}$, and

$$\|\nabla\boldsymbol{w}\| \leq C_\mathrm{L}\|\nabla\times\boldsymbol{v}\|. \tag{7.6}$$

Moreover, there hold the Poincaré–Friedrichs inequalities

$$\|\boldsymbol{v} - \boldsymbol{\Pi}_0(\boldsymbol{v})\|_{\omega_{\boldsymbol{a}}} \leq C_{\mathrm{PF},\omega_{\boldsymbol{a}}} h_{\omega_{\boldsymbol{a}}} \|\nabla\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}} \quad \text{and} \quad \|\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}} \leq C_{\mathrm{PF},\omega_{\boldsymbol{a}}} h_{\omega_{\boldsymbol{a}}} \|\nabla\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}} \tag{7.7}$$

for respectively 1) a vertex $\boldsymbol{a} \in \mathcal{V}_h$ which is interior or boundary such that all the faces sharing the vertex $\boldsymbol{a}$ lie in Γ_N, together with $\boldsymbol{v} \in \boldsymbol{H}^1(\omega_{\boldsymbol{a}})$, and 2) the Dirichlet vertices $\boldsymbol{a} \in \overline{\Gamma_\mathrm{D}}$ together with $\boldsymbol{v} \in \boldsymbol{H}^1(\omega_{\boldsymbol{a}})$ which vanishes on γ_D (recall this notation from Section 2.3) in the sense of traces. The values of the constats $C_{\mathrm{PF},\omega_{\boldsymbol{a}}}$, which at most depend on the mesh shape-regularity parameter $\kappa_{\mathcal{T}_h}$, are discussed in, e.g., [5, Section 2.1]; in the first case, $C_{\mathrm{PF},\omega_{\boldsymbol{a}}} \leq 1/\pi$ for convex $\omega_{\boldsymbol{a}}$, whereas in the second case, often $C_{\mathrm{PF},\omega_{\boldsymbol{a}}} \leq 1$.

Recalling notation (6.4), we can now establish our a posteriori error estimate:

Theorem 7.1 (Guaranteed, fully computable, and constant-free upper bound)**.** *Let $\boldsymbol{j}$ satisfy Assumption 3.1, let $\boldsymbol{A}$ be the weak solution to the curl–curl problem given by (7.2), and let $\boldsymbol{A}_h$ be its Nédélec finite element approximation given by (7.4). Let $\boldsymbol{j}_h^{\boldsymbol{a}}$ be given by Definition 5.1 for all vertices $\boldsymbol{a} \in \mathcal{V}_h$, and let $\boldsymbol{h}_h$ be given by Definition 6.1. Then*

$$\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\| \le \eta_{\mathrm{tot}} := \underbrace{\|\boldsymbol{h}_h - \nabla\times\boldsymbol{A}_h\|}_{\eta} + \underbrace{2C_{\mathrm{L}}\left\{\sum_{\boldsymbol{a}\in\mathcal{V}_h} C^2_{\mathrm{PF},\omega_{\boldsymbol{a}}}(\eta^{\boldsymbol{a}}_{\mathrm{osc},\boldsymbol{j}_h^{\boldsymbol{a}}})^2\right\}^{1/2}}_{\eta_{\mathrm{osc}}}.$$

Proof. For a piecewise polynomial current density $\boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$, Theorem 6.2 implies $\boldsymbol{h}_h \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$ with $\nabla\times\boldsymbol{h}_h = \boldsymbol{j}$. Thus, in this case the claim follows with $\eta_{\mathrm{osc}} = 0$ by the Prager–Synge theorem [33] in the $\boldsymbol{H}(\mathrm{curl})$-context, see, e.g., [8, Theorem 10] or [22, Theorem 3.1].

In general, we proceed as follows. Since $\boldsymbol{A}, \boldsymbol{A}_h \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$,

$$\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\| = \max_{\substack{\boldsymbol{v}\in\boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)\\ \|\nabla\times\boldsymbol{v}\|=1}} (\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h), \nabla\times\boldsymbol{v}).$$

Fix $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$ with $\|\nabla\times\boldsymbol{v}\| = 1$ and consider $\boldsymbol{w}$ from (7.6). Note that since $\boldsymbol{h}_h \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$ from Theorem 6.2, the Green theorem and $\nabla\times\boldsymbol{w} = \nabla\times\boldsymbol{v}$ give

$$(\nabla\times\boldsymbol{h}_h, \boldsymbol{w}) = (\boldsymbol{h}_h, \nabla\times\boldsymbol{w}) = (\boldsymbol{h}_h, \nabla\times\boldsymbol{v}).$$

Similarly, $\nabla\times\boldsymbol{w} = \nabla\times\boldsymbol{v}$ and the weak solution characterization (7.3) lead to

$$(\nabla\times\boldsymbol{A}, \nabla\times\boldsymbol{v}) = (\nabla\times\boldsymbol{A}, \nabla\times\boldsymbol{w}) = (\boldsymbol{j}, \boldsymbol{w}).$$

Thus

$$(\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h), \nabla\times\boldsymbol{v}) = (\boldsymbol{j} - \nabla\times\boldsymbol{h}_h, \boldsymbol{w}) + (\boldsymbol{h}_h - \nabla\times\boldsymbol{A}_h, \nabla\times\boldsymbol{v}).$$

The second term is trivially bounded by the estimator η via the Cauchy–Schwarz inequality, so that we are left with bounding the first one.

Recalling (5.5) and the decomposition (5.11c), there follows by virtue of (6.1b) and the constraint in (6.1a) that

$$(\boldsymbol{j} - \nabla\times\boldsymbol{h}_h, \boldsymbol{w}) = \sum_{\boldsymbol{a}\in\mathcal{V}_h} (\boldsymbol{j}_h^{\boldsymbol{a}} - \boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}(\boldsymbol{j}_h^{\boldsymbol{a}}), \boldsymbol{w})_{\omega_{\boldsymbol{a}}}.$$

Fix now $\boldsymbol{a} \in \mathcal{V}_h$. Let, first, $\boldsymbol{a} \in \overline{\Gamma_{\mathrm{D}}}$ be a Dirichlet vertex. Then trivially

$$(\boldsymbol{j}_h^{\boldsymbol{a}} - \boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}(\boldsymbol{j}_h^{\boldsymbol{a}}), \boldsymbol{w})_{\omega_{\boldsymbol{a}}} \le C_{\mathrm{PF},\omega_{\boldsymbol{a}}} \eta^{\boldsymbol{a}}_{\mathrm{osc},\boldsymbol{j}_h^{\boldsymbol{a}}} \|\nabla\boldsymbol{w}\|_{\omega_{\boldsymbol{a}}}$$

via the Cauchy–Schwarz inequality and the second bound in (7.7). Let, second, $\boldsymbol{a}$ be an interior vertex or a boundary vertex such that all the faces sharing the vertex $\boldsymbol{a}$ lie in Γ_{N}. Let $q_h \in \mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_*(\omega_{\boldsymbol{a}})$ be given. Then, by the Green theorem, since $\nabla\cdot(\boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}(\boldsymbol{j}_h^{\boldsymbol{a}})) = 0$ and using (5.11b),

$$(\boldsymbol{j}_h^{\boldsymbol{a}} - \boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}(\boldsymbol{j}_h^{\boldsymbol{a}}), \nabla q_h)_{\omega_{\boldsymbol{a}}} = (\nabla\cdot(\boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}(\boldsymbol{j}_h^{\boldsymbol{a}}) - \boldsymbol{j}_h^{\boldsymbol{a}}), q_h)_{\omega_{\boldsymbol{a}}} = (\boldsymbol{\Pi}_{p'}(\boldsymbol{j}) - \boldsymbol{j}, \underbrace{\nabla\psi^{\boldsymbol{a}} q_h}_{\in[\mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}})]^3})_{\omega_{\boldsymbol{a}}} = 0$$

by the definition of the projector $\boldsymbol{\Pi}$ from Section 2.6 and since $p' = \min\{p, 1\} \ge 1$. Thus, in particular the mean values can be subtracted and the Cauchy–Schwarz inequality together with the first bound in (7.7) lead to

$$(\boldsymbol{j}_h^{\boldsymbol{a}} - \boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}(\boldsymbol{j}_h^{\boldsymbol{a}}), \boldsymbol{w})_{\omega_{\boldsymbol{a}}} = (\boldsymbol{j}_h^{\boldsymbol{a}} - \boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}(\boldsymbol{j}_h^{\boldsymbol{a}}), \boldsymbol{w} - \boldsymbol{\Pi}_0(\boldsymbol{w}))_{\omega_{\boldsymbol{a}}} \le C_{\mathrm{PF},\omega_{\boldsymbol{a}}} \eta^{\boldsymbol{a}}_{\mathrm{osc},\boldsymbol{j}_h^{\boldsymbol{a}}} \|\nabla\boldsymbol{w}\|_{\omega_{\boldsymbol{a}}}.$$

Finally, the recovering estimate together with (7.6) show

$$\left\{\sum_{\boldsymbol{a}\in\mathcal{V}_h} \|\nabla\boldsymbol{w}\|^2_{\omega_{\boldsymbol{a}}}\right\}^{1/2} \le 2\|\nabla\boldsymbol{w}\| \le 2C_{\mathrm{L}}\|\nabla\times\boldsymbol{v}\| = 2C_{\mathrm{L}},$$

so that we conclude by the Cauchy–Schwarz inequality that $(\boldsymbol{j} - \nabla\times\boldsymbol{h}_h, \boldsymbol{w}) \le \eta_{\mathrm{osc}}$. □

7.4 p-robust local efficiency

Crucially, the a posteriori error estimate of Theorem 7.1 is locally efficient and p-robust:

Theorem 7.2 (p-robust local efficiency)**.** *Let the assumptions of Theorem 7.1 be satisfied. Admit Conjecture 6.3. Then*

$$\|\boldsymbol{h}_h - \nabla\times\boldsymbol{A}_h\|_K \lesssim \sum_{\boldsymbol{a}\in\mathcal{V}_K} \left[\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|_{\widetilde{\omega}_{\boldsymbol{a}}} + \eta^{\boldsymbol{a}}_{\mathrm{osc},\boldsymbol{j}^{\boldsymbol{a}}_h} + \widetilde{\eta}^{\boldsymbol{a}}_{\mathrm{osc},\boldsymbol{j}}\right].$$

Proof. Definition (6.1b) and the partition of unity (2.1) imply

$$\|\boldsymbol{h}_h - \nabla\times\boldsymbol{A}_h\|_K = \left\|\sum_{\boldsymbol{a}\in\mathcal{V}_K} (\boldsymbol{h}^{\boldsymbol{a}}_h - \psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}_h))\right\|_K \leq \sum_{\boldsymbol{a}\in\mathcal{V}_K} \|\boldsymbol{h}^{\boldsymbol{a}}_h - \psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}}.$$

Thus employing Theorem 6.4 concludes the proof. □

7.5 Remarks

Several remarks are again in order:

1. Even when $\boldsymbol{j}$ is non-polynomial, the projectors $\Pi_{p'}$ in (5.2) and $\boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}$ in (6.1a) are not seen in a practical implementation done via the Euler–Lagrange conditions with Lagrange multipliers, where only $-\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}$ and $\boldsymbol{j}^{\boldsymbol{a}}_h$ naturally appear, cf., e.g., [10, equations (3.6), (3.10b), and (3.11)].
2. When $\boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$, property (5.12) implies that $\eta^{\boldsymbol{a}}_{\mathrm{osc},\boldsymbol{j}^{\boldsymbol{a}}_h}$ from (6.4), and thus the corresponding data oscillation estimators in Theorems 7.1 and 7.2, vanish. Similarly, when $\boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_{p'}(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$, since $\nabla\cdot\boldsymbol{j} = 0$, it follows that $\boldsymbol{j} \in [\mathcal{P}_{p'}(\mathcal{T}_{\boldsymbol{a}})]^3$, see, e.g., [6, Corollary 2.3.1]. Thus $\eta^{\boldsymbol{a}}_{\mathrm{osc},\boldsymbol{j}}$ from (5.6) vanishes. Moreover, all these terms are higher-order with respect to $\|\nabla\times(\boldsymbol{A} - \boldsymbol{A}_h)\|$ if $\boldsymbol{j}$ is piecewise smooth.
3. It is possible to enhance the projector $\boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}$ from Section 6 by an additional constraint on elementwise orthogonality with respect to piecewise vector-valued constants as in (5.2). Then the data oscillation estimator η_{osc} in Theorem 7.1 can be modified to take an element-based form, in place of the curent vertex-patch-based one, with in particular all the Poincaré–Friedrichs constants brought down to $1/\pi$.
4. The alternative upper bound
$$\|\nabla\times(\boldsymbol{A} - \boldsymbol{A}_h)\| \leq \|\boldsymbol{h}_h - \nabla\times\boldsymbol{A}_h\| + C_{\mathrm{PFW}} h_\Omega \|\boldsymbol{j} - \nabla\times\boldsymbol{h}_h\|$$
may be employed, where C_{PFW} is any constant in the global Poincaré–Friedrichs–Weber inequality $\|\boldsymbol{v}\| \leq C_{\mathrm{PFW}} h_\Omega \|\nabla\times\boldsymbol{v}\|$ for all $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ such that $\nabla\cdot\boldsymbol{v} = 0$ and $(\boldsymbol{v},\boldsymbol{\varphi}) = 0$ for all $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{H}}(\Omega,\Gamma_{\mathrm{D}})$, cf. [21, Proposition 7.4].
5. The equilibration of Definition 6.1 is performed in local Nédélec spaces of order $p+1$. This is in agreement with p-robust flux equilibrations from [7, 19, 20]. Similarly to [7, 18], it is also possible to design a downgrade of the orders of the local problems (6.1a) from $p+1$ to p.

   Let us first discuss the case $p \geq 1$. The first step is to replace (5.4b) by (5.4a) with $\boldsymbol{\mathcal{RT}}_1$ replaced by $\boldsymbol{\mathcal{RT}}_p$. Then, according to Theorem B.1 with $q' = p$, we obtain $\boldsymbol{\delta}^{\boldsymbol{a}}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$ in place of (5.9a). Second, we employ (elementwise) $\boldsymbol{I}^{\boldsymbol{\mathcal{RT}}}_p(\psi^{\boldsymbol{a}}\boldsymbol{j})$ in (5.5). Then, when $\boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$, $\boldsymbol{j}^{\boldsymbol{a}}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$, $\nabla\cdot\boldsymbol{j}^{\boldsymbol{a}}_h = 0$, and $\sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{j}^{\boldsymbol{a}}_h = \boldsymbol{j}$ in place of (5.12) and (5.11c). Consequently, (6.1a) can be brought down to
$$\boldsymbol{h}^{\boldsymbol{a}}_h := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}}) \\ \nabla\times\boldsymbol{v}_h = \boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_p}(\boldsymbol{j}^{\boldsymbol{a}}_h)}} \|\boldsymbol{v}_h - \boldsymbol{I}^{\boldsymbol{\mathcal{N}}}_p(\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}_h))\|^2_{\omega_{\boldsymbol{a}}}, \tag{7.8}$$
where $\boldsymbol{I}^{\boldsymbol{\mathcal{N}}}_p$ is the elementwise canonical p-degree Nédélec interpolate, analogue to (2.5). This leads to a cheaper procedure where the guaranteed estimate of Theorem 7.1 still holds true, with merely $\boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_{p+1}}$ replaced by $\boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_p}$ and with η_{osc} that still vanishes when $\boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$. Similarly, the local efficiency of Theorem 7.2 is also preserved, with, however, the p-robustness theoretically lost. In particular, from (B.6b), estimate (5.10) still holds true up to a possibly p-dependent constant, whereas in (6.6), $\nabla\cdot(\boldsymbol{\Pi}_{\boldsymbol{V}^{\boldsymbol{a}}_p}(\boldsymbol{j}^{\boldsymbol{a}}_h)) = 0$ holds.

   For $p = 0$, because of $p' = p+1$ employed in (5.2), we need to replace (5.5) by
$$\boldsymbol{j}^{\boldsymbol{a}}_h := \boldsymbol{I}^{\boldsymbol{\mathcal{RT}}}_0(\psi^{\boldsymbol{a}}\boldsymbol{j} + \boldsymbol{\theta}^{\boldsymbol{a}}_h - \boldsymbol{\delta}^{\boldsymbol{a}}_h).$$

Let $\boldsymbol{j} \in \mathcal{RT}_0(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$. Then clearly $\boldsymbol{j}_h^{\boldsymbol{a}} \in \mathcal{RT}_0(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$. Moreover, from (2.6), $\nabla \cdot \boldsymbol{I}_0^{\mathcal{RT}}(\boldsymbol{\delta}_h^{\boldsymbol{a}}) = \mathcal{P}_0(\nabla \cdot \boldsymbol{\delta}_h^{\boldsymbol{a}}) = 0$, whereas $\nabla \cdot \boldsymbol{I}_0^{\mathcal{RT}}(\psi^{\boldsymbol{a}} \boldsymbol{j}) = \mathcal{P}_0(\nabla \cdot (\psi^{\boldsymbol{a}} \boldsymbol{j})) = \nabla \psi^{\boldsymbol{a}} \cdot \boldsymbol{j}$, also using that $\boldsymbol{j} \in [\mathcal{P}_0(\mathcal{T}_{\boldsymbol{a}})]^3$ as above in point 2, and similarly $\nabla \cdot \boldsymbol{I}_0^{\mathcal{RT}}(\boldsymbol{\theta}_h^{\boldsymbol{a}}) = \mathcal{P}_0(\nabla \cdot \boldsymbol{\theta}_h^{\boldsymbol{a}}) = -\nabla \psi^{\boldsymbol{a}} \cdot \boldsymbol{j}$. Thus $\nabla \cdot \boldsymbol{j}_h^{\boldsymbol{a}} = 0$. Finally, $\sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{j}_h^{\boldsymbol{a}} = \boldsymbol{I}_0^{\mathcal{RT}}(\sum_{\boldsymbol{a} \in \mathcal{V}_h} (\psi^{\boldsymbol{a}} \boldsymbol{j} + \boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}})) = \boldsymbol{I}_0^{\mathcal{RT}}(\boldsymbol{j}) = \boldsymbol{j}$ by the linearity of the Raviart–Thomas projector $\mathcal{RT}_0$. Then the above discussion for $p \geq 1$ applies.

6. The approach of [22, 23] includes solutions of local, a priori over-determined, problems on vertex patches in a multi-stage procedure. The present over-constrained problems (5.2) and consecutive steps in Definitions 5.1 and 6.1 share this spirit, though the minimizations directly determine the best-possible local energy error estimator contributions.

8 Numerical illustration

This section presents some numerical examples illustrating the key features of the estimator of Theorem 7.1. We impose the Dirichlet boundary condition on the whole boundary, i.e., $\Gamma_{\mathrm{D}} := \partial\Omega$. We consider both structured meshes and unstructured meshes generated with the software pacakge `MMG3D` [15]. When we speak about "a structured mesh", we mean a Cartesian partition of Ω into $N \times N \times N$ cubes where each cube is first subdivided into 6 pyramids (with the basis a face and the apex the barycenter of the cube) and then each pyramid into 4 tetrahedra. The corresponding mesh size is $h = \sqrt{3}/(2N)$. We consider the Nédélec finite element approximation (7.4) with varying degree $p \geq 1$.

8.1 $\boldsymbol{H}^3(\Omega)$ Solution with a polynomial right-hand side

We first consider the unit cube $\Omega := (0, 1)^3$ and a polynomial right-hand side $\boldsymbol{j} := (0, 0, 1)$, so that the data oscillation estimator η_{osc} vanishes. We can show that the solution is given by $\boldsymbol{A} = (0, 0, A_3)$ with

$$A_3(\boldsymbol{x}) := \frac{16}{\pi^4} \sum_{n,m \geq 1} \frac{1}{nm(n^2 + m^2)} \sin(n\pi \boldsymbol{x}_1) \sin(m\pi \boldsymbol{x}_2). \tag{8.1}$$

This function belongs to $H^3(\Omega)$ but not to $H^4(\Omega)$. In practice, we cut the series at $n = m = 100$, and obtain $\nabla \times \boldsymbol{A}$ by analytically differentiating (8.1).

We first fix the polynomial degree and consider a sequence of meshes. We use $p = 1$ and structured meshes with $N = 1, 2, 4, 8$, and then $p = 2$ and a sequence of unstructured meshes. Figure 1 presents the corresponding errors, estimates, and effectivity indices. We observe the expected convergence rate h^2 (recall that $\boldsymbol{A} \in \boldsymbol{H}^3(\Omega)$ merely). The estimator $\eta = \eta_{\mathrm{tot}}$ closely follows the error $\|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\|$, and the effectivity index is close to the optimal value 1; we actually numerically observe asymptotic exactness.

We then fix a mesh and increase the polynomial degree p from 1 to 6. We consider two configurations: a structured mesh where the unit cube is split into 24 tetrahedra as described above and an unstructured mesh consisting of 176 tetrahedra. Figure 2 reports the results. The convergence is not exponential, which is expected because of the solution's finite regularity. Also in this setting, the estimator closely follows the actual error, and the effectivity index always remains close to 1. In particular, the effectivity index does not increase with p, which illustrates the p-robustness of the estimator.

Although this is not reported in the figures, we also numerically check that the reconstructed flux $\boldsymbol{h}_h$ is indeed equilibrated, i.e., $\|\boldsymbol{j} - \nabla \times \boldsymbol{h}_h\| = 0$. Because of finite precision arithmetic, this value is not exactly zero, but ranges between 10^{-15} and 10^{-11}, which is perfectly reasonable compared to the actual error levels.

8.2 Analytical solution with a general right-hand side

We consider again the unit cube $\Omega := (0, 1)^3$, this time with a non-polynomial right-hand side $\boldsymbol{j} := 8\pi^2(\sin(2\pi \boldsymbol{x}_2) \sin(2\pi \boldsymbol{x}_3), 0, 0)$. The associated solution is analytic,

$$\boldsymbol{A} := (\sin(2\pi \boldsymbol{x}_2) \sin(2\pi \boldsymbol{x}_3), 0, 0). \tag{8.2}$$

Figure 3 presents an h convergence experiment with the same settings as above. The optimal convergence rate h^{p+1} is observed for $\|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\|$. The oscillation-free estimator η closely follows the actual error, with a slight underestimation. This is to be expected since the oscillation term, required to obtain a guaranteed bound, is not included. The contribution to the oscillation term is however expected to be small and rapidly diminishing, which is in agreement with the good effectivity index approaching the optimal value 1. We then consider a p convergence test. In Figure 4, we now observe the expected exponential convergence rate of the error and a perfect behavior of the effectivity index.

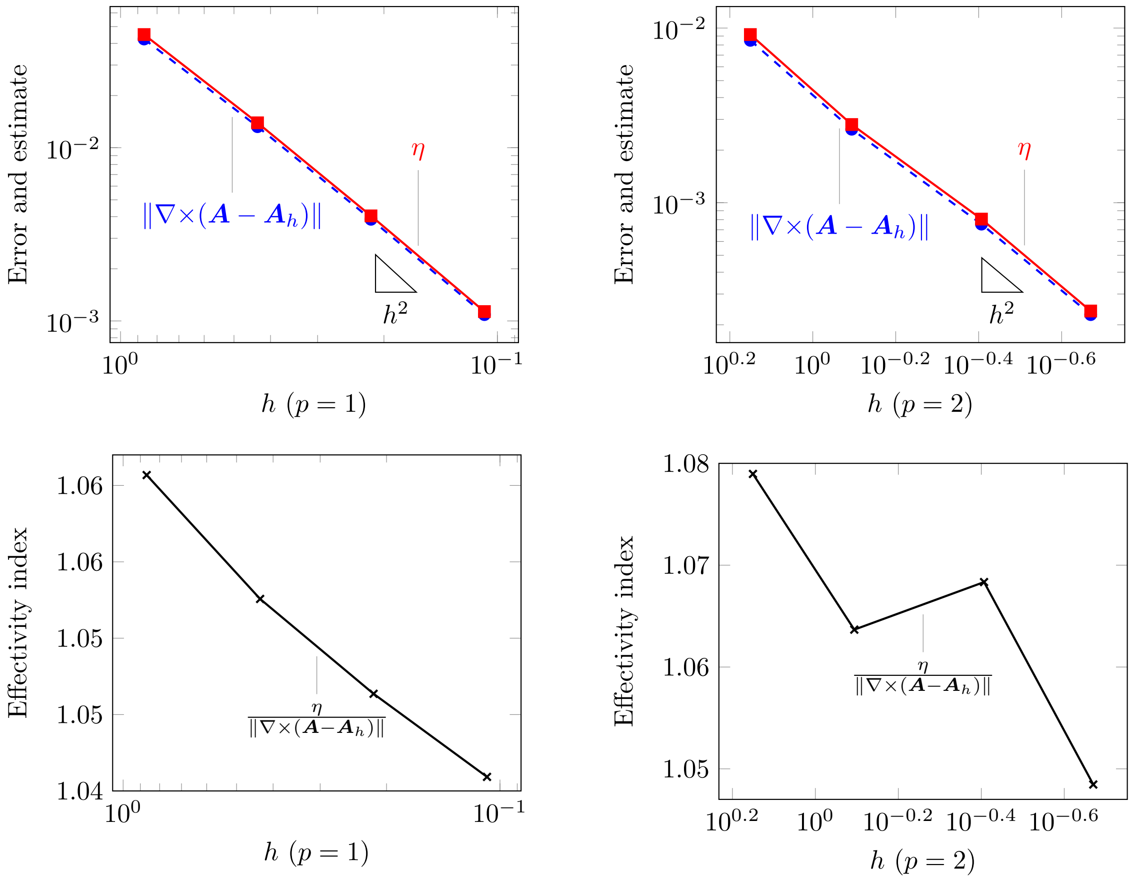

Figure 1: [Smooth solution (8.1)] Uniform mesh refinement.

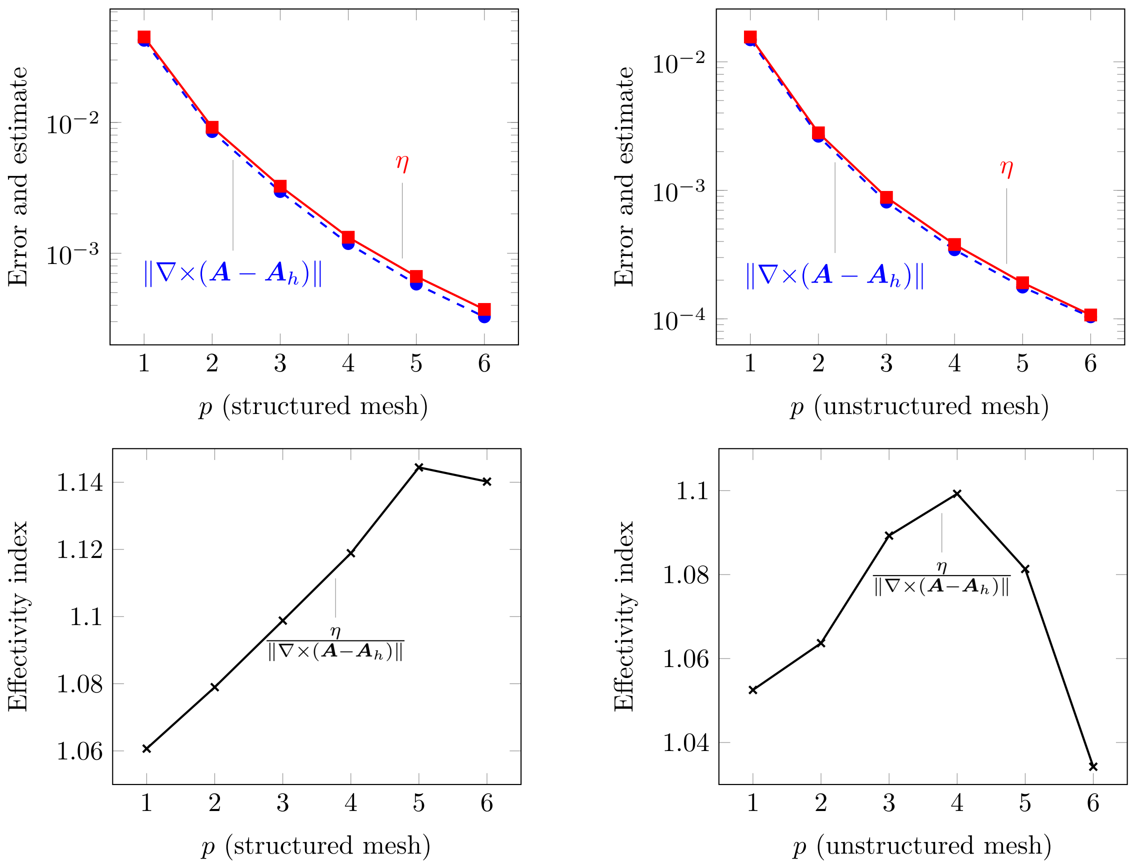

Figure 2: [Smooth solution (8.1)] Uniform polynomial degree refinement.

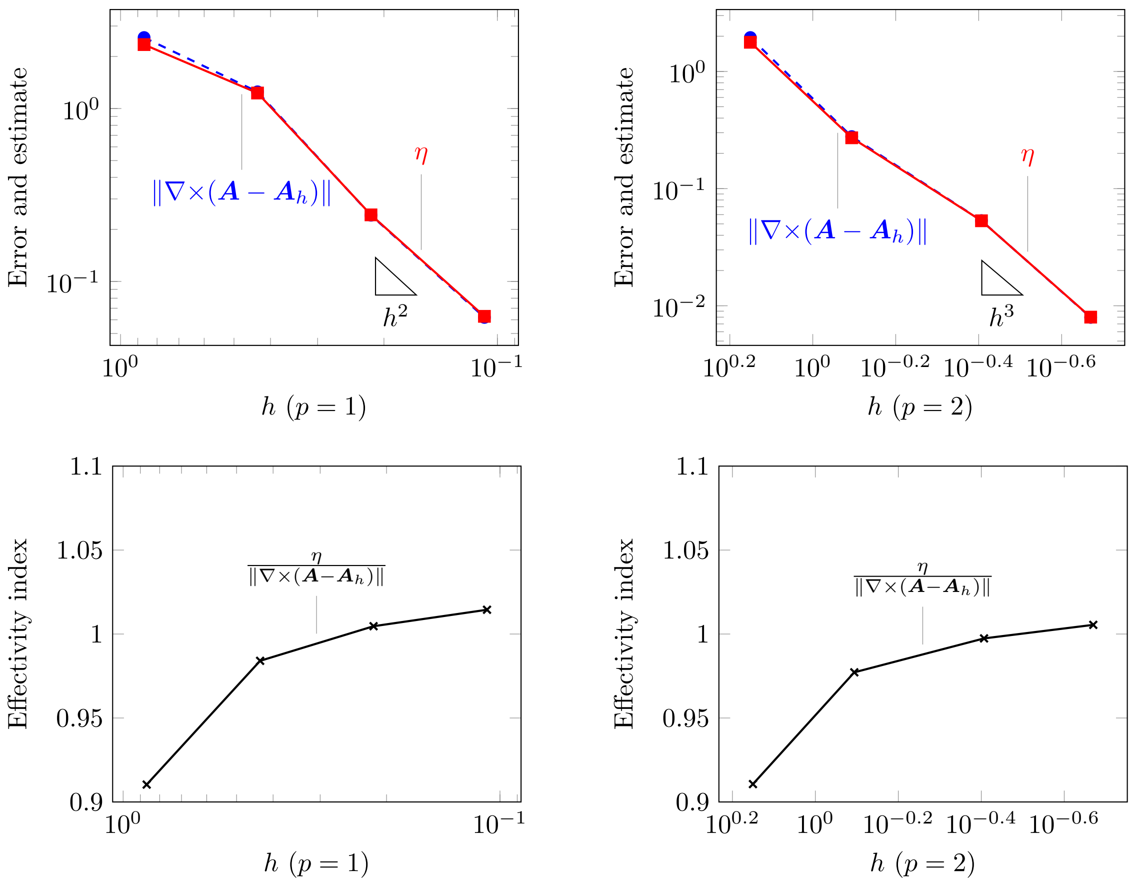

Figure 3: [Analytical solution (8.2)] Uniform mesh refinement.

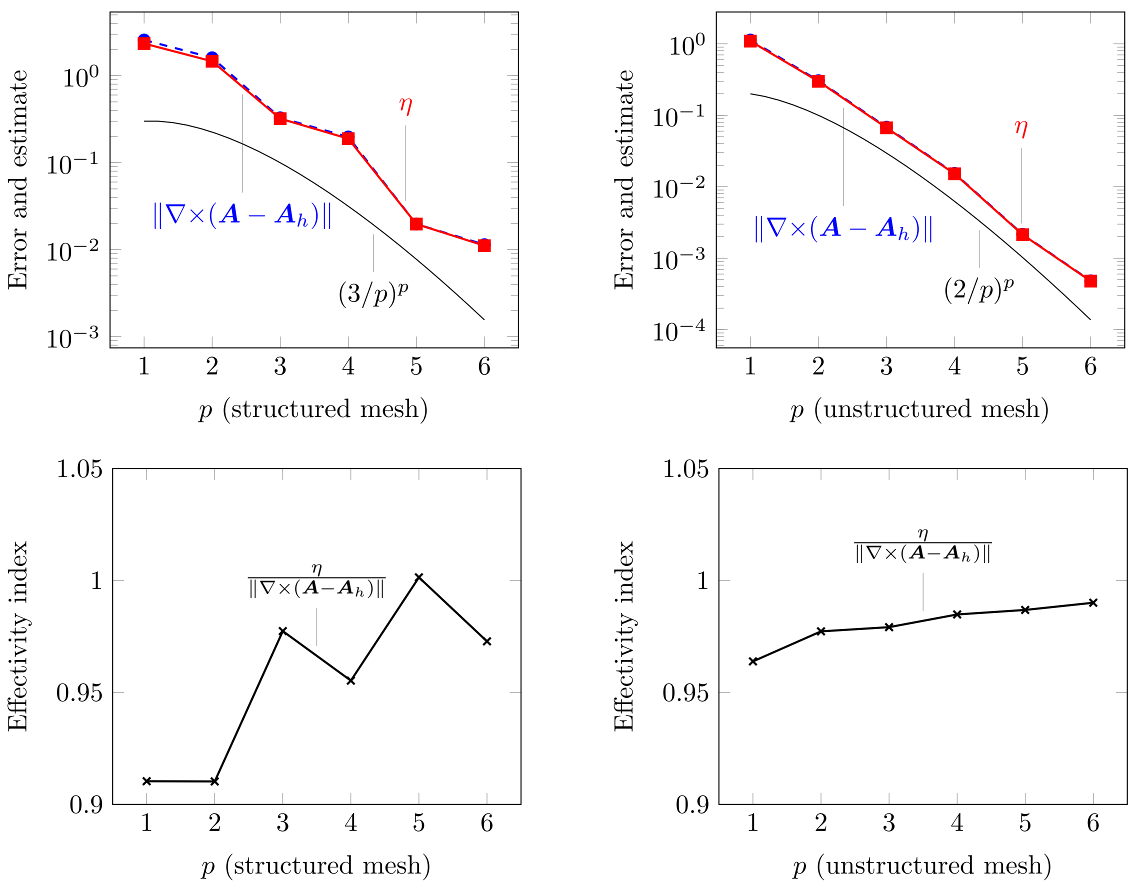

Figure 4: [Analytical solution (8.2)] Uniform polynomial degree refinement.

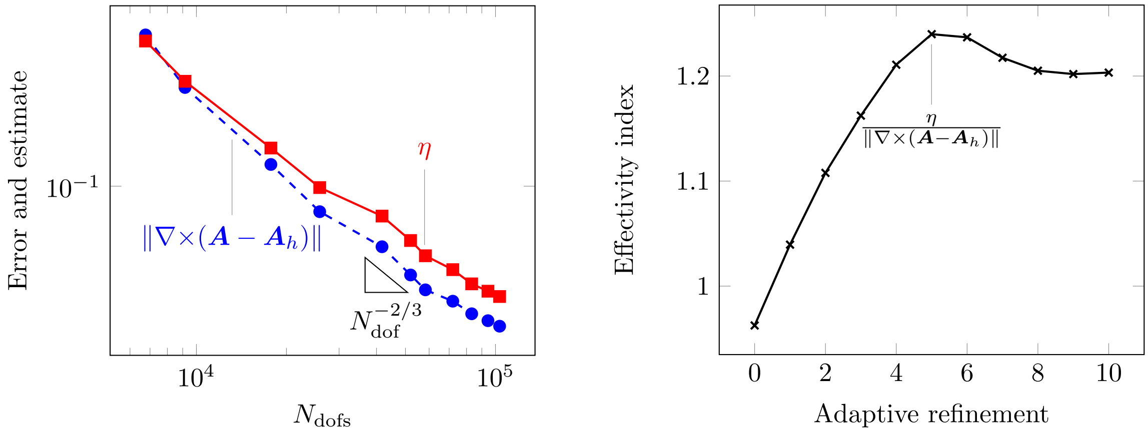

Figure 5: [Singular solution (8.3)] Adaptive mesh refinement.

8.3 Adaptivity with a singular solution

Our last experiment features a singular solution in a nonconvex domain, following [10, 22]. Specifically, we consider an L-shape example where $\Omega := L \times (0,1)$, with

$$L := \{\boldsymbol{x} = (r\cos\theta, r\sin\theta);\ |\boldsymbol{x}_1|, |\boldsymbol{x}_2| \leq 1, \quad 0 \leq \theta \leq 3\pi/2\}.$$

The right-hand side $\boldsymbol{j}$ is non-polynomial and chosen such that

$$\boldsymbol{A}(\boldsymbol{x}) = \big(0, 0, \chi(r) r^\alpha \sin(\alpha\theta)\big), \tag{8.3}$$

where $\alpha := 3/2$, $r^2 := |\boldsymbol{x}_1|^2 + |\boldsymbol{x}_2|^2$, $(\boldsymbol{x}_1, \boldsymbol{x}_2) = r(\cos\theta, \sin\theta)$, and $\chi : (0,1) \to \mathbb{R}$ is a smooth cutoff function such that $\chi = 0$ in a neighborhood of 1. One easily checks that $\nabla\cdot\boldsymbol{A} = 0$. Besides, since $\Delta\left(r^\alpha \sin(\alpha\theta)\right) = 0$ near the origin, the right-hand side is non-singular (i.e., $\boldsymbol{j} \in \boldsymbol{L}^2(\Omega)$), and the singularity appearing in the solution is solely due to the re-entrant edge.

We couple our estimator with an adaptive strategy based on Dörfler's marking [16] for $\eta_K := \|\boldsymbol{h}_h - \nabla\times\boldsymbol{A}_h\|_K$ and `MMG3D` [15] to build a series of adaptive meshes. We select $p = 2$ and an initial mesh made of 415 elements.

The behaviors of the error and of the estimator η with respect to the number of degrees of freedom $N_{\rm dofs}$ are represented in Figure 5. The effectivity index stays close to one, even on unstructured and locally refined meshes. Besides, the optimal convergence rate is observed (it is limited to $-2/3$ when using isotropic elements in the presence of an edge singularity). This seems to indicate that the estimator is perfectly suited to drive adaptive processes, and illustrates our local efficiency results.

Finally, Figures 6–7 present the meshes generated by the adaptive algorithm, the estimators $\eta_K = \|\boldsymbol{h}_h - \nabla\times\boldsymbol{A}_h\|_K$, and the elementwise errors $\|\nabla\times(\boldsymbol{A} - \boldsymbol{A}_h)\|_K$ (the top face and the faces sharing the re-entrant edge). The meshes are refined close to the re-entrant edge, as expected. The estimated error distribution closely matches the actual one, illustrating the local efficiency of the estimator.

A Over-constrained minimization in Raviart–Thomas spaces

In this appendix, we consider a fixed mesh vertex $\boldsymbol{a} \in \mathcal{V}_h$. Let an integer $q \geq 0$ be fixed and set $q' := \min\{q, 1\}$. We employ the notation of Section 2 and in particular recall that $\lesssim$ means smaller or equal to up to a constant only depending on the mesh shape-regularity parameter $\kappa_{\mathcal{T}_h}$. Recall the technical assumption of Section 2.7 for boundary patches. We also assume a polynomial form, mean value zero, and patchwise orthogonality conditions:

Assumption A.1 (Data $g^{\boldsymbol{a}}$ and $\boldsymbol{\tau}_h^{\boldsymbol{a}}$)**.** *The data $g^{\boldsymbol{a}}$ and $\boldsymbol{\tau}_h^{\boldsymbol{a}}$ satisfy*

$$g^{\boldsymbol{a}} \in L^2(\omega_{\boldsymbol{a}}) \quad \textit{and} \quad \boldsymbol{\tau}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_{q'}(\mathcal{T}_{\boldsymbol{a}}), \tag{A.1a}$$

$$(g^{\boldsymbol{a}}, 1)_{\omega_{\boldsymbol{a}}} = 0 \quad \textit{when } \boldsymbol{a} \notin \overline{\Gamma_{\rm D}}, \tag{A.1b}$$

$$(\boldsymbol{\tau}_h^{\boldsymbol{a}}, \nabla q_h)_{\omega_{\boldsymbol{a}}} + (g^{\boldsymbol{a}}, q_h)_{\omega_{\boldsymbol{a}}} = 0 \quad \forall q_h \in \mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_*(\omega_{\boldsymbol{a}}). \tag{A.1c}$$

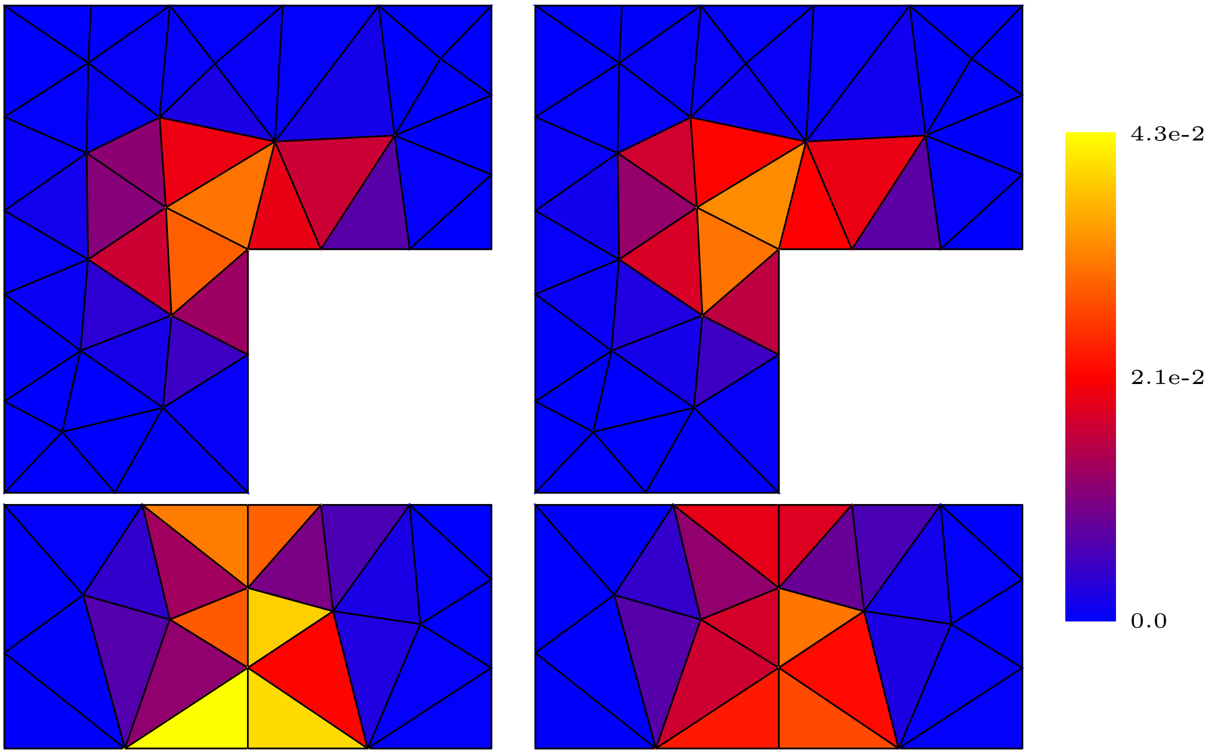

Figure 6: [Singular solution (8.3)]. Estimated (left) and actual (right) error distributions on the initial mesh. Top view (top) and side view (bottom).

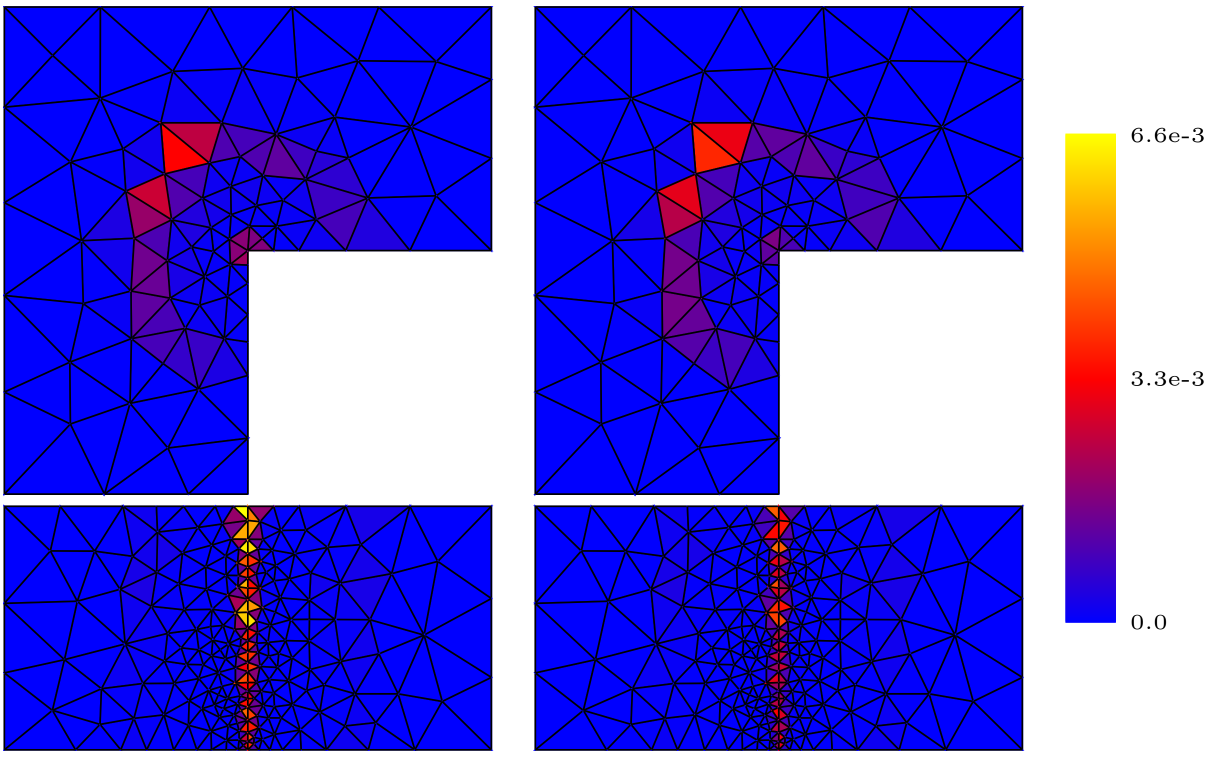

Figure 7: [Singular solution (8.3)] Estimated (left) and actual (right) error distributions at adaptive mesh refinement iteration #10. Top view (top) and side view (bottom).

We consider the following over-constrained minimization problem in the local Raviart–Thomas space $\boldsymbol{\mathcal{RT}}_{q'}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$:

$$\boldsymbol{\theta}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_{q'}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v}_h = \Pi_{q'}(g^{\boldsymbol{a}}) \\ (\boldsymbol{v}_h, \boldsymbol{r}_h)_K = (\boldsymbol{\tau}_h^{\boldsymbol{a}}, \boldsymbol{r}_h)_K \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^3, \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}_h - \boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}^2. \tag{A.2}$$

The following result is of independent interest:

Theorem A.2 (Over-constrained minimization in the Raviart–Thomas spaces)**.** *Let Assumption A.1 hold. Then there exists a unique solution $\boldsymbol{\theta}_h^{\boldsymbol{a}}$ to problem (A.2), satisfying the stability estimate*

$$\|\boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v} = g^{\boldsymbol{a}}}} \|\boldsymbol{v} - \boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} + \left\{ \sum_{K \in \mathcal{T}_{\boldsymbol{a}}} \left(\frac{h_K}{\pi} \|\Pi_{q'}(g^{\boldsymbol{a}}) - g^{\boldsymbol{a}}\|_K \right)^2 \right\}^{1/2}.$$

For the analysis of problem (A.2), it will be useful to consider

$$\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_{q'}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v}_h = \Pi_{q'}(g^{\boldsymbol{a}})}} \|\boldsymbol{v}_h - \boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}^2. \tag{A.3}$$

Minimizations (A.3) are in a conventional format in that the constraints only concern normal trace and divergence. Moreover, they fulfill the following important property:

Lemma A.3 (Existence, uniqueness, and stability of $\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}$ from (A.3))**.** *Let Assumption A.1 hold. Then there exists a unique solution $\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}$ to problem (A.3), satisfying the stability estimate*

$$\|\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}} - \boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v} = g^{\boldsymbol{a}}}} \|\boldsymbol{v} - \boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} + \left\{ \sum_{K \in \mathcal{T}_{\boldsymbol{a}}} \left(\frac{h_K}{\pi} \|\Pi_{q'}(g^{\boldsymbol{a}}) - g^{\boldsymbol{a}}\|_K \right)^2 \right\}^{1/2}. \tag{A.4}$$

Proof. Existence and uniqueness of $\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}$ from (A.3) are classical following, e.g., [6], thanks to the Neumann boundary compatibility condition (A.1b); note that this implies $(\Pi_{q'}(g^{\boldsymbol{a}}), 1)_{\omega_{\boldsymbol{a}}} = 0$ when $\boldsymbol{a} \notin \overline{\Gamma_{\mathrm{D}}}$. Moreover, since $\Pi_{q'}(g^{\boldsymbol{a}}) \in \mathcal{P}_{q'}(\mathcal{T}_{\boldsymbol{a}})$ and $\boldsymbol{\tau}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_{q'}(\mathcal{T}_{\boldsymbol{a}})$, taking $p = q'$, $\boldsymbol{\tau}_p = -\boldsymbol{\tau}_h^{\boldsymbol{a}}$, and $r_K = (\Pi_{q'}(g^{\boldsymbol{a}}) - \nabla \cdot \boldsymbol{\tau}_h^{\boldsymbol{a}})|_K$ in [20, Corollaries 3.3 and 3.6] for an interior vertex $\boldsymbol{a}$ and [20, Corollary 3.8] for a boundary vertex $\boldsymbol{a}$ leads to

$$\|\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}} - \boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v} = \Pi_{q'}(g^{\boldsymbol{a}})}} \|\boldsymbol{v} - \boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} = \|\nabla \tilde{r}^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}.$$

The equality above is a classical primal–dual equivalence, with $\tilde{r}^{\boldsymbol{a}} \in H^1_*(\omega_{\boldsymbol{a}})$ given by

$$(\nabla \tilde{r}^{\boldsymbol{a}}, \nabla v)_{\omega_{\boldsymbol{a}}} = (\boldsymbol{\tau}_h^{\boldsymbol{a}}, \nabla v)_{\omega_{\boldsymbol{a}}} + (\Pi_{q'}(g^{\boldsymbol{a}}), v)_{\omega_{\boldsymbol{a}}} \qquad \forall v \in H^1_*(\omega_{\boldsymbol{a}}).$$

Thus, as in, e.g., [19, Theorem 3.17],

$$\begin{aligned} \|\nabla \tilde{r}^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} &= \max_{\substack{v \in H^1_*(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \{ (\boldsymbol{\tau}_h^{\boldsymbol{a}}, \nabla v)_{\omega_{\boldsymbol{a}}} + (\Pi_{q'}(g^{\boldsymbol{a}}), v)_{\omega_{\boldsymbol{a}}} \} \\ &= \max_{\substack{v \in H^1_*(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \{ (\boldsymbol{\tau}_h^{\boldsymbol{a}}, \nabla v)_{\omega_{\boldsymbol{a}}} + (g^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} + (\Pi_{q'}(g^{\boldsymbol{a}}) - g^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} \}. \end{aligned}$$

The projection orthogonality and the elementwise Poincaré inequality then lead to

$$\begin{aligned} |(\Pi_{q'}(g^{\boldsymbol{a}}) - g^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}}| &= \left| \sum_{K \in \mathcal{T}_{\boldsymbol{a}}} (\Pi_{q'}(g^{\boldsymbol{a}}) - g^{\boldsymbol{a}}, v - \Pi_0 v)_K \right| \\ &\leq \left\{ \sum_{K \in \mathcal{T}_{\boldsymbol{a}}} \left(\frac{h_K}{\pi} \|\Pi_{q'}(g^{\boldsymbol{a}}) - g^{\boldsymbol{a}}\|_K \right)^2 \right\}^{1/2} \|\nabla v\|_{\omega_{\boldsymbol{a}}}, \end{aligned}$$

and (A.4) follows, since

$$\max_{\substack{v \in H^1_*(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \{ (\boldsymbol{\tau}_h^{\boldsymbol{a}}, \nabla v)_{\omega_{\boldsymbol{a}}} + (g^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} \} = \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v} = g^{\boldsymbol{a}}}} \|\boldsymbol{v} - \boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}$$

by the same primal–dual equivalence argument. □

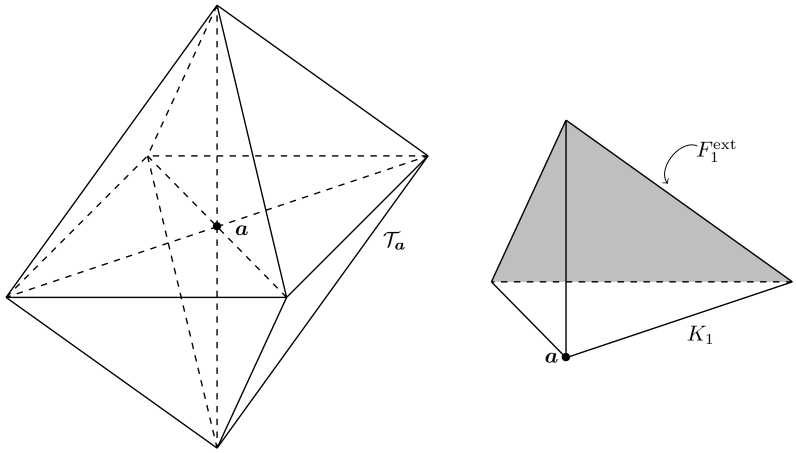

Figure 8: Interior vertex patch (left) and the element K_1 with the face $F_1^{\rm ext}$ (right)

Let, moreover, the *first-order* Raviart–Thomas piecewise polynomials $\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}$ be given by

$$\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_1(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = 0 \\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_K = (\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}},\boldsymbol{r}_h)_K \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^3, \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}_h - \boldsymbol{\tau}_h^{\boldsymbol{a}} + \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}^2. \tag{A.5}$$

The field $\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}$ can be seen as the correction of $\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}$ necessary to fulfill the constraints on the elementwise product with piecewise vector-valued constants in (A.2). As one might expect, the patchwise orthogonality assumption (A.1c), turns to be the key for the following crucial technical result:

Lemma A.4 (Existence, uniqueness, and stability of $\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}$ from (A.5))**.** *Let Assumption A.1 hold. Then there exists a unique solution $\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}$ to problem (A.5), and the following stability estimate holds true:*

$$\|\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \lesssim \|\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}. \tag{A.6}$$

Proof. Step 1. Existence and uniqueness of $\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}$. Owing to the convexity of the minimized functional in (A.5), it is enough to show that the minimization set in (A.5) is not empty to prove the existence and uniqueness of $\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}$. To do so, we will construct a particular divergence-free $\boldsymbol{\epsilon}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_1(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$ such that

$$(\boldsymbol{\epsilon}_h^{\boldsymbol{a}},\boldsymbol{r}_h)_K = (\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}},\boldsymbol{r}_h)_K \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^3, \forall K \in \mathcal{T}_{\boldsymbol{a}} \tag{A.7}$$

by an explicit run through the patch $\mathcal{T}_{\boldsymbol{a}}$ of tetrahedra sharing the vertex $\boldsymbol{a} \in \mathcal{V}_h$, similarly as in [7, 20].

We consider an interior patch in details, i.e., the case where the vertex $\boldsymbol{a} \in \mathcal{V}_h$ does not lie on the boundary $\partial\Omega$, cf. Figure 8, left. Then, following the concept of shelling of a polytopal complex, see [36, Theorem 8.12] and [20, Lemma B.1], there exists an enumeration K_i, $1 \le i \le |\mathcal{T}_{\boldsymbol{a}}|$, of the tetrahedra in the patch $\mathcal{T}_{\boldsymbol{a}}$ such that, except for the first tetrahedron in the enumeration K_1: (i) if there are at least two faces corresponding to the neighbors of K_i which have been already enumerated, then all the tetrahedra of $\mathcal{T}_{\boldsymbol{a}}$ sharing this edge come sooner in the enumeration; (ii) except for the last element $K_{|\mathcal{T}_{\boldsymbol{a}}|}$, there are one or two neighbors of K_i which have been already enumerated and correspondingly two or one neighbors of K_i which have not been enumerated yet. For a boundary patch, under the assumption in Section 2.7, a similar enumeration exists following [36, Theorem 8.12], upon invoking the "flattened patch" of [20, Section 7.2].

Consider a pass through the patch $\mathcal{T}_{\boldsymbol{a}}$ in the sense of the above enumeration. For the tetrahedron K_i, $1 \le i \le |\mathcal{T}_{\boldsymbol{a}}|$, let us denote by $\mathcal{F}_i^\sharp$ the faces of K_i corresponding to the neighbors of K_i which have been already passed through and $F^j = \partial K_i \cap \partial K_j \in \mathcal{F}_i^\sharp$ the face corresponding to the neighbor K_j. Also, let $F_i^{\rm ext}$ be the face of K_i lying on the patch boundary $\partial\omega_{\boldsymbol{a}}$. Consider the following problem:

$$\boldsymbol{\epsilon}_h^i := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_1(K_i) \\ \nabla\cdot\boldsymbol{v}_h = 0 \\ \boldsymbol{v}_h\cdot\boldsymbol{n}_{K_i} = 0 \text{ on } F_i^{\rm ext} \\ \boldsymbol{v}_h\cdot\boldsymbol{n}_{K_i} = \boldsymbol{\epsilon}_h^j\cdot\boldsymbol{n}_{K_i} \text{ on all } F^j \in \mathcal{F}_i^\sharp \\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_{K_i} = (\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}},\boldsymbol{r}_h)_{K_i} \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K_i)]^3}} \|\boldsymbol{v}_h - \boldsymbol{\tau}_h^{\boldsymbol{a}} + \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}\|_{K_i}^2; \tag{A.8}$$

this reduces (A.5) to the single tetrahedron K_i and ensures, if a solution to (A.8) exists, that $\boldsymbol{\epsilon}_h^{\boldsymbol{a}}$ defined as

$$\boldsymbol{\epsilon}_h^{\boldsymbol{a}}|_{K_i} := \boldsymbol{\epsilon}_h^i \in \boldsymbol{\mathcal{RT}}_1(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$$

satisfies (A.7). We are thus left to establish the existence and uniqueness of (A.8).

Step 1a: the first element K_1. Let us start with the first element K_1, cf. Figure 8, right. Then the set $\mathcal{F}_1^\sharp$ is empty, and we ask whether in the first-order Raviart–Thomas space $\boldsymbol{\mathcal{RT}}_1(K_1)$, one can impose simultaneously the divergence, the normal flux through one face, and moments against constant functions. We will reason by the canonical degrees of freedom, see, e.g., [6, Proposition 2.3.4 and Figure 2.14.c], and find a suitable $\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_1(K_1)$. First, we see that in $\boldsymbol{\mathcal{RT}}_1(K_1)$, the normal flux $\boldsymbol{v}_h{\cdot}\boldsymbol{n}_{K_1}$ on F_1^{ext} can be fixed to zero and the moments against constants $(\boldsymbol{v}_h, \boldsymbol{r}_h)_{K_1}$ can be fixed as in (A.8). We still have the freedom to choose the normal fluxes $\boldsymbol{v}_h{\cdot}\boldsymbol{n}_{K_1}$ on the faces of K_1 different from F_1^{ext}, and the question is whether this can be done so as to fix the divergence of $\boldsymbol{v}_h$ to zero. By [6, Proposition 2.3.3], there holds

$$\nabla{\cdot}\boldsymbol{v}_h = 0 \quad \Leftrightarrow \quad (\nabla{\cdot}\boldsymbol{v}_h, q_h)_{K_1} = 0 \qquad \forall q_h \in \mathcal{P}_1(K_1).$$

Employing the Green theorem and the fact that $\boldsymbol{v}_h{\cdot}\boldsymbol{n}_{K_1} = 0$ on F_1^{ext},

$$(\nabla{\cdot}\boldsymbol{v}_h, q_h)_{K_1} = \langle \boldsymbol{v}_h{\cdot}\boldsymbol{n}_{K_1}, q_h\rangle_{\partial K_1 \setminus F_1^{\mathrm{ext}}} - (\boldsymbol{v}_h, \nabla q_h)_{K_1}.$$

Now, since $\nabla q_h \in [\mathcal{P}_0(K_1)]^3$, the last term above is fixed from the last constraint in (A.8), so the question becomes: can one choose $\boldsymbol{v}_h{\cdot}\boldsymbol{n}_{K_1}$ on $\partial K_1 \setminus F_1^{\mathrm{ext}}$ such that

$$\langle \boldsymbol{v}_h{\cdot}\boldsymbol{n}_{K_1}, q_h\rangle_{\partial K_1 \setminus F_1^{\mathrm{ext}}} = (\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \nabla q_h)_{K_1} \qquad \forall q_h \in \mathcal{P}_1(K_1), \tag{A.9}$$

which gives 4 conditions for the 9 remaining degrees of freedom (there are 3 degrees of freedom per face in $\boldsymbol{\mathcal{RT}}_1(K_1)$ following [6, Proposition 2.3.4]).

We proceed as follows. Out of the three faces of K_1 different from F_1^{ext}, choose one and impose $\boldsymbol{v}_h{\cdot}\boldsymbol{n}_{K_1} = 0$ therein. Then we are left to set $\boldsymbol{v}_h{\cdot}\boldsymbol{n}_{K_1}$ on two faces, say F and $\widetilde{F}$. For F, consider the three hat basis functions ψ_F^k, $1 \le k \le 3$, as in Section 2.2, corresponding to its three vertices. Restricted to $\widetilde{F}$, which is necessary a face neighboring with F, they belong to $\mathcal{P}_1(\widetilde{F})$, and one of the restrictions, say ψ_F^3, is zero on $\widetilde{F}$. Thus, there holds

$$\langle \boldsymbol{v}_h{\cdot}\boldsymbol{n}_{K_1}, \psi_F^3\rangle_{\widetilde{F}} = 0,$$

and, following [6, Proposition 2.3.4], we can prescribe

$$\langle \boldsymbol{v}_h{\cdot}\boldsymbol{n}_{K_1}, \psi_F^k\rangle_{\widetilde{F}} := 0 \qquad 1 \le k \le 2.$$

Moreover, restricted to F, ψ_F^k create a basis of $\mathcal{P}_1(F)$, whereas restricted to K_1, they belong to $\mathcal{P}_1(K_1)$. Thus, we can also set

$$\langle \boldsymbol{v}_h{\cdot}\boldsymbol{n}_{K_1}, \psi_F^k\rangle_F := (\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \nabla \psi_F^k)_{K_1} \qquad 1 \le k \le 3.$$

With the choices made so far, we see that (A.9) holds for the three hat functions ψ_F^k, $1 \le k \le 3$. Finally, consider ψ_F^4, the hat basis function corresponding to the vertex opposite to the face F. Restricted to F, it is zero, so that

$$\langle \boldsymbol{v}_h{\cdot}\boldsymbol{n}_{K_1}, \psi_F^4\rangle_F = 0.$$

Moreover, restricted to $\widetilde{F}$, it completes ψ_F^1 and ψ_F^2 (restricted to $\widetilde{F}$) to create a basis of $\mathcal{P}_1(\widetilde{F})$, and restricted to K_1, it belongs to $\mathcal{P}_1(K_1)$, so that we can prescribe

$$\langle \boldsymbol{v}_h{\cdot}\boldsymbol{n}_{K_1}, \psi_F^4\rangle_{\widetilde{F}} := (\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \nabla \psi_F^4)_{K_1}.$$

Thus, (A.9) also holds for ψ_F^4, and since ψ_F^k, $1 \le k \le 4$, restricted to K_1 create a basis of $\mathcal{P}_1(K_1)$, (A.9) holds true, and a unique $\boldsymbol{\epsilon}_h^1$ from (A.8) exists.

Step 1b: any element K_i with $|\mathcal{F}_i^\sharp| = 1$. We now investigate those consecutive elements K_i which are such that two neighbors of K_i have not been passed through yet. This means that exactly one neighbor of K_i, say K_j, has already been passed through, so there is one face F^j in the set $\mathcal{F}_i^\sharp$. Since $\boldsymbol{v}_h{\cdot}\boldsymbol{n}_{K_i} = \boldsymbol{\epsilon}_h^j{\cdot}\boldsymbol{n}_{K_i}$ on F^j is requested in (A.8), (A.9) asks if can one choose $\boldsymbol{v}_h{\cdot}\boldsymbol{n}_{K_i}$ on $\partial K_i \setminus \{F_i^{\mathrm{ext}}, F^j\}$ such that

$$\langle \boldsymbol{v}_h{\cdot}\boldsymbol{n}_{K_i}, q_h\rangle_{\partial K_i \setminus \{F_i^{\mathrm{ext}}, F^j\}} = (\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \nabla q_h)_{K_i} - \langle \boldsymbol{\epsilon}_h^j{\cdot}\boldsymbol{n}_{K_i}, q_h\rangle_{F^j} \tag{A.10}$$

for all $q_h \in \mathcal{P}_1(K_i)$, which is still undetermined, giving 4 conditions for the 6 remaining degrees of freedom. The reasoning is similar as for K_1. Still denoting F and $\widetilde{F}$ the two remaining faces and ψ_F^k, $1 \le k \le 4$ the hat basis functions, we again have

$$\langle \boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i}, \psi_F^3 \rangle_{\widetilde{F}} = 0, \qquad \langle \boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i}, \psi_F^4 \rangle_F = 0.$$

Moreover, imposing

$$\begin{aligned}
&\langle \boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i}, \psi_F^k \rangle_{\widetilde{F}} := 0 \qquad 1 \le k \le 2,\\
&\langle \boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i}, \psi_F^4 \rangle_{\widetilde{F}} := (\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \nabla \psi_F^4)_{K_i} - \langle \boldsymbol{\epsilon}_h^j \cdot \boldsymbol{n}_{K_i}, \psi_F^4 \rangle_{F^j},\\
&\langle \boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i}, \psi_F^k \rangle_F := (\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \nabla \psi_F^k)_{K_i} - \langle \boldsymbol{\epsilon}_h^j \cdot \boldsymbol{n}_{K_i}, \psi_F^k \rangle_{F^j} \qquad 1 \le k \le 3
\end{aligned}$$

yields (A.10), and $\boldsymbol{\epsilon}_h^i$ exists.

Step 1c: any element K_i with $|\mathcal{F}_i^\sharp| = 2$. We now investigate those consecutive elements K_i which are such that only one neighbor of K_i has not been passed through yet, with K_{j_1} and K_{j_2} already passed through and faces F^{j_1}, F^{j_2} in the set $\mathcal{F}_i^\sharp$. Denote F the only remaining face, so that $F_i^{\rm ext}$, F^{j_1}, F^{j_2}, and F are the four faces of the tetrahedron K_i. As in (A.9) and (A.10), we need to ensure that

$$\langle \boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i}, q_h \rangle_F = (\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \nabla q_h)_{K_i} - \langle \boldsymbol{\epsilon}_h^{j_1} \cdot \boldsymbol{n}_{K_i}, q_h \rangle_{F^{j_1}} - \langle \boldsymbol{\epsilon}_h^{j_2} \cdot \boldsymbol{n}_{K_i}, q_h \rangle_{F^{j_2}} \tag{A.11}$$

for all $q_h \in \mathcal{P}_1(K_1)$. This time, the system is over-determined in that we request 4 conditions for the 3 remaining degrees of freedom of the normal components $\boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i}$ on the face F. As above, we can impose

$$\langle \boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i}, \psi_F^k \rangle_F := (\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \nabla \psi_F^k)_{K_i} - \langle \boldsymbol{\epsilon}_h^{j_1} \cdot \boldsymbol{n}_{K_i}, \psi_F^k \rangle_{F^{j_1}} - \langle \boldsymbol{\epsilon}_h^{j_2} \cdot \boldsymbol{n}_{K_i}, \psi_F^k \rangle_{F^{j_2}} \quad 1 \le k \le 3,$$

which fixes $\boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i}$ on the face F. Now, noting that $\langle \boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i}, \psi_F^4 \rangle_F = 0$, it follows that to prove (A.11), we need to show that

$$(\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \nabla \psi_F^4)_{K_i} - \langle \boldsymbol{\epsilon}_h^{j_1} \cdot \boldsymbol{n}_{K_i}, \psi_F^4 \rangle_{F^{j_1}} - \langle \boldsymbol{\epsilon}_h^{j_2} \cdot \boldsymbol{n}_{K_i}, \psi_F^4 \rangle_{F^{j_2}} = 0. \tag{A.12}$$

To prove (A.12), recall from property (i) of the enumeration (giving that all other elements sharing the edge e common to F^{j_1} and F^{j_2} have been already passed through) and the previous steps, see (A.9) and (A.10), that

$$\langle \boldsymbol{\epsilon}_h^j \cdot \boldsymbol{n}_{K_j}, \psi_F^4 \rangle_{\partial K_j} = (\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \nabla \psi_F^4)_{K_j} \tag{A.13}$$

for all the tetrahedra K_j sharing the edge e, different from K_i. Moreover, by the Green theorem, for $\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}$ given by (A.3), there holds

$$-(\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \nabla \psi_F^4)_{\omega_{\boldsymbol{a}}} = (\nabla \cdot \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \psi_F^4)_{\omega_{\boldsymbol{a}}} \overset{(A.3)}{=} (g^{\boldsymbol{a}}, \psi_F^4)_{\omega_{\boldsymbol{a}}}.$$

Thus, using the crucial patchwise orthogonality assumption (A.1c),

$$0 = (\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \nabla \psi_F^4)_{\omega_{\boldsymbol{a}}} = (\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \nabla \psi_F^4)_{\omega_e}, \tag{A.14}$$

where ω_e is the part of $\omega_{\boldsymbol{a}}$ corresponding to the elements sharing the edge e; the last equality holds since in the vertex patch subdomain $\omega_{\boldsymbol{a}}$, ψ_F^4 is only supported on the edge patch subdomain ω_e. Denote by $\widetilde{\omega}_e$ the part of ω_e without the element K_i. Then the normal traces orientation, the Green theorem first applied on $\widetilde{\omega}_e$ and later individually on K_j, the fact that all $\nabla \cdot \boldsymbol{\epsilon}_h^j = 0$, and the zero normal trace boundary conditions $\boldsymbol{\epsilon}_h^j \cdot \boldsymbol{n}_{K_j} = 0$ on the faces $F_j^{\rm ext}$ together with the zero values of ψ_F^4 give

$$\begin{aligned}
&- \langle \boldsymbol{\epsilon}_h^{j_1} \cdot \boldsymbol{n}_{K_i}, \psi_F^4 \rangle_{F^{j_1}} - \langle \boldsymbol{\epsilon}_h^{j_2} \cdot \boldsymbol{n}_{K_i}, \psi_F^4 \rangle_{F^{j_2}} = \langle \boldsymbol{\epsilon}_h^{j_1} \cdot \boldsymbol{n}_{\widetilde{\omega}_e}, \psi_F^4 \rangle_{F^{j_1}} + \langle \boldsymbol{\epsilon}_h^{j_2} \cdot \boldsymbol{n}_{\widetilde{\omega}_e}, \psi_F^4 \rangle_{F^{j_2}}\\
&= \sum_{j;\, K_j^\circ \subset \widetilde{\omega}_e} \big\{ (\boldsymbol{\epsilon}_h^j, \nabla \psi_F^4)_{K_j} + (\underbrace{\nabla \cdot \boldsymbol{\epsilon}_h^j}_{=0}, \psi_F^4)_{K_j} \big\} \sum_{j;\, K_j^\circ \subset \widetilde{\omega}_e} \langle \boldsymbol{\epsilon}_h^j \cdot \boldsymbol{n}_{K_j}, \psi_F^4 \rangle_{\partial K_j}\\
&\overset{(A.13)}{=} \sum_{j;\, K_j^\circ \subset \widetilde{\omega}_e} (\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \nabla \psi_F^4)_{K_j} \overset{(A.14)}{=} -(\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \nabla \psi_F^4)_{K_i},
\end{aligned} \tag{A.15}$$

which is (A.12). Thus, there exists a unique $\boldsymbol{\epsilon}_h^i$ from (A.8) also on this K_i.

Step 1d: the last element $K_{|\mathcal{T}_{\boldsymbol{a}}|}$. According to property (ii) of the enumeration, the last element $K_{|\mathcal{T}_{\boldsymbol{a}}|}$ is such that $|\mathcal{F}_{|\mathcal{T}_{\boldsymbol{a}}|}^\sharp| = 3$, so that all the neighbors have been already passed through. Consequently, all the degrees of freedom of $\boldsymbol{v}_h$ are fixed from the last three constraints in (A.8), and we need to show that $\nabla{\cdot}\boldsymbol{v}_h = 0$, i.e., that

$$(\nabla{\cdot}\boldsymbol{v}_h, q_h)_{K_{|\mathcal{T}_{\boldsymbol{a}}|}} = 0 \qquad \forall q_h \in \mathcal{P}_1(K_{|\mathcal{T}_{\boldsymbol{a}}|}),$$

since $\nabla{\cdot}\boldsymbol{v}_h \in \mathcal{P}_1(K_{|\mathcal{T}_{\boldsymbol{a}}|})$. This is equivalent to verifying that

$$\begin{aligned} 0 = (\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \nabla\psi_F^k)_{K_{|\mathcal{T}_{\boldsymbol{a}}|}} &- \langle \boldsymbol{\epsilon}_h^{j_1}{\cdot}\boldsymbol{n}_{K_{|\mathcal{T}_{\boldsymbol{a}}|}}, \psi_F^k\rangle_{F^{j_1}} \\ &- \langle \boldsymbol{\epsilon}_h^{j_2}{\cdot}\boldsymbol{n}_{K_{|\mathcal{T}_{\boldsymbol{a}}|}}, \psi_F^k\rangle_{F^{j_2}} - \langle \boldsymbol{\epsilon}_h^{j_3}{\cdot}\boldsymbol{n}_{K_{|\mathcal{T}_{\boldsymbol{a}}|}}, \psi_F^k\rangle_{F^{j_3}} \end{aligned} \tag{A.16}$$

for all $1 \le k \le 4$, where F^{j_1}, F^{j_2}, F^{j_3} are the three faces in $\mathcal{F}^{\sharp}_{|\mathcal{T}_{\boldsymbol{a}}|}$ and ψ_F^k are the hat basis functions associated with the four vertices of $K_{|\mathcal{T}_{\boldsymbol{a}}|}$. As in (A.14), (A.1c) implies

$$0 = (\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \nabla\psi_F^k)_{\omega_{\boldsymbol{a}}} \qquad 1 \le k \le 4. \tag{A.17}$$

Moreover, as in (A.13),

$$\langle \boldsymbol{\epsilon}_h^{j}{\cdot}\boldsymbol{n}_{K_j}, \psi_F^k\rangle_{\partial K_j} = (\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \nabla\psi_F^k)_{K_j} \qquad 1 \le k \le 4 \tag{A.18}$$

is satisfied on all elements K_j of the patch $\mathcal{T}_{\boldsymbol{a}}$ other than $K_{|\mathcal{T}_{\boldsymbol{a}}|}$. Let $\widetilde{\omega}_{\boldsymbol{a}}$ correspond to the patch subdomain $\omega_{\boldsymbol{a}}$ without the element $K_{|\mathcal{T}_{\boldsymbol{a}}|}$. Then, as in (A.15),

$$\begin{aligned} &- \langle \boldsymbol{\epsilon}_h^{j_1}{\cdot}\boldsymbol{n}_{K_{|\mathcal{T}_{\boldsymbol{a}}|}}, \psi_F^k\rangle_{F^{j_1}} - \langle \boldsymbol{\epsilon}_h^{j_2}{\cdot}\boldsymbol{n}_{K_{|\mathcal{T}_{\boldsymbol{a}}|}}, \psi_F^k\rangle_{F^{j_2}} - \langle \boldsymbol{\epsilon}_h^{j_3}{\cdot}\boldsymbol{n}_{K_{|\mathcal{T}_{\boldsymbol{a}}|}}, \psi_F^k\rangle_{F^{j_3}} \\ &= \langle \boldsymbol{\epsilon}_h^{j_1}{\cdot}\boldsymbol{n}_{\widetilde{\omega}_{\boldsymbol{a}}}, \psi_F^k\rangle_{F^{j_1}} + \langle \boldsymbol{\epsilon}_h^{j_2}{\cdot}\boldsymbol{n}_{\widetilde{\omega}_{\boldsymbol{a}}}, \psi_F^k\rangle_{F^{j_2}} + \langle \boldsymbol{\epsilon}_h^{j_3}{\cdot}\boldsymbol{n}_{\widetilde{\omega}_{\boldsymbol{a}}}, \psi_F^k\rangle_{F^{j_3}} \\ &= \sum_{j;\, K_j^\circ \subset \widetilde{\omega}_{\boldsymbol{a}}} \big\{ (\boldsymbol{\epsilon}_h^j, \nabla\psi_F^k)_{K_j} + \underbrace{(\nabla{\cdot}\boldsymbol{\epsilon}_h^j}_{=0}, \psi_F^k)_{K_j} \big\} = \sum_{j;\, K_j^\circ \subset \widetilde{\omega}_{\boldsymbol{a}}} \langle \boldsymbol{\epsilon}_h^{j}{\cdot}\boldsymbol{n}_{K_j}, \psi_F^k\rangle_{\partial K_j} \\ &\overset{(A.18)}{=} \sum_{j;\, K_j^\circ \subset \widetilde{\omega}_{\boldsymbol{a}}} (\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \nabla\psi_F^k)_{K_j} \overset{(A.17)}{=} -(\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \nabla\psi_F^k)_{K_{|\mathcal{T}_{\boldsymbol{a}}|}} \end{aligned}$$

for all $1 \le k \le 4$, which is (A.16). Thus, there exists a unique $\boldsymbol{\epsilon}_h^{|\mathcal{T}_{\boldsymbol{a}}|}$ of (A.8) on $K_{|\mathcal{T}_{\boldsymbol{a}}|}$.

Step 2. Stability bound. We now proceed with the proof of (A.6).

Step 2a: generic stability bound. Set $\overline{\boldsymbol{\rho}}_h^{\boldsymbol{a}} := \boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}$, and denote by $\boldsymbol{\rho}_h^{\boldsymbol{a}}$ the $\boldsymbol{L}^2(\omega_{\boldsymbol{a}})$-orthogonal projection of $\overline{\boldsymbol{\rho}}_h^{\boldsymbol{a}}$ onto $[\mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}})]^3$. Considering the Euler–Lagrange equations associated with (A.5), it is clear that we can equivalently replace $\overline{\boldsymbol{\rho}}_h^{\boldsymbol{a}}$ by $\boldsymbol{\rho}_h^{\boldsymbol{a}}$ in the definition of $\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}$ (reasoning as in points 1 and 2 of Section 7.5). Furthermore, because (A.8) is a quadratic minimization problem with linear constraints, the operator $T : [\mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}})]^3 \ni \boldsymbol{\rho}_h^{\boldsymbol{a}} \to \bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_1(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ (well-defined from *Step 1*) is linear. Since both $[\mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}})]^3$ and $\boldsymbol{\mathcal{RT}}_1(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ are finite-dimensional spaces, the operator T is continuous, and there exists a constant $C(\mathcal{T}_{\boldsymbol{a}})$ such that

$$\|\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \le C(\mathcal{T}_{\boldsymbol{a}}) \|\boldsymbol{\rho}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \le C(\mathcal{T}_{\boldsymbol{a}}) \|\overline{\boldsymbol{\rho}}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}, \tag{A.19}$$

where we used the fact that $\boldsymbol{\rho}_h^{\boldsymbol{a}}$ is defined from $\overline{\boldsymbol{\rho}}_h^{\boldsymbol{a}}$ by projection in the last inequality. The constant $C(\mathcal{T}_{\boldsymbol{a}})$ is independent of the polynomial degree q but depends on the patch $\mathcal{T}_{\boldsymbol{a}}$ in an unspecified way. To make the dependence explicit, we resort in the next step to a reference patch $\widehat{\mathcal{T}}$ and a divergence-preserving Piola mapping.

Step 2b: explicit stability bound. For a fixed shape-regularity parameter $\kappa_{\mathcal{T}_h}$ from Section 2.2, there exists a maximal number of elements $N(\kappa_{\mathcal{T}_h})$ allowed in any patch $\mathcal{T}_{\boldsymbol{a}}$. In turn, for any $N(\kappa_{\mathcal{T}_h})$, there exists a finite set of reference patches $\{\widehat{\mathcal{T}}\}$ such that for all vertex patches $\mathcal{T}_{\boldsymbol{a}}$, there exists a reference patch $\widehat{\mathcal{T}}$ and a bilipschitz mapping $\mathcal{M} : \omega_{\boldsymbol{a}} \to \widehat{\omega}$ ($\widehat{\omega}$ being the open domain associated with $\widehat{\mathcal{T}}$) such that $\mathcal{M}|_K$ is an affine mapping between the tetrahedron $K \in \mathcal{T}_{\boldsymbol{a}}$ and a tetrahedron $\widehat{K} \in \widehat{\mathcal{T}}$. Given $\mathcal{M}$, we then define the Piola mapping $\boldsymbol{\phi}^{\mathrm{d}} : \boldsymbol{L}^2(\omega_{\boldsymbol{a}}) \to \boldsymbol{L}^2(\widehat{\omega})$ such that $\boldsymbol{\phi}^{\mathrm{d}} : \boldsymbol{\mathcal{RT}}_1(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \to \boldsymbol{\mathcal{RT}}_1(\widehat{\mathcal{T}}) \cap \boldsymbol{H}_0(\mathrm{div}, \widehat{\omega})$, see [17, Chapter 7.2].

Crucially, we observe that for all $\widehat{K} \in \widehat{\mathcal{T}}$, $\boldsymbol{v} \in \boldsymbol{L}^2(K)$, and $\widehat{\boldsymbol{r}}_h \in [\mathcal{P}_0(\widehat{K})]^3$ there exists $\boldsymbol{r}_h \in [\mathcal{P}_0(K)]^3$ such that $(\boldsymbol{\phi}^{\mathrm{d}}(\boldsymbol{v}), \widehat{\boldsymbol{r}}_h)_{\widehat{K}} = (\boldsymbol{v}, \boldsymbol{r}_h)_K$, since, elementwise, the Piola transform amounts to a multiplication by a constant matrix and a change of coordinates. It follows that $\boldsymbol{\phi}^{\mathrm{d}}$ maps the minimization set of (A.5) on $\mathcal{T}_{\boldsymbol{a}}$ into the minimization set of the equivalent problem set on $\widehat{\mathcal{T}}$ with constraints $\boldsymbol{\phi}^{\mathrm{d}}(\overline{\boldsymbol{\rho}}_h^{\boldsymbol{a}})$.

Now, on the reference patch $\widehat{\mathcal{T}}$, if $\widehat{\boldsymbol{\epsilon}}$ is the minimizer of (A.5) with the datum $\boldsymbol{\phi}^{\mathrm{d}}(\overline{\boldsymbol{\rho}}_h^{\boldsymbol{a}})$, we conclude from *Step 2a* that

$$\|\widehat{\boldsymbol{\epsilon}}\|_{\widehat{\omega}} \le C(\kappa_{\mathcal{T}_h}) \|\boldsymbol{\phi}^{\mathrm{d}}(\overline{\boldsymbol{\rho}}_h^{\boldsymbol{a}})\|_{\widehat{\omega}} \le C(\kappa_{\mathcal{T}_h}) \|\boldsymbol{\phi}^{\mathrm{d}}\| \|\overline{\boldsymbol{\rho}}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}},$$

where $\|\boldsymbol{\phi}^{\mathrm{d}}\|$ denotes the operator norm of $\boldsymbol{\phi}^{\mathrm{d}}$ mapping $\boldsymbol{L}^2(\omega_{\boldsymbol{a}})$ into $\boldsymbol{L}^2(\widehat{\omega})$. On the other hand, since $(\boldsymbol{\phi}^{\mathrm{d}})^{-1}(\widehat{\boldsymbol{\epsilon}})$ belongs the minimization set on $\mathcal{T}_{\boldsymbol{a}}$, we have

$$\|\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}} - \overline{\boldsymbol{\rho}}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \leq \|(\boldsymbol{\phi}^{\mathrm{d}})^{-1}(\widehat{\boldsymbol{\epsilon}}) - \overline{\boldsymbol{\rho}}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \leq \|(\boldsymbol{\phi}^{\mathrm{d}})^{-1}\|\|\widehat{\boldsymbol{\epsilon}}\|_{\widehat{\omega}} + \|\overline{\boldsymbol{\rho}}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}},$$

so that

$$\|\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}} - \overline{\boldsymbol{\rho}}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \leq (1 + C(\kappa_{\mathcal{T}_h})\|(\boldsymbol{\phi}^{\mathrm{d}})^{-1}\|\|\boldsymbol{\phi}^{\mathrm{d}}\|)\|\overline{\boldsymbol{\rho}}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}.$$

At that point, we conclude the proof since $\|(\boldsymbol{\phi}^{\mathrm{d}})^{-1}\|\|\boldsymbol{\phi}^{\mathrm{d}}\|$ only depends on $\kappa_{\mathcal{T}_h}$ (see [17, Chapter 7.2]) and $\|\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \leq \|\overline{\boldsymbol{\rho}}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} + \|\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}} - \overline{\boldsymbol{\rho}}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}$. □

With the help of Lemmas A.3 and A.4, the proof of Theorem A.2 follows:

Proof of Theorem A.2. It follows straightforwardly from (A.3) and (A.5) that $\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}} + \bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}$ lies in the minimization set of (A.2). Consequently, the existence and uniqueness of (A.2) follows since the minimized functional in (A.2) is convex. Moreover, the triangle inequality together with Lemma A.4 implies

$$\|\boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \leq \|\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}} + \bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}} - \boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \leq \|\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} + \|\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}} - \boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \lesssim \|\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}} - \boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}},$$

and we conclude by Lemma A.3. □

B Decomposition of a divergence-free piecewise polynomial with an elementwise orthogonality into local divergence-free contributions

Let $q \geq 0$ be a fixed integer and recall the notation of Section 2; namely, $\boldsymbol{I}_q^{\mathcal{RT}}$ is the canonical elementwise q-degree Raviart–Thomas interpolate from (2.5) and $\lesssim$ means smaller or equal to up to a constant only depending on the mesh shape-regularity parameter $\kappa_{\mathcal{T}_h}$. The following result is of independent interest:

Theorem B.1 (Decomposition of a divergence-free Raviart–Thomas piecewise polynomial with an elementwise orthogonality constraint into local divergence-free contributions)**.** *Let*

$$\boldsymbol{\delta}_h \in \boldsymbol{\mathcal{RT}}_q(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) \quad \text{with} \quad \nabla\cdot\boldsymbol{\delta}_h = 0 \tag{B.1}$$

be a divergence-free q-degree Raviart–Thomas piecewise polynomial that is elementwise orthogonal to vector-valued constants,

$$(\boldsymbol{\delta}_h, \boldsymbol{r}_h)_K = 0 \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^3, \forall K \in \mathcal{T}_h. \tag{B.2}$$

Then there exists a unique solution to the q'-degree Raviart–Thomas elementwise minimizations, $q' = q$ or $q' = q+1$,

$$\boldsymbol{\delta}_h^{\boldsymbol{a}}|_K := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_{q'}(K) \\ \nabla\cdot\boldsymbol{v}_h = 0 \\ \boldsymbol{v}_h\cdot\boldsymbol{n}_K = \boldsymbol{I}_{q'}^{\mathcal{RT}}(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)\cdot\boldsymbol{n}_K \text{ on } \partial K}} \|\boldsymbol{v}_h - \boldsymbol{I}_{q'}^{\mathcal{RT}}(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)\|_K^2 \tag{B.3}$$

for all tetrahedra $K \in \mathcal{T}_h$ and all vertices $\boldsymbol{a} \in \mathcal{V}_K$. This yields patchwise divergence-free contributions

$$\boldsymbol{\delta}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_{q'}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \quad \text{with} \quad \nabla\cdot\boldsymbol{\delta}_h^{\boldsymbol{a}} = 0 \qquad \forall \boldsymbol{a} \in \mathcal{V}_h, \tag{B.4}$$

decomposing $\boldsymbol{\delta}_h$ as

$$\boldsymbol{\delta}_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\delta}_h^{\boldsymbol{a}}. \tag{B.5}$$

Moreover, for all tetrahedra $K \in \mathcal{T}_h$ and all vertices $\boldsymbol{a} \in \mathcal{V}_K$, there hold the local stability estimates

$$\|\boldsymbol{\delta}_h^{\boldsymbol{a}} - \boldsymbol{I}_{q'}^{\mathcal{RT}}(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)\|_K \lesssim \|\boldsymbol{\delta}_h\|_K, \tag{B.6a}$$

$$\|\boldsymbol{\delta}_h^{\boldsymbol{a}}\|_K \lesssim_{q'} \|\boldsymbol{\delta}_h\|_K, \tag{B.6b}$$

where $\lesssim_{q'}$ means $\lesssim$ for $q' = q+1$ and up to a constant only depending on the mesh shape-regularity parameter $\kappa_{\mathcal{T}_h}$ and the degree q when $q' = q$.

Remark B.2 (The two settings $q' = q$ or $q' = q+1$ in Theorem B.1)**.** *With the choice $q' = q$, the contributions $\boldsymbol{\delta}_h^{\boldsymbol{a}}$ in Theorem B.1 stay in the same degree Raviart–Thomas space as the datum $\boldsymbol{\delta}_h$, but, unfortunately, the stability* (B.6b) *is not necessarily q-robust. For q-robustness, the choice $q' = q+1$, increasing the degree of $\boldsymbol{\delta}_h^{\boldsymbol{a}}$ by one, is to be used. Note that in this case, the Raviart–Thomas interpolator $\boldsymbol{I}_{q'}^{\mathcal{RT}}$ can be disregarded, since then $\boldsymbol{I}_{q'}^{\mathcal{RT}}(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h) = \psi^{\boldsymbol{a}}\boldsymbol{\delta}_h$.*

Proof. Let $\boldsymbol{\delta}_h$ satisfy (B.1) and (B.2). We address (B.3)–(B.6) in four steps.

Step 1. Proof of the well-posedness of (B.3). Fix $K \in \mathcal{T}_h$ and $\boldsymbol{a} \in \mathcal{V}_K$. The existence and uniqueness of $\boldsymbol{\delta}_h^{\boldsymbol{a}}|_K$ from (B.3) are classical following, e.g., [6], when the Neumann compatibility condition $\langle \boldsymbol{I}_{q'}^{\mathcal{RT}}(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)\cdot\boldsymbol{n}_K, 1\rangle_{\partial K} = 0$ is satisfied. This can be shown via (2.5a), the Green theorem, the assumption $\nabla\cdot\boldsymbol{\delta}_h = 0$ in (B.1), and the elementwise orthogonality assumption (B.2) (note that $(\nabla\psi^{\boldsymbol{a}})|_K \in [\mathcal{P}_0(K)]^3$) as

$$\langle \boldsymbol{I}_{q'}^{\mathcal{RT}}(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)\cdot\boldsymbol{n}_K, 1\rangle_{\partial K} = \langle \psi^{\boldsymbol{a}}\boldsymbol{\delta}_h\cdot\boldsymbol{n}_K, 1\rangle_{\partial K} = \langle \boldsymbol{\delta}_h\cdot\boldsymbol{n}_K, \psi^{\boldsymbol{a}}\rangle_{\partial K} = (\nabla\cdot\boldsymbol{\delta}_h, \psi^{\boldsymbol{a}})_K + (\boldsymbol{\delta}_h, \nabla\psi^{\boldsymbol{a}})_K = 0.$$

Step 2. Proof of the stability estimates (B.6). Still for a fixed $K \in \mathcal{T}_h$ and $\boldsymbol{a} \in \mathcal{V}_K$, consider the problem

$$\hat{\boldsymbol{\delta}}_h^{\boldsymbol{a}}|_K := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_{q'}(K) \\ \nabla\cdot\boldsymbol{v}_h = -\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\delta}_h \\ \boldsymbol{v}_h\cdot\boldsymbol{n}_K = 0 \text{ on } \partial K}} \|\boldsymbol{v}_h\|_K^2. \tag{B.7}$$

This problem is again well-posed since, from (B.2), $(\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\delta}_h, 1)_K = (\boldsymbol{\delta}_h, \nabla\psi^{\boldsymbol{a}})_K = 0$; moreover, $\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\delta}_h \in \mathcal{P}_q(K) \subset \mathcal{P}_{q'}(K)$, since from $\nabla\cdot\boldsymbol{\delta}_h = 0$, it follows that $\boldsymbol{\delta}_h|_K \in [\mathcal{P}_q(K)]^3$ (see, e.g., [6, Corollary 2.3.1]). It follows that $\hat{\boldsymbol{\delta}}_h^{\boldsymbol{a}}|_K = \boldsymbol{\delta}_h^{\boldsymbol{a}}|_K - \boldsymbol{I}_{q'}^{\mathcal{RT}}(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)|_K$; indeed, crucial for this is the commuting property (2.6) yielding, on the simplex K, $\nabla\cdot(\boldsymbol{I}_{q'}^{\mathcal{RT}}(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)) = \mathcal{P}_{q'}(\nabla\cdot(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)) = \mathcal{P}_{q'}(\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\delta}_h) = \nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\delta}_h$. Problem (B.7) fits the framework of [20, Lemma A.3] with $r_F = 0$, $r_K = -\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\delta}_h$, and $p = q'$, so that

$$\|\boldsymbol{\delta}_h^{\boldsymbol{a}} - \boldsymbol{I}_{q'}^{\mathcal{RT}}(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)\|_K = \|\hat{\boldsymbol{\delta}}_h^{\boldsymbol{a}}\|_K = \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_{q'}(K) \\ \nabla\cdot\boldsymbol{v}_h = -\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\delta}_h \\ \boldsymbol{v}_h\cdot\boldsymbol{n}_K = 0 \text{ on } \partial K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},K) \\ \nabla\cdot\boldsymbol{v} = -\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\delta}_h \\ \boldsymbol{v}\cdot\boldsymbol{n}_K = 0 \text{ on } \partial K}} \|\boldsymbol{v}\|_K = \|\nabla\zeta_K\|_K.$$

Here, by primal–dual equivalence, $\zeta_K \in H^1_*(K)$ is such that

$$(\nabla\zeta_K, \nabla v)_K = -(\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\delta}_h, v)_K \qquad \forall v \in H^1_*(K)$$

with $H^1_*(K) := \{v \in H^1(K);\ (v,1)_K = 0\}$. On this space, the Poincaré inequality gives $\|v\|_K \lesssim h_K\|\nabla v\|_K$, so that the Cauchy–Schwarz inequality and mesh shape regularity yield

$$\|\nabla\zeta_K\|_K = \max_{\substack{v \in H^1_*(K) \\ \|\nabla v\|_K = 1}} (\nabla\zeta_K, \nabla v)_K = \max_{\substack{v \in H^1_*(K) \\ \|\nabla v\|_K = 1}} -(\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\delta}_h, v)_K \lesssim \|\nabla\psi^{\boldsymbol{a}}\|_{\infty,K}\|\boldsymbol{\delta}_h\|_K h_K \lesssim \|\boldsymbol{\delta}_h\|_K.$$

Combining the two above estimates gives the desired stability result (B.6a). The other stability result (B.6b) follows immediately from (B.6a) by the triangle inequality together with the non-q-robust stability bound $\|\boldsymbol{I}_{q'}^{\mathcal{RT}}(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)\|_K \lesssim_{q'} \|\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h\|_K \le \|\boldsymbol{\delta}_h\|_K$ when $q' = q$, whereas $\|\boldsymbol{I}_{q'}^{\mathcal{RT}}(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)\|_K = \|\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h\|_K \le \|\boldsymbol{\delta}_h\|_K$ when $q' = q+1$.

Step 3. Proof of the patchwise properties (B.4). The first property in (B.4) follows from the prescription of the normal components in (B.3), whereas the second one is the divergence prescription in (B.3).

Step 4. Proof of the decomposition (B.5). Finally, in order to prove (B.5), set $\tilde{\boldsymbol{\delta}}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\delta}_h^{\boldsymbol{a}}$. Now fix an element $K \in \mathcal{T}_h$ and remark that from the normal trace constraint in (B.3) and the linearity of the interpolator $\boldsymbol{I}_{q'}^{\mathcal{RT}}$, on ∂K,

$$\tilde{\boldsymbol{\delta}}_h\cdot\boldsymbol{n}_K = \sum_{\boldsymbol{a}\in\mathcal{V}_K} \boldsymbol{I}_{q'}^{\mathcal{RT}}(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)\cdot\boldsymbol{n}_K = \boldsymbol{I}_{q'}^{\mathcal{RT}}\left[\sum_{\boldsymbol{a}\in\mathcal{V}_K}(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)\right]\cdot\boldsymbol{n}_K = \boldsymbol{I}_{q'}^{\mathcal{RT}}(\boldsymbol{\delta}_h)\cdot\boldsymbol{n}_K = \boldsymbol{\delta}_h\cdot\boldsymbol{n}_K$$

also using the partition of unity (2.1). Similarly, by the divergence constraint in (B.3) and $\nabla\cdot\boldsymbol{\delta}_h = 0$ from (B.1), on K,

$$\nabla\cdot\tilde{\boldsymbol{\delta}}_h = \sum_{\boldsymbol{a}\in\mathcal{V}_K} \nabla\cdot\boldsymbol{\delta}_h^{\boldsymbol{a}} = 0 = \nabla\cdot\boldsymbol{\delta}_h.$$

Consequently, $(\tilde{\boldsymbol{\delta}}_h - \boldsymbol{\delta}_h)|_K \in \boldsymbol{\mathcal{RT}}_{q'}(K)$ has zero normal trace and divergence. Moreover, the Euler–Lagrange conditions of problem (B.3) state

$$(\boldsymbol{\delta}_h^{\boldsymbol{a}} - \boldsymbol{I}_{q'}^{\mathcal{RT}}(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h), \boldsymbol{v}_h)_K = 0 \quad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_{q'}(K) \text{ with } \nabla\cdot\boldsymbol{v}_h = 0 \text{ and } \boldsymbol{v}_h\cdot\boldsymbol{n}_K = 0 \text{ on } \partial K.$$

Summing this over all vertices $\boldsymbol{a} \in \mathcal{V}_K$ and using again the linearity of $\boldsymbol{I}_{q'}^{\mathcal{RT}}$, we infer

$$(\tilde{\boldsymbol{\delta}}_h - \boldsymbol{\delta}_h, \boldsymbol{v}_h)_K = 0 \qquad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_{q'}(K) \text{ with } \nabla\cdot\boldsymbol{v}_h = 0 \text{ and } \boldsymbol{v}_h\cdot\boldsymbol{n}_K = 0 \text{ on } \partial K,$$

so that indeed $\tilde{\boldsymbol{\delta}}_h = \boldsymbol{\delta}_h$ on any mesh element $K \in \mathcal{T}_h$. □

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