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# The Central Valuations Monad

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## Abstract

We give a commutative valuations monad  $\mathcal{Z}$  on the category **DCPO** of dcpo's and Scott-continuous functions. Compared to the commutative valuations monads given in [2], our new monad  $\mathcal{Z}$  is larger and it contains all push-forward images of valuations on the unit interval  $[0, 1]$  along lower semi-continuous maps. We believe that this new monad will be useful in giving domain-theoretic denotational semantics for statistical programming languages with continuous probabilistic choice.

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## 1 Introduction

The valuations monad  $\mathcal{V}$  on the category **DCPO** of dcpo's and Scott-continuous functions is a staple of the domain-theoretic approach for denotational semantics of programming languages with probabilistic choice and recursion [3, 4]. For a dcpo  $D$ ,  $\mathcal{V}D$  consists of *subprobability valuations* on  $D$ , which are the Scott-continuous functions  $\nu$  from the set  $\sigma D$  of Scott open subsets of  $D$  to  $[0, 1]$  satisfying *strictness* ( $\nu(\emptyset) = 0$ ) and *modularity* ( $\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$ ). The set  $\mathcal{V}D$  is a dcpo in the *stochastic order*:  $\nu_1 \leq \nu_2$  if and only if  $\nu_1(U) \leq \nu_2(U)$  for all  $U \in \sigma D$ . The *unit* of  $\mathcal{V}$  at dcpo  $D$  is the map  $\eta_D: D \rightarrow \mathcal{V}D :: x \mapsto \delta_x$ , where  $\delta_x$  is the *Dirac valuation* at  $x$ , defined by  $\delta_x(U) = 1$  if  $x \in U$  and  $\delta_x(U) = 0$  otherwise. For a Scott-continuous map  $f: D \rightarrow \mathcal{V}E$ , the *Kleisli extension*  $f^\dagger$  of  $f$  is defined by  $f^\dagger(\nu)(U) = \int_{x \in X} f(x)(U) d\nu$  for  $\nu \in \mathcal{V}D$  and  $U \in \sigma E$ . The integral in this definition is a Choquet type integral: for a general Scott-continuous function  $h: D \rightarrow [0, 1]$ , the value of  $\int_{x \in X} h d\nu$  is defined to be the Riemann integral  $\int_0^1 \nu(h^{-1}(t, 1)) dt$ . Following this, the action of  $\mathcal{V}$  on a Scott-continuous function  $g: D \rightarrow E$  between dcpo's  $D$  and  $E$  is  $\mathcal{V}(g) \stackrel{\text{def}}{=} (\eta_E \circ g)^\dagger$ ; concretely, for  $\nu \in \mathcal{V}D$  and  $U \in \sigma E$ ,  $\mathcal{V}(g)(\nu)(U) = \nu(g^{-1}(U))$ . Subprobability valuations on general topological spaces and the corresponding integral of lower semi-continuous functions against subprobability valuations can be defined similarly [3].

While it is well-known that  $\mathcal{V}$  can be restricted to a commutative monad on the category **DOM** of domains and Scott-continuous functions, it is unknown whether  $\mathcal{V}$  can be restricted to any Cartesian closed full subcategory of **DOM**. This is known as the *Jung-Tix problem* [5].

One may note that the category **DCPO** itself is Cartesian closed and  $\mathcal{V}$  is a monad on it. What does one lose if we use the category **DCPO** and monad  $\mathcal{V}$  for semantics? A short answer is that compared to **DOM**,  $\mathcal{V}$  is not known to be *commutative* over **DCPO**, which is an important property for the denotational semantics of programming languages.



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Commutativity of  $\mathcal{V}$  over **DCPO** is equivalent to showing the following Fubini-style equation

$$\int_{x \in D} \int_{y \in E} h(x, y) d\mu d\nu = \int_{y \in E} \int_{x \in D} h(x, y) d\nu d\mu \quad (1)$$

holds for all dcpo's  $D$  and  $E$ , all Scott-continuous functions  $h: D \times E \rightarrow [0, 1]$  and all  $\nu \in \mathcal{V}D, \mu \in \mathcal{V}E$ . As pointed out in [2], the main difficulty in establishing (1) over **DCPO** is that the Scott topology on the product dcpo  $D \times E$  may be different from the product topology  $\sigma D \times \sigma E$ . Actually, we do know that Equation (1) holds for those functions  $h$  that are continuous when  $D \times E$  is given the product topology  $\sigma D \times \sigma E$  ([4, Lemma 2.37]).

Instead of directly proving (1), we showed (together with Lindenhovius) how to construct three submonads of  $\mathcal{V}$  that are commutative on **DCPO**, and used each one to give a sound and (strongly) adequate semantics to PFPC (Probabilistic FixPoint Calculus) [2]. The simplest of those three monads is the monad  $\mathcal{M}$ . For each dcpo  $D$ ,  $\mathcal{M}D$  is defined to be the smallest sub-dcpo of  $\mathcal{V}D$  that contains  $\mathcal{S}D$ , the family of *simple valuations* on  $D$ , where a simple valuation is a finite convex sum of Dirac valuations. The other two commutative monads are denoted  $\mathcal{W}$  and  $\mathcal{P}$  and the following inclusions hold for each dcpo  $D: \mathcal{S}D \subseteq \mathcal{M}D \subseteq \mathcal{W}D \subseteq \mathcal{P}D \subseteq \mathcal{V}D$ .

Each of our three monads is large enough to interpret *discrete* probabilistic choice in PFPC [2]. However, it is unclear if any of these monads is large enough to interpret *continuous* probabilistic choice. In this note, we define a new commutative valuations monad  $\mathcal{Z}$  on the category **DCPO** which is larger than  $\mathcal{M}, \mathcal{W}$  and  $\mathcal{P}$  with the hope of addressing this problem.

## 2 Central Valuations

Our idea for defining  $\mathcal{Z}$  is inspired by the notion of centre in group theory (which always forms an abelian subgroup) and the notion of centre of a premonoidal category (which always forms a monoidal subcategory) [6].

► **Definition 1.** A subprobability valuation  $\nu$  on a dcpo  $D$  is called a *central valuation* if for any dcpo  $E$ , any valuation  $\mu$  on  $E$ , and any Scott-continuous function  $h: D \times E \rightarrow [0, 1]$ , we have

$$\int_{x \in D} \int_{y \in E} h(x, y) d\mu d\nu = \int_{y \in E} \int_{x \in D} h(x, y) d\nu d\mu.$$

We shall write  $\mathcal{Z}D$  for the set of all central valuations on a dcpo  $D$ .

It is easy to see that simple valuations are central, and that the central valuations are closed under directed suprema under the stochastic order. Thus, for each dcpo  $D$ ,  $\mathcal{Z}D$  is a sub-dcpo of  $\mathcal{V}D$  containing  $\mathcal{S}D$ . Moreover, we have the following theorem, which can be proved using the *disintegration formula* in [1].

► **Theorem 2.** The assignment  $\mathcal{Z}(-)$  extends to a commutative monad over the category **DCPO** when equipped with the (co)restricted monad operations of  $\mathcal{V}$ . In other words,  $\mathcal{Z}$  is a commutative submonad of  $\mathcal{V}$ .

**Proof.** The unit of  $\mathcal{Z}$  at dcpo  $D$  sends each  $x \in D$  to  $\delta_x$  which is obviously a central valuation.

Let  $f: C \rightarrow \mathcal{Z}D$  be a Scott-continuous function. Then  $f$  can also be viewed as a Scott-continuous map from  $C$  to  $\mathcal{V}D$ , since  $\mathcal{Z}D$  is a sub-dcpo of  $\mathcal{V}D$ . We prove that  $f^\dagger: \mathcal{V}C \rightarrow \mathcal{V}D$  maps central valuations on  $C$  to central valuations on  $D$ . Towards this end, we pick  $\mu$

from  $\mathcal{ZC}$ , and assume that  $E$  is a dcpo,  $\nu$  is an arbitrary subprobability valuation on  $E$  and  $h: D \times E \rightarrow [0, 1]$  is a Scott-continuous map. Then by the disintegration formula (see Lemma 3.1(iii) in [1]) we have that

$$\int_{y \in E} \int_{x \in D} h(x, y) d(f^\dagger(\mu)) d\nu = \int_{y \in E} \int_{t \in C} \int_{x \in D} h(x, y) df(t) d\mu d\nu,$$

and the right side of the equation is equal to

$$\int_{t \in C} \int_{x \in D} \int_{y \in E} h(x, y) d\nu df(t) d\mu$$

by the fact that  $f(t), t \in D$  and  $\mu$  are central valuations. Again, by the disintegration formula that is just  $\int_{x \in D} \int_{y \in E} h(x, y) d\nu df^\dagger(\mu)$ . Hence we have proved that  $f^\dagger(\mu)$  is indeed a central valuation provided that  $\mu$  is. Similar arguments show that the monadic strength also (co)restricts as required. The corresponding (co)restrictions of the monadic operations of  $\mathcal{V}$  to  $\mathcal{Z}$  validate that  $\mathcal{Z}$  is a strong monad on **DCPO**. The commutativity of  $\mathcal{Z}$ , which is equivalent to Equation (1) holding for all dcpo's  $D$  and  $E$  and central valuations  $\mu$  and  $\nu$  on them, is then obvious by definition of  $\mathcal{Z}$ . ◀

In fact, it is proved in [2] that all *point-continuous valuations* are central and therefore  $SD \subseteq MD \subseteq WD \subseteq PD \subseteq ZD \subseteq VD$  for each dcpo  $D$ . Therefore  $\mathcal{Z}$  is the largest commutative submonad of  $\mathcal{V}$  known so far. Furthermore, observe that  $\mathcal{Z} = \mathcal{V}$  iff  $\mathcal{V}$  is a commutative monad on **DCPO**. The latter has been an open problem since 1989, and our simple observation leads us to believe  $\mathcal{Z}$  is a very large commutative submonad of  $\mathcal{V}$ .

It is not difficult to see that, in order to model sampling against continuous probability distributions on the interval  $[0, 1]$ , the monad used for the semantics should at least contain the push-forward images of the Lebesgue valuation on  $[0, 1]$  (equipped with the metric topology) along lower semi-continuous maps. We can demonstrate even more is true of our new monad  $\mathcal{Z}$  (see Theorem 4 below). For this, let us first recall that a space  $X$  is called *core-compact* if the set  $\mathcal{O}X$  of all open subsets of  $X$  is a continuous lattice in the inclusion order. Equivalently,  $X$  is core-compact if and only if for each open subset  $U$  of  $X$  and  $x \in U$ , there exists an open subset  $V$  such that  $x \in V \ll U$ , where  $V \ll U$  means that  $V$  is *way-below*  $U$  in the sense of domain theory. Many important spaces are core-compact. For example, each locally compact space is core-compact, and in particular, the unit interval with the usual topology is compact Hausdorff, hence locally compact hence core-compact.

► **Lemma 3.** *Let  $X$  be a core-compact topological space. Let  $D, E$  be dcpos, and  $f: X \rightarrow D$  a lower semi-continuous map, i.e.,  $f$  is continuous when  $D$  is equipped with the Scott topology. Then the map  $f \times \text{id}_E: X \times \Sigma E \rightarrow D \times E$  is also lower semi-continuous, where  $\Sigma E$  denotes the topological space  $(E, \sigma E)$  and  $X \times \Sigma E$  is the topological product of  $X$  and  $\Sigma E$ .*

**Proof.** First, we assume that  $X$  is core-compact and prove that  $f \times \text{id}_E$  is lower semi-continuous. Towards this end, we pick a Scott open subset  $O$  of  $D \times E$ , and assume that  $f \times \text{id}_E((x_0, e_0)) \in O$ , that is  $(f(x_0), e_0) \in O$ . We must find an open neighbourhood  $U$  of  $x_0$  in  $X$  and a Scott open neighbourhood  $V$  of  $e_0$  in  $E$  such that  $f \times \text{id}_E(U \times V) \subseteq O$ . We let  $A = \{x \in X \mid (f(x), e_0) \in O\}$ . Then  $A$  is just  $f^{-1}(O_{e_0})$ , where  $O_{e_0} = \{d \in D \mid (d, e_0) \in O\}$ . Since  $f: X \rightarrow D$  is lower semi-continuous and  $O_{e_0}$  is Scott open in  $D$ , we know that  $A$  is an open neighbourhood of  $x_0$  in  $X$ . Now the core-compactness of  $X$  enables us to find, in  $X$ , an open subset  $U$  and a sequence of open subsets  $U_i, i \in \mathbb{N}$  such that  $x_0 \in U \ll \dots \ll U_n \dots \ll U_1 \ll A$ . For each  $U_n, n \in \mathbb{N}$ , we define  $V_n = \{e \mid f(x, e) \in O \text{ for all } x \in U_n\}$  and

let  $V = \bigcup_{n \in \mathbb{N}} V_n$ . Since for each  $n \in \mathbb{N}$ ,  $U_n \subseteq A$ , we have for all  $x \in U_n$ ,  $(f(x), e_0) \in O$ . Thus we know that  $e_0 \in V_n$  for each  $n \in \mathbb{N}$ , and hence  $e_0 \in V$ . Moreover, for any  $(x, e) \in U \times V$ , there exists a natural number  $n$  such that  $e \in V_n$ , then it follows that  $f \times \text{id}_E((x, e)) = (f(x), e) \in f(U) \times V_n \subseteq f(U_n) \times V_n \subseteq O$ . The last inclusion is due to the construction of  $V_n$ . To sum up, it is true that  $f \times \text{id}_E(U \times V) \subseteq O$ . Since  $U$  is an open subset of  $X$  which contains  $x_0$  and  $e_0 \in V$ , we finish the proof by showing that  $V$  is Scott open in  $E$ . To this end we let  $\{e_i\}_{i \in I}$  be a directed subset of  $E$  with  $\sup_{i \in I} e_i \in V$ . For each  $i \in I$ , set  $W_i = \{x \in X \mid (f(x), e_i) \in O\}$ . It is easy to see that  $\{W_i \mid i \in I\}$  is a directed family of open subsets of  $X$ . Since  $\sup_{i \in I} e_i \in V = \bigcup_{n \in \mathbb{N}} V_n$ ,  $\sup_{i \in I} e_i$  is in some  $V_n$ . This means that for each  $x \in U_n$ ,  $(f(x), \sup_{i \in I} e_i) \in O$ . Because  $O$  is Scott open, for each  $x \in U_n$ , there exists  $i \in I$  such that  $(f(x), e_i) \in O$ , i.e.,  $x \in W_i$ . Hence we have that  $\sup_{i \in I} e_i \in V_n \subseteq \bigcup_{i \in I} W_i$ . Remember that  $U_{n+1} \ll U_n$ , it follows that  $U_{n+1} \subseteq W_j$  for some  $j \in I$ . By definition of  $W_j$ , we know that  $f(U_{n+1}) \times \{e_j\} \subseteq O$ , which means that  $e_j \in V_{n+1}$ , this time by definition of  $V_{n+1}$ . So we find  $j \in I$  with  $e_j \in V_{n+1} \subseteq V$ , and indeed  $V$  is Scott open in  $E$ . ◀

► **Theorem 4.** *Let  $X$  be a core-compact space and  $f$  be a lower semi-continuous map from  $X$  to a dcpo  $D$ . If  $\nu$  is a valuation on  $X$ , then  $f_*(\nu) \stackrel{\text{def}}{=} \lambda O \in \sigma D. \nu(f^{-1}(O))$ , the push-forward valuation along  $f$ , is a central valuation on  $D$ . In particular, for a core-compact dcpo  $D$ , all valuations on  $D$  are central, i.e.,  $\mathcal{V}D = \mathcal{Z}D$ .*

**Proof.** By definition, we prove for any dcpo  $E$ , continuous valuations  $\mu$  on  $E$  and Scott-continuous map  $h: D \times E \rightarrow [0, 1]$  the equation

$$\int_{x \in D} \int_{y \in E} h(x, y) d\mu df_*(\nu) = \int_{y \in E} \int_{x \in D} h(x, y) df_*(\nu) d\mu$$

holds.

Note that for each  $y \in E$ , the map  $g \stackrel{\text{def}}{=} (x \mapsto \int_{y \in E} h(x, y) d\mu): D \rightarrow [0, 1]$  is Scott-continuous, and  $f: X \rightarrow D$  is lower semi-continuous. Hence for the left side of the above equation we have

$$\int_{x \in D} \int_{y \in E} h(x, y) d\mu df_*(\nu) = \int_{x \in X} g(f(x)) d\nu = \int_{x \in X} \int_{y \in E} h(f(x), y) d\mu d\nu.$$

The first equality follows from the so-called *change-of-variable* formula, which can be found in [4]. As a consequence of it, we also have that

$$\int_{y \in E} \int_{x \in D} h(x, y) df_*(\nu) d\mu = \int_{y \in E} \int_{x \in X} h(f(x), y) d\nu d\mu.$$

Since  $X$  is core-compact and the function  $f: X \rightarrow D$  is lower semi-continuous, by Lemma 3 we know that  $f \times \text{id}_E: X \times \Sigma E \rightarrow X \times Y$  is lower semi-continuous. This implies that the map  $(x, y) \mapsto h(f(x), y): X \times \Sigma E \rightarrow [0, 1]$  is lower semi-continuous. Hence by Lemma 2.37 in [4] we know that  $\int_{x \in X} \int_{y \in E} h(f(x), y) d\mu d\nu = \int_{y \in E} \int_{x \in X} h(f(x), y) d\nu d\mu$ , which finishes the proof.

The second claim is a straightforward consequence of the first one. ◀

► **Theorem 5.** *Let  $f: [0, 1] \rightarrow D$  be a lower semi-continuous map into a dcpo  $D$ . If  $\nu$  is any continuous valuation on  $[0, 1]$ , then  $f_*(\nu)$  is a central valuation on  $D$ .*

**Proof.** Since  $[0, 1]$  is core-compact in the usual topology, the result follows from Theorem 4. ◀

We have not been able to establish the above theorem for any of the monads  $\mathcal{M}$ ,  $\mathcal{W}$  or  $\mathcal{P}$ , so we believe that  $\mathcal{Z}$  is a promising candidate for modeling *continuous* probabilistic choice. We plan to address this in future work.

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