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Konstantin Zimenko[†], Andrey Polyakov[‡,†], and Denis Efimov[‡,†]

***Abstract*— The paper presents methods for nonlinear homogeneous systems representation in a canonical form allowing stability conditions to be given by a linear matrix inequality. The main restriction for a system to admit the required representation is that its right-hand side has to be bounded on a unit sphere. It is shown that some nonhomogeneous systems can also be presented in the canonical form. Based on canonical representation a stabilizing control algorithm for affine in control nonlinear systems is presented with LMI-based tuning procedure. The results are supported with numerical examples.**

I. Introduction

Homogeneity is a dilation symmetry, which is widely used for system analysis, control and observer design (see, for example, [1]-[25]). Linear systems and a lot of essentially nonlinear models of physical plants are homogeneous. Homogeneous differential equations/inclusions also appear as approximations [18], [20] and set-valued extensions [3], [23] of nonlinear systems. Additional interest to the class of homogeneous systems is based on various useful stability and robustness features (nonasymptotic convergence, robustness with respect to external perturbations and time delays, etc.).

In the paper [14] a necessary and sufficient stability condition (stabilizability criterion) is presented. In some particular cases this condition can be formulated in terms of linear matrix inequalities (LMIs) that simultaneously simplifies the control parameter tuning. However, in general, this effect is achieved only for linear and close to linear systems (for example, [22], [24], [25], etc.). In [15] canonical representation of homogeneous systems is given. It allows stability conditions to be presented in the form of LMI even for sufficiently nonlinear systems. However, the class of homogeneous systems considered in [15] is too restrictive since the differentiability of the right-hand side of the systems is required.

In this paper several methods for representation in canonical homogeneous form are presented. It is demonstrated that to admit a representation in canonical homogeneous form the right-hand side of the system has to be bounded on a unit sphere. It is a significant extension of the class of homogeneous systems in comparison with [15]. In some cases stability analysis and control design problems for nonhomogeneous systems can be sufficiently simplified with the use of homogeneous representations (for example, based on homogeneous extension a system can be embedded in a homogeneous differential inclusion [19]). In this paper it is shown that some nonhomogeneous systems can also be presented in the canonical form. The representation is based on a combination of homogeneity with convex embedding. The procedure is conceptually similar to representation of nonlinear systems in the form of linear differential inclusions (see, for example, [26], [27], [28]). A new LMI-based control design procedure is developed for a class of nonlinear homogeneous systems using the canonical representation. Due to homogeneity the closed-loop system has many properties useful in practice.

The paper is organized in the following way. Notation used in the paper is given in Section II. Section III presents preliminaries employed in the paper. Sections IV and V describe the main results on canonical representation of homogeneous systems and control design method, respectively. Finally, concluding remarks are summarized in Section VI.

II. Notation

Through the paper the following notation will be used:

- $\mathbb{R}_+ = \{x \in \mathbb{R} \colon x > 0\}$, where $\mathbb{R}$ is the field of real numbers;
- the symbol $\overline{1,m}$ is used to denote a sequence of integers $1,...,m$;
- $\text{diag}\{a_i\}_{i=1}^n$ is the diagonal matrix with the elements a_i on the main diagonal;
- the eigenvalues of a matrix $G \in \mathbb{R}^{n\times n}$ are denoted by $\lambda_i(G)$, $i = \overline{1,n}$;
- $\mathcal{C}^n(X,Y)$ is the set of continuously differentiable (at least up to the order n) functions $X \to Y$, where X and Y are open subsets of $\mathbb{R}^n$;
- $e_s(i) = \Bigg(\underbrace{0 \cdots 0 \overbrace{\;1\;}^{i\text{th}} 0 \cdots 0}_{s \text{ components}}\Bigg)^T \in \mathbb{R}^s$, $s \geq 1$, is a vector of the canonical basis of $\mathbb{R}^s$;
- the inequality $P > 0$ ($P \geq 0$) means that a symmetric matrix $P = P^T \in \mathbb{R}^{n\times n}$ is positive definite (positive semi-definite);
- $\|\cdot\|$ denotes a norm;
- $\mathbb{S} = \{x \in \mathbb{R}^n \colon \|x\| = 1\}$ is the unit sphere in $\mathbb{R}^n$;
- $\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\right)$.

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III. Preliminaries

A. Stability Notions

Consider the system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, $f(0) = 0$.

Definition 1 [21], [23] *The origin of (1) is said to be globally finite-time stable if it is globally asymptotically stable and any solution $x(t, x_0)$ of the system (1) reaches the equilibrium point at some finite time moment, i.e., $x(t, x_0) = 0\ \forall t \geq T(x_0)$ and $x(t, x_0) \neq 0\ \forall t \in [0, T(x_0))$, $x_0 \neq 0$, where $T\colon \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$, $T(0) = 0$ is the settling-time function.*

Definition 2 [17] *A set $M \subset \mathbb{R}^n$ is said to be globally finite-time attractive for (1) if any solution $x(t, x_0)$ of (1) reaches M in some finite time moment $t = T(x_0)$ and remains there $\forall t \geq T(x_0)$, $T\colon \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is the settling-time function. It is fixed-time attractive if in addition the settling-time function $T(x_0)$ is globally bounded by some number $T_{\max} > 0$.*

Theorem 1 [21] *Suppose there exist a positive definite C^1 function V defined on an open neighborhood of the origin $D \subset \mathbb{R}^n$ and real numbers $C > 0$ and $\sigma \geq 0$, such that the following condition is true for the system (1)*

$$\dot{V}(x) \leq -CV^{\sigma}(x), \quad x \in D \setminus \{0\}.$$

Then depending on the value σ the origin is stable with different types of convergence:

- *if $\sigma = 1$, the origin is asymptotically stable;*
- *if $0 \leq \sigma < 1$, the origin is finite-time stable and*

$$T(x_0) \leq \frac{1}{C(1-\sigma)} V^{1-\sigma}(x_0);$$

- *if $\sigma > 1$ the origin is nearly fixed-time stable, i.e., it is asymptotically stable and, for every $\varepsilon \in \mathbb{R}_+$, the set $B = \{x \in D\colon V(x) < \varepsilon\}$ is fixed-time (independent on the initial values) attractive with*

$$T_{\max} = \frac{1}{C(\sigma - 1)\varepsilon^{\sigma-1}}.$$

If $D = \mathbb{R}^n$ and function V is radially unbounded, then the system (1) admits these properties globally.

B. Generalized Homogeneity

Homogeneity is a certain invariance of a mathematical object with respect to a group of transformations called dilations. In this paper we deal with the one-parameter group $\mathbf{d}(s) : \mathbb{R}^n \to \mathbb{R}^n$ of linear dilations given by

$$\tfrac{\partial}{\partial s}\mathbf{d}(s) = G_{\mathbf{d}}\mathbf{d}(s) = \mathbf{d}(s)G_{\mathbf{d}}, \\ \mathbf{d}(s) = e^{G_{\mathbf{d}}s} = \textstyle\sum_{i=0}^{+\infty} \frac{s^i G_{\mathbf{d}}^i}{i!}, \ \ s \in \mathbb{R},$$

where $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$ is an anti-Hurwitz matrix called the generator of the dilation [29].

Definition 4 [19] *The dilation $\mathbf{d}$ is said to be strictly monotone if $\exists \beta > 0 : \|\mathbf{d}(s)\| < e^{\beta s}$ for $s \leq 0$.*

Theorem 2 [14] *If $\mathbf{d}$ is a dilation in $\mathbb{R}^n$, then*

- *the generator matrix $G_{\mathbf{d}}$ is anti-Hurwitz and there exists a matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$PG_{\mathbf{d}} + G_{\mathbf{d}}^T P > 0, \quad P = P^T > 0. \tag{2}$$

- *the dilation $\mathbf{d}$ is strictly monotone with respect to the norm $\|x\|_P = \sqrt{x^T P x}$ for $x \in \mathbb{R}^n$ and P satisfying (2):*

$$\begin{array}{lll} e^{\alpha s} \leq \|\mathbf{d}(s)\|_P \leq e^{\beta s} & if & s \leq 0, \\ e^{\beta s} \leq \|\mathbf{d}(s)\|_P \leq e^{\alpha s} & if & s \geq 0, \end{array} \tag{3}$$

where $\alpha = \frac{1}{2}\lambda_{\max}\left(P^{\frac{1}{2}}G_{\mathbf{d}}P^{-\frac{1}{2}} + P^{-\frac{1}{2}}G_{\mathbf{d}}^T P^{\frac{1}{2}}\right)$, $\beta = \frac{1}{2}\lambda_{\min}\left(P^{\frac{1}{2}}G_{\mathbf{d}}P^{-\frac{1}{2}} + P^{-\frac{1}{2}}G_{\mathbf{d}}^T P^{\frac{1}{2}}\right)$ and $\|\mathbf{d}(s)\|_P = \sup_{x\neq 0} \frac{\|\mathbf{d}(s)x\|_P}{\|x\|_P}$.

Theorem 2 shows that any dilation $\mathbf{d}$ is strictly monotone if $\mathbb{R}^n$ is equipped with the weighted Euclidean norm $\|x\|_P = \sqrt{x^T P x}$, provided that the matrix $P > 0$ satisfies (2).

Definition 5 [9] *A vector field $f\colon \mathbb{R}^n \to \mathbb{R}^n$ (a function $h\colon \mathbb{R}^n \to \mathbb{R}$) is said to be $\mathbf{d}$-homogeneous of degree $\nu \in \mathbb{R}$ if*

$$f(\mathbf{d}(s)x) = e^{\nu s}\mathbf{d}(s)f(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R}. \tag{4}$$

$$(\textit{resp. } h(\mathbf{d}(s)x) = e^{\nu s}h(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R}.) \tag{5}$$

Let $\mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ (respectively $\mathbb{H}_{\mathbf{d}}(\mathbb{R}^n)$) be the set of vector fields $\mathbb{R}^n \to \mathbb{R}^n$ (respectively functions $\mathbb{R}^n \to \mathbb{R}$) satisfying the identity (4) (respectively (5)), which are continuous on $\mathbb{R}^n \setminus \{0\}$. Let $\deg_{\mathbb{F}_{\mathbf{d}}}(f)$ (respectively $\deg_{\mathbb{H}_{\mathbf{d}}}(f)$) denote the homogeneity degree of $f \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ (respectively $h \in \mathbb{H}_{\mathbf{d}}(\mathbb{R}^n)$).

A special case of homogeneous function is a homogeneous norm [14], [8]: a continuous positive definite $\mathbf{d}$-homogeneous function of degree 1. For monotone dilations we define the canonical homogeneous norm $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ as

$$\|x\|_{\mathbf{d}} = e^{s_x} \text{ for } x \neq 0, \text{ where } s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\| = 1 \tag{6}$$

and, by continuity, we assign $\|0\|_{\mathbf{d}} = 0$. Note that $\|\mathbf{d}(s)x\|_{\mathbf{d}} = e^s\|x\|_{\mathbf{d}}$ and

$$\|\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x\| = 1. \tag{7}$$

Lemma 1 [14] *If $\mathbf{d}$ is a strictly monotone dilation, then*

- *the homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is Lipschitz continuous outside the origin;*
- *if the norm $\|\cdot\|$ is smooth outside the origin, then the homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is also smooth outside the origin, $\frac{\partial\|\mathbf{d}(-s)x\|}{\partial s} < 0$ if $s \in \mathbb{R}$, $x \in \mathbb{R}^n \setminus \{0\}$ and*

$$\frac{\partial \|x\|_{\mathbf{d}}}{\partial x} = \left.\frac{\|x\|_{\mathbf{d}} \left.\frac{\partial\|z\|}{\partial z}\right|_{z=\mathbf{d}(-s)x}}{\left.\frac{\partial\|z\|}{\partial z}\right|_{z=\mathbf{d}(-s)x} G_{\mathbf{d}}\mathbf{d}(-s)x}\right|_{s=\ln\|x\|_{\mathbf{d}}}. \tag{8}$$

Lemma 2 [14] *The vector field $f \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ is Lipschitz continuous (smooth) on $\mathbb{R}^n \setminus \{0\}$ if and only if it satisfies a Lipschitz condition (it is smooth) on the unit sphere $\mathbb{S}$, provided that $\mathbf{d}$ is strictly monotone on $\mathbb{R}^n$.*

If a function (or a vector field) is smooth, then homogeneity is inherited by its derivatives in a certain way.

Lemma 3 [1], [14], [15] *If $f \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$ ($h \in \mathbb{H}_{\mathbf{d}}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$), then*

$$e^{\deg_{\mathbb{H}_{\mathbf{d}}}(h)s} \frac{\partial h(x)}{\partial x} = \left.\frac{\partial h(z)}{\partial z}\right|_{z=\mathbf{d}(s)x} \mathbf{d}(s), \tag{9}$$

$$e^{\deg_{\mathbb{F}_{\mathbf{d}}}(f)s} \mathbf{d}(s) \frac{\partial f(x)}{\partial x} = \left.\frac{\partial f(z)}{\partial z}\right|_{z=\mathbf{d}(s)x} \mathbf{d}(s), \tag{10}$$

$$\frac{\partial h(x)}{\partial x} G_{\mathbf{d}} x = \deg_{\mathbb{H}_{\mathbf{d}}}(h) h(x), \tag{11}$$

$$\frac{\partial f(x)}{\partial x} G_{\mathbf{d}} x = (\deg_{\mathbb{F}_{\mathbf{d}}}(f) I_n + G_{\mathbf{d}}) f(x) \tag{12}$$

for $x \in \mathbb{R}^n \setminus \{0\}$ and $s \in \mathbb{R}$.

Due to topological equivalence of any $\mathbf{d}$-homogeneous system to a standard homogeneous one [14], all results about stability and robustness of standard and weighted homogeneous systems hold for $\mathbf{d}$-homogeneous systems as well. For example, the rate of convergence of homogeneous systems can be assessed by its homogeneity degree:

Theorem 3 [16] *An asymptotically stable $\mathbf{d}$-homogeneous system $\dot{x} = f(x)$, $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is*

- *finite-time stable if $\deg_{\mathbb{F}_{\mathbf{d}}}(f) < 0$;*
- *nearly fixed-time stable if $\deg_{\mathbb{F}_{\mathbf{d}}}(f) > 0$.*

The homogeneity provides many other advantages to analysis and design of nonlinear control system (see, for example, [13], [7], [5]).

IV. Convex embedding for homogeneous systems

A. Convex Embedding for Differentiable Homogeneous Systems

Let us consider a system in the form

$$\dot{x} = f(x), \tag{13}$$

where $f \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$, $\deg_{\mathbb{F}_{\mathbf{d}}}(f) = \nu \in \mathbb{R}$, $f(0) = 0$.

Let $f \in \mathcal{C}^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$. Define the bounds

$$\overline{g}_{ij} = \sup_{y \in \mathbb{R}^n : \|y\|=1} \left.\frac{\partial f_i(z)}{\partial z_j}\right|_{z=y},$$

$$\underline{g}_{ij} = \inf_{y \in \mathbb{R}^n : \|y\|=1} \left.\frac{\partial f_i(z)}{\partial z_j}\right|_{z=y},$$

that are always exist due to $f \in \mathcal{C}^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$. Denote the set of vertices defined by

$$\mathcal{V}_g = \left\{ \vartheta = \begin{pmatrix} \vartheta_{11} & \cdots & \vartheta_{1n} \\ \vdots & \ddots & \vdots \\ \vartheta_{n1} & \cdots & \vartheta_{nn} \end{pmatrix} \in \mathbb{R}^{n\times n} : \vartheta_{jl} \in \{\underline{g}_{jl}, \overline{g}_{jl}\}, j, l = \overline{1, n} \right\}.$$

Proposition 1 *Let the matrix $\nu I_n + G_{\mathbf{d}}$ be invertible. There exist $\alpha_i \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}_+ \cup \{0\}) : \sum_{i=1}^N \alpha_i(x) = 1$ and $M_i \in \mathcal{V}_g$, $i = 1, ..., N \leq 2^{n^2}$ such that every trajectory of (13) is also a trajectory of*

$$\dot{x} = \|x\|_{\mathbf{d}}^{\nu} \sum_{i=1}^{N} \alpha_i(\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x) \mathbf{d}(\ln \|x\|_{\mathbf{d}}) A_i \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x, \tag{14}$$

where $A_i = (\nu I_n + G_{\mathbf{d}})^{-1} M_i G_{\mathbf{d}}$.

Example 1 Consider the system (13) with

$$f(x) = \begin{pmatrix} -x_1 x_3 \\ -x_1 \\ x_2 \end{pmatrix},$$

that is $\mathbf{d}$-homogeneous of degree 1, where $G_{\mathbf{d}} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a generator of the dilation. According to Proposition 1 the function $f(x)$ can be represented in the form (14) with $\|x\| = \sqrt{x^T x}$ and the matrices

$$A_1 = \begin{pmatrix} 0.75 & 0 & 0.25 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.75 & 0 & -0.25 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} -0.75 & 0 & 0.25 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} -0.75 & 0 & -0.25 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In [15] the representation of the function f in the form like (14) was obtained by another constructions. Comparing with [15] the given result allows to obtain vertices A_i significantly closer to each other, that may simplify the stability analysis and control design for homogeneous systems.

Another advantage with respect to [15] is that the differentiability at the origin is not required. However, the condition $f \in \mathcal{C}^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$ also restricts sufficiently the class of homogeneous functions for representation in the form (14). This condition is relaxed below.

B. Convex Embedding for Non-Smooth Homogeneous Systems

Let $f(x) \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ in (13) be just bounded on a unit sphere $\mathbb{S}$ and $P \in \mathbb{R}^{n\times n}$ satisfies (2). Define the bounds

$$\overline{q}_{ij} = \sup_{y \in \mathbb{R}^n : \|y\|_P = 1} \frac{e_n^T(i) f(y) y_j}{y^T P G_{\mathbf{d}} y},$$

$$\underline{q}_{ij} = \inf_{y \in \mathbb{R}^n : \|y\|_P = 1} \frac{e_n^T(i) f(y) y_j}{y^T P G_{\mathbf{d}} y}$$

and denote the corresponding set of vertices $\mathcal{V}_q$ as above using the sets $\{\underline{q}_{ij}, \overline{q}_{ij}\}$ instead of $\{\underline{g}_{ij}, \overline{g}_{ij}\}$.

Proposition 2 *There exist $\alpha_i \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}_+ \cup \{0\})$: $\sum_{i=1}^N \alpha_i(x) = 1$ and $M_i \in \mathcal{V}_q$, $i = 1, ..., N \leq 2^{n^2}$ such that every trajectory of (13) is also a trajectory of (14), where $A_i = M_i P G_{\mathbf{d}}$.*

C. Extension to Some Non-Homogeneous Systems

The results of Propositions 1, 2 can be used for some nonhomogeneous dynamics representation in the form (14). Consider the system

$$\dot{x} = b(x) f(x), \tag{15}$$

where $b : \mathbb{R}^n \to \mathbb{R}$ is bounded on $\mathbb{R}^n$ and $f \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ is bounded on $\mathbb{S}$, $f(0) = 0$.

Define the bounds

$$\overline{p}_{ij} = \sup_{x \in \mathbb{R}^n} b(x) \left.\frac{e_n^T(i) f(z) z_j}{z^T P G_{\mathbf{d}} z}\right|_{z=\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x},$$

$$\underline{p}_{ij} = \inf_{x \in \mathbb{R}^n} b(x) \left.\frac{e_n^T(i) f(z) z_j}{z^T P G_{\mathbf{d}} z}\right|_{z=\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x}$$

and denote the corresponding set of vertices $\mathcal{V}_p$ (if $f(x)$ is differentiable on the unit sphere define the bounds as $\overline{p}_{ij} = \sup_{x\in\mathbb{R}^n} b(x)\frac{\partial f_i(z)}{\partial z_j}\Big|_{z=\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x}$, $\underline{p}_{ij} = \inf_{x\in\mathbb{R}^n} b(x)\frac{\partial f_i(z)}{\partial z_j}\Big|_{z=\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x}$).

Corollary 1 *Every trajectory of (15) is also a trajectory of (14), where A_i is defined as in Proposition 2 (Proposition 1), if $f(x)$ is bounded on the unit sphere ($f(x)$ is differentiable on the unit sphere).*

Remark 1 If one can represent the system under consideration

$$\dot{x} = \sum_{i=1}^{M} b_i(x) f_i(x), \tag{16}$$

with bounded $b_i : \mathbb{R}^n \to \mathbb{R}$, $f_i \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$, $M \le n$, then in order to obtain vertices closer to each other the given results can be used for each term independently (see the following example). Similarly, the systems $\dot{x} = \mathrm{diag}\{b_i(x)\}_{i=1}^n f(x)$ with bounded $b_i : \mathbb{R}^n \to \mathbb{R}$, $f_i \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ can be considered under the commutativity conditions that

$$\mathrm{diag}\{b_i(x)\}_{i=1}^n \mathbf{d}(\ln\|x\|_{\mathbf{d}}) = \mathbf{d}(\ln\|x\|_{\mathbf{d}})\mathrm{diag}\{b_i(x)\}_{i=1}^n.$$

Example 2 Consider the system

$$\dot{x} = \begin{pmatrix} \cos(x_1)x_2 \\ -x_3 \\ |x_3|^{0.5} \end{pmatrix}.$$

Let us rewrite the system in the form (16), where $M = 3$, $b_1(x) = \cos(x_1)$, $b_2(x) = b_3(x) = 1$,

$$f_1(x) = \begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix}, \; f_2(x) = \begin{pmatrix} 0 \\ -x_3 \\ 0 \end{pmatrix}, \; f_3(x) = \begin{pmatrix} 0 \\ 0 \\ |x_3|^{0.5} \end{pmatrix},$$

and $f_i(x)$, $i = \overline{1,3}$ are $\mathbf{d}$-homogeneous of degrees -0.5, where $G_{\mathbf{d}} = \left(\begin{smallmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Then using the results of Corollary 1, Proposition 1 and Proposition 2 for representing $f_1(x)$, $f_2(x)$ and $f_3(x)$ correspondingly the following matrices can be obtained

$$\begin{gathered}
A_{11} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; A_{12} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
A_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; A_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \\
A_{31} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.398 & 0.477 & 1 \end{pmatrix}, \; A_{32} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.398 & 0.477 & 1 \end{pmatrix}, \\
A_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.398 & -0.477 & 1 \end{pmatrix}, \; A_{34} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.398 & 0.477 & -1 \end{pmatrix}, \\
A_{35} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.398 & -0.477 & 1 \end{pmatrix}, \; A_{36} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.398 & 0.477 & -1 \end{pmatrix}, \\
A_{37} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.398 & -0.477 & -1 \end{pmatrix}, \; A_{38} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.398 & -0.477 & -1 \end{pmatrix}.
\end{gathered}$$

V. Control Design

The system representation (14) can be useful for control design (stability analysis), where parameter tuning (stability conditions) are presented in the form of linear matrix inequalities. In [15] it is shown how the canonical homogeneous form can be used for stabilization of the affine control system $\dot{x} = f_0(x) + \sum_{i=1}^m u_i f_i(x)$ with homogeneous functions f_0, f_i. In this section, we demonstrate how the canonical form can be utilized for control design for the system

$$\dot{x} = f(x) + Bu, \tag{17}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $B \in \mathbb{R}^{n\times m}$, and f can be represented in the form

$$f(x) = \|x\|_{\mathbf{d}}^{\nu} \sum_{i=1}^{N} \alpha_i(\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x)\mathbf{d}(\ln\|x\|_{\mathbf{d}})A_i\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x, \tag{18}$$

with corresponding matrices $A_i \in \mathbb{R}^{n\times n}$, $i = \overline{1,N}$, a generator $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$, $\nu \in \mathbb{R}$, where the dilation $\mathbf{d}$ is strictly monotone on $\mathbb{R}^n$ equipped with the norm $\|x\|_P = \sqrt{x^T P x}$, for $P > 0$ satisfying (2).

Theorem 4 *Let $G_{\mathbf{d}}B = \gamma B$, $\gamma \in \mathbb{R}$. Let $K \in \mathbb{R}^{m\times n}$, $H \in \mathbb{R}^{n\times n}$, $\eta, \delta \in \mathbb{R}_+$ be a solution of the bilinear matrix inequalities (BMIs)*

$$\begin{aligned}
&\eta P \ge H > 0, \\
&\theta_{\min}(PG_{\mathbf{d}} + G_{\mathbf{d}}^T P) \le H + HG_{\mathbf{d}} + G_{\mathbf{d}}^T H \le \theta_{\max}(PG_{\mathbf{d}} + G_{\mathbf{d}}^T P), \\
&(\theta_{\max}P + H)(A_i + BK) + (A_i + BK)^T(\theta_{\max}P + H) \le -\delta P, \\
&(\theta_{\min}P + H)(A_i + BK) + (A_i + BK)^T(\theta_{\min}P + H) \le -\delta P,
\end{aligned} \tag{19}$$

for $i = \overline{1,N}$ and some $\theta_{\min}, \theta_{\max} \in \mathbb{R}_+$, then the system (17) with

$$u(x) = \|x\|_{\mathbf{d}}^{\gamma+\nu} K\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x \tag{20}$$

is finite-time (nearly fixed-time) stable if $\nu < 0$ ($\nu > 0$) with the settling-time function estimation $T(x_0) \le -\frac{\eta^{1+\nu}V_0^{-\nu}}{\delta\nu}$ (for every $\varepsilon \in \mathbb{R}_+$, the set $B = \{x \in D\colon V(x) < \varepsilon\}$ is fixed-time attractive with $T_{\max} = \frac{\eta^{1+\nu}}{\delta\nu\varepsilon^{\nu}}$).

Remark 2 The BMIs (19) can be replaced by a more conservative LMIs for simplicity of parameters tuning. Indeed, with the use of Young relation, one can obtain that the inequalities

$$\begin{aligned}
&(\theta_{\max}P + H)(A_i + BK) + (A_i + BK)^T(\theta_{\max}P + H) \le -\delta P, \\
&(\theta_{\min}P + H)(A_i + BK) + (A_i + BK)^T(\theta_{\min}P + H) \le -\delta P
\end{aligned}$$

are satisfied if

$$\begin{aligned}
&(\theta_{\max}P + H)A_i + A_i^T(\theta_{\max}P + H) + \theta_{\max}PBK + \theta_{\max}K^T B^T P \\
&\quad + \tfrac{1}{2}(B^T H + K)^T(B^T H + K) \le -\delta P, \\
&(\theta_{\min}P + H)A_i + A_i^T(\theta_{\min}P + H) + \theta_{\min}PBK + \theta_{\min}K^T B^T P \\
&\quad + \tfrac{1}{2}(B^T H + K)^T(B^T H + K) \le -\delta P.
\end{aligned} \tag{21}$$

Then using Schur complement for (21) one can obtain that the BMIs (19) hold if the system of LMIs

$$\begin{aligned}
&\eta P \ge H > 0, \\
&\theta_{\min}(PG_{\mathbf{d}} + G_{\mathbf{d}}^T P) \le H + HG_{\mathbf{d}} + G_{\mathbf{d}}^T H \le \theta_{\max}(PG_{\mathbf{d}} + G_{\mathbf{d}}^T P), \\
&\begin{pmatrix} S_1 & HB + K^T \\ B^T H + K & 2I_m \end{pmatrix} \ge 0, \\
&\begin{pmatrix} S_2 & HB + K^T \\ B^T H + K & 2I_m \end{pmatrix} \ge 0, \quad i = \overline{1,N}
\end{aligned} \tag{22}$$

is satisfied, where

$$\begin{aligned}
S_1 = &-(\theta_{\max}P + H)A_i - A_i^T(\theta_{\max}P + H) \\
&-\theta_{\max}PBK - \theta_{\max}K^T B^T P - \delta P, \\
S_2 = &-(\theta_{\min}P + H)A_i - A_i^T(\theta_{\min}P + H) \\
&-\theta_{\min}PBK - \theta_{\min}K^T B^T P - \delta P.
\end{aligned}$$

Example 3 Consider the system (17) with $x \in \mathbb{R}^2$, $u \in \mathbb{R}$,

$$f(x) = \begin{pmatrix} x_2 + 0.5|x_1|^{1/3}|x_2|^{1/2} \\ 0.5|x_2|^{1/2} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The system is homogeneous with $\deg_{\mathbb{F}}(f) = -0.5$ and $G_{\mathbf{d}} = \begin{pmatrix} 1.5 & 0 \\ 0 & 1 \end{pmatrix}$. The condition $G_{\mathbf{d}}B = \gamma B$ is satisfied for $\gamma = 1$. Let us represent $f(x) = f_1(x) + f_2(x)$, where

$$f_1(x) = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, \quad f_2(x) = 0.5 \begin{pmatrix} |x_1|^{1/3}|x_2|^{1/2} \\ |x_2|^{1/2} \end{pmatrix}.$$

Then, applying Proposition 1 and Corollary 1 we obtain

$$f_1(x) = \|x\|_{\mathbf{d}}^{-0.5}\mathbf{d}(\ln\|x\|_{\mathbf{d}}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{d}(-\ln\|x\|_{\mathbf{d}})x,$$
$$f_2(x) = \|x\|_{\mathbf{d}}^{-0.5}\mathbf{d}(\ln\|x\|_{\mathbf{d}}) \times \begin{pmatrix} \{-0.037, 0.18\} & \{-0.24, 1.007\} \\ \{-0.062, 0.246\} & \{-1.524, 1.524\} \end{pmatrix} \mathbf{d}(-\ln\|x\|_{\mathbf{d}})x,$$

where the dilation is strictly monotone on $\mathbb{R}^n$ equipped with the weighted Euclidean norm $\|\cdot\|_P$, where

$$P = \begin{pmatrix} 9.73 & 1.5997 \\ 1.5997 & 0.5064 \end{pmatrix},$$

i.e., $f(x) = \|x\|_{\mathbf{d}}^{-0.5}\mathbf{d}(\ln\|x\|_{\mathbf{d}})A_i\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x$ with

$$A_i \in \begin{pmatrix} \{-0.037, 0.18\} & \{0.76, 2.007\} \\ \{-0.062, 0.246\} & \{-1.524, 1.524\} \end{pmatrix}, \quad i = \overline{1, 16}.$$

Solving (22) with $\theta_{\min} = 20.5$ and $\theta_{\max} = 21.5$ we obtain $K = \begin{pmatrix} -55.7383 & -22.2543 \end{pmatrix}$, and the closed-loop system is finite-time stable. Numerical simulations have been done using explicit Euler method with the step $h = 10^{-3}$ and the following initial conditions $x_1(0) = x_2(0) = 1$.

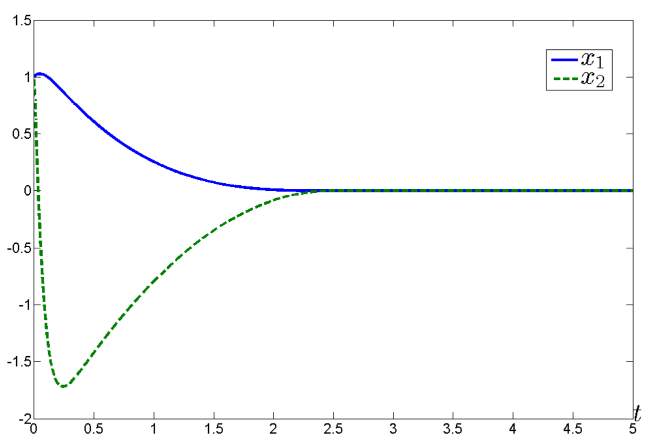

Fig. 1. Evolution of the system states

Remark 3 Note that the matrix P cannot be used as a solution of the inequalities (19) or (22) while it is used for representation of $f(x)$ in the form (18), i.e. the matrices A_i depend on P satisfying (2). In this regard, feasibility of (19) and (22) may depend on the choice of the matrix P.

Note, that the given control is a generalized version of [25]. Indeed, if $f(x)$ in (17) is linear, then the proposed result coincides with [25].

VI. CONCLUSIONS

The paper presents methods for homogeneous systems representation in the canonical form (14). In comparison with the closest analogue [15] the proposed result allows to significantly expand the class of systems possessing the form (14). The only restriction is that a homogeneous system should be bounded on a unit sphere. Also it is shown that some nonhomogeneous systems can be presented in the canonical form.

If a homogeneous system is presented in the canonical form it can be used for control design and stability analysis with parameter tuning (stability conditions) in the form of LMI (even for sufficiently nonlinear systems). In the paper the stabilizing control algorithm for affine in control nonlinear systems is presented. The settling time estimates are obtained. Due to homogeneity the closed-loop system has a number of robust properties (ISS, robustness with respect to delays, etc.).

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