ASYMPTOTIC ANALYSIS OF ABSTRACT TWO-SCALE WAVE PROPAGATION PROBLEMS *

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Abstract.
This work addresses the mathematical analysis, by means of asymptotic analysis, of linear wave propagation problems involving two scales, represented by a single small parameter, and written in an abstract setting. This abstract setting is defined using linear operators in Hilbert spaces and also enters the framework of semi-group theory. In this setting, we show, under some assumptions on the structure of the wave propagation problems, weak and strong convergence of solutions with respect to the small parameter towards the solution of a well-defined limit problem.

Key word. Wave equation, Asymptotic analysis, Operator theory

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1. Introduction. In the analysis of life science problems or industrial problems by means of partial differential equations (PDE), a class of problem that is frequently encountered is the class of two-scales problems. These problems – after non-dimensionnalisation – involve a non-dimensionnalised small parameter – denoted δ in this work. This parameter represents a ratio of two length scales or of two velocity scales for instance. In the most interesting cases, when δ is small, the behavior of the underlying PDE is not trivially understood and some kind of asymptotic analysis process is required. More precisely, one study the limit of the solution to the PDE when δ goes to zero to obtain at the limit simpler PDEs in which the small parameter δ is absent.

In applied mathematics such problems are studied for linear problems since decades, see for instance [3] in the context of periodic structures and homogenization, [8] for the study of the mechanical behavior of plates (i.e. thin domains) or [18, 10] for the study of stokes equations in nearly incompressible fluids. Many generalization of these works has been published since then, see for instances [7, 5, 1].

Although in semi-group theory the definition of an abstract framework based on linear operator theory [15] is now very standard (see [21, 2, 9] or [19, 4] in the context of control theory) we have find very few work in the literature that construct and analyze a similar abstract framework (see however [12]) that try to unify (part of) the previously mentioned problems.

This is precisely the objective of this work. Of course the mentioned class of problems is too large to be meaningfully represented in a single abstract framework, therefore we will restrict our analysis to wave propagation problems that possesses what is denoted later a two-scale structure. The abstract framework that is presented in this work encompasses many relevant applications in elastodynamics, electromagnetic or piezoelectric problems (see [13, 14, 6]). In the present paper we only focus on the description of the abstract framework and its rigorous analysis, while we leave its application to the mentioned electro-mechanical problems for a forthcoming work.

The paper is organized as follows.

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- Section 2 is devoted to the abstract definition of two-scale wave propagation problems.
  - We first define and study what is denoted a sequence of two-scale operators in Hilbert spaces. Formally these operators have the following form
$$G_\delta \;=\; \overline{G_0 + \delta\, G_1},$$
    where G_0 and G_1 are $\delta-$independent, densely defined and closed operators. In this section we also introduce the notion of weak, strong and strict scale separation properties. These definitions corresponds to assumptions on G_0 and G_1 that enable an analysis of the limit process when δ goes to zero.
  - We then define the family of wave propagation problems under study. In particular, a simplified formal definition of these problems is
$$\ddot{u}_\delta + G_\delta^*\, G_\delta u_\delta = 0.$$
    We provide existence, uniqueness and stability results for the solutions of these problems.
- In Section 3 we define the limit problem, i.e. the problem solved at the limit by the sequence of solutions of the family of the wave propagation problems. We give in this section existence, uniqueness and stability results as well as other properties. In particular, we define the (self-explanatory) notion of propagating and non-propagating solutions.
- Section 4 is devoted to the convergence analysis of the solutions of the family of problems towards the limit problems.
  - We first study the case where only the weak scale separation property holds. In that case we show a weak convergence property.
  - We show strong convergence property in the cases where the strict or the strong scale separation property holds.

To easy the reading all the proofs of Section 2 and 3 are given Section 5. Finally two appendices are devoted to more classical – yet not find in the literature – proofs. The first one is the proof of the existence, uniqueness and stability results for the family of wave propagation problems, and second one is a density proof that is required at some point in the analysis in order to show the convergence of solutions of the family of wave propagation problems to a limit problem.

2. Two-scale wave propagation problems.

2.1. Preliminary definition: Two-scale operators.

We assume given Hilbert spaces
$$\big(\mathcal{H};\, (\cdot,\cdot)_{\mathcal{H}}\big) \quad \text{and} \quad \big(\mathcal{G};\, (\cdot,\cdot)_{\mathcal{G}}\big)$$
as well as a sequence of unbounded operators $G_\delta : D(G_\delta) \subset \mathcal{H} \mapsto \mathcal{G}$ for $\delta \in (0,1)$.

Definition 2.1. *The sequence of operators $G_\delta : D(G_\delta) \subset \mathcal{H} \mapsto \mathcal{G}$ is said to be a sequence of two-scale operators if*

$$G_\delta \;=\; \overline{G_0 + \delta\, G_1}, \tag{2.1}$$

where the operators G_0 and G_1 are $\delta-$independent, densely defined and closed operator from $\mathcal{H}$ to $\mathcal{G}$.

In Definition 2.1 it is implicitly assumed that $G_0 + \delta G_1$ is closable hence G_δ is the closure of this operator. Without additional assumptions one can not deduce interesting properties from sequence of two-scale operators. In what follows we introduce three different properties, denoted respectively weak, strong and strict scale separation properties. These properties describe in more details the relation between the operators G_0 and G_1. We begin with the notion of weak scale separation.

Definition 2.2. *A sequence of two-scale operators G_δ satisfies the weak scale separation property if*

$$D(G_0) \cap D(G_1) \text{ dense in } \big(D(G_0) \,;\, \|\cdot\|_{\mathcal{H}} + \|G_0 \cdot\|_{\mathcal{G}} \big) \tag{2.2}$$

and

$$\begin{cases} D(G_0^*) \cap D(G_1^*) \text{ dense in } \big(D(G_0^*) \,;\, \|\cdot\|_{\mathcal{G}} + \|G_0^* \cdot\|_{\mathcal{H}} \big), & (a) \\ \operatorname{Ker} G_0^* \cap D(G_1^*) \text{ dense in } \big(\operatorname{Ker} G_0^* \,;\, \|\cdot\|_{\mathcal{G}} \big). & (b) \end{cases} \tag{2.3}$$

Note that the definition above straightforwardly implies that $D(G_0) \cap D(G_1)$ is dense in $D(G_0)$ for the norm $\|\cdot\|_{\mathcal{H}}$ hence $D(G_0) \cap D(G_1)$ is dense in $\mathcal{H}$, meaning that G_δ is densely defined since $D(G_0) \cap D(G_1)$ is a subspace of $D(G_\delta)$. We now explain the important consequences of the weak scale separation property.

Theorem 2.3. *For any sequence of two-scale operators G_δ such that property (2.3a) holds, let $\{u_\delta\}$ be a sequence in $D(G_\delta)$ satisfying, for some scalar C independent of δ,*

$$\|u_\delta\|_{\mathcal{H}} + \delta^{-1} \|G_\delta\, u_\delta\|_{\mathcal{G}} \leqslant C, \tag{2.4}$$

then, up to a subsequence, there exists $u_0 \in \operatorname{Ker} G_0$ such that $u_\delta \underset{\mathcal{H}}{\rightharpoonup} u_0$.

This property gives the intuitive result that, for a sequence u_δ converging towards u_0, if $\overline{G_0 + \delta\, G_1}\, u_\delta$ goes to zero then in some sense $G_0\, u_\delta$ also vanishes and we expect at the limit that $G_0\, u_0 = 0$.

To continue analyzing the benefit granted by the weak scale separation property we introduce the orthogonal projections operators in the kernel of G_0^* denoted Q_0 and defined by,

$$Q_0 \in \mathcal{L}(\mathcal{G}), \quad Q_0^2 = Q_0, \quad (g - Q_0\, g, h)_{\mathcal{G}} = 0 \quad \forall\, (g, h) \in \mathcal{G} \times \operatorname{Ker} G_0^*.$$

Note that this definition make sense since G_0^* is densely defined and closed hence its kernel is a closed subspace of $\mathcal{G}$.

Lemma 2.4. *For any sequence of two-scale operators G_δ satisfying (2.3) the operator $G_1^*\, Q_0$ is densely defined in $\mathcal{G}$ and $Q_0\, G_1$ is closable, densely defined in $\mathcal{H}$.*

The fact that $Q_0\, G_1$ is closable is essential in the analysis of problems involving the corresponding two-scale operator G_δ. The reason being that one can – thanks to the theorem below – construct a $\delta-$independent space that embeds $D(G_\delta)$ and that is, in general, a proper subspace $\mathcal{H}$.

Theorem 2.5. *For any sequence of two-scale operators G_δ satisfying (2.2) and (2.3) we have*

$$Q_0\, G_\delta \;\subset\; \delta\, \overline{Q_0\, G_1} \tag{2.5}$$

moreover

$$\forall\, u \in D(G_\delta), \quad \|\overline{Q_0\, G_1}\, u\|_{\mathcal{G}} \leqslant \delta^{-1} \|\, G_\delta\, u\|_{\mathcal{G}}$$

Note that the inclusion (2.5) implies first that $D(G_\delta) \subset D(\overline{Q_0\, G_1})$, this latter space being a Banach space when equipped with its graph norm, second, that $Q_0\, G_\delta$ and $\delta\, \overline{Q_0\, G_1}$ coincides on $D(G_\delta)$.

Theorem 2.6. *For any sequence of two-scale operators G_δ satisfying (2.2) and for any sequence $\{u_\delta\}$ in $D(G_\delta)$ satisfying, for some scalar C independent of δ,*

$$\|u_\delta\|_{\mathcal{H}} + \delta^{-1} \|G_\delta\, u_\delta\|_{\mathcal{G}} \leqslant C \tag{2.6}$$

and

$$(\delta^{-1} G_\delta\, u_\delta,\, G_0\, v)_{\mathcal{G}} \underset{\delta \to 0}{\longrightarrow} 0, \quad \forall\, v \in \mathcal{D}(G_0) \cap D(G_1), \tag{2.7}$$

then $\delta^{-1}(I - Q_0)\, G_\delta\, u_\delta \underset{\mathcal{G}}{\rightharpoonup} 0.$

To understand why the result above is important notice that, for any $u_\delta \in D(G_\delta)$ we have, thanks to Theorem 2.5,

$$\delta^{-1} G_\delta u_\delta = \delta^{-1}\big(Q_0 + (I - Q_0)\big)\, G_\delta u_\delta = \overline{Q_0\, G_1}\, u_\delta + \delta^{-1}(I - Q_0)\, G_\delta u_\delta, \tag{2.8}$$

hence, thanks to Theorem 2.6 the last term of (2.8) can be dealt with while the second one involves only a $\delta-$independent operator.

We introduce now the property of strong scale separation.

Definition 2.7. *A sequence of two-scale operators G_δ satisfies the strong scale separation property if*

$$D(G_0) \cap D(G_1) \text{ dense in } \big(\, D(G_0)\,;\, \|\cdot\|_{\mathcal{H}} + \|G_0\, \cdot\|_{\mathcal{G}}\,\big) \tag{2.9}$$

and

$$\forall\, (u, v) \in D(G_0) \times D(G_1) \quad (G_0\, u, G_1\, v)_{\mathcal{G}} = 0. \tag{2.10}$$

Many simplifications occur if the strong scale separation property holds as shown in the theorem below.

Theorem 2.8. *Any sequence of two-scale operators G_δ that satisfies the strong scale separation property satisfies the weak two-scale separation property,*

$$G_\delta = G_0 + \delta G_1 \quad \text{and} \quad \overline{Q_0\, G_1} = G_1.$$

Finally, we introduce the notion of strict scale separation property.

Definition 2.9. *A sequence of two-scale operators G_δ satisfies the strict scale separation property if it satisfies*

$$\forall\,(g,h) \in D(G_0^*) \times D(G_1^*), \quad (G_0^*\, g, G_1^*\, h)_{\mathcal{H}} = 0, \tag{2.11}$$

and

$$\forall\,(u,v) \in D(G_0) \times D(G_1) \quad (G_0\, u, G_1\, v)_{\mathcal{G}} = 0. \tag{2.12}$$

Theorem 2.10. *Any sequence of two-scale operators G_δ that satisfies the strict scale separation property satisfies the strong scale separation property and*

$$\operatorname{Ker} G_0 \cap D(G_1) \text{ dense in } \big(\operatorname{Ker} G_0\,;\, \|\cdot\|_{\mathcal{H}}\big).$$

Remark 2.11. If a sequence of two-scale operators G_δ satisfies the weak scale separation property and if

$$\forall\,(g,h) \in D(G_0^*) \times D(G_1^*), \quad (G_0^*\, g, G_1^*\, h)_{\mathcal{H}} = 0,$$

then the operator $H_\delta = \overline{G_0^* + \delta G_1^*} : D(H_\delta) \subset \mathcal{G} \mapsto \mathcal{H}$ is a sequence of two-scale operator satisfying the strong scale separation property (with the role of $\mathcal{H}$ and $\mathcal{G}$ reverse).

Remark 2.12. When the strict scale separation property holds one has

$$D(G_0) \cap D(G_1) = \big(\operatorname{Ker} G_0 \cap D(G_1)\big) \oplus \big(\operatorname{Ker} G_1 \cap D(G_0)\big)$$

where the orthogonality holds for the scalar product $(\cdot,\cdot)_{\mathcal{H}}$ and

$$(\cdot,\cdot)_{\mathcal{H}} + (G_0\,\cdot, G_0\,\cdot)_{\mathcal{G}} + (G_1\,\cdot, G_1\,\cdot)_{\mathcal{G}}.$$

Such property shows that, when the strict scale separation, functions belonging to $D(G_0) \cap D(G_1)$ (typically solutions of a given set of equations) can be naturally decomposed into two orthogonal elements that could potentially be computed independently.

2.2. A family of wave propagation problems.

Functional spaces. We assume given a vector space $\mathcal{V}_\delta$ (that depends on δ for the sake of generality) such that

$$\mathcal{V}_\delta \subset \mathcal{H} \quad \text{and} \quad \mathcal{V}_\delta \text{ is dense in } \mathcal{H}, \tag{2.13}$$

where the density holds with the norm $\|\cdot\|_{\mathcal{H}}$. We assume that $\mathcal{V}_\delta$ is a Banach space when equipped with a norm denoted $\|\cdot\|_{\mathcal{V}_\delta}$. Finally we introduce another space used later to define a bilinear form responsible for lower order effects (dissipation, couplings, ...) in the wave propagation problem. This space is independent of δ and denoted $\mathcal{B}$, it is a dense subspace of $\mathcal{H}$ and

$$\mathcal{V}_\delta \subset \mathcal{B} \subset \mathcal{H}.$$

The space $\mathcal{B}$ is a Hilbert space equipped with the scalar product $(\cdot,\cdot)_{\mathcal{B}}$ and corresponding norm $\|\cdot\|_{\mathcal{B}}$. We assume that there exists $C_{\mathcal{I}}$ – independent of δ – such that

$$\forall\, v \in \mathcal{V}_\delta, \quad \|v\|_{\mathcal{B}} \leqslant C_{\mathcal{I}}\, \|v\|_{\mathcal{V}_\delta} \quad \text{and} \quad \forall\, v \in \mathcal{B}, \quad \|v\|_{\mathcal{H}} \leqslant C_{\mathcal{I}}\, \|v\|_{\mathcal{B}}. \tag{2.14}$$

The family of problems. We introduce two bilinear forms denoted a_δ and b_0,

$$a_\delta : \mathcal{V}_\delta \times \mathcal{V}_\delta \mapsto \mathbb{R}, \quad b_0 : \mathcal{B} \times \mathcal{B} \mapsto \mathbb{R}.$$

(a_δ) The bilinear form a_δ is symmetric, nonnegative, continuous in $\mathcal{V}_\delta$ and coercive. More precisely, we assume that, for all u and v in $\mathcal{V}_\delta$,

$$a_\delta(u,v) = a_\delta(v,u), \qquad a_\delta(u,u) \geqslant 0, \tag{2.15}$$

moreover we assume that

$$\|u\|^2_{\mathcal{V}_\delta} = (u,u)_{\mathcal{V}_\delta} \quad \text{with} \quad (u,v)_{\mathcal{V}_\delta} := (u,v)_{\mathcal{H}} + a_\delta(u,v) \tag{2.16}$$

and therefore, the Cauchy-Schwartz inequality implies that

$$|a_\delta(u,v)| \leqslant \|u\|_{\mathcal{V}_\delta} \|v\|_{\mathcal{V}_\delta}.$$

The space $(\mathcal{V}_\delta; (\cdot,\cdot)_{\mathcal{V}_\delta})$ is a Hilbert space, and, by inspection, one can see that the injection in $\mathcal{H}$ is continuous, indeed, by definition,

$$\|u\|_{\mathcal{H}} \leqslant \|u\|_{\mathcal{V}_\delta}, \quad \forall\, u \in \mathcal{V}_\delta. \tag{2.17}$$

(b_0) The bilinear form b_0 is nonnegative and continuous in $\mathcal{B}$. More precisely for all $u \in \mathcal{V}_\delta$ and $v \in \mathcal{V}_\delta$,

$$0 \leqslant b_0(u,u), \quad |b_0(u,v)| \leqslant \|u\|_{\mathcal{B}} \|v\|_{\mathcal{B}}. \tag{2.18}$$

For all scalar $\delta \in (0,1)$, the abstract family of problems reads: find

$$u_\delta \in C^1([0,T];\mathcal{H}) \cap C^0([0,T];\mathcal{V}_\delta)$$

solution to, for all $v \in \mathcal{V}_\delta$,

$$\begin{cases} \dfrac{d^2}{dt^2}(u_\delta, v)_{\mathcal{H}} + a_\delta(u_\delta, v) + \dfrac{d}{dt} b_0(u_\delta, v) = (f_\delta, v)_{\mathcal{H}}, \quad \mathcal{D}'(0,T), \\ u_\delta(0) = u^0_\delta, \\ \dot{u}_\delta(0) = u^1_\delta, \end{cases} \tag{2.19}$$

where f_δ is a sufficiently smooth source term and (u^0_δ, u^1_δ) are sufficiently smooth initial data. Note that we consider for now general δ-dependent initial data and source terms. These quantities must be well-prepared in a sense given later.

Existence and uniqueness result. We introduce in this section the notion of weak solutions. These solutions are defined using minimal (to our knowledge) regularity assumptions on the data. More precisely, we assume that

$$(u_\delta^0, u_\delta^1) \in \mathcal{V}_\delta \times \mathcal{H} \quad \text{and} \quad f_\delta \in L^1\big(0,T;\mathcal{H}\big). \tag{2.20}$$

DEFINITION 2.13. *A weak solution is a function*

$$u_\delta \in L^2\big(0,T;\mathcal{V}_\delta\big),$$

that satisfies, for all $v \in H^2\big(0,T;\mathcal{H}\big) \cap H^1\big(0,T;\mathcal{V}_\delta\big)$ *such that* $v(T) = 0$ *and* $\dot{v}(T) = 0$,

$$\begin{aligned} \int_0^T (u_\delta(t), \ddot{v}(t))_{\mathcal{H}}\,\mathrm{d}t + \int_0^T a_\delta(u_\delta(t), v(t))\,\mathrm{d}t - \int_0^T b_0(u_\delta(t), \dot{v}(t))\,\mathrm{d}t \\ = (u_\delta^1, v(0))_{\mathcal{H}} + b_0(u_\delta^0, v(0)) - (u_\delta^0, \dot{v}(0))_{\mathcal{H}} \\ + \int_0^T (f_\delta(t), v(t))_{\mathcal{H}}\,\mathrm{d}t. \end{aligned} \tag{2.21}$$

We have the following existence, uniqueness and stability results.

THEOREM 2.14. *There exists a unique weak solution* u_δ. *It satisfies, up tp modifications on zero-measure sets,*

$$u_\delta \in C^1([0,T];\mathcal{H}) \cap C^0([0,T];\mathcal{V}_\delta)$$

and, for all $t \in [0,T]$,

$$\sqrt{\mathcal{E}_\delta}(t) \leqslant \sqrt{\mathcal{E}_\delta}(0) + \frac{1}{\sqrt{2}} \int_0^t \|f_\delta(\tau)\|_{\mathcal{H}}\,\mathrm{d}\tau, \tag{2.22}$$

where

$$\mathcal{E}_\delta(t) = \frac{1}{2}\Big(\|\dot{u}_\delta(t)\|_{\mathcal{H}}^2 + a_\delta\big(u_\delta(t), u_\delta(t)\big) \Big). \tag{2.23}$$

Moreover u_δ *is the unique solution to* (2.19).

The stability result (2.22) give some first useful informations on the analysis of the family of problems (2.19). Indeed if

$$\mathcal{E}_\delta(0) \quad \text{and} \quad \int_0^T \|f_\delta(t)\|_{\mathcal{H}}\,\mathrm{d}t$$

are bounded uniformly with respect to δ then so is the energy functional $\mathcal{E}_\delta(t)$. Although the energy functional corresponds only to a $\delta-$dependent semi-norm for the solution it is possible to deduce that the norm in $L^2(0,T;\mathcal{H})$ of u_δ is uniformly bounded (if u_δ^0 is uniformly bounded in $\mathcal{H}$) hence u_δ converges weakly, up to subsequences, to some u_0 in the space $L^2(0,T;\mathcal{H})$. Remark however that, first, even when the initial data u_0^δ and u_1^δ are independent on δ, the initial energy $\mathcal{E}_\delta(0)$ may still depend on δ (see (2.23)), second, it is yet not clear at this stage, which equations, if any, are satisfied by the weak limit. To clarify this we need additional assumption on the structure of the bilinear form a_δ.

The two-scale structure. Our analysis of the sequence of solutions $\{u_\delta\}$ rely on the fundamental assumption that the bilinear form a_δ can be rewritten in such a way that the dependence in δ is explicit and has the following specific structure.

Assumption 2.15. *There exists a sequence of two-scale operators $G_\delta : D(G_\delta) \subset \mathcal{H} \to \mathcal{G}$ satisfying, the weak scale separation property, $D(G_\delta) = \mathcal{V}_\delta$ and*

$$\forall\, (u,v) \in \mathcal{V}_\delta \times \mathcal{V}_\delta, \quad a_\delta(u,v) = \delta^{-2}\big(G_\delta\, u, G_\delta\, v\big)_{\mathcal{G}}. \tag{2.24}$$

3. The limit problem.

Some notations: the spaces $\mathcal{V}_\infty$ and $\mathcal{V}_0$. For the sake of conciseness we introduce here several notations use extensively in what follows. We set

$$\mathcal{V}_\infty := D(G_0) \cap D(G_1),$$

and define the following scalar product and norm: for all u and v in $\mathcal{V}_\infty$,

$$(u,v)_{\mathcal{V}_\infty} := (u,v)_{\mathcal{H}} + (G_0\, u, G_0\, v)_{\mathcal{G}} + (G_1\, u, G_1\, v)_{\mathcal{G}}, \quad \|u\|^2_{\mathcal{V}_\infty} := (u,u)_{\mathcal{V}_\infty}.$$

We also introduce

$$\mathcal{V}_0 := D(\overline{Q_0\, G_1}),$$

and, define the following scalar product and norm: for all u and v in $\mathcal{V}_0$,

$$(u,v)_{\mathcal{V}_0} := (u,v)_{\mathcal{H}} + (\overline{Q_0\, G_1}\, u, \overline{Q_0\, G_1}\, v)_{\mathcal{G}}, \quad \|u\|^2_{\mathcal{V}_0} := (u,u)_{\mathcal{V}_0}. \tag{3.1}$$

Proposition 3.1. *The following inclusions $\mathcal{V}_\infty \subset \mathcal{V}_\delta \subset \mathcal{V}_0$ hold. Moreover,*

$$\forall\, u \in \mathcal{V}_\infty \quad \|u\|_{\mathcal{V}_\delta} \leqslant \sqrt{2}\,\delta^{-1}\|u\|_{\mathcal{V}_\infty} \quad \text{and} \quad \forall\, u \in \mathcal{V}_\delta, \quad \|u\|_{\mathcal{V}_0} \leqslant \|u\|_{\mathcal{V}_\delta}. \tag{3.2}$$

Finally we introduce the following subspaces of $\mathcal{V}_\infty$ and $\mathcal{V}_0$,

$$\mathcal{V}^0_\infty := \mathcal{V}_\infty \cap \operatorname{Ker} G_0 \quad \text{and} \quad \mathcal{V}^0_0 := \mathcal{V}_0 \cap \operatorname{Ker} G_0.$$

In order to study the asymptotic limit of the sequence $\{u_\delta\}$ we use the following additional assumption.

Assumption 3.2. *The following density properties hold*

$$\mathcal{V}^0_\infty \text{ is dense in } (\,\mathcal{V}^0_0\,;\ \|\cdot\|_{\mathcal{V}_0}\,). \tag{3.3}$$

Let us now comment Assumption 3.2. One of the difficulty that is encountered when studying the variational formulation (2.19) is that, formally, to be able to pass to the limit in (2.19) one should use a test function that belongs to the δ-independent $\mathcal{V}_\infty^0$ hence obtaining a weak formulation in this space while the limit solution belongs (assuming the time t fixed) to the larger space $\mathcal{V}_0^0$. The density result (3.3) solves this lack of symmetry.

When dealing with dissipation or coupling terms, i.e., when the bilinear form b_0 is not trivial, one has to relate the space $\mathcal{V}_0^0$ and $\mathcal{B}$. Hopefully we have the following theorem.

THEOREM 3.3. *When Assumption 3.2 holds, then* $\mathcal{V}_0^0 \subset \mathcal{B}$ *and*

$$\forall\, v \in \mathcal{V}_0^0, \quad \|v\|_{\mathcal{B}} \leqslant C_{\mathcal{I}}\, \|v\|_{\mathcal{V}_0}.$$

Finally we introduce a closed subspace of $\mathcal{H}$ defined by

$$\mathcal{H}_0 = \overline{\mathcal{V}_0^0}^{\|\cdot\|_{\mathcal{H}}}.$$

The space $\mathcal{H}_0$ if the closure of $\mathcal{V}_0^0$ for the norm of $\mathcal{H}$, it satisfies $\mathcal{H}_0 \subset \operatorname{Ker} G_0$ and is a Hilbert space when equipped with the scalar product and norm of $\mathcal{H}$.

We have now all the necessary material to define the limit problem.

Formal limit equation. We introduce the following symmetric, non-negative bilinear form

$$\forall\, (u,v) \in \mathcal{V}_0 \times \mathcal{V}_0, \quad a_0(u,v) = (\overline{Q_0\, G_1}\, u, \overline{Q_0\, G_1}\, v)_{\mathcal{G}}. \tag{3.4}$$

Now remark that with the definition (3.1) we have, for all u and v in $\mathcal{V}_0$,

$$\|u\|_{\mathcal{V}_0}^2 = \|u\|_{\mathcal{H}}^2 + a_0(u,u) \quad \text{and} \quad |a_0(u,v)| \leqslant \|u\|_{\mathcal{V}_0} \|v\|_{\mathcal{V}_0}, \tag{3.5}$$

hence a_0 is continuous in $\mathcal{V}_0$ and coercive. Therefore thanks to (3.5) it make sense to define the following $\delta-$independent wave propagation problem: find

$$u_0 \in C^1([0,T]; \mathcal{H}_0) \cap C^0([0,T]; \mathcal{V}_0^0)$$

solution to, for all $v \in \mathcal{V}_0^0$,

$$\begin{cases} \dfrac{d^2}{dt^2}(u_0, v)_{\mathcal{H}} + a_0(u_0, v) + \dfrac{d}{dt} b_0(u_0, v) = (f_0, v)_{\mathcal{H}}, \quad \mathcal{D}'(0,T), \\ u_0(0) = u^0, \\ \dot{u}_0(0) = P_0\, u^1, \end{cases} \tag{3.6}$$

where f_0 is a sufficiently smooth source term and (u^0, u^1) sufficiently smooth initial data and where P_0 is the orthogonal projection in $\mathcal{H}_0$. It is defined by

$$P_0 \in \mathcal{L}(\mathcal{H}), \quad P_0^2 = P_0, \quad (u - P_0\, u, v)_{\mathcal{H}} = 0 \quad \forall\, (u,v) \in \mathcal{H} \times \mathcal{H}_0. \tag{3.7}$$

Note that the solution is sought in the subspace $\mathcal{V}_0^0$ of functions in $\mathcal{V}_0$. As shown below in Section 4.1, under adequate assumptions, problem (3.6) turns out to be the limit problem satisfied by the weak limit of solutions to (2.19).

Existence and uniqueness results. We define weak solutions for the limit problem. We assume that the source term f_0 and initial data (u^0, u^1) satisfy

$$(u^0, u^1) \in \mathcal{V}_0^0 \times \mathcal{H} \quad \text{and} \quad f_0 \in L^1(0,T;\mathcal{H}). \tag{3.8}$$

Existence, uniqueness and stability results for the limit problem are obtained as a special case of the theory presented Section 2.2.

Definition 3.4. *A weak solution is a function*

$$u_0 \in L^2\big(0,T;\mathcal{V}_0^0\big),$$

that satisfies, for all $v \in H^2\big(0,T;\mathcal{H}_0\big) \cap H^1\big(0,T;\mathcal{V}_0^0\big)$ such that $v(T) = 0$ and $\dot v(T) = 0$,

$$\int_0^T (u_0(t), \ddot v(t))_{\mathcal{H}} \,\mathrm{d}\, t + \int_0^T a_0(u_0(t), v(t)) \,\mathrm{d}t - \int_0^T b_0(u_0(t), \dot v(t)) \,\mathrm{d}t \tag{3.9}$$
$$= (u^0, \dot v(0))_{\mathcal{H}} - (u^1, v(0))_{\mathcal{H}} - b_0(u^0, v(0))_{\mathcal{H}} + \int_0^T (f_0(t), v(t))_{\mathcal{H}} \,\mathrm{d}t.$$

As a straightforward consequence of Theorem 2.14 one can show the following result.

Proposition 3.5. *There exists a unique limit weak solution u_0. It satisfies, up to modifications on zero-measure sets,*

$$u_0 \in C^1([0,T];\mathcal{H}_0) \cap C^0([0,T];\mathcal{V}_0^0)$$

and, for all $t \in [0,T]$,

$$\sqrt{\mathcal{E}_0}(t) \leqslant \sqrt{\mathcal{E}_0}(0) + \frac{1}{\sqrt{2}} \int_0^t \|f_0(\tau)\|_{\mathcal{H}} \,\mathrm{d}\tau$$

where

$$\mathcal{E}_0(t) = \frac{1}{2}\Big(\|\dot u_0(t)\|_{\mathcal{H}}^2 + \|\overline{Q_0\, G_1}\, u_0(t)\|_{\mathcal{G}}^2 \Big).$$

Moreover u_0 is the unique solution to (3.6).

The space $\mathcal{R}_0$ and $\mathcal{S}_0$. For the analysis of the limit problem we shall introduce the space $\mathcal{R}_0$ and $\mathcal{S}_0$ related to the kernel of $\overline{Q_0\, G_1}$. We define

$$\mathcal{R}_0 := \mathcal{V}_0^0 \,\cap\, \operatorname{Ker} \overline{Q_0\, G_1} = \operatorname{Ker} G_0 \cap \operatorname{Ker} \overline{Q_0\, G_1}. \tag{3.10}$$

Since by definition $\overline{Q_0\, G_1}$ is closed and densely defined in $\mathcal{H}$ its kernel is closed in $\mathcal{H}$. It is also closed in $\mathcal{V}_0^0$ since, for all $r \in \operatorname{Ker} \overline{Q_0\, G_1}$ we have $\|r\|_{\mathcal{V}_0} = \|r\|_{\mathcal{H}}$. We deduce then that the space $\mathcal{R}_0$ is a closed subspace of $\mathcal{V}_0^0$. We can therefore define $\mathcal{S}_0$ as the orthogonal complement of $\mathcal{R}_0$ in $\mathcal{V}_0^0$. We have

$$\mathcal{V}_0^0 = \mathcal{R}_0 \oplus \mathcal{S}_0, \tag{3.11}$$

where the decomposition is orthogonal with respect to the scalar product in $\mathcal{V}_0$, moreover, by definition we have

$$\forall\, (r,s) \in \mathcal{R}_0 \times \mathcal{S}_0, \quad 0 = (r,s)_{\mathcal{V}_0} = (r,s)_{\mathcal{H}} + (\overline{Q_0\, G_1}\, r, \overline{Q_0\, G_1}\, s)_{\mathcal{G}} = (r,s)_{\mathcal{H}},$$

hence $\mathcal{R}_0$ and $\mathcal{S}_0$ are also orthogonal with respect to the scalar product in $\mathcal{H}$. This means that we also have

$$\mathcal{H}_0 = \mathcal{R}_0 \oplus \mathcal{R}_0^{\perp} \quad \text{and} \quad \mathcal{S}_0 \subset \mathcal{R}_0^{\perp}$$

where the orthogonality relations stands with the scalar product in $\mathcal{H}$.

The space $\mathcal{R}_0$ is called the space of non-propagating solutions whereas the space $\mathcal{S}_0$ is called the space of propagating solutions. This terminology is justified in the case where b_0 vanish or is proportional to the scalar product in $\mathcal{H}$. Indeed in that case it is shown below that the projection of the limit solution u_0 in $\mathcal{R}_0$ satisfies a simple differential problem while its projection in $\mathcal{S}_0$ satisfies an abstract wave propagation problem.

Reduced equation on the space of propagating solutions. To simplify this section we assume that initial data vanishes and that the source term belongs at almost all time to $\mathcal{S}_0$

$$u_\delta^0 = 0, \quad u_\delta^1 = 0 \quad \text{and} \quad f_0 \in L^1(0,T;\mathcal{S}_0). \tag{3.12}$$

Thanks to the orthogonal decomposition (3.11) we deduce the following property.

PROPOSITION 3.6. *Any limit weak solution u_0 satisfies the decomposition*

$$u_0 = r_0 + s_0$$

with

$$r_0 \in C^1([0,T];\mathcal{R}_0) \quad \textit{and} \quad s_0 \in C^1([0,T];\mathcal{R}_0^{\perp}) \cap C^0([0,T];\mathcal{S}_0).$$

Our objective is now to obtain an equation only for s_0 by eliminating the term r_0 in (3.9). To do so, we choose as a test function $r \in H^2\big(0,T;\mathcal{R}_0\big)$ such that $r(T) = 0$ and $\dot r(T) = 0$ in (3.9). We obtain, after integrating by parts the term involving $\ddot r$,

$$\int_0^T (\dot r_0(t), \dot r(t))_{\mathcal{H}}\, \mathrm{d}\,t + \int_0^T b_0(r_0(t), \dot r(t))\, \mathrm{d}t \; = -\int_0^T b_0(s_0(t), \dot r(t))\, \mathrm{d}t \tag{3.13}$$

where we have used the property that $a_0(u_0(t), r(t)) = 0$, that $(s_0(t), \ddot r(t))_{\mathcal{H}} = 0$ and $f_0 \in L^1(0,T;\mathcal{S}_0)$. Equation (3.13) can be written in a strong form in time, i.e. it is equivalent to, for all $r \in \mathcal{R}_0$,

$$\begin{cases} (\dot r_0(t), r)_{\mathcal{H}} + b_0(r_0(t), r) = -b_0(s_0(t), r), \quad C^0([0,T]), \\ r_0(0) = 0, \end{cases} \tag{3.14}$$

where $r_0(0) = 0$ is a consequence of (3.12) and Proposition 3.6. We recall now the properties of the bilinear form b_0, we have, for all $(u,v) \in \mathcal{V}_0^0 \times \mathcal{V}_0^0$

$$|b_0(u,v)| \leqslant \|u\|_{\mathcal{B}} \|u\|_{\mathcal{B}} \leqslant C_{\mathcal{I}}^2 \|u\|_{\mathcal{V}_0} \|u\|_{\mathcal{V}_0} \quad \text{and} \quad 0 \leqslant b_0(u,u),$$

Hence, from the bilinear form b_0 we define bounded linear operators

$$B_{\mathcal{R}} : \mathcal{R}_0 \mapsto \mathcal{R}_0 \quad \text{and} \quad B_{\mathcal{S}}^{\mathcal{R}} : \mathcal{S}_0 \mapsto \mathcal{R}_0$$

where $\mathcal{R}_0$ and $\mathcal{S}_0$ are Hilbert spaces equipped with the scalar product in $\mathcal{V}_0$ and the corresponding norm $\|\cdot\|_{\mathcal{V}_0}$ (we recall that $\|r\|_{\mathcal{V}_0} = \|r\|_{\mathcal{H}}$ for $r \in \mathcal{R}_0$) by

$$\begin{cases} \forall\, (r,\tilde{r}) \in \mathcal{R}_0 \times \mathcal{R}_0, & (B_{\mathcal{R}}\, r, \tilde{r})_{\mathcal{V}_0} = (B_{\mathcal{R}}\, r, \tilde{r})_{\mathcal{H}} = b_0(r,\tilde{r}), \\ \forall\, (s,r) \in \mathcal{S}_0 \times \mathcal{R}_0, & (B_{\mathcal{S}}^{\mathcal{R}}\, s, r)_{\mathcal{V}_0} = (B_{\mathcal{S}}^{\mathcal{R}}\, s, r)_{\mathcal{H}} = b_0(s,r), \end{cases}$$

where $B_{\mathcal{R}}$ is a non-negative operator. Then equations (3.14) can be recast as

$$\begin{cases} \dot{r}_0 + B_{\mathcal{R}}\, r_0 = -B_{\mathcal{S}}^{\mathcal{R}}\, s_0, \\ r_0(0) = 0, \end{cases} \tag{3.15}$$

where the first equation is written in $C^0([0,T];\mathcal{R}_0)$. From (3.15) and from [2] we deduce that r_0 is given by,

$$r_0(t) = -\int_0^t e^{B_{\mathcal{R}}(t-\tau)} B_{\mathcal{S}}^{\mathcal{R}}\, s_0(\tau)\, \mathrm{d}\tau \;\in C^1([0,T];\mathcal{R}_0), \tag{3.16}$$

where $e^{-B_{\mathcal{R}}t}$ is the uniformly continuous semi-group (see [2]) generated by the bounded operator $-B_{\mathcal{R}}$.

Remark 3.7. Note that if a more general source term $f_0 \in L^1(0,T;\mathcal{H})$ is considered then one can show that the first equation of (3.15) becomes

$$\dot{r}_0(\tau) + B_{\mathcal{R}}\, r_0(\tau) = -B_{\mathcal{S}}^{\mathcal{R}}\, s_0(\tau) + \int_0^t Q_{\mathcal{R}} f(\tau)\, \mathrm{d}\tau,$$

where $Q_{\mathcal{R}}$ is the orthogonal projection operator $\mathcal{R}_0$ with respect to the scalar product in$\mathcal{H}$. Then observe that if $B_{\mathcal{S}}^{\mathcal{R}} = 0$ (for instance either because b_0 vanishes or is the scalar product in $\mathcal{H}$) then r_0 satisfies a differential equation that is not an abstract wave propagation problem (we recall that $B_{\mathcal{R}}$ is a non-negative operator).

From the expression (3.16) we deduce that

$$\dot{r}_0(t) = \Psi_t B_{\mathcal{S}}^{\mathcal{R}}\, s_0$$

with $\Psi_t : C^0([0,T];\mathcal{R}_0) \to C^0([0,T];\mathcal{R}_0)$ given by

$$\forall\, r \in C^0([0,T];\mathcal{R}_0), \quad \Psi_t\, r = B_{\mathcal{R}} \int_0^t e^{B_{\mathcal{R}}(t-\tau)} r(\tau)\, \mathrm{d}\tau - r(t).$$

With this expression and with the weak formulation (3.6) one can writes a formulation involving only s_0. Observing that, for all $s \in \mathcal{S}_0$ we have

$$b_0(\dot{r}_0(t), s) \; = b_0(\Psi_t B_{\mathcal{S}}^{\mathcal{R}} s_0, s)$$

and we can show that, for all $s \in \mathcal{S}_0$,

$$\begin{cases} \dfrac{d^2}{dt^2}(s_0,s)_{\mathcal{H}} + a_0(s_0,s) + \dfrac{d}{dt} b_0(s_0,s) + b_0(\Psi_t B_{\mathcal{S}}^{\mathcal{R}} s_0, s) = (f_0,s)_{\mathcal{H}}, \;\; \mathcal{D}'(0,T), \\ s_0(0) = 0, \\ \dot{s}_0(0) = 0. \end{cases} \tag{3.17}$$

This system corresponds to a wave equation with a non-local term in time that corresponds to dissipative and other coupling effects. The non-local operator in time is in fact falsely non-local since we have seen that it is derived, rather simply, from a local in time evolution problem. Note that in application (see for instance [14]) this may not be as straightforwardly apparent. Note also that we have shown existence of a solution for this problem by construction. Uniqueness of smooth solutions is obtained – by energy estimates – as a consequence of the following positivity property.

THEOREM 3.8. *For all* $s \in H^1([0,T];\mathcal{S}_0)$ *such that* $s(0) = 0$, *we have, for all* $t \in [0,T]$,

$$\int_0^t b_0(\dot{s}(\tau),\dot{s}(\tau))\,\mathrm{d}\tau + \int_0^t b_0(\Psi_\tau B_{\mathcal{S}}^{\mathcal{R}} s, \dot{s}(\tau))_{\mathcal{H}}\,\mathrm{d}\tau \geqslant 0.$$

Proof. The proof is done by rewinding some of the arguments given above to obtain (3.17). Assuming first that $s \in C^1([0,T];\mathcal{S}_0)$, we have

$$\mathcal{I} = \int_0^t b_0(\dot{s}(\tau),\dot{s}(\tau))\mathrm{d}\tau - \int_0^t b_0(\Psi_\tau B_{\mathcal{S}}^{\mathcal{R}} s, \dot{s}(\tau))_{\mathcal{H}} \\ = \int_0^t b_0(\dot{s}(\tau),\dot{s}(\tau))\,\mathrm{d}\tau + \int_0^t b_0(r(\tau),\dot{s}(\tau))_{\mathcal{H}}\,\mathrm{d}\tau \tag{3.18}$$

where $r \in C^2([0,T];\mathcal{R}_0)$ is defined by

$$\dot{r} + B_{\mathcal{R}}\, r = -B_{\mathcal{S}}^{\mathcal{R}}\, s \quad \text{and} \quad r(0) = 0, \tag{3.19}$$

which implies $\dot{r}(t) = \Psi_t B_{\mathcal{S}}^{\mathcal{R}} s \in C^0([0,T];\mathcal{R}_0)$. From Eq. (3.19) we deduce that $\ddot{r} + B_{\mathcal{R}}\,\dot{r} = -B_{\mathcal{S}}^{\mathcal{R}}\,\dot{s}$ as well as the energy identity

$$\|\dot{r}(t)\|_{\mathcal{H}}^2 + \int_0^t b_0(\dot{r}(\tau),\dot{r}(\tau))\,\mathrm{d}\tau = -\int_0^t b_0(\dot{s}(\tau),\dot{r}(\tau))\,\mathrm{d}\tau. \tag{3.20}$$

Using (3.20) in (3.18) we obtain

$$\mathcal{I} = \int_0^t b_0(\dot{r}(\tau)+\dot{s}(\tau),\dot{r}(\tau)+\dot{s}(\tau))\,\mathrm{d}\tau + \|\dot{r}(t)\|_{\mathcal{H}}^2 \geqslant 0.$$

The final statement of the theorem is obtained by density of $C^1([0,T];\mathcal{S}_0)$ in the space $H^1([0,T];\mathcal{S}_0)$ (c.f. [17] Chap. 1). ∎

The strong and strict scale separation assumptions. To conclude this section, we summarize in the proposition below all the straightforward simplifications – consequences of Theorem 2.8 and Theorem 2.10 – obtained when the sequence of two-scale operators G_δ satisfies the strong or the strict scale separation property.

PROPOSITION 3.9. *If the sequence of two-scale operators* G_δ *satisfies the strong scale separation property then*

$$\begin{cases} \mathcal{V}_\infty = \mathcal{V}_\delta = D(G_0) \cap D(G_1), \\ \mathcal{V}_0 = D(G_1), \\ \mathcal{V}_\infty^0 = \mathcal{V}_0^0 = D(G_1) \cap \operatorname{Ker} G_0, \\ \mathcal{R}_0 = \operatorname{Ker} G_1 \cap \operatorname{Ker} G_0 \end{cases}$$

and

$$\forall\,(u,v) \in \mathcal{V}_0 \times \mathcal{V}_0, \quad a_0(u,v) = (G_1\,u, G_1\,v)_{\mathcal{G}}. \tag{3.21}$$

Moreover if the strict two-scale separation holds

$$\left\{\begin{array}{l} \mathcal{V}_\infty = \mathcal{V}_\infty^0 \oplus \big(D(G_0) \cap \operatorname{Ker} G_1\big), \\ \mathcal{H}_0 = \operatorname{Ker} G_0, \\ \mathcal{R}_0 = \{0\}, \\ \mathcal{S}_0 = \mathcal{V}_\infty^0, \end{array}\right. \tag{3.22}$$

Note that the decomposition in (3.22) *holds is orthogonal for the scalar product in* $\mathcal{H}$ *and in* $\mathcal{V}_\infty$.

The proposition above shows that Assumption 3.2 is automatically satisfied when the strong scale separation holds and, moreover, when the strict scale separation holds we have a simple characterization of $\mathcal{H}_0$, namely $\mathcal{H}_0 = \operatorname{Ker} G_0$.

4. Convergence analysis. In this Section we study the asymptotic limit, when δ goes to zero, of the family of the $\delta-$dependent wave propagation problems (2.19) towards the solution of the limit problem (3.6). In Section 4.1 we investigate the condition on which the convergence is weak whereas in Section 4.3 we study strong convergence properties.

4.1. Weak convergence. The objective of this section is to give the minimal assumption that guaranty that the sequence $\{u_\delta\}$ of weak solutions of problem (2.19) converges weakly towards the weak solution of the limit problem (3.6). In the sequel we assume that the weak scale separation holds and we shall also assume that the data are well-prepared as define below.

Assumption 4.1. **Well-prepared data.** *The initial data and the source terms of problem* (2.19) *satisfy*

$$(u_\delta^0, u_\delta^1) \in \mathcal{V}_\delta \times \mathcal{H} \quad \text{and} \quad f_\delta \in L^1\big(0,T;\mathcal{H}\big)$$

and we assume that there exists $(u^0,u^1) \in \mathcal{V}_0 \times \mathcal{H}$ *and* $f_0 \in L^1\big(0,T;\mathcal{H}\big)$ *such that*

$$u_\delta^0 \underset{\mathcal{V}_0}{\rightharpoonup} u^0, \quad u_\delta^1 \underset{\mathcal{H}}{\rightharpoonup} u^1 \quad \text{and} \quad f_\delta \underset{L^1(0,T;\mathcal{H})}{\rightharpoonup} f_0. \tag{4.1}$$

Moreover we assume that there exists $C_0 > 0$ *independent of* δ *such that*

$$\|G_\delta\, u_\delta^0\|_{\mathcal{G}} \leqslant \delta\, C_0. \tag{4.2}$$

It has to be noted that the only strong assumption here is on u_δ^0. Indeed we impose that $G_0\, u_\delta^0$ vanishes sufficiently fast with δ going to zero. This can be interpreted as the assumption that the term u_δ^0 must be adapted to the structure of the solution $u_0(t)$ of the limit problem (3.6) that belongs to $\mathcal{V}_0^0 = \mathcal{V}_0 \cap \operatorname{Ker} G_0$. The first consequences of Assumption 4.1 are given below.

Lemma 4.2. *If the data are well-prepared then* $u^0 \in \mathcal{V}_0^0$ *and there exists a scalar* $C_u > 0$ *independent of* δ *such that the weak solutions* u_δ *of problem* (2.19) *satisfies,*

$$\sup_{t\in[0,T]} \big(\, \|\dot u_\delta(t)\|_{\mathcal{H}} + \|u_\delta(t)\|_{\mathcal{V}_\delta}\,\big) \leqslant C_u. \tag{4.3}$$

Proof. Since the weak convergence (4.1) hold we have that

$$\|u_\delta^0\|_{\mathcal{V}_0}, \quad \|u_\delta^1\|_{\mathcal{H}} \quad \text{and} \quad \int_0^T \|f_\delta(t)\|_{\mathcal{H}}\,\mathrm{d}t,$$

are bounded sequences. Therefore, the first statement – $u^0 \in \mathcal{V}_0^0$ – is a consequence of the estimate (4.2) and Theorem 2.3. Moreover, the second statement is a straightforward consequence of the energy estimates given by (2.22) as well as the property that the energy at the initial time $\mathcal{E}_\delta(0)$ satisfies

$$\mathcal{E}_\delta(0) = \|u_\delta^1\|_{\mathcal{H}}^2 + a_\delta(u_\delta^0, u_\delta^0) \leqslant \|u_\delta^1\|_{\mathcal{H}}^2 + 2\,\delta^{-2}\,\|G_0\,u_\delta^0\|_{\mathcal{G}}^2 + 2\,\|G_1\,u_\delta^0\|_{\mathcal{G}}^2$$

and is bounded uniformly. ∎

Thanks to Lemma 4.2 and in particular to estimate (4.3) we have the existence of a weak limit of the sequence $\{u_\delta\}$ in $L^2(0,T;\mathcal{H})$. This weak limit is still not characterized as the solution of the limit problem (3.6). This is the object of the theorem below that is the main result of this section.

THEOREM 4.3. *Let the weak scale separation and Assumption 3.2 hold and assume the data of problem (2.19) are well-prepared. Then the weak solutions u_δ of problem (2.19) satisfy,*

$$u_\delta \underset{H^1(0,T;\mathcal{H})}{\rightharpoonup} u_0 \quad \textit{and} \quad u_\delta \underset{L^2(0,T;\mathcal{V}_0)}{\rightharpoonup} u_0,$$

where u_0 is the unique weak solution of the limit problem (3.6) with initial data (u^0, u^1) and source term f_0.

Proof.
Step 1: Weak limit of u_δ. Thanks to the assumption that the data are well prepared there exists for each δ a unique weak solution u_δ of problem (2.19) and the uniform estimate of the solutions (4.3) holds. In particular, one can show, using (3.2), that

$$\sup_{t\in[0,T]} \|\dot{u}_\delta(t)\|_{\mathcal{H}}^2, \quad \sup_{t\in[0,T]} \|u_\delta(t)\|_{\mathcal{V}_0}^2 \quad \text{and} \quad \sup_{t\in[0,T]} \delta^{-2}\,\|G_\delta\,u_\delta(t)\|_{\mathcal{G}}^2$$

are bounded uniformly with respect to δ. From this estimate we deduce two results. First, there exits $u_0 \in H^1(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V}_0)$ such that, up to a subsequence, u_δ weakly converges to u_0. Second, using a similar reasoning used to prove Theorem 2.3 we can prove that $u_0 \in L^2(0,T;\operatorname{Ker} G_0)$ hence $u_0 \in L^2(0,T;\mathcal{V}_0^0)$

Step 2: Weak limit of $\delta^{-1}\,G_\delta\,u_\delta$. Since $\delta^{-1}\,G_\delta\,u_\delta$ is bounded uniformly in δ and $t \in [0,T]$, multiplying (2.21) by δ one can see that

$$\int_0^T (\delta^{-1} G_\delta\,u_\delta(t), G_0\,v(t))\,\mathrm{d}t \underset{\delta\to 0}{\longrightarrow} 0.$$

Applying Theorem 2.6 (in fact one should use a time version of this theorem that is left to the reader) we obtain, since $\overline{Q_0\,G_1}\,u_\delta$ weakly converges to $\overline{Q_0\,G_1}\,u_0$, that

$$\delta^{-1} G_\delta\,u_\delta \underset{L^2(0,T;\mathcal{G})}{\rightharpoonup} \overline{Q_0\,G_1} u_0.$$

Step 3: Limit equation. We now choose

$$v \in D([0,T);\mathcal{V}_\infty^0) \tag{4.4}$$

in the variational formulation (2.21) and pass to the limit. We obtain that u_0 satisfies the limit weak variational formulation (3.9) for all v satisfying (4.4). Note that in particular we use the implication

$$u_\delta \underset{L^2(0,T;\mathcal{V}_0)}{\rightharpoonup} u_0 \quad \text{and} \quad \int_0^T \|u_\delta(t)\|_{\mathcal{V}_\delta}\,\mathrm{d}t \leqslant T\,C_u$$
$$\Rightarrow \quad \int_0^T b_0(u_\delta(t),\dot v(t))\,\mathrm{d}t \underset{\delta\to 0}{\longrightarrow} \int_0^T b_0(u_0(t),\dot v(t))\,\mathrm{d}t,$$

that is a consequence of the property that $\mathcal{V}_\delta$ is (uniformly) continuously embedded in $\mathcal{B}$ (see Eq. (2.14)). Finally using Assumption 3.2 one can show that the space $D([0,T);\mathcal{V}_\infty^0)$ is dense in the space

$$\mathcal{W}_0 := \{v \in H^2\big([0,T];\mathcal{H}_0\big) \cap H^1\big([0,T];\mathcal{V}_0^0\big) \ \mid\ \dot v(T) = 0, \quad v(T) = 0\}.$$

Thanks to this density result we can then show that u_0 satisfies the limit weak variational formulation (3.9) for all test function in $\mathcal{W}_0$ (the proof of this density result is given in Appendix 7 for the sake of completeness).

Conclusion. We have shown that sequence u_δ converges weakly, up to a subsequence to a function $u_0 \in L^2(0,T;\mathcal{V}_0^0)$ that satisfies the weak variational formulation (3.9). The function u_0 is therefore the unique weak solution of the limit problem corresponding to data (u^0,u^1) and f_0. This last statement implies that all the converging subsequences converge to the same element u_0. □

Generalization. The weak convergence that we have shown can be easily extended to more general situations that correspond to low order perturbation in δ of the equations. In particular the same convergence result (towards the solution of the same limit problem) can be proven in the case where, instead of Assumption 2.15 one has,

$$\forall\ (u,v) \in \mathcal{V}_\delta \times \mathcal{V}_\delta, \quad a_\delta(u,v) = \delta^{-2}\big((I + \gamma_a(\delta)\,S_\delta)\,G_\delta\,u, G_\delta\,v\big)_{\mathcal{G}},$$

where S_δ is a bounded (uniformly with respect to δ) self-adjoint operator in $\mathcal{G}$ and $\gamma_a(\delta)$ a continuous scalar function vanishing when δ goes to zero. Of course, in the same way, one could define, a $\delta-$dependent perturbation in of $b_0(u_\delta,v)$, namely

$$\forall\ (u,v) \in \mathcal{V}_\delta \times \mathcal{V}_\delta, \quad b_\delta(u,v) = b_0(u,v) + \gamma_b(\delta)\,b_{1,\delta}(u,v)$$

where $\gamma_b(\delta)$ is another a continuous scalar function vanishing when δ goes to zero and $b_{1,\delta}$ a uniformly (with respect to δ) continuous bilinear form in $\mathcal{B}$ such that

$$\forall\ (u,v) \in \mathcal{V}_\delta \times \mathcal{V}_\delta, \quad 0 \leqslant b_\delta(u,u), \quad |b_\delta(u,v)| \leqslant \|u\|_{\mathcal{B}}\|v\|_{\mathcal{B}}.$$

We could have then considered instead of (2.19) the following equation

$$\begin{cases} \dfrac{d^2}{dt^2}((I + \gamma_m(\delta)\,M_\delta)u_\delta, v)_{\mathcal{H}} + a_\delta(u_\delta,v) + \dfrac{d}{dt}b_\delta(u_\delta,v) = (f_\delta,v)_{\mathcal{H}}, \quad \mathcal{D}'(0,T), \\ u_\delta(0) = u_\delta^0, \\ \dot u_\delta(0) = u_\delta^1, \end{cases} \tag{4.5}$$

where, again $\gamma_m(\delta)$ is a continuous scalar function vanishing when δ goes to zero and M_δ a bounded (uniformly with respect to δ) self-adjoint operator in $\mathcal{H}$.

4.2. Strong convergence. In this section we assume that the strong scale separation holds, we recall that this implies in particular, see Proposition 3.9,

$$\mathcal{V}_\infty = \mathcal{V}_\delta = D(G_0) \cap D(G_1), \quad \mathcal{V}_0 = D(G_1), \quad \mathcal{V}_\infty^0 = \mathcal{V}_0^0 = D(G_1) \cap \operatorname{Ker} G_0$$

and $a_\delta(u,v) = \delta^{-2}(G_0\,u, G_0\,v)_{\mathcal{G}} + (G_1\,u, G_1\,v)_{\mathcal{G}}$. Since $a_\delta(u,u) = \|u\|^2_{\mathcal{V}_\delta}$, from Lemma 4.3, we deduce that if the data are well-prepared then

$$\sup_{t\in[0,T]} \|G_0\,u_\delta(t)\| \leqslant \delta\, C_u. \tag{4.6}$$

This simple observation shows that when the strong scale separation holds and if the data are well-prepared then $\|G_0\,u_\delta(t)\|$ vanishes at the limit.

ASSUMPTION 4.4. *$\mathcal{V}_0^0$ is compactly embedded in $\mathcal{H}_0$.*

Thanks to Assumption 4.4 and the observation that $\|G_0\,u_\delta(t)\|$ vanishes at the limit allows to state a straightforward corollary of Theorem 4.3 based on Lions' compactness lemma.

COROLLARY 4.5. *Let the strong scale separation, Assumption 3.2 and Assumption 4.4 hold and assume that the data of problem (2.19) are well-prepared. Then the weak solutions u_δ of problem (2.19) satisfy,*

$$P_0\,u_\delta \underset{L^2(0,T;\mathcal{H})}{\to} u_0. \tag{4.7}$$

where u_0 is the unique weak solution of the limit problem (3.6) with initial data (u^0, u^1) and source term f_0.

Proof. We first observe that

$$P_0\,u_\delta \in \mathcal{Z} := H^1(0,T;\mathcal{H}_0) \cap L^2(0,T;\mathcal{V}_0^0)$$

and, from Theorem 4.3,

$$P_0\,u_\delta \underset{\mathcal{Z}}{\rightharpoonup} P_0 u_0 = u_0. \tag{4.8}$$

Lion's compactness lemma (see [16]) states that, thanks to Assumption (4.4), the space $L^2(0,T;\mathcal{H}_0)$ is compactly embedded in Z hence the strong convergence (4.7) towards u_0 holds (note that we use the property here that $\mathcal{H}_0$ is equipped with the norm in $\mathcal{H}$). ∎

The previous result shows that $P_0 u_\delta$ strongly converges towards u_0 if some compactness properties are verified We give now an assumption that, if verified, allows to show that $(I - P_0)\,u_\delta$ converges towards 0.

Assumption 4.6. *There exits $C_p > 0$ such that*

$$\forall\, u \in \mathcal{H}_0^\perp \cap D(G_0), \quad \|u\|_{\mathcal{H}} \leqslant C_p\, \|G_0\, u\|_{\mathcal{G}}, \tag{4.9}$$

where the orthogonality holds for the scalar product in $\mathcal{H}$.

Corollary 4.7. *If the hypotheses of Corollary 4.5 hold and Assumption 4.6 holds then*

$$\sup_{t\in[0,T]} \|(I-P_0)\, u_\delta(t)\|_{\mathcal{H}} \leqslant C_p\, C_u\, \delta \quad \text{and} \quad u_\delta \underset{L^2(0,T;\mathcal{H})}{\rightarrow} u_0. \tag{4.10}$$

Proof. Observe that $(I-P_0)\, u_\delta(t) \in \mathcal{H}_0^\perp$ and, writing,

$$u_\delta = P_0\, u_\delta + (I-P_0)\, u_\delta \tag{4.11}$$

we have, since, $u_\delta(t) \in \mathcal{V}_\infty \subset D(G_0)$ and $P_0\, u_\delta(t) \in \operatorname{Ker} G_0$, that

$$(I-P_0)\, u_\delta(t) \in D(G_0).$$

Therefore the estimate in (4.10) is a direct consequence of (4.6) and (4.9) while the convergence result follows easily using Corollary 4.5 and the decomposition (4.11). ☐

4.3. Error estimates. In this section we assume again that the strong scale separation holds, moreover to simplify the statement of the results of this section we assume that

$$u_\delta^0 = 0 \quad \text{and} \quad u_\delta^1 = 0. \tag{4.12}$$

There is no theoretical difficulties in considering non-vanishing initial data, however the energy estimates given below are less concise. In this section we use capital letter to refer to primitive in time of quantities, typically,

$$\mathtt{U}_\delta(t) = \int_0^t u_\delta(\tau)\,\mathrm{d}\tau, \quad \mathtt{F}_\delta(t) = \int_0^t f_\delta(\tau)\,\mathrm{d}\tau, \quad \ldots.$$

The equation satisfied by the unknown $\mathtt{U}_\delta$ and $\mathtt{U}_0$ are easily deduced, they are given by, for all $t \in [0,T]$ and $v \in \mathcal{V}_\infty$,

$$\begin{aligned}(\ddot{\mathtt{U}}_\delta(t), v)_{\mathcal{H}} + \delta^{-2}(G_0\, \mathtt{U}_\delta(t), G_0\, v)_{\mathcal{G}} \\ + (G_1\, \mathtt{U}_\delta(t), G_1\, v)_{\mathcal{G}} + b_0(\dot{\mathtt{U}}_\delta(t), v) = (\mathtt{F}_\delta(t), v)_{\mathcal{H}}\end{aligned} \tag{4.13}$$

and for all $t \in [0,T]$ and $v \in \mathcal{V}_0^0$,

$$(\ddot{\mathtt{U}}_0(t), v)_{\mathcal{H}} + (G_1\, \mathtt{U}_\delta(t), G_1\, v)_{\mathcal{G}} + b_0(\dot{\mathtt{U}}_0(t), v) = (\mathtt{F}_0(t), v)_{\mathcal{H}}. \tag{4.14}$$

From these equations and from Theorem 2.14 and Proposition 3.5 one can deduce the following energy estimates, for all $t \in [0,T]$,

$$\left\{\begin{aligned} &\big(\, \|\dot{u}_\delta(t)\|_{\mathcal{H}}^2 + \delta^{-2}\|G_0\, u_\delta(t)\|_{\mathcal{G}}^2 + \|G_1\, u_\delta(t)\|_{\mathcal{G}}^2\, \big)^{\frac{1}{2}} \leqslant \int_0^T \|f_\delta(t)\|_{\mathcal{H}}\,\mathrm{d}t \leqslant C_f, \\ &\big(\, \|u_\delta(t)\|_{\mathcal{H}}^2 + \delta^{-2}\|G_0\, \mathtt{U}_\delta(t)\|_{\mathcal{G}}^2 + \|G_1\, \mathtt{U}_\delta(t)\|_{\mathcal{G}}^2\, \big)^{\frac{1}{2}} \leqslant \int_0^T \|\mathtt{F}_\delta(t)\|_{\mathcal{H}}\,\mathrm{d}t \leqslant C_{\mathtt{F}}, \end{aligned}\right.$$

and

$$\left\{\begin{aligned} &\big(\,\|\dot u_0(t)\|^2_{\mathcal H}+\|G_1\,u_0(t)\|^2_{\mathcal G}\,\big)^{\frac12}\leqslant\int_0^T\|f_0(t)\|_{\mathcal H}\,\mathrm dt\leqslant C_f,\\ &\big(\,\|u_0(t)\|^2_{\mathcal H}+\|G_1\,\mathtt U_0(t)\|^2_{\mathcal G}\,\big)^{\frac12}\leqslant\int_0^T\|\mathtt F_0(t)\|_{\mathcal H}\,\mathrm dt\leqslant C_F,\end{aligned}\right. \tag{4.15}$$

where

$$C_f=\sup_{\delta\in(0,1)}\int_0^T\|f_\delta(t)\|_{\mathcal H}\,\mathrm dt\quad\text{ and }\quad C_{\mathtt F}=\sup_{\delta\in(0,1)}\int_0^T\|\mathtt F_\delta(t)\|_{\mathcal H}\,\mathrm dt.$$

Below we study first the case where the strict scale separation holds (this implies in particular that the strong scale separation holds).

4.3.1. Strict scale separation. We consider in this section that the strict scale separation holds (this implies in particular that $\mathcal H_0=\operatorname{Ker}G_0$).

THEOREM 4.8. *Assume that the strict scale separation, Assumption 3.2 and Assumption 4.6 hold and that the data of problem (2.19) are well-prepared and satisfy (4.12). Then the weak solutions u_δ of problem (2.19) satisfy*

$$\sup_{t\in[0,T]}\big(\,\|u_\delta(t)-u_0(t)\|^2_{\mathcal H}+\|G_1(\mathtt U_\delta(t)-\mathtt U_0(t))\|^2_{\mathcal G}\,\big)^{\frac12}\leqslant\int_0^T\|\mathtt F_\delta(t)-\mathtt F_0(t)\|_{\mathcal H}\,\mathrm dt+\delta\,C(T), \tag{4.16}$$

where u_0 is the unique weak solution of the limit problem (3.6) with vanishing initial data and source term f_0 and

$$C(T)=\sqrt2\,C_p\left(C_{\mathtt F}+T\,C_{\mathcal I}^2\,(C_f+C_{\mathtt F})\right)$$

Proof. **Step 0: Preliminary observation.** Since G_0 is densely defined and closed, Assumption 4.6 is equivalent to require that the range of G_0 is closed in $\mathcal G$, which is also equivalent to require that the range of G_0^* is closed. Therefore Assumption 4.6 implies in particular that

$$(\operatorname{Ker}G_0)^\perp=\operatorname{Im}G_0^*. \tag{4.17}$$

Step 1: Decomposition of the solution. Since $\operatorname{Ker}G_0$ is a closed subspace of $\mathcal H$ and since $\mathtt U_\delta\in C^2([0,T];\mathcal H)\cap C^1([0,T];\mathcal V_\infty)$ we have the orthogonal decomposition (the decomposition being orthogonal with respect to the scalar product in $\mathcal H$),

$$\mathtt U_\delta=\mathtt U_{\delta,0}+\mathtt Z_\delta, \tag{4.18}$$

where $\mathtt U_{\delta,0}\in C^2([0,T];\operatorname{Ker}G_0)$ and $\mathtt Z_\delta\in C^2([0,T];(\operatorname{Ker}G_0)^\perp)$. From (4.17) and the strict scale separation property (2.11) we deduce that

$$(\operatorname{Ker}G_0)^\perp=\operatorname{Im}G_0^*\subset\operatorname{Ker}G_1,$$

hence in particular $\mathtt Z_\delta\in C^2([0,T];D(G_1))$, from which we deduce that,

$$\mathtt U_{\delta,0}\in C^2([0,T];\operatorname{Ker}G_0)\cap C^1([0,T];\mathcal V_0^0)$$

where we recall that $\mathcal{V}_0^0 = \mathcal{V}_\infty^0 = \operatorname{Ker} G_0 \cap D(G_1)$. Finally, this also gives

$$\mathtt{Z}_\delta \in C^2([0,T]; (\operatorname{Ker} G_0)^\perp \cap \operatorname{Ker} G_1) \cap C^1([0,T]; V_\infty).$$

Step 2: Energy identity for the error equation. We define the error term

$$\mathtt{E}_\delta = \mathtt{U}_{\delta,0} - \mathtt{U}_0 \in C^2([0,T]; \operatorname{Ker} G_0) \cap C^1([0,T]; \mathcal{V}_0^0)$$

From the decomposition (4.18) we deduce that, for all $v \in \mathcal{V}_0^0$,

$$(\ddot{\mathtt{U}}_\delta(t), v)_{\mathcal{H}} + (G_1\, \mathtt{E}_\delta(t), G_1\, v) + b_0(\dot{\mathtt{E}}_\delta(t) + \dot{\mathtt{Z}}_\delta(t), v) = (\mathtt{F}_\delta(t) - \mathtt{F}_0(t), v)_{\mathcal{H}}. \tag{4.19}$$

For every time $t \in [0,T]$, choosing $v = \dot{\mathtt{E}}_\delta(t)$ in (4.19) yields, using the orthogonality of $\mathtt{E}_\delta(t)$ and $\mathtt{Z}_\delta(t)$ in $\mathcal{H}$,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big(\|\dot{\mathtt{E}}_\delta(t)\|_{\mathcal{H}}^2 + \|G_1\, \mathtt{E}_\delta(t)\|_{\mathcal{G}}^2\Big) + b_0(\dot{\mathtt{E}}_\delta(t) + \dot{\mathtt{Z}}_\delta(t), \dot{\mathtt{E}}_\delta(t)) \\ = (\mathtt{F}_\delta(t) - \mathtt{F}_0(t), \dot{\mathtt{E}}_\delta(t))_{\mathcal{H}}. \end{aligned} \tag{4.20}$$

Moreover, choosing $v = \dot{\mathtt{Z}}_\delta(t)$ in (4.13) yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big(\|\dot{\mathtt{Z}}_\delta(t)\|_{\mathcal{H}}^2 + \delta^{-2}\, \|G_0\, \mathtt{Z}_\delta(t)\|_{\mathcal{G}}^2\Big) + b_0(\dot{\mathtt{U}}_{\delta,0}(t) + \dot{\mathtt{Z}}_\delta(t), \dot{\mathtt{Z}}_\delta(t)) \\ = (\mathtt{F}_\delta(t), \dot{\mathtt{Z}}_\delta(t))_{\mathcal{H}}. \end{aligned} \tag{4.21}$$

Step 3: Error estimate. Our next step is to sum (4.20) and (4.21), we get

$$\begin{aligned} \frac{d}{dt}\mathcal{J}_\delta + b_0(\dot{\mathtt{E}}_\delta(t) + \dot{\mathtt{Z}}_\delta(t), \dot{\mathtt{E}}_\delta(t) + \dot{\mathtt{Z}}_\delta(t)) = (\mathtt{F}_\delta(t) - \mathtt{F}_0(t), \dot{\mathtt{E}}_\delta(t))_{\mathcal{H}} \\ + (\mathtt{F}_\delta(t), \dot{\mathtt{Z}}_\delta(t))_{\mathcal{H}} - b_0(\dot{\mathtt{U}}_0(t), \dot{\mathtt{Z}}_\delta(t)). \end{aligned} \tag{4.22}$$

where we have define the energy $\mathcal{J}_\delta(t)$ associated with the error by

$$\mathcal{J}_\delta(t) = \frac{1}{2}\Big(\|\dot{\mathtt{E}}_\delta(t)\|_{\mathcal{H}}^2 + \|\dot{\mathtt{Z}}_\delta(t)\|_{\mathcal{H}}^2 + \|G_1\, \mathtt{E}_\delta(t)\|_{\mathcal{G}}^2 + \delta^{-2}\, \|G_0\, \mathtt{Z}_\delta(t)\|_{\mathcal{G}}^2\Big)$$

We now integrate in time (4.22) and we need to estimate the terms on the right hand side. We have, using the Cauchy-Schwartz inequality,

$$\left|\int_0^t (\mathtt{F}_\delta(\tau) - \mathtt{F}_0(\tau), \dot{\mathtt{E}}_\delta(\tau))_{\mathcal{H}}\, \mathrm{d}\tau\right| \leqslant \int_0^t \|\mathtt{F}_\delta(\tau) - \mathtt{F}_0(\tau)\|_{\mathcal{H}}\, \|\dot{\mathtt{E}}_\delta(\tau)\|_{\mathcal{H}}\, \mathrm{d}\tau. \tag{4.23}$$

Again we use the Cauchy-Schwartz inequality to show that

$$\left|\int_0^t (\mathtt{F}_\delta(\tau), \dot{\mathtt{Z}}_\delta(\tau))_{\mathcal{H}}\, \mathrm{d}\tau\right| \leqslant \int_0^t \|\mathtt{F}_\delta(\tau)\|_{\mathcal{H}}\, \|\dot{\mathtt{Z}}_\delta(\tau)\|_{\mathcal{H}}\, \mathrm{d}\tau. \tag{4.24}$$

Moreover,

$$\begin{aligned} \left|\int_0^t b_0(\dot{\mathtt{U}}_0(\tau), \dot{\mathtt{Z}}_\delta(\tau))\, \mathrm{d}s\right| &\leqslant \int_0^t \|\dot{\mathtt{U}}_0(\tau)\|_{\mathcal{B}}\, \|\dot{\mathtt{Z}}_\delta(\tau)\|_{\mathcal{B}}\, \mathrm{d}\tau \\ &\leqslant C_{\mathcal{I}}^2 \int_0^t \|\dot{\mathtt{U}}_0(\tau)\|_{\mathcal{V}_0}\, \|\dot{\mathtt{Z}}_\delta(\tau)\|_{\mathcal{V}_0}\, \mathrm{d}\tau = C_{\mathcal{I}}^2 \int_0^t \|\dot{\mathtt{U}}_0(\tau)\|_{\mathcal{V}_0}\, \|\dot{\mathtt{Z}}_\delta(\tau)\|_{\mathcal{H}}\, \mathrm{d}\tau. \end{aligned} \tag{4.25}$$

Now observe that

$$\|\dot{\mathtt{Z}}_\delta(t)\|_{\mathcal{H}} = \|z_\delta(t)\|_{\mathcal{H}} \leqslant C_p\,\|G_0\,z_\delta(t)\|_{\mathcal{G}} \leqslant \sqrt{2}\,C_p\,\delta\,\mathcal{J}_\delta^{\frac{1}{2}}.$$

From the estimate above, (4.22), (4.23), (4.24) and (4.25) we deduce that,

$$\frac{d}{dt}\mathcal{J}_\delta(t) \leqslant \int_0^t \|\mathtt{F}_\delta(\tau) - \mathtt{F}_0(\tau)\|_{\mathcal{H}}\,\mathcal{J}_\delta^{\frac{1}{2}}\,\mathrm{d}\tau + \sqrt{2}\,C_p\,\delta\int_0^t \big(\|\mathtt{F}_\delta(\tau)\|_{\mathcal{H}} + C_{\mathcal{I}}^2\,\|\dot{\mathtt{U}}_0(\tau)\|_{\mathcal{V}_0}\big)\,\mathcal{J}_\delta^{\frac{1}{2}}\,\mathrm{d}\tau.$$

We conclude the proof of the theorem using Gronwall's lemma as well as the following estimate

$$\|\dot{\mathtt{U}}_0(t)\|_{\mathcal{V}_0}^2 = \|u_0(t)\|_{\mathcal{H}}^2 + \|G_1\,u_0(t)\|_{\mathcal{G}}^2 \leqslant C_{\mathtt{F}}^2 + C_f^2$$ ∎

that is a direct consequence of (4.15).

4.3.2. Strong two-scale separation.

When only the strong scale separation holds an additional assumption is required. To state this assumption we introduce the bounded operator $\mathbf{G}_0 : \mathcal{V}_\infty \mapsto \mathcal{G}$ defined by

$$\forall\, v \in \mathcal{V}_\infty, \quad \mathbf{G}_0 v = G_0 v.$$

Assumption 4.9. *There exits $C_c > 0$ such that*

$$\forall\, u \in (\operatorname{Ker}\mathbf{G}_0)^\perp \cap \mathcal{V}_\infty, \quad \|u\|_{\mathcal{V}_\infty} \leqslant C_c\,\|G_0\,u\|_{\mathcal{G}}, \tag{4.26}$$

where the orthogonality holds for the scalar product in $\mathcal{V}_\infty$.

This last assumption is in fact very related to the so-called *inf-sup condition* (see [5] for more details). Note that one can shown that if Assumption 4.9 holds then Assumption 4.6 holds. The proof of this statement is given at the end of this section.

Theorem 4.10. *Assume that the strong scale separation, Assumption 3.2 and Assumption 4.9 hold and that the data of problem (2.19) are well-prepared. Then the weak solutions u_δ of problem (2.19) satisfy,*

$$\sup_{t\in[0,T]} \Big(\|\mathtt{U}_\delta(t) - \mathtt{U}_0(t)\|_{\mathcal{H}}^2 + \|G_1\int_0^t (\mathtt{U}_\delta(\tau) - \mathtt{U}_0(\tau))\,\mathrm{d}\tau\|_{\mathcal{G}}^2\Big)^{\frac{1}{2}} \leqslant T\int_0^T \|\mathtt{F}_\delta(t) - \mathtt{F}_0(t)\|_{\mathcal{H}}\,\mathrm{d}t + \delta\,C(T). \tag{4.27}$$

where u_0 is the unique weak solution of the limit problem (3.6) with vanishing initial data and source term f_0 and

$$C(T) = C_c\,(1 + C_{\mathcal{I}}^2)\,(1+T)\,\big(C_f + C_{\mathtt{F}}\big)$$

Proof.

Step 1: Introduction of a Lagrange multiplier. For all $v \in \mathcal{V}_\infty$ and $t \in [0,T]$, we define a time dependent functional $L \in C^0([0,T];\mathcal{L}(V_\infty))$ by

$$L(t,v) = (\mathtt{F}_0(t),v)_{\mathcal{H}} - (\ddot{\mathtt{U}}_0(t),v)_{\mathcal{H}} - (G_1\,\mathtt{U}_0(t),G_1\,v)_{\mathcal{G}} - b_0(\dot{\mathtt{U}}_0(t),v). \tag{4.28}$$

From the Riesz representation theorem we can define an element $\ell \in C^0([0,T];\mathcal{V}_\infty)$ such that

$$\forall\, v \in \mathcal{V}_\infty, \quad (\ell(t),v)_{\mathcal{V}_\infty} = L(t,v).$$

Observe now that, since (4.14) holds we have $(\ell(t),v)_{V_\infty} = 0$ for all $v \in \mathcal{V}_0^0$. Hence, we see that $\ell(t) \in (\operatorname{Ker}\mathbf{G}_0)^\perp$ (here the orthogonality relation holds for the scalar product in $\mathcal{V}_\infty$). Now, observe that

$$(\operatorname{Ker}\mathbf{G}_0)^\perp = \overline{\operatorname{Im}\mathbf{G}_0^*}.$$

We can not conclude, yet, that $\ell(t)$ belongs to the range of $\mathbf{G}_0^*$. For the that we need to show that the range is closed, which is equivalent to show that the range of $\mathbf{G}_0$ is closed. This property is guaranteed by Assumption 4.9. Hence we have

$$\ell(t) \in \operatorname{Im}\mathbf{G}_0^* \quad \Rightarrow \quad \exists\, g(t) \in D(\mathbf{G}_0^*) \text{ such that } \ell(t) = \mathbf{G}_0^*\, g(t).$$

We can slightly refine the result above. Since $\mathbf{G}_0^*$ has closed range, there exists $C_0 > 0$ such that

$$\forall\, g \in (\operatorname{Ker}\mathbf{G}_0^*)^\perp, \quad \|g\|_{\mathcal{G}} \leqslant C_0\|\mathbf{G}_0^*\, g\|_{\mathcal{V}_\infty},$$

showing that $\mathbf{G}_0^*$ is injective on $(\operatorname{Ker}\mathbf{G}_0^*)^\perp = \overline{\operatorname{Im}\mathbf{G}_0} = \operatorname{Im}\mathbf{G}_0$. Hence we obtain the following unique characterization of $\ell(t)$,

$$\ell(t) \in \operatorname{Im}\mathbf{G}_0^* \quad \Rightarrow \quad \exists!\; g_0(t) \in \operatorname{Im}\mathbf{G}_0 \text{ such that } \ell(t) = \mathbf{G}_0^*\, g(t).$$

To conclude, using (4.28) and the property of the adjoint

$$\forall\, v \in \mathcal{V}_\infty, \quad L(t,v) = (\ell(t),v)_{\mathcal{V}_\infty} = (\mathbf{G}_0^*\, g_0(t),v)_{\mathcal{V}_\infty} = (g_0(t),\mathbf{G}_0\, v)_{\mathcal{G}} = (g_0(t),G_0\, v)_{\mathcal{G}},$$

we have shown that there exists $g_0 \in C^0([0,T];\operatorname{Im}\mathbf{G}_0)$ such that, for all $v \in \mathcal{V}_\infty$, and all $t \in [0,T]$,

$$\begin{cases} (\ddot{\mathtt{U}}_0(t),v)_{\mathcal{H}} + (G_1\,\mathtt{U}_0(t),G_1\,v) + b_0(\dot{\mathtt{U}}_0(t),v) + (g_0(t),G_0\,v)_{\mathcal{G}} = (\mathtt{F}_0(t),v)_{\mathcal{H}}, \\ G_0\,\mathtt{U}_0(t) = 0, \\ \mathtt{U}_0(0) = 0, \\ \dot{\mathtt{U}}_0(0) = 0. \end{cases} \tag{4.29}$$

Step 2: Mixed formulation for the family of problems. Our objective here is to write a formulation for the family of problems that is similar to the one obtained just above. We introduce the function

$$g_\delta = \delta^{-2} G_0\,\mathtt{U}_\delta \in C^1([0,T];\operatorname{Im}\mathbf{G}_0)$$

and observe that problem (4.13) is equivalent to: for all $v \in \mathcal{V}_\delta$, and all $t \in [0,T]$,

$$\begin{cases} (\ddot{\mathtt{U}}_\delta(t),v)_{\mathcal{H}} + (G_1\,\mathtt{U}_\delta(t),G_1\,v)_{\mathcal{G}} + b_0(\dot{\mathtt{U}}_\delta(t),v) + (g_\delta(t),G_0\,v)_{\mathcal{G}} = (\mathtt{F}_\delta(t),v)_{\mathcal{H}}, \\ G_0\,\mathtt{U}_\delta(t) = \delta^2\, g_\delta, \\ \mathtt{U}_\delta(0) = 0, \\ \dot{\mathtt{U}}_0(0) = 0. \end{cases} \tag{4.30}$$

Step 3: Energy identity for the error equation. We define the error term

$$\mathtt{E}_\delta = \mathtt{U}_\delta - \mathtt{U}_0 \in C^2([0,T];\mathcal{H}) \cap C^1([0,T];\mathcal{V}_\infty), \quad h_\delta = g_\delta - g_0 \in C^0([0,T];\operatorname{Im}\mathbf{G}_0).$$

Using (4.29) and (4.30) one finds that

$$\left\{\begin{aligned} &(\ddot{\mathtt{E}}_\delta(t), v)_\mathcal{H} + (G_1\,\mathtt{E}_\delta(t), G_1\,v) + b_0(\dot{\mathtt{E}}_\delta(t), v)_\mathcal{G} + (h_\delta(t), G_0\,v)_\mathcal{G} \\ &\qquad\qquad\qquad\qquad = (\mathtt{F}_\delta(t) - \mathtt{F}_0(t), v)_\mathcal{H}, \\ &G_0\,\mathtt{E}_\delta(t) = \delta^2\,h_\delta + \delta^2\,g_0, \\ &\mathtt{E}_\delta(0) = 0, \\ &\dot{\mathtt{E}}_\delta(0) = 0, \end{aligned}\right. \tag{4.31}$$

In a first step we integrate the first equation of (4.31) with respect to time and, in a second step, choose for every time $t \in [0,T]$, $v = \mathtt{E}_\delta(t)$, we obtain

$$\begin{aligned} &(\dot{\mathtt{E}}_\delta(t), \mathtt{E}_\delta(t))_\mathcal{H} + (G_1 \int_0^t \mathtt{E}_\delta(s)\,\mathrm{d}s, G_1\,\mathtt{E}_\delta(t))_\mathcal{G} + b_0(\mathtt{E}_\delta(t), \mathtt{E}_\delta(t)) \\ &\qquad + (\mathtt{H}_\delta(t), G_0\,\mathtt{E}_\delta(t))_\mathcal{G} = (\int_0^t (\mathtt{F}_\delta(\tau) - \mathtt{F}_0(\tau))\,\mathrm{d}\tau, \mathtt{E}_\delta(t))_\mathcal{H}. \end{aligned} \tag{4.32}$$

We can now use the second equation of (4.31) to simplify the identity just above, indeed,

$$\begin{aligned} (\mathtt{H}_\delta(t), G_0\,\mathtt{E}_\delta(t))_\mathcal{G} &= \delta^2\,(\mathtt{H}_\delta(t), h_\delta(t))_\mathcal{G} + \delta^2\,(\mathtt{H}_\delta(t), g_0(t))_\mathcal{G} \\ &= \frac{\delta^2}{2}\frac{d}{dt}\|\mathtt{H}_\delta(t)\|_\mathcal{G}^2 + \delta^2\,(\mathtt{H}_\delta(t), g_0(t))_\mathcal{G}. \end{aligned} \tag{4.33}$$

Using (4.33) in (4.32) we obtain the energy identity

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|\mathtt{E}_\delta(t)\|_\mathcal{H}^2 + \|G_1 \int_0^t \mathtt{E}_\delta(\tau)\mathrm{d}\tau\|_\mathcal{G}^2 + \delta^2\|\mathtt{H}_\delta(t)\|_\mathcal{G}^2\Big) \\ &\qquad = (\int_0^t (\mathtt{F}_\delta(\tau) - \mathtt{F}_0(\tau))\,\mathrm{d}\tau, \mathtt{E}_\delta(t))_\mathcal{H} - \delta^2\,(\mathtt{H}_\delta(t), g_0(t))_\mathcal{G}. \end{aligned} \tag{4.34}$$

Using Gronwall's Lemma, one deduces that

$$\begin{aligned} &\Big(\,\|\mathtt{E}_\delta(t)\|_\mathcal{H}^2 + \|G_1 \int_0^t \mathtt{E}_\delta(\tau)\mathrm{d}\tau\|_\mathcal{G}^2 + \delta^2\|\mathtt{H}_\delta(t)\|_\mathcal{G}^2\,\Big)^{\frac{1}{2}} \\ &\qquad \leqslant \int_0^t \Big(\|\int_0^\tau (\mathtt{F}_\delta(s) - \mathtt{F}_0(s))\,\mathrm{d}s\|_\mathcal{H} + \delta\,\|g_0(\tau)\|_\mathcal{G}\Big)\mathrm{d}\tau. \end{aligned} \tag{4.35}$$

Step 4: Estimate of the Lagrange multiplier. To conclude the proof we provide an estimate of g_0 with respect to the source term f_0. From the first equation of (4.29) we have, for all $v \in \mathcal{V}_\infty$,

$$(g_0(t), G_0\,v)_\mathcal{G} = (\mathtt{F}_0(t), v)_\mathcal{H} - (\dot{u}_0(t), v)_\mathcal{H} - (G_1\,\mathtt{U}_0(t), G_1\,v) - b_0(u_0(t), v).$$

Using the Cauchy-Scwartz inequality, (2.18) and (2.14) we find that

$$|(g_0(t), G_0\, v)_{\mathcal{G}}| \leqslant \big(\|\mathtt{F}_0(t)\|_{\mathcal{H}} + \|\dot{u}_0(t)\| + \|G_1\, \mathtt{U}_0(t)\|_{\mathcal{G}} \big) \|v\|_{\mathcal{V}_0} + C_{\mathcal{I}}\, \|u_0(t)\|_{\mathcal{V}_0}\, \|v\|_{\mathcal{B}}.$$

Note that, since (2.14) is true in particular for $\delta = 1$ we have $\|v\|_{\mathcal{B}} \leqslant C_{\mathcal{I}} \|v\|_{\mathcal{V}_\infty}$, therefore we deduce from the inequality above the estimate

$$|(g_0(t), G_0\, v)_{\mathcal{G}}| \leqslant \big(\|\mathtt{F}_0(t)\|_{\mathcal{H}} + \|\dot{u}_0(t)\| + \|G_1\, \mathtt{U}_0(t)\|_{\mathcal{G}} + C_{\mathcal{I}}^2\, \|u_0(t)\|_{\mathcal{V}_0} \big)\, \|v\|_{\mathcal{V}_\infty}.$$

Since $g_0(t) \in \operatorname{Im} \mathbf{G}_0$ and $\operatorname{Ker} \mathbf{G}_0$ is a closed subspace of $\mathcal{V}_\infty$ one can find $v_0(t)$ belonging to $(\operatorname{Ker} \mathbf{G}_0)^\perp \subset \mathcal{V}_\infty$ such that $g_0(t) = G_0\, v_0(t)$, and thanks to Assumption 4.9,

$$\|v_0(t)\|_{\mathcal{V}_\infty}^2 \leqslant C_c^2\, \|G_0\, v_0(t)\|_{\mathcal{G}}^2 = C_c^2\, \|g_0(t)\|_{\mathcal{G}}^2,$$

therefore

$$\|g_0(t)\|_{\mathcal{G}} \leqslant C_c \,\big(\|\mathtt{F}_0(t)\|_{\mathcal{H}} + \|\dot{u}_0(t)\| + \|G_1\, \mathtt{U}_0(t)\|_{\mathcal{G}} + C_{\mathcal{B}}^2\, \|u_0(t)\|_{\mathcal{V}_0} \big). \tag{4.36}$$

Using the estimate (4.15) and (4.36) into (4.35) we arrive at the statement of the theorem. ∎

We conclude this section by showing that Assumption 4.9 is stronger than Assumption 4.6.

Theorem 4.11. *If the strong-scale separation property holds and if Assumption 4.9 holds then Assumption 4.6 holds with $C_c = C_p$.*

Proof. Note that from (4.26) we deduce that

$$\forall\ v \in (\operatorname{Ker} \mathbf{G}_0)^\perp \cap \mathcal{V}_\infty, \quad \|v\|_{\mathcal{H}} \leqslant C_c\, \|G_0\, v\|_{\mathcal{G}}. \tag{4.37}$$

Because of the weak-scale separation property we have $\mathcal{V}_\infty$ dense in $D(G_0)$ hence, for all $u \in \mathcal{H}_0^\perp \cap D(G_0)$ we introduce a sequence $\{v_n\}$ such that $v_n \in \mathcal{V}_\infty$ and

$$\|v_n - u\|_{\mathcal{H}}^2 + \|G_0\, v_n - G_0\, u\|_{\mathcal{G}}^2 \underset{n\to+\infty}{\longrightarrow} 0\,. \tag{4.38}$$

Moreover, denoting $\mathbf{P}_0$ the orthogonal projection – for the scalar product in $\mathcal{V}_\infty$ – in $\operatorname{Ker} \mathbf{G}_0$, we have

$$(I - \mathbf{P}_0)\, v_n \in (\operatorname{Ker} \mathbf{G}_0)^\perp \cap \mathcal{V}_\infty \quad \text{and} \quad \mathbf{P}_0\, v_n \in \mathcal{V}_\infty^0 \quad \text{hence} \quad (\mathbf{P}_0\, v_n, u)_{\mathcal{H}} = 0, \tag{4.39}$$

where we have used the property that $\mathcal{H}_0 = \overline{\mathcal{V}_\infty^0}^{\|\cdot\|_{\mathcal{H}}}$ when the strong-scale separation holds to show the last property. From (4.39) and (4.38) we deduce that

$$\begin{aligned}\|v_n - u\|_{\mathcal{H}}^2 &+ \|G_0\, v_n - G_0\, u\|_{\mathcal{G}}^2 \\ &= \|\mathbf{P}_0\, v_n\|_{\mathcal{H}}^2 + \|(I - \mathbf{P}_0) v_n - u\|_{\mathcal{H}}^2 + \|G_0\, (I - \mathbf{P}_0) v_n - G_0\, u\|_{\mathcal{G}}^2 \underset{n\to+\infty}{\longrightarrow} 0\,.\end{aligned}$$

Using the inequality (4.37) for the sequence $\{(I - \mathbf{P}_0) v_n\}$ and passing to the limit we obtain the result of the theorem. ∎

5. Proofs of Section 2 and 3.

Proof of Theorem 2.3. Since u_δ is uniformly bounded in $\mathcal{H}$ it converges, up to a subsequence, weakly to u_0 in $\mathcal{H}$ we have, for all $g \in D(G_0^*) \cap D(G_1^*)$,

$$(G_\delta\, u_\delta, g)_{\mathcal{G}} = (u_\delta, G_\delta^*\, g)_{\mathcal{G}} = (u_\delta, G_0^*\, g)_{\mathcal{G}} + \delta(u_\delta, G_1^*\, g)_{\mathcal{G}} \underset{\delta\to 0}{\longrightarrow} (u_0, G_0^*\, g)_{\mathcal{G}}.$$

Note that in the previous equation we have used the property that $G_0^* + \delta G_1^* \subset G_\delta^*$ (see [11] Eq. (12.12)). Now since $G_\delta\, u_\delta$ vanishes at the limit we have

$$(u_0, G_0^*\, g)_{\mathcal{G}} = 0, \quad \forall\, g \in D(G_0^*) \cap D(G_1^*).$$

By density of $D(G_0^*) \cap D(G_1^*)$ into $D(G_0^*)$ for the graph norm of G_0^* we deduce that the equality above holds for all $g \in D(G_0^*)$. Since G_0 is closed and densely defined, this shows that $u_0 \in D(G_0)$ and $G_0\, u_0 = 0$.

Proof of Lemma 2.4.
The domain of $G_1^*\, Q_0$. By definition we have

$$D(G_1^*\, Q_0) = \{g \in D(Q_0) = \mathcal{G} \;|\; Q_0\, g \in D(G_1^*)\}$$

For all $g \in \mathcal{G}$ we have the orthogonal decomposition in $\mathcal{G}$ given by $g = g_0 + g_\perp$ with $g_0 \in \operatorname{Ker} G_0^*$ and $Q_0\, g_\perp = 0$. Therefore

$$D(G_1^*\, Q_0) = \{g = g_0 + g_\perp,\ g_0 \in \operatorname{Ker} G_0,\ g_\perp \in (\operatorname{Ker} G_0)^\perp \;|\; Q_0\, g = g_0 \in D(G_1^*)\},$$

which shows that

$$D(G_1^*\, Q_0) = \operatorname{Ker} G_0^* \cap D(G_1^*) \oplus (\operatorname{Ker} G_0^*)^\perp. \tag{5.1}$$

The operator $G_1^*\, Q_0$ is densely defined. This is a consequence of (5.1), the assumption that $\operatorname{Ker} G_0^* \cap D(G_1^*)$ is dense in $\operatorname{Ker} G_0^*$ and the property that

$$\mathcal{G} = \operatorname{Ker} G_0^* \oplus (\operatorname{Ker} G_0^*)^\perp.$$

The operator $Q_0\, G_1$ is closable. Since we have shown that $G_1^*\, Q_0$ is densely defined it has an adjoint. We conclude that $Q_0 G_1$ is closable since Q_0 is self-adjoint and

$$Q_0 G_1 = Q_0^* G_1^{**} \subset (G_1^*\, Q_0)^*$$

which is closed (we have use here several important relation in operator theory, we refer the reader to p. 168 of [15] for more informations).

Proof of Theorem 2.5. From Lemma 2.4 we know that $Q_0\, G_1$ is closable. Hence in a first step we show the inclusion $\delta^{-1}\, Q_0\, G_\delta \subset \overline{Q_0\, G_1}$ and the inequality in a second step.

The inclusion. Since $\overline{Q_0\, G_1}$ is densely defined its adjoint is well-defined and is given by $G_1^*\, Q_0$. This adjoint operator is densely defined by Lemma 2.4. Moreover, since $G_\delta = G_0 + \delta G_1$ is densely defined thanks to (2.2), we have $G_0^* + \delta\, G_1^* \subset G_\delta^*$, we have (see Problem 5.3 and 5.26 of [15], p.163 and p. 168),

$$G_1^*\, Q_0 \;=\; \delta^{-1}\,(G_0^* + \delta\, G_1^*)\, Q_0 \;\subset\; \delta^{-1}\, G_\delta^*\, Q_0 \;=\; \delta^{-1}\,(Q_0\, G_\delta)^*. \tag{5.2}$$

Hence, since $G_1^* Q_0$ is densely-defined we deduce (see Problem 5.25 and 5.26 of [15], p. 168) two properties. First we deduce from (5.2) that $G_\delta^* Q_0$ is also densely defined, its adjoint exists and since Q_0 is bounded and self-adjoint,

$$Q_0 G_\delta = Q_0^* G_\delta^{**} \subset (G_\delta^* Q_0)^*,$$

hence $Q_0 G_\delta$ is closable and $Q_0 G_\delta \subset (Q_0 G_\delta)^{**}$ (we use here Theorem 5.29 of [15]). Second we deduce from (5.2) that

$$\delta^{-1} (Q_0 G_\delta)^{**} \subset (G_1^* Q_0)^*.$$

Now since $G_1^* Q_0$ is densely defined and closable, we have (again Theorem 5.29 of [15]) $\overline{Q_0 G_1} = (Q_0 G_1)^{**}$ and since Q_0 is bounded and self-adjoint (Problem 5.26 of [15])

$$(Q_0 G_1)^{**} = (G_1^* Q_0)^*.$$

We deduce that

$$\delta^{-1} (Q_0 G_\delta)^{**} \subset \overline{Q_0 G_1} \quad \Rightarrow \quad \delta^{-1} Q_0 G_\delta \subset \overline{Q_0 G_1}.$$

The inequality. Using the orthogonal projection operator Q_0, we deduce that for all $u \in D(G_\delta)$,

$$\|\delta^{-1} G_\delta u_\delta\|_{\mathcal{G}}^2 = \delta^{-1}\|Q_0 G_\delta u_\delta\|_{\mathcal{G}}^2 + \delta^{-1}\|(I - Q_0) G_\delta u_\delta\|_{\mathcal{G}}^2$$

From the equality above we deduce, since $\delta \in (0, 1]$, that

$$\delta^{-1}\|Q_0 G_\delta u_\delta\|_{\mathcal{G}}^2 \leqslant \|\delta^{-1} G_\delta u_\delta\|_{\mathcal{G}}^2.$$

Hence, since $\delta^{-1} Q_0 G_\delta \subset \overline{Q_0 G_1}$ we obtain

$$\|\overline{Q_0 G_1} u_\delta\|_{\mathcal{G}} \leqslant \|\delta^{-1} G_\delta u_\delta\|_{\mathcal{G}},$$

which concludes the proof.

Proof of Theorem 2.6. Defining $g_\delta = \delta^{-1} G_\delta u_\delta$ we have, thanks to the estimate (2.6)

$$\|g_\delta\|_{\mathcal{G}} \leqslant C,$$

and there exists $g_0 \in \mathcal{G}$ that is the weak limit, up to a subsequence, of g_δ in $\mathcal{G}$. We have moreover, thanks to (2.7),

$$(g_0, G_0 v)_{\mathcal{G}} = 0, \quad \forall v \in D(G_0) \cap D(G_1).$$

Since, by assumption $D(G_0) \cap D(G_1)$ is dense in $D(G_0)$ for the graph norm of G_0 one can deduce that

$$(g_0, G_0 v)_{\mathcal{G}} = 0, \quad \forall v \in D(G_0).$$

and therefore $g_0 \in \operatorname{Ker} G_0^*$. This implies that, up to a subsequence,

$$\delta^{-1} (I - Q_0) G_\delta u_\delta \underset{\mathcal{G}}{\rightharpoonup} (I - Q_0) g_0 = 0.$$

This result being valid independently of the chosen subsequence the above weak convergence towards 0 is valid for the complete sequence.

Proof of Theorem 2.8.

The equality $\overline{Q_0\, G_1} = G_1$. First observe that (2.10) implies that

$$\overline{\operatorname{Im} G_0} \subset \operatorname{Ker} G_1^*.$$

Now, since $I - Q_0$ is, by definition, the orthogonal projection operator on the closure of $\operatorname{Im} G_0$, we have

$$\overline{\operatorname{Im} G_0} \subset \operatorname{Ker} G_1^* \quad \Rightarrow \quad G_1^*(I - Q_0) = 0 \quad \Rightarrow \quad \overline{Q_0\, G_1} = G_1,$$

since $(Q_0 G_1)^{**} = \overline{Q_0\, G_1} = (G_1^* Q_0)^*$.

Density results. Since $G_1^*\, Q_0 = G_1^*$ and G_1^* is densely defined in $\mathcal{G}$ we have

$$D(G_1^*\, Q_0) = \operatorname{Ker} G_0^* \cap D(G_1^*) \oplus (\operatorname{Ker} G_0^*)^\perp \text{ dense in } \mathcal{G} = \operatorname{Ker} G_0^* \oplus (\operatorname{Ker} G_0^*)^\perp.$$

From the statement above we deduce that $\operatorname{Ker} G_0^* \cap D(G_1^*)$ is dense in $\operatorname{Ker} G_0^*$. We must now show that $D(G_0^*) \cap D(G_1^*)$ is dense in $D(G_0^*)$ for the graph norm of G_0^*. We define the space

$$\mathcal{Z} = \operatorname{Ker} G_0^* \cap D(G_1^*) \oplus (\operatorname{Ker} G_0^*)^\perp \cap D(G_0^*),$$

where the decomposition is orthogonal with respect to the scalar product in $\mathcal{H}$. Observe that the space $\mathcal{Z}$ is a subspace of $D(G_0^*) \cap D(G_1^*)$ since by assumption

$$(\operatorname{Ker} G_0^*)^\perp = \overline{\operatorname{Im} G_0} \subset \operatorname{Ker} G_1^*.$$

Observe moreover that

$$D(G_0^*) = \operatorname{Ker} G_0^* \oplus (\operatorname{Ker} G_0^*)^\perp \cap D(G_0^*),$$

where the decomposition is orthogonal with respect to the scalar product in $\mathcal{H}$. However we have already shown that $\operatorname{Ker} G_0^* \cap D(G_1^*)$ is dense in $\operatorname{Ker} G_0^*$ for the norm $\mathcal{H}$, it is therefore dense for the graph norm of G_0^*. This shows that $\mathcal{Z}$ (hence $D(G_0^*) \cap D(G_1^*)$) is dense in $D(G_0^*)$ for the same norm.

The equality $G_\delta = G_0 + \delta G_1$. By definition we have

$$G_0 + \delta G_1 \subset G_\delta.$$

Let us show the reverse inclusion. Note that from Theorem 2.5 we have

$$D(G_\delta) = D(Q_0 G_\delta) \subset D(\overline{Q_0 G_1}) = D(G_1).$$

Now, for all $g \in D(G_0^*) \cap D(G_1^*)$ and $v \in D(G_\delta) \subset D(G_1)$ we have

$$\begin{aligned}(G_\delta\, v, g)_{\mathcal{G}} &= (v, G_\delta^*\, g)_{\mathcal{H}} \\ &= (v, (G_0 + \delta G_1)^*\, g)_{\mathcal{H}} = (v, G_0^*\, g + \delta G_1^*\, g)_{\mathcal{H}} = (v, G_0^*\, g) + \delta (G_1 v,\, g)_{\mathcal{G}}.\end{aligned}$$

By density of $D(G_0^*) \cap D(G_1^*)$ in $D(G_0^*)$ we deduce that, for all $v \in D(G_\delta) \subset D(G_1)$,

$$(v, G_0^*\, g) = (G_\delta\, v, g)_{\mathcal{G}} - \delta (G_1\, v, g)_{\mathcal{G}}, \quad \forall g \in D(G_0^*).$$

This implies that $v \in D(G_0^{**}) = D(G_0)$ therefore $D(G_\delta) \subset D(G_1) \cap D(G_0)$.

Proof of Theorem 2.10. Let us first introduce the orthogonal projector in $\operatorname{Ker} G_0$, denoted P_0, which exact definition is given Eq. (3.7). We give below the four steps of the proof omitting details as they are carbon copies of arguments given in the proof of Lemma 2.4 and Theorem 2.8.

Step 1. We prove that $P_0\, G_1^*$ is closable.

Step 2. We prove that $G_1 P_0 = G_1$ which shows that $D(G_1 P_0)$ is dense in $\mathcal{G}$.

Step 3. We write that $D(G_1 P_0) = \operatorname{Ker} G_0 \cap D(G_1) \oplus (\operatorname{Ker} G_0)^\perp$ is dense in $\mathcal{G}$, which implies $\operatorname{Ker} G_0 \cap D(G_1)$ dense in $\operatorname{Ker} G_0$.

Step 4. Observing that $D(G_0) \cap D(G_1) \subset \operatorname{Ker} G_0 \cap D(G_1) \oplus (\operatorname{Ker} G_0)^\perp \cap D(G_0)$ we show that this latter space is dense in $D(G_0)$.

Proof of Theorem 2.14. Although the result stated is new it is expected and classically obtained. Indeed the proof combined classical result in semi-group theory [19] and variational method [9], for these reasons the proof is delayed to the appendix.

Proof of Theorem 3.3. For any $v_0 \in \mathcal{V}_0^0$ we define $u_\delta \in \mathcal{V}_\delta$ as the unique solution of

$$(u_\delta, v)_{\mathcal{V}_\delta} = (v_0, v)_{\mathcal{V}_0}, \quad \forall\, v \in \mathcal{V}_\delta. \tag{5.3}$$

Because of Proposition 3.1 we find $\|u_\delta\|_{\mathcal{V}_0} \leqslant \|u_\delta\|_{\mathcal{V}_\delta} \leqslant \|v_0\|_{\mathcal{V}_0}$ hence the sequence $\{u_\delta\}$ is uniformly bounded in $\mathcal{V}_0$. The sequence converges weakly and up to a subsequence to some element $u_0 \in \mathcal{V}_0$ and in particular it converges weakly in $\mathcal{H}$ to the same element. Moreover the estimation (2.4) of Theorem 2.3 is satisfied hence we have $u_0 \in \operatorname{Ker} G_0$ and therefore $u_0 \in \mathcal{V}_0^0$. Additionally the convergence (2.7) in Theorem 2.6 holds hence

$$\delta^{-1}\,(I - Q_0)\, G_\delta\, u_\delta \underset{\mathcal{G}}{\rightharpoonup} 0. \tag{5.4}$$

Choosing now a test function $v \in \mathcal{V}_\infty^0$ in (5.3) we have

$$(u_\delta, v)_{\mathcal{H}} + (\delta^{-1} G_\delta\, u_\delta, G_1\, v)_{\mathcal{G}} = (u_\delta, v)_{\mathcal{H}} + (\,\overline{Q_0\, G_1}\, v_0, Q_0\, G_1\, v\,)_{\mathcal{G}}. \tag{5.5}$$

Since $\delta^{-1}\, Q_0\, G_\delta \subset \overline{Q_0\, G_1}$ we have

$$(\delta^{-1} G_\delta\, u_\delta, G_1\, v)_{\mathcal{G}} = (\delta^{-1}\,(I - Q_0)\, G_\delta\, u_\delta, G_1\, v)_{\mathcal{G}} + (\overline{Q_0\, G_1}\, u_\delta, G_1\, v)_{\mathcal{G}}.$$

Using (5.4) and the property that u_δ weakly converges to u_0 in $\mathcal{V}_0$ we can pass to the limit δ goes to zero in (5.5). We obtain, since $Q_0\, \overline{Q_0\, G_1} = \overline{Q_0\, G_1}$,

$$(u_0, v)_{\mathcal{V}_0} = (v_0, v)_{\mathcal{V}_0}, \quad \forall\, v \in \mathcal{V}_\infty^0.$$

When Assumption 3.2 holds then $\mathcal{V}_\infty^0$ is dense in $\mathcal{V}_0^0$ for the norm $\|\cdot\|_{\mathcal{V}_0}$ which implies that the equality above holds for all $v \in \mathcal{V}_0^0$ and therefore $u_0 = v_0$ independently of the chosen subsequence. Now, thanks to (2.14),

$$\|u_\delta\|_{\mathcal{B}} \leqslant C_{\mathcal{I}} \|u_\delta\|_{\mathcal{V}_\delta} \leqslant C_{\mathcal{I}}\, \|v_0\|_{\mathcal{V}_0}, \tag{5.6}$$

meaning that u_δ weakly converges, up to a subsequence, towards some u_0 in $\mathcal{B}$ and in particular in $\mathcal{H}$. Since u_δ also weakly converges to v_0 in $\mathcal{V}_0^0$ and in particular in $\mathcal{H}$. This shows that $u_0 = v_0$ hence $v_0 \in \mathcal{B}$. Passing to the (weak) limit in (5.6) we obtain

$$\|v_0\|_{\mathcal{B}} \leqslant C_{\mathcal{I}}\, \|v_0\|_{\mathcal{V}_0}.$$

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6. Appendix: Existence and uniqueness of solutions. The objective of this appendix is to prove Theorem 2.14. The proof is divided in three main parts. First, we prove that existence and uniqueness of strong solutions of (2.19) (this requires to define what are these strong solutions). Then by a limit process we show that weak solution, as given by Definition 2.13 exists and the energy estimate holds. We show uniqueness of a weak solution by a duality argument. Finally, we show that solutions of (2.14) are unique and that the weak solutions are indeed the solutions.

6.1. Operator equation and strong solutions. To the bilinear forms a_δ and b_0 we associate operators A_δ and B_0 in $\mathcal{H}$ in the usual way. These operators inherits the properties of the corresponding bilinear forms. More precisely:

(A_δ) The operator A_δ is the variational operator associated to the triple $(\mathcal{H}, \mathcal{V}_\delta, a_\delta)$. Its domain is the subspace of $\mathcal{V}_\delta$ denoted $D(A_\delta)$, defined as

$$D(A_\delta) = \{u \in \mathcal{V}_\delta \mid \exists\, w \in \mathcal{H},\ a_\delta(u,v) = (w,v)_{\mathcal{H}},\ \forall\, v \in \mathcal{V}_\delta\}. \tag{6.1}$$

For every element u in the domain one can define by density of $\mathcal{V}_\delta$ in $\mathcal{H}$ a unique $A_\delta u \in \mathcal{H}$ such that, for all $v \in \mathcal{V}_\delta$, $(A_\delta u, v)_\mathcal{H} = a_\delta(u,v)$. Doing so a linear operator $A_\delta : D(A_\delta) \subset \mathcal{H} \to \mathcal{H}$ has been defined. The operator A_δ satisfies the following properties inherited from (2.15): for all $u \in D(A_\delta)$,

$$(A_\delta u, u)_\mathcal{H} \geqslant 0, \quad (A_\delta u, u)_\mathcal{H} + \|u\|^2_\mathcal{H} = \|u\|^2_{\mathcal{V}_\delta}$$

and, moreover, for all $v \in \mathcal{V}_\delta$,

$$|(A_\delta u, v)_\mathcal{H}| \leqslant \|u\|_{\mathcal{V}_\delta} \|v\|_{\mathcal{V}_\delta}.$$

As a direct consequence of the properties above one can show – using Lax-Milgram theorem – that $A_\delta + \lambda_A I$ is surjective from $D(A_\delta)$ into $\mathcal{H}$ hence $D(A_\delta)$ is dense in $\mathcal{H}$ (see Proposition 3.1.6 of [20]). To conclude with the definition of A_δ we mention that it is self-adjoint (e.g. Corollary 12.19 of [11]), i.e., $D(A_\delta) = D(A_\delta^*)$ and for all u and v in $D(A_\delta)$,

$$(A_\delta u, v)_\mathcal{H} = (u, A_\delta v)_\mathcal{H}.$$

(B_0) The domain of the operator B_0 is defined as

$$D(B_0) = \{u \in \mathcal{B} \mid \exists\, w \in \mathcal{H},\ b_0(u,v) = (w,v)_\mathcal{H},\ \forall\, v \in \mathcal{V}_\delta\}, \tag{6.2}$$

and the operator $B_0 : D(B_0) \subset \mathcal{H} \to \mathcal{H}$ is defined following the same procedure used to defined A_δ. It inherits the non-negativity property of b_0, for all $u \in D(B_0)$,

$$0 \leqslant c_B \|u\|^2_\mathcal{H} \leqslant (B_0 u, u)_\mathcal{H}.$$

In applications, the operator A_δ corresponds to the "wave-propagation operator" whereas B_0 accounts for losses and/or couplings. The fact that B_0 is potentially unbounded involves additional difficulties that is not encounter when studying simpler wave equations (e.g. when B_0 is bounded). One difficulty is to define precisely the space in which a strong solution exists. To do so, we introduce the subspace $D(\mathbf{A}_\delta)$ of $\mathcal{V}_\delta \times \mathcal{H}$,

$$D(\mathbf{A}_\delta) = \big\{(u_0,u_1) \in \mathcal{V}_\delta \times \mathcal{B} \mid \exists\, w \in \mathcal{H}, \quad a_\delta(u_0,v) + b_0(u_1,v) = (w,v)_\mathcal{H},\ \forall\, v \in \mathcal{V}_\delta\big\}. \tag{6.3}$$

Then for two functions u_0 and u_1 such that $(u_0,u_1) \in \mathcal{D}(\mathbf{A}_\delta)$ we define the operator $\mathbf{A}_\delta : D(\mathbf{A}_\delta) \mapsto \mathcal{H}$ by

$$\mathbf{A}_\delta \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = w, \tag{6.4}$$

where w is the unique element in $\mathcal{H}$ such that, for all $v \in \mathcal{V}_\delta$, we have $a_\delta(u_0,v) + b_0(u_1,v) = (w,v)_\mathcal{H}$. With these definitions we define the following family of problems: assume that (u^0_δ, u^1_δ) and $f_\delta)$ are sufficiently regular, find $u_\delta(t)$ such that

$$\begin{cases} \ddot{u}_\delta + \mathbf{A}_\delta \begin{pmatrix} u_\delta \\ \dot{u}_\delta \end{pmatrix} = f_\delta, \quad t \in [0,T] \\ u_\delta(0) = u_0, \\ \dot{u}_\delta(0) = u_1, \end{cases} \tag{6.5}$$

Remark 6.1. If B_0 is a bounded operator then: $D(B_0) = \mathcal{H}$ and the domain $D(\mathbf{A}_\delta)$ takes a simpler form, $D(\mathbf{A}_\delta) = D(A_\delta) \times \mathcal{H}$ moreover

$$\forall\, (u_0, u_1) \in D(A_\delta) \times \mathcal{H}, \quad \mathbf{A}_\delta \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = A_\delta\, u_0 + B_0\, u_1.$$

We assume now that

$$(u_\delta^0, u_\delta^1) \in D(\mathbf{A}_\delta), \qquad \text{and} \qquad f_\delta \in W^{1,1}\big(0, T; \mathcal{H}\big) \cap C^0\big([0,T]; \mathcal{H}\big).$$

Definition 6.2. (**Strong solution**) *A strong solution is a function*

$$u_\delta \in C^2\big([0,T]; \mathcal{H}\big) \cap C^1\big([0,T]; \mathcal{V}_\delta\big), \quad (u_\delta, \dot{u}_\delta) \in C^0\big([0,T]; D(\mathbf{A}_\delta)\big) \tag{6.6}$$

satisfying (6.5).

To prove existence and uniqueness of a strong solution we use Hille-Yoshida theory, and, in particular we write problem (6.5) as a first order system. To do so, we define an unbounded linear operator in $\mathcal{V}_\delta \times \mathcal{H}$,

$$\mathcal{A}_\delta : \mathcal{V}_\delta \times \mathcal{H} \to \mathcal{V}_\delta \times \mathcal{H}, \qquad \mathcal{A}_\delta \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} u_1 \\ -\mathbf{A}_\delta \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \end{pmatrix}$$

where the domain of this operator is

$$D(\mathcal{A}_\delta) = \{(u_0, u_1) \in D(\mathbf{A}_\delta) \ | \ u_1 \in \mathcal{V}_\delta\}. \tag{6.7}$$

Then we construct a first order formulation of problem (6.5). Setting

$$y_\delta = \begin{pmatrix} u_\delta \\ \dot{u}_\delta \end{pmatrix} \quad \text{and} \quad z_\delta = \begin{pmatrix} 0 \\ f_\delta \end{pmatrix},$$

we look for $y_\delta(t)$ such that

$$\begin{cases} \dot{y}_\delta = \mathcal{A}_\delta\, y_\delta + z_\delta, \quad t \in [0,T], \\[2ex] y_\delta(0) = \begin{pmatrix} u_\delta^0 \\ u_\delta^1 \end{pmatrix}. \end{cases} \tag{6.8}$$

Observe that any strong solutions of (6.5) is a solution of (6.8).

The space $\mathcal{V}_\delta \times \mathcal{H}$ is equipped with the scalar product $(\cdot,\cdot)_{\mathcal{V}_\delta \times \mathcal{H}}$ given by, for all $y = (u_0, u_1) \in \mathcal{V}_\delta \times \mathcal{H}$ and for all $z = (v_0, v_1) \in \mathcal{V}_\delta \times \mathcal{H}$,

$$(y, z)_{\mathcal{V}_\delta \times \mathcal{H}} := a_\delta(u_0, v_0) + (u_0, v_0)_\mathcal{H} + (\, u_1, v_1)_\mathcal{H}. \tag{6.9}$$

Note that Equation (6.9) indeed defines a scalar product in $\mathcal{V}_\delta \times \mathcal{H}$ because of (2.16). An important property require here is the notion of dissipative operators (see e.g. Chapter 3 of [20]).

LEMMA 6.3. *The operator $\mathcal{A}_\delta - \mathcal{I}/2$ is dissipative,*

$$\forall y \in D(\mathcal{A}_\delta), \quad \Big((\mathcal{A}_\delta - \mathcal{I}/2)\, y, y \Big)_{\mathcal{V}_\delta \times \mathcal{H}} \leqslant 0$$

and $\mathcal{A}_\delta - \mathcal{I}$ is surjective in $\mathcal{V}_\delta \times \mathcal{H}$.

Proof.
Step 1. We show that $\mathcal{A}_\delta - \mathcal{I}/2$ is dissipative. For all $y = (u_0, u_1) \in D(\mathcal{A}_\delta)$ (observe that $D(\mathcal{A}_\delta)$ is a subspace of $\mathcal{V}_\delta \times \mathcal{V}_\delta$) we have,

$$\big((\mathcal{A}_\delta - \mathcal{I}/2)\, y, y \big)_{\mathcal{V}_\delta \times \mathcal{H}} = \left(\begin{pmatrix} u_1 - \dfrac{u_0}{2} \\ -\mathbf{A}_\delta \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} - \dfrac{u_1}{2} \end{pmatrix}, y \right)_{\mathcal{V}_\delta \times \mathcal{H}}.$$

Using the definition of the scalar product in $\mathcal{V}_\delta \times \mathcal{H}$ given by (6.9) we obtain

$$\big((\mathcal{A}_\delta - \mathcal{I})\, y, y \big)_{\mathcal{V}_\delta \times \mathcal{H}} = a_\delta(u_1, u_0) + (u_1, u_0)_{\mathcal{H}} - \left(\mathbf{A}_\delta \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, u_1 \right)_{\mathcal{H}} - \frac{1}{2} \|y\|^2_{\mathcal{V}_\delta \times \mathcal{H}}.$$

By definition of the operator $\mathbf{A}_\delta$ – see Equation (6.3) – we obtain that

$$\big((\mathcal{A}_\delta - \mathcal{I})\, y, y \big)_{\mathcal{V}_\delta \times \mathcal{H}} = (u_1, u_0)_{\mathcal{H}} - b_0(u_1, u_1) - \frac{1}{2} \|y\|^2_{\mathcal{V}_\delta \times \mathcal{H}} \leqslant 0.$$

Step 2. We now show that $\mathcal{A}_\delta - \mathcal{I}$ is surjective. For any $z = (v_0, v_1) \in \mathcal{V}_\delta \times \mathcal{H}$ we introduce the following variational problem: find $u_0 \in \mathcal{V}_\delta$ such that for all $u \in \mathcal{V}_\delta$,

$$a_\delta(u_0, u) + b_0(u_0, u) + (u_0, u)_{\mathcal{H}} = -(v_1, u)_{\mathcal{H}} - b_0(v_0, u). \tag{6.10}$$

Thanks to (2.16) and Lax-Milgram theorem, it can be easily shown that there exists a unique solution of problem (6.10). Moreover one can verify that

$$(u_0, u_0 + v_0) \in D(\mathcal{A}_\delta).$$

Then setting $u_1 = u_0 + v_0$ we have $u_1 \in \mathcal{V}_\delta$ and

$$\mathbf{A}_\delta \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = -v_1 \quad \Rightarrow \quad \big(\mathcal{A}_\delta - \mathcal{I} \big) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix},$$

which shows the desired surjectivity property. ∎

THEOREM 6.4. *There exists a unique strong solution.*

Proof. First we observe that since $\mathcal{A}_\delta - \mathcal{I}$ is surjective, from Proposition 3.1.6 of [20], $D(\mathcal{A}_\delta - \mathcal{I}) = D(\mathcal{A}_\delta)$ is dense in $\mathcal{V}_\delta \times \mathcal{H}$. Moreover since $\mathcal{A}_\delta - \mathcal{I}/2$ is dissipative we have that $\mathcal{A}_\delta - \mathcal{I}/2$ is maximal dissipative. We can then apply the Lumer–Phillips theorem (see for instance Theorem 3.8.4 of [20]) to show that $\mathcal{A}_\delta - \mathcal{I}/2$ is the generator of a contraction semi-group on $\mathcal{V}_\delta \times \mathcal{H}$. Now, starting from (6.8) we consider the following problem in $C^0\big([0,T]; \mathcal{V}_\delta \times \mathcal{H}\big)$,

$$\dot{\tilde{y}}_\delta = \mathcal{A}_\delta\, \tilde{y}_\delta - \frac{\tilde{y}_\delta}{2} + z_\delta e^{-\frac{t}{2}}, \quad \tilde{y}_\delta(0) = y_\delta(0). \tag{6.11}$$

Then, since $\mathcal{A}_\delta - \mathcal{I}/2$ is the generator of a contraction semi-group we have that if

$$z_\delta \in W^{1,1}\big(0,T;\mathcal{V}_\delta \times \mathcal{H}\big) \cap C^0([0,T];\mathcal{V}_\delta \times \mathcal{H}) \quad \text{and} \quad \tilde{y}_\delta(0) \in D(\mathcal{A}_\delta), \tag{6.12}$$

from Proposition 3.3, Part II of [4], there exits a unique solution $\tilde{y}_\delta$ of (6.11) belonging to the space

$$C^1\big([0,T];\mathcal{V}_\delta \times \mathcal{H}\big) \cap C^0\big([0,T];D(\mathcal{A}_\delta)\big). \tag{6.13}$$

Then one can see that $y_\delta(t) = \tilde{y}_\delta(t)\, e^{\frac{t}{2}}$ is a solution of (6.8) also belonging to the space (6.13). Reciprocally, if the regularity (6.12) holds, any solution of (6.8) belonging in the space given by (6.13) is a solution of (6.11), hence it is unique.

Finally, any strong solution u_δ of (6.5) defines a solution $y_\delta = (u_\delta, \dot{u}_\delta)$ of (6.8) with the regularity (6.13). Hence, the unique solution of (6.5), if it exists, is unique. Reciprocally, the first component of y_δ – solution of (6.8) with the regularity (6.13) – is a strong solution of (6.5). ∎

Now from existence and uniqueness result we deduce energy estimates that are fundamental in the following analysis.

Corollary 6.5. *Let u_δ be a strong solution of* (6.5)*, then one can define the energy functional $\mathcal{E}_\delta(t)$ by*

$$\mathcal{E}_\delta(t) = \frac{1}{2}\Big(\, \|\dot{u}_\delta(t)\|^2_{\mathcal{H}} + a_\delta\big(\, u_\delta(t), u_\delta(t)\big)\Big), \tag{6.14}$$

and we have the estimates, for all $t \in [0,T]$,

$$\sqrt{\mathcal{E}_\delta}(t) \leqslant \sqrt{\mathcal{E}_\delta}(0) + \frac{1}{\sqrt{2}} \int_0^t \|f_\delta(\tau)\|_{\mathcal{H}}\, \mathrm{d}\tau. \tag{6.15}$$

Proof. Taking the scalar product in $\mathcal{H}$ of the first equation of (6.5) with $\dot{u}_\delta$ and integrating in time, one obtains,

$$\mathcal{E}_\delta(t) + \int_0^t b_0(\dot{u}_\delta(s), \dot{u}_\delta(s))_{\mathcal{H}}\, \mathrm{d}s = \mathcal{E}_\delta(0) + \int_0^t (f_\delta(s), \dot{u}_\delta(s))_{\mathcal{H}}\, \mathrm{d}s.$$

It is then rather standard to show (6.15) using Gronwall's Lemma. ∎

6.2. Existence and uniqueness of weak solutions. Existence and uniqueness results are proven rather classically (only the **Step 3** below is a small modification of existing results), we reproduce the main steps for the sake of completeness.

Step 1: Construction of a Cauchy sequence. By construction we recall that $D(\mathcal{A}_\delta)$ is dense in $\mathcal{V}_\delta \times \mathcal{H}$ and $W^{1,1}(0,T;\mathcal{H})$ is dense in $L^1(0,T;\mathcal{H})$, therefore, every initial data and source term

$$(u^0_\delta, u^1_\delta) \in \mathcal{V}_\delta \times \mathcal{H}, \quad \text{and} \quad f_\delta \in L^1\big(0,T;\mathcal{H}\big).$$

is the limit of a converging sequence $(u^0_{\delta,n}, u^1_{\delta,n}, f_{\delta,n})$ in $D(\mathcal{A}_\delta) \times W^{1,1}(0,T;\mathcal{H})$. Thanks to Theorem 6.4, we can associate to this sequence the strong solution of problem (6.5) that we denote

$$u_{\delta,n} \in C^2\big([0,T];\mathcal{H}\big) \cap C^1\big([0,T];\mathcal{V}_\delta\big).$$

Thanks to the estimate of Corollary 6.5 one can show that the difference $u_{\delta,m} - u_{\delta,n}$ is a Cauchy-sequence in the space

$$\mathcal{E} := C^1\big([0,T];\mathcal{H}\big) \cap C^0\big([0,T];\mathcal{V}_\delta\big), \quad \|u\|_{\mathcal{E}} := \sup_{t\in[0,T]} \big(\|u(t)\|_{\mathcal{H}} + \|\dot{u}(t)\|_{\mathcal{H}} + \|u(t)\|_{\mathcal{V}_\delta}\big),$$

that is a Banach space when equipped with the norm $\|\cdot\|_{\mathcal{E}}$ therefore it converges towards $u_\delta \in \mathcal{E}$.

Step 2: Existence of a weak solution. Multiplying (6.5) by a test function

$$v \in H^2\big(0,T;\mathcal{H}\big) \cap H^1\big(0,T;\mathcal{V}_\delta\big)$$

we obtain that every term of the sequence $\{u_{n,\delta}\}$ is a weak solution with initial data and source term given by $(u^0_{\delta,n}, u^1_{\delta,n}, f_{\delta,n})$. Passing to the limit in the weak formulation we obtain the existence of a weak solution for the data $(u^0_\delta, u^1_\delta, f_\delta)$. Note that, this solution belongs to $\mathcal{E}$.

Step 3: Energy relation. We can obtain an energy relation for the weak solution constructed by the limit process above. Using a converging sequence of strong solutions (see **Step 1** and **Step 2**) the limit solution necessary belongs to $\mathcal{E}$ and passing to limit in the energy relation of 6.5 we obtain the desired energy relation for the weak solution. Note that such energy identity allows to prove the uniqueness of the solution constructed as the limit of strong solutions. This is not helpful to show uniqueness of weak solution in general.

Step 4: Uniqueness of the weak solution. Duality argument. To show the uniqueness of the weak solution we show that every solution associated to vanishing data are zero. We have that

$$u_\delta \in L^2\big(0,T;\mathcal{V}_\delta\big),$$

satisfies, for all $v \in H^2\big([0,T];\mathcal{H}\big) \cap H^1\big([0,T];\mathcal{V}_\delta\big)$ such that $v(T) = 0$ and $\dot{v}(T) = 0$,

$$\int_0^T (u_\delta(t), \ddot{v}(t))_{\mathcal{H}}\,\mathrm{d}t + \int_0^T a_\delta(u_\delta(t), v(t))\,\mathrm{d}t - \int_0^T b_0(u_\delta(t), \dot{v}(t))\,\mathrm{d}t \,= 0.$$

We choose $v(t) = w(T-t)$ for some test function $w \in H^2\big([0,T];\mathcal{H}\big) \cap H^1\big([0,T];\mathcal{V}_\delta\big)$ satisfying $w(0) = 0$ and $\dot{w}(0) = 0$. We have

$$\begin{aligned}\text{(6.16)}\quad \int_0^T (u_\delta(T-t), \ddot{w}(t))_{\mathcal{H}}\,\mathrm{d}t + \int_0^T a_\delta(u_\delta(T-t), w(t))\,\mathrm{d}t \\ + \int_0^T b_0(u_\delta(T-t), \dot{w}(t))\,\mathrm{d}t \,= 0.\end{aligned}$$

Now we choose $w = w_\delta$ where w_δ is the strong solution of the wave propagation problem

$$\ddot{w} + \mathbf{A}_{\delta,*}\begin{pmatrix} w^\delta \\ \dot{w}^\delta \end{pmatrix} = u_\delta(T-\cdot),$$

with vanishing initial data and where $\mathbf{A}_{\delta,*}$ is defined as $\mathbf{A}_\delta$ with the bilinear form $b_0^*(u,v) = b_0(v,u)$ instead of b_0 (note that b_0^* satisfies the exact same properties (2.18) as b_0). The strong solution exists as the theory developed for strong solutions applied with $\mathbf{A}_{\delta,*}$, moreover we have, for all $v \in \mathcal{V}_\delta$,

$$(\ddot{w}_\delta(t), v)_{\mathcal{H}} + a_\delta(w_\delta(t), v) + b_0^*(\dot{w}_\delta(t), v) = (u_\delta(T-t), v)_{\mathcal{H}}. \tag{6.17}$$

Choosing $v = u_\delta(T-t)$ in the equation above and integrating with respect to time we obtain, thanks to (6.16),

$$\int_0^T \|u_\delta(T-t)\|_{\mathcal{H}}^2 \,\mathrm{d}t = 0,$$

which shows the uniqueness of the weak solution.

6.3. Existence and uniqueness for problem (2.19)**.** Choosing $v(t) = w\varphi(t)$ with $\varphi \in \mathcal{D}(0,T)$ and $w \in \mathcal{V}_\delta$ in (2.21) shows that weak solutions satisfy (2.19). To obtain uniqueness of a solution to problem (2.19) one classically consider the same problem with vanishing data: for all $v \in \mathcal{V}_\delta$,

$$\begin{cases} \dfrac{d^2}{dt^2}(u_\delta, v)_{\mathcal{H}} + a_\delta(u_\delta, v) + \dfrac{d}{dt} b_0(u_\delta, v) = 0, \quad \mathcal{D}'(0,T), \\ u_\delta(0) = 0, \\ \dot{u}_\delta(0) = 0, \end{cases}$$

and shows that u_δ is zero. Defining $\mathtt{U}_\delta(t) = \int_0^t u_\delta(\tau)\,\mathrm{d}\tau$ we have

$$\mathtt{U}_\delta \in C^2\big([0,T];\mathcal{H}\big) \cap C^1\big([0,T];\mathcal{V}_\delta\big),$$

and one can show that, for all $v \in \mathcal{V}_\delta$,

$$\begin{cases} (\ddot{\mathtt{U}}_\delta, v)_{\mathcal{H}} + a_\delta(\mathtt{U}_\delta, v) + b_0(\dot{\mathtt{U}}_\delta, v) = 0, \quad \mathcal{C}^0[0,T], \\ \mathtt{U}_\delta(0) = 0, \\ \dot{\mathtt{U}}_\delta(0) = 0, \end{cases}$$

hence $\mathtt{U}_\delta$ is a strong solution of (6.5) with vanishing data therefore it is necessarily zero. We then conclude that u_δ also vanishes.

7. Appendix: Density results. Below $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are Hilbert spaces equipped with scalar products $(\cdot,\cdot)_{\mathcal{X}}$, $(\cdot,\cdot)_{\mathcal{Y}}$ and $(\cdot,\cdot)_{\mathcal{Z}}$ and associated norms $\|\cdot\|_{\mathcal{X}}$, $\|\cdot\|_{\mathcal{Y}}$ and $\|\cdot\|_{\mathcal{Z}}$. We further assume that, $\mathcal{X} \subset \mathcal{Y} \subset \mathcal{Z}$ with continuous injection and that $\mathcal{X}$ is dense in $\mathcal{Y}$ and $\mathcal{Y}$ is dense in $\mathcal{Z}$.

LEMMA 7.1. *For every $y \in D([0,T);\mathcal{Y})$ there exists a sequence $\{x_n\}$ in $D([0,T);\mathcal{X})$ such that, for all positive integer m,*

$$\|y(t) - x_n(t)\|_{H^m(0,T;\mathcal{Y})} \underset{n\to+\infty}{\longrightarrow} 0. \tag{7.1}$$

Proof. This result is proven by introducing a Yosida regularizing operator $J_n \in \mathcal{L}(\mathcal{Y}, \mathcal{X}) \subset \mathcal{L}(Y)$, defined by, for all $y \in \mathcal{Y}$ and $x \in \mathcal{X}$,

$$(J_n y, x)_{\mathcal{Y}} + \frac{1}{n}(J_n y, x)_{\mathcal{X}} = (x, y)_{\mathcal{Y}}.$$

It is standard to show the norm estimates $\|J_n\|_{\mathcal{L}(\mathcal{Y})} \leqslant 1$ and $\|J_n\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} \leqslant C\sqrt{n/2}$ where C is such that,

$$\forall\, x \in \mathcal{X}, \quad \|x\|_{\mathcal{X}} \leqslant C\,\|x\|_{\mathcal{Y}}.$$

Moreover, one can show strong convergence property (that holds since $\mathcal{X}$ is dense in $\mathcal{Y}$ by assumption)

$$\forall\, y \in \mathcal{Y}, \quad \|y - J_n y\|_{\mathcal{Y}} \underset{n\to+\infty}{\longrightarrow} 0.$$

For every $y \in D([0,T); \mathcal{Y})$ we define the sequence $\{x_n\}$ as, for every $t \in [0,T]$,

$$x_n(t) = J_n y(t).$$

Since J_n is a bounded linear operator one can verifies that $x_n \in D([0,T); \mathcal{X})$. Then the convergence property (7.1) is a consequence of Lebesgue's dominated convergence theorem: indeed, thanks to the property of J_n, the sequence $\{x_n\}$ is uniformly bounded in $C^m([0,T]; \mathcal{Y})$ and

$$\Big\| \frac{d^p}{dt^p}\big(x_n(t) - y(t)\big)\Big\|_{\mathcal{Y}} \quad \text{with} \quad 0 \leqslant p \leqslant m,$$

vanishes at the limit for all $t \in [0,T]$. □

We now state a result that can be proven with the now standard tools given in [17] Chapter 1.

Proposition 7.2. *The space $D([0,T); \mathcal{Y})$ is dense in*

$$\{v \in H^2(0,T; \mathcal{Z}) \cap H^1(0,T; \mathcal{Y}) \mid v(T) = 0,\ \dot{v}(T) = 0\}.$$

With such proposition we can state the main result of this appendix.

Theorem 7.3. *The space $D([0,T); \mathcal{X})$ is dense in*

$$\{v \in H^2(0,T; \mathcal{Z}) \cap H^1(0,T; \mathcal{Y}) \mid v(T) = 0,\ \dot{v}(T) = 0\}.$$

Proof. To show this density result one simply need to use the results of Lemma 7.1 and Proposition 7.2. □