On energetically optimal finite-time stabilization

Andrey Polyakov, Denis Efimov, Xubin Ping

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***Abstract*—The problem of finite-time stabilization of a linear plant with an optimization of both a settling time and an weighted/averaged control energy is studied using the concept of generalized homogeneity. It is shown that the optimal finite-time stabilizing control in this case can be designed solving a simple linear algebraic equation. Some issues of a practical applicability and a robustness of the obtained control law are studied.**

I. Introduction

The linear quadratic regulator (LQR) on an infinite horizon seems to be the most famous example of an optimal control design, when the infinite dimensional optimization problem can be reduced to solving a finite-dimensional algebraic (Riccati) equation. This LQR regulator has the form of a static linear feedback which stabilizes exponentially the origin of a linear system and optimizes both the system trajectory and the control energy. This paper answers the question how a similar design can be developed for an *optimal finite-time stabilization*.

If the settling time $T > 0$ is prescribed (fixed), the problem of an energetically optimal finite-time control is an LQR problem on the finite horizon $(0, T)$. An optimal solution in this case is a time-varying linear feedback known as minimum energy control (see, e.g., [1]). The minimum energy control is one of classical examples of prescribed-time stabilization algorithms studied in [2]. To guarantee the prescribed-time convergence, the time-varying gain of the linear feedback tends to ∞ as the time t tends to the settling time $T > 0$. Such a time-varying controller cannot be utilized for a set-point tracking if $t > T$. The infinite gain also complicates a practical application of the minimum energy control and the robustness analysis of the closed-loop system. The model predictive control (MPC) [3] suggests a possible way to implement the optimal control algorithms in practice, to improve the robustness of the feedforward optimal control and to fulfill additional state constrains.

Seemly, the first algorithm of the finite-time stabilization were introduced in the mid of 1930s as a solution to the so-called minimum time control problem (see, e.g., [4] for more details). In [5], a family of optimal finite-time stabilization algorithms was introduced for the planar control system. Several generalizations and extensions of the mentioned results can be found in the literature (see, e.g., [6], [7]). The optimization of a settling time together with a quadratic cost functional was also studied in [8, Chapter 6]. All the mentioned finite-time control algorithms are relay or more complicated discontinuous regulators, which are difficult to implement for the set-point tracking due to the chattering phenomenon [9], [10]. In addition, the switching surface of the minimum time relay control algorithms usually admits an explicit representation only for a two dimensional (planar) system. The construction of high order algorithms leads to nonlinear equations, which do not have explicit solutions in general cases.

Continuous finite-time controllers are more appreciated for the set-point tracking than discontinuous ones. The design of such control algorithms is a rather popular research direction last years [11], [12], [13], [14]. The continuous finite-time controllers can also be related with solutions of some optimal control problems [15]. In this paper, we deal with a particular optimal control problem on non-fixed horizon with a mixed cost functional, which optimizes both a settling time and a weighted/averaged control energy. Similarly to LQR case, we are interested in the optimal control design by solving an algebraic equation. We show that such a design is always possible for a class of the so-called generalized homogeneous linear plants [16].

The generalized homogeneity is a symmetry of an object (function, set, vector field, etc) with respect to a group of the so-called generalized dilations. Recall that the standard (classical) homogeneity were introduced by Leonhard Euler in 18th century as the symmetry of a function $x \to f(x)$ with respect to the uniform dilation of its argument $x \to \lambda x$, namely, $\exists \nu \in \mathbb{R} : f(\lambda x) = \lambda^{\nu} f(x), \forall \lambda > 0, \forall x$. It seems that the first paper, where the symmetry of a differential equation with respect to a

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generalized (non-uniform) dilation of the initial state is studied, were published in 1958 by Vladimir Zubov [17]. Some extensions of the homogeneity theory of finite-dimensional and infinite-dimensional dynamical models can be found in [18], [19], [20], [16]. Homogeneous differential equations/inclusions form an important class of control systems models [21], [22], [23], [24], [25]. They appear as local approximations [26] or set-valued extensions [27] of nonlinear systems and include models of process control [28], mechanical models with frictions [30], etc. Stability and stabilizability problems were studied for both standard [31], [32] and weighted homogeneous [33], [34], [35], [36], [37], [38], [39] systems which are the most popular today [30], [27], [22], [23], [25]. An important feature of homogeneous systems is that asymptotic stability and negative homogeneity degree of the system always imply its finite-time stability [13]. An introduction to homogeneous optimal control can be found in [16, Chapter 12]. The homogeneous MPC design is studied in [40]. This paper presents an interesting application of the homogeneity theory to minimum energy finite-time stabilization.

Notation: $\mathbb{N}$ is the set of natural numbers; $\mathbb{R}$ is the field of real numbers; $\mathbb{C}$ is the field of complex number; $\|\cdot\|$ denotes a norm in $\mathbb{R}^n$; $\mathbf{0}$ is the zero of a vector space (e.g., the zero vector in $\mathbb{R}^n$ or the zero matrix in $\mathbb{R}^{n\times m}$) ; $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix; $P \succ 0$ denotes positive definiteness of a matrix $P = P^\top \in \mathbb{R}^{n\times n}$; $\lambda_{\max}(P)$ is a maximum eigenvalue of the symmetric matrix P; $\dim S$ denotes the dimension of a manifold S.

II. Problem Statement

Let us consider the following optimal control problem: find $u \in L^2((0,T),\mathbb{R})$ such that

$$\dot{x}(t)=Ax(t)+Bu(t), \quad x(0)=x_0\neq\mathbf{0}, \quad x(T)=\mathbf{0}, \tag{1}$$

$$J(u,T) := T + \tfrac{R}{T^\eta}\int_0^T \tfrac{u^2(\tau)}{(T-\tau)^\gamma}d\tau \to \min \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, the time horizon $T > 0$ *is non-fixed*, $R \succ 0$ and $\eta, \gamma \in \mathbb{R}$ are a constant parameters.

The cost functional J allows minimization of both a (possibly averaged/weighted) control energy and a time required for stabilization of the system state at zero. The parameters η and γ specify some restrictions to the control. For example, for $\eta = 1, \gamma = 0$ the second term of the cost functional defines an averaged control energy, while a proper selection of $\gamma > 0$ could guarantee $|u(\tau)| \to 0$ as $\tau \to T$. Below we find a solution to (1), (2) for $\gamma > -1$ and $\eta + \gamma \geq 0$. Notice that the case $\eta = \gamma = 0$ can also be studied using the results presented in [8, Chapter 8].

For the non-fixed horizon, the functional J may have a finite value only if $T < +\infty$, i.e., the optimal trajectory (if it exists) always reaches $\mathbf{0}$ in a finite time $T^* = T^*(x_0)$. To guarantee the existence of an optimal solution to (1), (2) for all $x_0 \in \mathbb{R}^n$ the following assumption is necessary.

Assumption 1: The pair $\{A, B\}$ is controllable.

In this paper, we deal only with the so-called generalized homogeneous plants (see below). Recall [41] that a linear vector field $x \to Ax$ is homogeneous with a non-zero degree if and only if the matrix A is nilpotent.

Assumption 2: The matrix A is nilpotent.

Note that under Assumption 1 a preliminary linear feedback can be realized to fulfill Assumption 2 (see, [25] or [41] for more details).

In practice, the solution of the optimization problem (1)-(2) can be utilized for energetically optimal finite-time stabilization and MPC design. Our goal is to derive the corresponding controllers as well as to analyze stability and robustness properties of the closed-loop system.

III. Homogeneous optimal finite-time control

A. *Homogeneous mappings*

By definition, the homogeneity is a dilation symmetry. In this paper, we deal with the so-called linear dilations [16, Chapter 6] given by

$$\mathbf{d}(s) = e^{sG_{\mathbf{d}}} = \sum_{i=0}^{\infty} \tfrac{(sG_{\mathbf{d}})^i}{i!}, \quad s \in \mathbb{R} \tag{3}$$

where $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ is an anti-Hurwitz matrix (called *the generator of the dilation* $\mathbf{d}$.) The latter guarantees that $\mathbf{d}$ satisfies the *limit property*:

$$\lim_{s\to-\infty}\|\mathbf{d}(s)x\|=0, \quad \lim_{s\to+\infty}\|\mathbf{d}(s)x\|=+\infty, \quad \forall x \neq \mathbf{0}$$

required for a group $\mathbf{d}$ to be a dilation in $\mathbb{R}^n$ [19].

A dilation introduces an alternative norm topology in $\mathbb{R}^n$ using the so-called canonical $\mathbf{d}$-homogeneous norm [42] given by

$$\|x\|_{\mathbf{d}} = e^s \text{ with } s \in \mathbb{R} : \|\mathbf{d}(-s)x\| = 1 \tag{4}$$

and $\|\mathbf{0}\|_{\mathbf{d}} = 0$, where $\mathbf{d}$ is a linear monotone[1] dilation. The function $\|\cdot\|_{\mathbf{d}}$ is continuous on $\mathbb{R}^n$, locally Lipschitz continuous on $\mathbb{R}^n\backslash\{\mathbf{0}\}$ and $\mathbf{d}$-homogeneous of degree 1. Moreover, if $\|\cdot\|\in C^1(\mathbb{R}^n\backslash\{\mathbf{0}\})$ then $\|\cdot\|_{\mathbf{d}}\in C^1(\mathbb{R}^n\backslash\{\mathbf{0}\})$.

Definition 1: [19] A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ (resp. a function $h : \mathbb{R}^n \to \mathbb{R}$) is said to be $\mathbf{d}$-homogeneous

[1] A dilation is monotone if $s \to \|\mathbf{d}(s)x\|$ is a monotone function for any x. Any linear dilation in $\mathbb{R}^n$ is monotone provided that the norm $\|\cdot\|$ is properly selected [43].

of degree $\nu \in \mathbb{R}$ if $f(\mathbf{d}(s)x) = e^{\nu s}\mathbf{d}(s)f(x)$ (resp. $h(\mathbf{d}(s)x) = e^{\nu s}h(x)$), $\forall x \in \mathbb{R}^n$, $s \in \mathbb{R}$.

If a vector field f is $\mathbf{d}$-homogeneous of degree ν then solutions of $\dot{x} = f(x)$ possess a certain symmetry as well [19]: $x(t, \mathbf{d}(s)x_0) = \mathbf{d}(s)x(e^{\nu s}t, x_0)$, where $x(\cdot, z)$ denotes a solution with the initial condition $x(0) = z$.

Example 1: [43] The linear vector field $x \to Ax$, $A \in \mathbb{R}^{n\times n}$ is $\mathbf{d}$-homogeneous of degree $\nu \neq 0$ if and only if $AG_{\mathbf{d}} = (\nu I_n + G_{\mathbf{d}})A$. For $\nu \neq 0$, the latter identity may hold if and only if A is nilpotent.

B. Auxiliary Results and Observations

Under Assumption 2 the system (1), (2) is homogeneous and its optimal solution is symmetric with respect to a dilation of the initial condition (see [16, Chapter 12] for more details). Recall [13] that any asymptotically stable homogeneous system of negative degree is finite-time stable. This means that an optimal solution of (1), (2) may have a form of a homogeneous stabilizing feedback. Below we prove this rigorously and show that the settling time (the optimal stabilization time $T^* = T^*(x_0)$) of the problem (1), (2) is a homogeneous function $T^*: \mathbb{R}^n \to [0, +\infty)$ given by

$$T^*(x) := \min\left\{\underset{r>0}{\operatorname{argmin}} \tfrac{x^\top \mathbf{d}^\top(-\ln r)Q\mathbf{d}(-\ln r)x}{r^{\mu-1}} + r\right\}, \quad (5)$$

where the parameters $Q = Q^\top \in \mathbb{R}^{n\times n}$, $\mu \in \mathbb{R}$ and the dilation $\mathbf{d}$ are defined below. Under certain restrictions on Q and μ, the function (5) becomes the canonical homogeneous norm (4).

Lemma 1: Let $\mathbf{d}$ be a linear dilation in $\mathbb{R}^n$ with a generator $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ and $\mathbf{d}_\mu$ be a linear dilation with the generator $G_{\mathbf{d}_\mu} := G_{\mathbf{d}} + \frac{\mu}{2}I_n$ such that $G_{\mathbf{d}_\mu} - \frac{1}{2}I_n$ is an anti-Hurwitz matrix. If $Q = Q^\top \in \mathbb{R}^{n\times n}$ is a positive definite matrix then

1) T^* is well-defined on $\mathbb{R}^n$, positive definite and $\mathbf{d}_\mu$-homogeneous of degree 1.
2) T^* is continuously differentiable on $\mathbb{R}^n\backslash\Sigma$,
$$\Sigma \subset \{\mathbf{0}\}\textstyle\bigcup_{s\in\mathbb{R}}\mathbf{d}_\mu(s)\Sigma_0,$$
$$\Sigma_0 := \left\{x \in \mathbb{R}^n : x^\top Z x = 1, x^\top(ZG_{\mathbf{d}_\mu} + G_{\mathbf{d}_\mu}^\top Z)x = 0\right\} \quad (6)$$
where $Z := QG_{\mathbf{d}_\mu} + G_{\mathbf{d}_\mu}^\top Q - Q$ and Σ is a set of measure zero in $\mathbb{R}^n$ provided that $\dim \Sigma_0 \leq n-2$.
3) $T^* \in C(\mathbb{R}^n) \cap C^1(\mathbb{R}^n\backslash\{\mathbf{0}\})$ provided that
$$ZG_{\mathbf{d}_\mu} + G_{\mathbf{d}_\mu}^\top Z \succ 0, \quad Z \succ 0; \quad (7)$$
4) T^* is locally bounded and continuous at $\mathbf{0}$ if $\mu \geq 0$ is an integer number and $G_{\mathbf{d}} \in \mathbb{R}^n$ is diagonalizable:
$$\Phi^{-1}G_{\mathbf{d}}\Phi = \begin{pmatrix} n & 0 & \dots & 0 \\ 0 & n-1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}, \quad (8)$$
where $\Phi \in \mathbb{R}^{n\times n}$ is some non-singular matrix.

C. Optimal homogeneous finite-time stabilization

The following lemma is given without proof since it is a straightforward corollary of the canonical block decomposition for linear control systems [45].

Lemma 2: Under Assumptions 1 and 2 there exists a non-singular matrix $\Phi \in \mathbb{R}^{n\times n}$:

$$\Phi^{-1}A\Phi = \begin{pmatrix} \mathbf{0} & A_{12} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_{23} & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & A_{n-1\,n} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \ \Phi^{-1}B = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \dots \\ A_{n\,n+1} \end{pmatrix}, \quad (9)$$

where $0 \neq A_{i\,i+1} \in \mathbb{R}$, $i = 1, ..., n$.

The latter means that A is $\mathbf{d}$-homogeneous of the degree -1 provided that the generator $G_{\mathbf{d}}$ satisfies (8) with Φ given in Lemma 2.

Theorem 1: Let Assumptions 1 and 2 be fulfilled . Let $\gamma > -1$ and $\mathbf{d}$ be a dilation in $\mathbb{R}^n$ with $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ given by (8) and $\Phi \in \mathbb{R}^{n\times n}$ satisfying (9). The optimal solution to the problem (1), (2) with a *fixed horizon* $T > 0$ is

$$u^*(t) = K\mathbf{d}(-\ln(T-t))x(t), \quad K = -R^{-1}B^\top P^{-1}, \quad (10)$$

and the optimal cost is

$$J^*(x_0, T) = T + \tfrac{x_0^\top \mathbf{d}(-\ln T)P^{-1}\mathbf{d}(-\ln T)x_0}{T^{\eta+\gamma-1}}, \quad (11)$$

where $P \in \mathbb{R}^{n\times n}$ is the unique positive definite solution of the linear matrix equation

$$(A + G_{\mathbf{d}})P + P(A + G_{\mathbf{d}})^\top = (1-\gamma)P + BR^{-1}B^\top. \quad (12)$$

Theorem 1 presents the minimum energy control for homogeneous systems in the form (10).

Corollary 1: If all conditions of Theorem 1 are fulfilled then the optimal control for (1), (2) with *non-fixed horizon* is given by

$$u^*(t) = K\mathbf{d}(-\ln(T^*(x_0) - t))x, \quad (13)$$

where $T^* : \mathbb{R}^n \to \mathbb{R}$ has the form (5) with $Q = P^{-1}$ and $\mu = \eta + \gamma$.

Proof. The cost for the fixed horizon is given by (11). In the case of the non-fixed horizon, the optimal time T^* corresponds to a minimum of $J^*(T, x_0)$, i.e. $T^* = T^*(x_0)$ is given by (5). In this case, the time-varying optimal control has the form (13). ■

D. Analysis of the feedback algorithm

The MPC methodology [3] suggests to define a feedback law in the form $u(x_0) := u^*(0)$, where $t \to u^*(t)$ is a continuous-time optimal feedforward control. In the view of Corollary 1, the corresponding feedback in our case has the form

$$u(x) = K\mathbf{d}(-\ln T^*(x))x. \quad (14)$$

If $\eta = \gamma = 0$ then the optimal control problem (1), (2) satisfies the Bellman principle[2] and (14) is the *optimal feedback control* for (1), (2). In the general case, the control (14) defines just a feedback law inspired by MPC ideas. The stability and robustness of the closed-loop system (1), (14) as well as properties of the feedback law need to be studied additionally.

Corollary 2: If all conditions of Theorem 1 are fulfilled then the control $u : \mathbb{R}^n \to \mathbb{R}$ given by (14) is $\mathbf{d}_\mu$-homogeneous of degree $\frac{\mu}{2}$, namely,

$$u(\mathbf{d}_\mu(s)x) = e^{\frac{\mu}{2}s}u(x), \quad \forall x \neq 0, s \in \mathbb{R} \tag{15}$$

where $\mathbf{d}_\mu(s) = e^{\frac{\mu}{2}s}\mathbf{d}(s)$ and $\mu = \eta + \gamma$. Moreover, the control u is

- continuously differentiable on $\mathbb{R}^n \backslash \{\Sigma\}$ and
$$u^\top(x)u(x) \leq (T^*(x))^\mu \lambda_{\max}(Z^{-\frac{1}{2}}K^\top K Z^{-\frac{1}{2}}) \tag{16}$$
provided that $Z := P^{-1}G_{\mathbf{d}_\mu} + G_{\mathbf{d}_\mu}^\top P^{-1} - P^{-1} \succ 0$, where Σ is a set of measure zero in $\mathbb{R}^n$ (defined in Lemma 1);
- globally bounded for $\mu = 0$;
- continuous at $\mathbf{0}$ and locally bounded for $\mu \in \mathbb{N}$;
- in $C(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \backslash \{\mathbf{0}\})$ provided that (7) holds.

According to the latter corollary, the feedback control (14) may be a discontinuous function of the state. The Filippov's theory [47] of differential equations with discontinuous right-hand sides is the conventional tool for the stability and robustness analysis of the closed-loop system (1), (14) in this case. However, the Filippov's method extends the discontinuous differential equation (1), (14) to a differential inclusion, which may have a larger set of solutions than (1), (13). In [8, Chapter 8] an alternative method of solution definition was introduced. In fact, it can be re-formulated as follows: among all possible Filippov solutions of (1), (14) we select the one, which coincides with the unique solution of (1), (13). Such a definition of solution does not look consistent with the conventional ODE theory as well as with the conventional stability and robustness (ISS) analysis. In addition, the feedback (14) corresponds to (13) only if the optimal control problem satisfies the Bellman principle. The stability and robustness analysis of the closed-loop system (1), (14) is a difficult problem in the general case. In this paper we study only continuous controllers (14).

Corollary 3 (Finite-time stability): Let all conditions of Theorem 1 are fulfilled and the condition (7) hold. If the matrix $A + BK$ is Hurwitz then the closed-loop system (1), (14) is globally uniformly finite-time stable and $x \to \|x\|_{\mathbf{d}_\mu}$ is its Lyapunov function satisfying

$$\tfrac{d}{dt}\|x\|_{\mathbf{d}_\mu} \leq -q, \quad \forall x \neq \mathbf{0},$$

where $\|\cdot\|_{\mathbf{d}_\mu}$ is the canonical homogeneous norm induced by $\|x\|_X = \sqrt{x^\top X x}$ with $q > 0, X \in \mathbb{R}^{n\times n}$ satisfying

$$X(A+BK+qG_{\mathbf{d}_\mu})+(A+BK+qG_{\mathbf{d}_\mu})^\top X \preceq 0, \quad X \succ 0. \tag{17}$$

The robustness of the control system can be investigated using the concept of Input-to-State Stability (ISS) [48]. Recall that the system

$$\dot{x} = f(x, d), \tag{18}$$

is ISS if there exists $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ such that

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma(\|d\|_{L^\infty_{(0,t)}}), \quad \forall t \geq 0$$

Corollary 4 (Robustness): Let all conditions of Corollary 3 be fulfilled and $u : \mathbb{R}^n \to \mathbb{R}$ be given by (14). Let $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ be a continuous function such that

$$f(x, \mathbf{0}) = Ax + Bu(x), \quad \forall x \in \mathbb{R}^n.$$

If there exists a dilation $\tilde{\mathbf{d}}$ in $\mathbb{R}^k$ such that

$$f(\mathbf{d}_\mu(s)x, \tilde{\mathbf{d}}(s)d) = e^{-s}\mathbf{d}_\mu(s)f(x, d),$$

$\forall x \in \mathbb{R}^n$, $\forall d \in \mathbb{R}^k$, $\forall s \in \mathbb{R}$, then the system (18) is ISS.

The latter result follows from [23, Corollary 2.21] or [49]. Corollary 4 allows robustness of a rather large class of systems to be analyzed. For example, selecting

$$f(x, d) = Ax + Bu(x + d_1) + d_2, \quad d = (d_1^\top, d_2^\top)^\top$$

we easily conclude the ISS of the nonlinear control system (1), (14) with respect to measurement errors and exogenous perturbations.

Remark 1: Under conditions of Corollary 3 we have $T^*(x) = \|x\|_{\mathbf{d}_\mu,Z}$, where $\|x\|_{\mathbf{d}_\mu,Z}$ is the canonical homogeneous norm induced by the norm $\|x\|_Z = \sqrt{x^\top Z x}$ and Z defined in Lemma 1. Some issues of the digital implementation of such a controller (14) with $T^*(x) = \|x\|_{\mathbf{d}_\mu,Z}$ are already studied in [50]. The similar implicit controller was successfully implemented in quadrotor in order to demonstrate an improvement of a control quality in comparison with a linear feedback [51].

IV. Example

Let us consider the planar optimal control problem

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad x_1(T) = x_2(T) = 0,$$
$$J(T, u) = T + \int_0^T \tfrac{u(\tau)^2}{T-t} d\tau \to \min,$$

where $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2, t > 0$ is the system state and $u(t) \in \mathbb{R}$ is the control input.

Obviously, the considered problem admits the representation (1), (2) with $n=2$, $A=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, $B=\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, $R=1$. In this case, we have $G_{\mathbf{d}} = \left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)$.

[2]A tail of any optimal trajectory is optimal too.

The considered cost functional corresponds to (2) with $\eta=0, \gamma=1, R=1$. The solution of (12) is given by $P=\begin{pmatrix} \frac{1}{12} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{2} \end{pmatrix}$. Since $\mu=\eta+\gamma=1$ then $G_{\mathbf{d}_\mu}=G_{\mathbf{d}}+\frac{1}{2}I_2$,

$$Z=P^{-1}G_{\mathbf{d}_\mu}^\top+G_{\mathbf{d}_\mu}P^{-1}-P^{-1}=P^{-1}\begin{pmatrix} \frac{1}{3} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}P^{-1}\succ 0.$$

One can be checked numerically that $ZG_{\mathbf{d}_\mu}+G_{\mathbf{d}_\mu}^\top Z\succ 0$.

Hence, by Corollary 1 the optimal control is given by

$$u^*(t)=K\mathbf{d}(-\ln(\|x(0)\|_{\mathbf{d}_{\mu,Z}}-t))x \tag{19}$$

where $K=-B^\top P^{-1}$ and $\|\cdot\|_{\mathbf{d}_{\mu,Z}}$ is the canonical $\mathbf{d}_\mu$-homogeneous norm induced by the norm $\|x\|=\sqrt{x^\top Zx}$. The feedback (14) in this case has the form

$$\tilde{u}(x)=K\mathbf{d}(-\ln(\|x\|_{\mathbf{d}_{\mu,Z}}))x. \tag{20}$$

For comparison reasons we assume that the control laws are implemented in a digital device as follows:

$$u(t)=u^*(t_i), \quad t\in[t_i,t_{i+1}),$$
$$u(t)=\tilde{u}(x(t_i)), \quad t\in[t_i,t_{i+1}),$$

where $t_i=ih, i=0,1....$ and $h>0$ is a sampling period.

Let $x(0)=Z^{-\frac{1}{2}}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. The optimal settling time in this case $T^*(x(0))=1$.

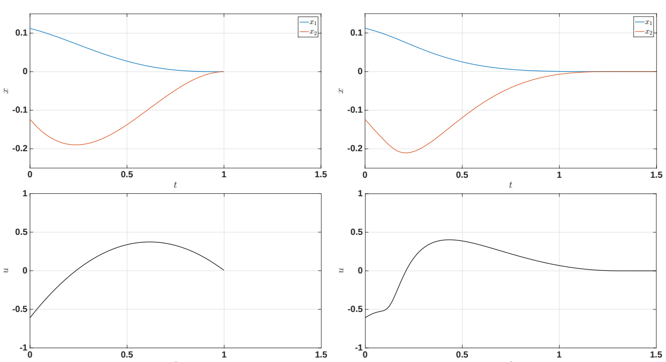

Fig. 1. Numerical simulations for the time-varying linear control (19) and the nonlinear control (20) in the disturbance-free case

The numerical simulations for the double integrator controlled by (19) and (20) are presented at Fig. 1 and Fig. 2 for $h=0.001$. First, the ideal (non-perturbed) case is considered (see Fig. 1). Next, the control under measurement errors is studied (see Fig. 2). In the latter case, it was assumed that the state x is measured with the error: $\hat{x}=x+\left(\begin{smallmatrix}0.001\\0.001\end{smallmatrix}\right)(\sin(10t)+\cos(100t))$. The controller (19) is defined only on $[0,1]$, so it is simulated only on this time interval. The feedback law (20) is defined globally and simulated on the interval $[0,1.5]$.

The non-linear feedback control (20) is not optimal, but the closed-loop system has a trajectory close to the optimal one generated by the control (19). The advantages of the non-linear feedback law are: 1) it can be utilized for set-point tracking; 2) the closed-loop system with the control (20) is robust (ISS) with respect of large class of perturbations. The disadvantage is the necessity to compute the canonical homogeneous norm numerically and on-line [50].

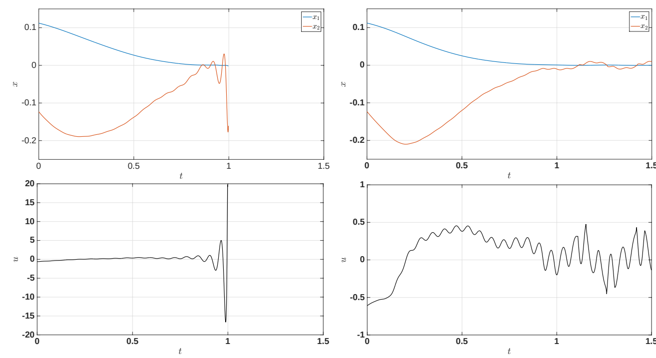

Fig. 2. Numerical simulations for the time-varying linear control (19) and the nonlinear control (20) in the perturbed case

V. Conclusions

The problem of energetically optimal finite-time stabilization of linear generalized homogeneous systems is studied. The considered optimization problem has a mixed cost functional aimed at minimization of both a settling time and a weighted/averaged control energy. For a class of generalized homogeneous linear systems, the optimal control is obtained in the form of a linear feedback with time-varying gain defined by a solution of the linear algebraic equation. A nonlinear homogeneous feedback (14) inspired by MPC ideas is studied as well. More general MPC design using the obtained results is the interesting problem for future investigations.

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