A Bayesian Framework for Poisson Process Characterization of Extremes with Objective Prior

Théo Moins, Julyan Arbel, Anne Dutfoy, Stéphane Girard

A Bayesian Framework for Poisson Process Characterization of Extremes with Objective Prior

Théo Moins[1], Julyan Arbel[1], Anne Dutfoy[2], Stéphane Girard[1]

[1]Statify, Inria Grenoble Rhône-Alpes

[2]EDF R&D dept. Périclès

2021 World Meeting of the International Society for Bayesian Analysis

July 01, 2021

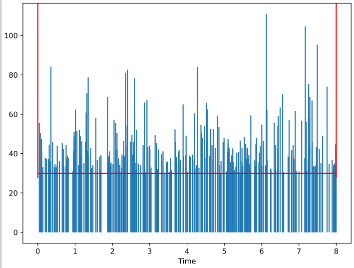
Motivations

Let $(X_1, \ldots, X_n)$ be i.i.d. r.v. and the associated point process N_n, evaluated on $I_u = [u, +\infty)$.

Theorem 1 (S. Coles 2001)

Under mild conditions and for a sufficiently large u, N_n can be approximated by a non-homogeneous Poisson process N of intensity measure Λ with parameters $\boldsymbol{\theta} = (\mu, \sigma, \xi) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}$, such that for all $x > u$,

$$\Lambda(I_x) = \int_x^{+\infty} \lambda(t)dt = \begin{cases} \left\{1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right\}_+^{-\frac{1}{\xi}} & \text{if } \xi \neq 0\,, \\ \exp(-\frac{x-\mu}{\sigma}) & \text{if } \xi = 0. \end{cases}$$



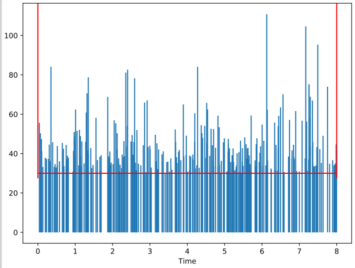
Motivations

Let $(X_1, \ldots, X_n)$ be i.i.d. r.v. and the associated point process N_n, evaluated on $I_u = [u, +\infty)$.

Theorem 1 (S. Coles 2001)

Under mild conditions and for a sufficiently large u, N_n can be approximated by a non-homogeneous Poisson process N of intensity measure Λ with parameters $\boldsymbol{\theta} = (\mu, \sigma, \xi) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}$, such that for all $x > u$,

$$\Lambda(I_x) = \int_x^{+\infty} \lambda(t)dt = \begin{cases} \left\{1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right\}_+^{-\frac{1}{\xi}} & \text{if } \xi \neq 0\,, \\ \exp\left(-\frac{x-\mu}{\sigma}\right) & \text{if } \xi = 0. \end{cases}$$



Likelihood: $$p_{PP}(\mathbf{x} \mid \mu, \sigma, \xi) = \exp\left\{-m\left(1 + \xi\left(\frac{u-\mu}{\sigma}\right)\right)^{-\frac{1}{\xi}}\right\}\sigma^{-n}\prod_{i=1}^{n}\left(1 + \xi\left(\frac{x_i-\mu}{\sigma}\right)\right)^{-\frac{1}{\xi}-1}$$

The Bayesian Point of view

Inference on $\boldsymbol{\theta} = (\mu, \sigma, \xi)$ using Bayesian algorithms such as MCMC.

The main advantages are (S. G. Coles et al. 1996):

- Consideration of expert information with informative prior,
- Can be used in any case, even when the likelihood is irregular (for $\xi < -1$),
- Access to the posterior predictive distribution:

$$p(\tilde{x} \mid \boldsymbol{x}) = \int \underbrace{p(\tilde{x} \mid \boldsymbol{\theta})}_{\substack{\text{randomness} \\ \text{of future obs.}}} \quad \underbrace{p(\boldsymbol{\theta} \mid \boldsymbol{x})}_{\substack{\text{parameter} \\ \text{uncertainty}}} \, d\boldsymbol{\theta}.$$

The Bayesian Point of view

Inference on $\boldsymbol{\theta} = (\mu, \sigma, \xi)$ using Bayesian algorithms such as MCMC.

The main advantages are (S. G. Coles et al. 1996):

- Consideration of expert information with informative prior,
- Can be used in any case, even when the likelihood is irregular (for $\xi < -1$),
- Access to the posterior predictive distribution:

$$p(\tilde{x} \mid \boldsymbol{x}) = \int \underbrace{p(\tilde{x} \mid \boldsymbol{\theta})}_{\substack{\text{randomness} \\ \text{of future obs.}}} \underbrace{p(\boldsymbol{\theta} \mid \boldsymbol{x})}_{\substack{\text{parameter} \\ \text{uncertainty}}} \, d\boldsymbol{\theta}.$$

Computational Bayesian issues: prior elicitation, reparameterization, etc.

Uninformative prior for extremes

Uninformative prior for extremes

Theorem 2 (Moins et al.)

The uniform prior on $(\mu, \log \sigma, \xi)$ leads to a proper posterior for the Poisson process model for extremes as soon as $n \geq 3$.

Uninformative prior for extremes

Theorem 2 (Moins et al.)

The uniform prior on $(\mu, \log \sigma, \xi)$ *leads to a proper posterior for the Poisson process model for extremes as soon as* $n \geq 3$.

Theorem 3 (Moins et al.)

Jeffreys prior for a Poisson process for extremes with parameters $\boldsymbol{\theta} = (\mu, \sigma, \xi)$ *exists provided* $\xi > -1/2$*, and can be written as*

$$\pi_{J,PP}(\mu, \sigma, \xi) \propto \frac{\left(1 + \xi\left(\frac{u-\mu}{\sigma}\right)\right)^{-\frac{3}{2\xi}-1}}{\sigma^2(1+\xi)(1+2\xi)^{\frac{1}{2}}}.$$

It is improper and leads to a proper posterior as soon as $\xi > -1/2$.

An orthogonal parameterisation

Alternative parameterisation: instead of (μ, σ, ξ), run MCMC on an orthogonal parameterisation suggested for another purpose by (Chavez-Demoulin et al. 2005):

$$(r, \nu, \xi) = \left(m \left(1 + \xi \left(\frac{u - \mu}{\sigma} \right) \right)^{-1/\xi}, (1 + \xi)(\sigma + \xi(u - \mu)), \xi \right).$$

r: intensity of PP, expected number of exceedances.
This ensures full orthogonality, which means that $I(r, \nu, \xi)$ is diagonal.

An orthogonal parameterisation

Alternative parameterisation: instead of (μ, σ, ξ), run MCMC on an orthogonal parameterisation suggested for another purpose by (Chavez-Demoulin et al. 2005):

$$(r, \nu, \xi) = \left(m\left(1 + \xi\left(\frac{u-\mu}{\sigma}\right)\right)^{-1/\xi}, (1+\xi)(\sigma + \xi(u-\mu)), \xi \right).$$

r: intensity of PP, expected number of exceedances.
This ensures full orthogonality, which means that $I(r, \nu, \xi)$ is diagonal.

Jeffreys prior:

$$\pi_{J,PP}(r, \nu, \xi) \propto \frac{r^{1/2}}{\nu(1+\xi)(1+2\xi)^{1/2}}.$$

An orthogonal parameterisation

Alternative parameterisation: instead of (μ, σ, ξ), run MCMC on an orthogonal parameterisation suggested for another purpose by (Chavez-Demoulin et al. 2005):

$$(r, \nu, \xi) = \left(m \left(1 + \xi \left(\frac{u - \mu}{\sigma} \right) \right)^{-1/\xi}, (1 + \xi)(\sigma + \xi(u - \mu)), \xi \right).$$

r: intensity of PP, expected number of exceedances.
This ensures full orthogonality, which means that $I(r, \nu, \xi)$ is diagonal.

Jeffreys prior:

$$\pi_{J,PP}(r, \nu, \xi) \propto \frac{r^{1/2}}{\nu(1 + \xi)(1 + 2\xi)^{1/2}}.$$

Our experiments show better convergence with orthogonal parameters.

References

- Chavez-Demoulin, V. and A.C. Davison (2005). “Generalized additive modelling of sample extremes”. In: *Journal of the Royal Statistical Society: Series C (Applied Statistics)* 54.1, pp. 207–222.
- Coles, S. G. and E. A. Powell (1996). “Bayesian Methods in Extreme Value Modelling: A Review and New Developments”. In: *International Statistical Review / Revue Internationale de Statistique* 64.1, pp. 119–136.
- Coles, S.G. (2001). *An introduction to statistical modeling of extreme values*. Springer Series in Statistics. London: Springer-Verlag. ISBN: 1-85233-459-2.