On the complexity of overlaying a hypergraph with a graph with bounded maximum degree

Frédéric Havet, Dorian Mazauric, Viet-Ha Nguyen

On the complexity of overlaying a hypergraph with a graph with bounded maximum degree.

Frédéric Havet[1], Dorian Mazauric[2], and Viet-Ha Nguyen[1]

[1]CNRS, Université Côte d'Azur, Inria, I3S, France
[2]Inria, Université Côte d'Azur, France

January 11, 2022

Abstract

Let G and H be respectively a graph and a hypergraph defined on a same set of vertices, and let F be a graph. We say that G *F-overlays* a hyperedge S of H if the subgraph of G induced by S contains F as a spanning subgraph, and that *G F-overlays H* if it F-overlays every hyperedge of H. For a fixed graph F and a fixed integer k, the problem $(\Delta \leq k)$-F-Overlay consists in deciding whether there exists a graph with maximum degree at most k that F-overlays a given hypergraph H. In this paper, we prove that for any graph F which is neither complete nor anticomplete, there exists an integer $\mathrm{np}(F)$ such that $(\Delta \leq k)$-F-Overlay is NP-complete for all $k \geq \mathrm{np}(F)$.

1 Introduction

In order to obtain the low resolution structure of molecule-macro assemblies the following problem arises : given a list of complexes, determine the plausible contacts between subunits of an assembly. A convenient way of modelling this uses graphs and hypergraphs: we are given a hypergraph H whose vertices represent the subunits and whose hyperedges represents complexes; the aim is then to find a graph G on the same vertex set whose edges represent contacts between subunits and satisfying some properties.

One of the properties is that the subgraph of G induced by each hyperedge must belong to a family $\mathcal{F}$ of admissible graphs. Precisely, a graph G **$\mathcal{F}$-overlays** a hyperedge S if there exists $F \in \mathcal{F}$ such that F is a spanning subgraph of $G[S]$, and G **$\mathcal{F}$-overlays** H if G $\mathcal{F}$-overlays every hyperedge of H. In a typical example, the family $\mathcal{F}$ is the set of trees (or equivalently connected graphs) and the goal is to minimize the number of edges in G. This was studied by Agarwal et al. [1] in the aforementioned context of structural biology, but also by several authors for various applications like the design of vacuum systems [6, 7], scalable overlay networks [4, 13], and reconfigurable interconnection networks [8, 9]. Some variants have also been considered in the contexts of inferring a most likely social network [2], determining winners of combinatorial auctions [5], as well as drawing hypergraphs [3, 12].

Motivated by the fact that a subunit (e.g. a protein) cannot be connected to many other subunits, Havet et al. [10] studied the problem in which the sought graph G must have bounded maximum degree. Therefore they introduced the following problem where $\mathcal{F}$ is a fixed family of graphs, k a fixed integer and $\Delta(G)$ denotes the maximum degree of G.

> $(\Delta \leq k)$-$\mathcal{F}$-Overlay
> Input: A hypergraph H.
> Question: Does there exist a graph G $\mathcal{F}$-overlaying H such that $\Delta(G) \leq k$?

They studied the complexity of this problem and the associated maximization problem $(\Delta \leq k)$-$\mathcal{F}$-Overlay and Max $(\Delta \leq k)$-$\mathcal{F}$-Overlay which, given a hypergraph H and an integer p, consists in deciding whether or not there exists a graph G of maximum degree at most k and F-overlays at least p hyperedges of H. Special attention was paid to the particular case when the family $\mathcal{F}$ contains a unique graph F and H is then an $|F|$-uniform hypergraph (*i.e.* every hyperedge has $|F|$ vertices). In this case, we abbreviate $\{F\}$-overlay into F-overlay, and $(\Delta \le k)$-$\{F\}$-Overlay into $(\Delta \le k)$-F-Overlay. For convenience, a graph F-overlaying H and with maximum degree at most k is called an **(F, H, k)-graph**. Examples of (F, H, k)-graphs are given in Figure 1 when F is O_3 or P_3, the graphs with three vertices and respectively one edge and two edges.

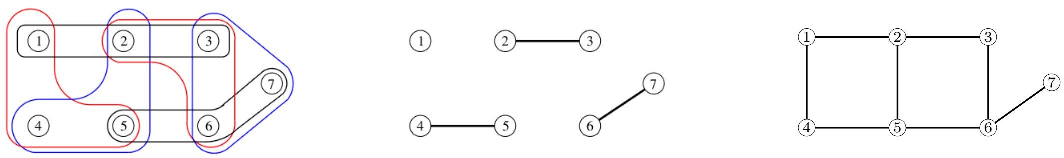

Figure 1: A hypergraph H (left), an $(O_3, H, 1)$-graph (middle) and a $(P_3, H, 3)$-graph(right).

Observe that if F is a graph with maximum degree greater than k, then solving $(\Delta \le k)$-F-Overlay or Max $(\Delta \le k)$-F-Overlay is trivial as the answer is always 'No'. Havet et al. [10] proved a complete polynomial/NP-complete dichotomy for Max $(\Delta \le k)$-$\mathcal{F}$-Overlay depending on the pairs (F, k). They proved that, except in a few exceptions, Max $(\Delta \le k)$-F-Overlay is NP-complete if and only if $\Delta(F) \le k$. The exceptions are when F is either an anticomplete graph $\overline{K}_p$ or the complete graph on two vertices K_2 in which case Max $(\Delta \le k)$-F-Overlay is always polynomial-time solvable, or when F is the graph O_3 with three vertices and one edge and $k = 1$ with Max $(\Delta \le 1)$-O_3-Overlay being polynomial-time solvable.

Regarding $(\Delta \le k)$-F-Overlay, establishing such a dichotomy seems more complicated. Indeed, Havet et al. [10] showed several pairs (F, k) (with $\Delta(F) \le k$) such that $(\Delta \le k)$-F-Overlay is polynomial-time and some such that $(\Delta \le k)$-F-Overlay is NP-complete, and posed the following problem.

Problem 1 (Havet et al. [10])**.** Characterize the pairs (F, k) for which $(\Delta \le k)$-F-Overlay is polynomial-time solvable and those for which it is NP-complete.

In order to attack this problem, they propose the following conjecture.

Conjecture 2 (Havet et al. [10])**.** If $(\Delta \le k)$-F-Overlay is $\mathcal{NP}$-complete, then $(\Delta \le k+1)$-F-Overlay is also $\mathcal{NP}$-complete.

In this paper, we give some partial answers to Problem 1 and some evidences for Conjecture 2. We prove that except when F is complete or anticomplete, if k is large enough (with respect to F), then $(\Delta \le k)$-F-Overlay is NP-complete. Recall that a graph is **complete** (resp. **anticomplete**) if its vertices are pairwise adjacent (resp. non-adjacent). The complete (resp. anticomplete) graph on p vertices is denoted by K_p (resp. $\overline{K}_p$).

We define $\mathrm{np}(F)$ as the minimum integer k_0 such that $(\Delta \le k)$-F-Overlay is NP-complete for all $k \ge k_0$ or $+\infty$ if no such k_0 exists. The aim of this article is to prove the following theorem.

Theorem 3. $\mathrm{np}(F) = +\infty$ *if and only if F is complete or anticomplete.*

Let H be a p-uniform hypergraph. The anticomplete graph on $V(H)$ vertices $\overline{K}_p$-overlays H. Thus, for any non-negative integer k, the answer to $(\Delta \le k)$-$\overline{K}_p$-Overlay is always affirmative, and so this problem can be trivially solved in polynomial time. Thus $\mathrm{np}(\overline{K}_p) = +\infty$ for all positive integer p.

If K_p is complete, then let G be the graph with vertex $V(H)$ in which two vertices are adjacent if and only if they belong to a same hyperedge of H. Obviously, a graph K_p-overlays H if and only if it contains G as a subgraph. Hence, to solve $(\Delta \le k)$-K_p-Overlay it suffices to build G and to check whether $\Delta(G) \le k$, which can be done in polynomial time. Thus $\mathrm{np}(K_p) = +\infty$ for all positive integer p.

Therefore to prove Theorem 3, it only remains to prove its sufficiency part, which is the following theorem.

Theorem 4. *If F is neither a complete graph nor an anticomplete graph, then* $\mathrm{np}(F) < +\infty$.

A possibility to prove this theorem would be to prove Conjecture 2 and that, for every graph F which is not complete, there exists k such that $(\Delta \leq k)$-F-Overlay is NP-complete. Unfortunately, we do not prove Conjecture 2. However, first show in Corollary 10 that it is sufficient to prove Theorem 4 for F with no isolated vertices. We then establish a weaker statement than Conjecture 2 for such graphs: in Lemma 12 we show that, for a graph F with no isolated vertices, as soon as there exists k such that $(\Delta \leq k)$-F-Overlay is NP-complete, then $\mathrm{np}(F) < +\infty$. This lemma, together with the following theorem, directly implies Theorem 4.

Theorem 5. *Let F be a graph with no isolated vertex and which is not complete. There exists k such that $(\Delta \leq k)$-F-Overlay is NP-complete.*

In Section 4, we first prove Theorem 5 when F belongs to some particular classes of graphs : F is **regular** (Theorem 13), and F is a **complete graph minus an edge**, denoted by K^- (Theorem 14) and F is a **disjoint union of the complete bipartite graph** $K_{a,a+1}$ (Theorem 15). Then, in Section 5, we prove Theorem 5 in full. Its proof requires the previously established particular cases and uses the techniques introduced in proving them. Finally, in Section 6, we give some final remarks and present some open questions for further research.

Remark 6. In all the paper, our aim is to prove that $(\Delta \leq k)$-F-Overlay is NP-complete under some assumptions on k and F. Observe that, given an graph G, we can easily check whether G F-overlays H or not in polynomial-time solvable, thus the problem is clearly in NP. Therefore, we only need to prove that the problem is NP-hard.

All our NP-hardness proofs are reductions from either (3,4)-Sat or 3-Colorability of 4-regular graphs.

In $(3,4)$-Sat, an instance is a set of clauses, each of which being a conjuction of three literals on variables, such that every variable appears in at most 4 clauses; the problem consists in deciding whether there is a truth assignment to the variables such that every clause is satisified. (3,4)-Sat has been proved NP-complete by Tovey [14].

3-Colorability consists in deciding whether a given graph admits a proper 3-coloring. It has been proved NP-complete for 4-regular graphs by Holyer [11].

2 Notations and definitions

For a positive integer p, let $[p] = \{1, \dots, p\}$.

2.1 Graphs

Let G be graph. We denote by $V(G)$ and $E(G)$ its sets of vertices and edges, respectively. The **neighborhood** of a vertex v, denoted by $N_G(v)$, or simply $N(v)$ when G is clear from the context, is the set of vertices adjacent to v and its **degree**, denoted by $d_G(v)$ or simply $d(v)$, is the cardinality of $N_G(v)$. A vertex is **isolated** in G if it has degree 0. The minimum and maximum degree of G are respectively denoted by $\delta(G)$ and $\Delta(G)$. Hence a graph F has no isolated vertices if and only if $\delta(F) \geq 1$. We denote by V_i (resp. $V_{\leq i}$, $V_{\geq i}$) the set of vertices of G that has degree exactly (resp. at most, at least) i in G.

For $S \subseteq V(G)$, the **subgraph induced by** S, denoted by $G[S]$, is the graph with vertex set S and edge set $\{uv \mid u \in S, v \in S \text{ and } uv \in E(G)\}$.

The **degree sequence** of a graph F is the non-decreasing sequence $\mathbf{d} = \{d_1, d_2, \dots, d_p\}$ such that there exists an ordering $(v_1, \dots, v_p)$ of the vertices of F such that $d(v_i) = d_i$ for all $i \in [p]$. We denote by $\lambda_1 < \lambda_2 < \cdots < \lambda_t$ the different values of $\mathbf{d}$ (that are the integers λ in which there exists j such that $d_j = \lambda$). We also denote by α_i the **multiplicity** or number of occurrences of λ_i in $\mathbf{d}$: $\alpha_i = |\{j \mid d_j = \lambda_i\}|$. Observe that $d_1 = \lambda_1 = \delta(F)$ and $d_p = \lambda_t = \Delta(F)$.

We denote by P_t the path on t vertices.

We denote by $G_1 + G_2$ the disjoint union of the two graphs G_1 and G_2.

2.2 Hypergraphs

Let H be a hypergraph. We denote by $V(H)$ and $E(H)$ its sets of vertices and hyperedges (a hyperedge is a subset of vertices of $V(H)$), respectively.
A hypergraph is **p-uniform** for some $p \in \mathbb{N}$, if all its hyperedge have exactly p vertices. Observe that $(\Delta \leq k)$-F-Overlay only makes sense for $|V(F)|$-uniform hypergraphs. Therefore in the paper, we only work with hypergraphs that are uniform, often without specifying it.

In a hypergraph H, a hyperedge S is **pendant** at a vertex x, if S is the unique hyperedge containing v for all $v \in S \setminus \{x\}$.

Let F be a graph, H a hypergraph, and G a graph F-overlaying H. For each hyperedge $S \in E(H)$, one can choose a copy F_S of F which is a subgraph of $G[S]$. We then say that v is a **λ-vertex** in S if v has degree λ in F_S. If H' is a sub-hypergraph of H (typically a gadget in an NP-hardness proof), we denote by abbreviate $G[V(H')]$ into $G[H']$. We also say that G has degree d in H' if it has degree d in $G[H']$.

3 Reduction to Theorem 5

3.1 Graphs with isolated vertices

Lemma 7. *Let F be a graph. If $(\Delta \leq k)$-F-Overlay is NP-complete, then $(\Delta \leq k)$-$(F+\overline{K}_1)$-Overlay is also NP-complete.*

Proof. We shall give a reduction from $(\Delta \leq k)$-F-Overlay to $(\Delta \leq k)$-$(F+\overline{K}_1)$-Overlay.
Let $\mathbf{d}$ be the (non-decreasing) degree sequence of F, and let λ^+ be the first non-zero value in this sequence. ($\lambda^+ = \lambda_1$ if F has no isolated vertex, and $\lambda^+ = \lambda_2$ otherwise.)
Let H be an $|F|$-uniform hypergraph. We construct an $(|F|+1)$-uniform hypergraph H' as follows.

- Let $H_1, \dots, H_t$ be $t = \lfloor \frac{k}{\lambda^+} \rfloor |E(H)| + 1$ disjoint copies of H. We add $V(H_i)$ to $V(H')$ for all $i \in [t]$.
- For any $S \in E(H)$, we add a new vertex v_S to $V(H')$. For all $i \in [t]$, denoting by S_i the copy of S in H_i, we add the hyperedge $S'_i = S_i \cup \{v_S\}$ to H'.

We shall prove that there is an (F, H, k)-graph G if and only if there exists an $(F+\overline{K}_1, H', k)$-graph G'.

Assume first that there is an (F, H, k)-graph. We build a graph G' by taking $G'[H_i] = G$ for any $i \in [t]$. Observe that $G'[S'_i]$ is $(F+\overline{K}_1)$-overlaid since $G[S_i]$ is F-overlaid and v_S is an isolated vertex. Furthermore G' has at most degree k. Thus, G' is an $(F+\overline{K}_1, H', k)$-graph.

Conversely, assume that there exists an $(F+\overline{K}_1, H', k)$-graph G'. We will prove that there exists a copy H_i of H such that $G'[H_i]$ is an (F, H, k)-graph. Observe that, for any $S \in E(H)$, the vertex v_S is either isolated or has degree at least λ^+ in each $G'[S'_i]$ for $i \in [t]$. Thus, v_S is not a 0-vertex in at most $\lfloor \frac{k}{\lambda^+} \rfloor$ hyperedges. Since there are $|E(H)|$ such vertices, there exists a copy H_i of H such that for any $S \in E(H)$, v_S is a 0-vertex in all hyperedges $G'[S'_i]$. Thus $G'[H_i]$ is an (F, H, k)-graph. ☐

Applying the lemma several times, we get the following.

Corollary 8. *Let F be a graph and q a positive integer. If $(\Delta \leq k)$-F-Overlay is NP-complete, then $(\Delta \leq k)$-$(F+\overline{K}_q)$-Overlay is also NP-complete. Hence $\mathrm{np}(F+\overline{K}_q) \leq \mathrm{np}(F)$.*

The family of graphs with isolated vertices to which this result does not apply is $K_p + \overline{K}_q$ because $(\Delta \leq k)$-K_p-Overlay is in P. We then need the following.

Theorem 9. $\mathrm{np}(K_p + \overline{K}_1) \leq 2p-2$ *for all* $p \geq 2$.

Proof. Let $p \geq 2$ and $k \geq 2p-2$. Let q and r be the integers such that $k = (p-1)q + r$ with $0 \leq r < p-1$. Note that $q \geq 2$ since $k \geq 2(p-1)$.

We shall prove that $(\Delta \leq k)$-$(K_p+\overline{K}_1)$-Overlay is NP-complete with a reduction from 3-Colorability on 4-regular graphs.

We need the following gadget. Let u be a vertex. A **$(p-1)$-gagdget at** u is the hypergraph H_u constructed as follows. The vertex set of H_u is the disjoint union of $\{u, v\}$ and $q+1$ sets $U_1, \dots, U_{q+1}$ of $p-1$ vertices, and its hyperedges are $\{u, v\} \cup U_i$ for $i \in [q+1]$.

Claim 9.1. *Let H_u be a $(p-1)$-gadget at u.*
(i) *u has degree at least $p-1$ in every $(K_p+\overline{K}_1, H_u, k)$-graph.*
(ii) *There is a $(K_p+\overline{K}_1, H_u, k)$-graph in which u has degree $p-1$.*

Proof of Claim. (i) Let G_u be a $(K_p+\overline{K}_1, H_u, k)$-graph. Assume for a contradiction that u has degree less than $p-1$ in G_u. Then u must be a 0-vertex in each S_i, $i \in [q+1]$. Hence v must be adjacent to the $p-1$ vertices of U_i in each S_i. Thus v as degree at least $(p-1)(q+1) > k$ in G_u, a contradiction.

(ii) For $i \in [q]$, let F_i be a copy of $K_p+\overline{K}_1$ in which u is isolated, and let F_{q+1} be a copy of $K_p+\overline{K}_1$ in which v is isolated, and let $G_u = \bigcup_{i\in[q+1]} F_i$. Clearly, G_u $(K_p+\overline{K}_1)$-overlays H_u, v has degree $q(p-1) \le k$ in G_u and u has degree $p-1$ in G_u. So G_u is the desired $(K_p+\overline{K}_1, H_u, k)$-graph. $\Diamond$

Given a 4-regular graph G, we build a p-uniform hypergraph H as follows.

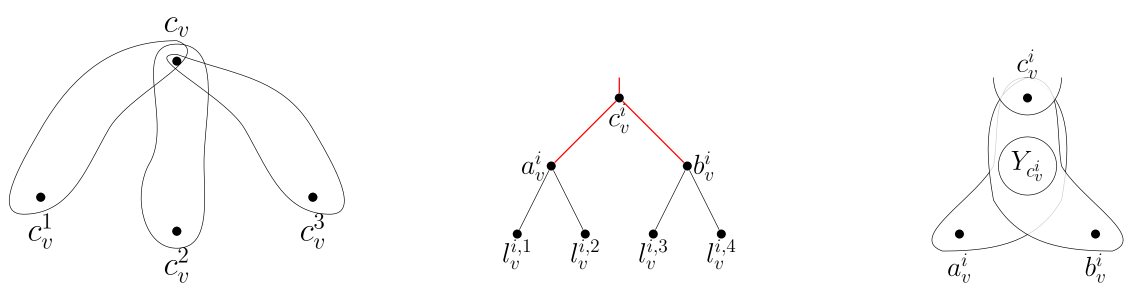

Figure 2: **Constructing the hypergraph H.** Left : the vertex gadget VG_v. At each c_v^i of this gadget, we add a binary tree (center). Each pair of edges joining a vertex x to its two children in this tree is replaced by an x-edge-gadget. For example, the two red edges $c_v^i a_v^i$ and $c_v^i a_v^i$ are replaced by the c_v^i-edge-gadget to the right.

- For each vertex $v \in V(G)$, we create a *vertex gadget* VG_v with three hyperedges $S_v^i = \{c_v, c_v^i\} \cup X_v^i$ for $i \in [3]$ where $|X_v^i| = p-2$. We add $q-2$ $(p-1)$-gadgets at c_v. We say that S_v^i is the *parent hyperedge* of c_v^i for each $i \in [3]$.
- For each vertex v and each $i \in [3]$, we construct a *color gadget* CG_v^i for $i \in [3]$ as follows.
  - We create a binary tree T_v^i with vertex set $\{c_v^i, a_v^i, b_v^i, \ell_v^{i,1}, \ell_v^{i,2}, \ell_v^{i,3}, \ell_v^{i,4}\}$ and edge set $\{c_v^i a_v^i, c_v^i b_v^i, a_v^i \ell_v^{i,1}, a_v^i \ell_v^{i,2}, b_v^i \ell_v^{i,3}, b_v^i \ell_v^{i,4}\}$, rooted at c_v^i. In this tree, a_v^i and b_v^i are the children of c_v^i, $\ell_v^{i,1}$ and $\ell_v^{i,2}$ are the children of a_v^i, and $\ell_v^{i,3}$ and $\ell_v^{i,4}$ are the children of b_v^i.
  - For any vertex $x \in \{c_v^i, a_v^i, b_v^i\}$, let y_1, y_2 be its children in T_v^i, and let $e_1 = xy_1, e_2 = xy_2$. We construct an *x-edge-gadget* as follows: we add a set Y_x of $p-2$ new vertices, the hyperedges $S(e_1) = \{x, y_1\} \cup Y_x$ and $S(e_2) = \{x, y_2\} \cup Y_x$. For convenience, we say that $S(xy_1)$ (resp. $S(xy_2)$) is the *parent hyperedge* of y_1 (resp. y_2). Moreover, for any leaf $\ell_v^{i,j}$, we denote by $S_v^{i,j}$ the hyperedge containing the vertex $\ell_v^{i,j}$. We then add $q-1$ $(p-1)$-gadgets at x.
- For every vertex $v \in V(G)$, let $e_v^1, e_v^2, e_v^3, e_v^4$, be an ordering of the edges incident to v. For each edge $uv \in E(G)$, let j_u and j_v be the indices such that $uv = e_u^{j_u} = e_v^{j_v}$. Then, for all $i \in [3]$, we identify the vertices ℓ_u^{i,j_u} and ℓ_v^{i,j_v} and we add $q-1$ $(p-1)$-gadgets at this vertex.

Let us now prove that there is a proper 3-coloring of G if and only if there is a $(K_p+\overline{K}_1, H, k)$-graph G^*.

Assume first that there is a $(K_p+\overline{K}_1, H, k)$-graph G^*.
Let $v \in V(G)$. By Claim 9.1 (i), the vertex c_v has degree at least $p-1$ in each of its $(p-1)$-gadgets. So it has at most $2(p-1)+r$ neighbours in $S_v^1 \cup S_v^2 \cup S_v^3$. But those hyperedges pairwise intersect in $\{c_v\}$. Thus there is $i \in [3]$ such that c_v is a 0-vertex in S_v^i. Since there is only one 0-vertex in S_v^i, c_v^i must be a $(p-1)$-vertex in S_v^i. Therefore, we can define a 3-coloring ϕ by $\phi(v) = i$ where i is an index such that c_v^i is a $(p-1)$-vertex in S_v^i. Let us now prove that ϕ is proper. We need the following claim.

Claim 9.2. *Let $v \in V(G)$ and $i \in [3]$. If c_v^i is a $(p-1)$-vertex in S_v^i, then so is the leaf $\ell_v^{i,j}$ in $S_v^{i,j}$ for all $j \in [4]$.*

Proof of Claim. It suffices to prove that for any $x \in \{c_v^i, b_v^i, a_v^i\}$, if x is a $(p-1)$-vertex in its parent hyperedge, then so are both y_1, y_2 in their parent hyperedges.

Assume that x is a $(p-1)$-vertex in its parent hyperedge. Since x has degree at least $p-1$ in each of its $(p-1)$-gadgets by Claim 9.1 (i), it has at most r neighbors in $S(xy_1) \cup S(xy_2)$. It implies that x is a 0-vertex in both $S(xy_1), S(xy_2)$. Hence, the vertex y_1 (resp. y_2) must be a $(p-1)$-vertex in $S(xy_1)$ (resp. $S(xy_2)$). $\diamondsuit$

Consider an edge $uv \in E(G)$, $i \in [3]$. By Claim 9.1 (i), the vertex $\ell = \ell_u^{i,j_u} = \ell_v^{i,j_v}$ has degree at least $p-1$ in each of its $q-1$ $(p-1)$-gadgets. Thus it has at most $(p-1)+r$ neighbors in $S_u^{i,j_u} \cup S_v^{i,j_v}$. As ℓ is the unique common vertex of S_u^{i,j_u} and S_v^{i,j_v}, it is a $(p-1)$-vertex in at most one of those. Hence, by Claim 9.2, at most one of c_u^i, c_v^i is a $(p-1)$-vertex in its parent hyperedge. Thus at most one of u, v is colored i by ϕ. Therefore, ϕ is a proper 3-coloring of G.

Conversely, let ϕ be a proper 3-coloring of G. We construct a graph G^* as follows.

- For any vertex gadget VG_v, $i \in [3]$, let $G^*[S_v^i]$ be a copy of $K_p + \overline{K}_1$ in which every vertex in X_v^i is a $(p-1)$-vertex, and c_v is a 0-vertex (resp. $(p-1)$-vertex) and c_v^i is a $(p-1)$-vertex (resp. 0-vertex) in S_v^i if $\phi(v) = i$ (resp. $\phi(v) \neq i$).
- In every color gadget CG_v^i, for $x \in \{c_v^i, b_v^i, a_v^i\}$ with children y_1 and y_2, let $G^*[S(xy_1)]$ and $G^*[S(xy_2)]$ be two similar copies of $K_p + \overline{K}_1$ such that:
  - if $i \neq \phi(v)$, then x has degree $p-1$ in $G^*[S(xy_1)]$ and $G^*[S(xy_2)]$; y_1 and y_2 are 0-vertices in $S(xy_1)$ and $S(xy_2)$ respectively (so x has degree $p-1$ in $G^*[S(xy_1) \cup S(xy_2)]$).
  - if $i = \phi(v)$, then x has degree 0 in $G^*[S(xy_1)]$ and $G^*[S(xy_2)]$; y_1 and y_2 are $(p-1)$-vertices in $S(xy_1)$ and $S(xy_2)$ respectively (so x has degree at most $p-1$ in $G^*[S(xy_1) \cup S(xy_2)]$).
  - every vertex in Y_x is a $p-1$ vertex in both $S(xy_1)$ and $S(xy_2)$ and so has degree at most p in $G^*[S(xy_1) \cup S(xy_2)]$;
- For any $(p-1)$-gadget H_x at vertex some x, we let $G^*[H_x]$ be a $(K_p + \overline{K}_1, H_x, k)$-graph in which v has degree $p-1$. Such a copy exists by Claim 9.1 (ii).

By construction, G^* $(K_p + \overline{K}_1)$-overlays H. Let us check that $\Delta(G^*) \leq k$. Let u be a vertex of G^*.

- If u is in at most two hyperedges (in particular, if u is in X_v^i or u is in Y_x for x internal vertex in some T_v^i or u is only in a $(p-1)$-gadget), then u has degree at most $2(p-1)$, and so at most k.
- Assume now that $u \in \{c_v^i, a_v^i, b_v^i\}$ for $i \in [3]$ with u parent of y_1, y_2. Then u has degree $p-1$ in each of its $q-2$ $(p-1)$-gadgets. Moreover if $i = \phi(v)$ (resp. $i \neq \phi(v)$), then u has degree $p-1$ (resp. 0) in its parent hyperedge and $p-1$ (resp. 0) in $G^*[S_{uy_1}^1 \cup S_{uy_2}^1]$. Hence u has degree at most $(q-1)(p-1) + (p-1) = q(p-1) \leq k$.
- Assume that u is the identification of ℓ_v^{i,j_v} and ℓ_w^{i,i_w} for an edge $vw \in E(G)$. First, u has degree $p-1$ in each of its $q-1$ $(p-1)$-gadgets. Moreover, since either $\phi(v) \neq i$ or $\phi(w) \neq i$, then u has degree $p-1$ in at most one of S_v^{i,j_v}, S_w^{i,j_w} and 0 in the other. Therefore, u has degree at most $q(p-1) \leq k$ in G^*.

Consequently, G^* is a $(K_p + \overline{K}_1, H, k)$-graph. $\square$

Corollary 8 and Theorem 9 directly imply the following.

Corollary 10. *Theorem 4 holds if and only if it holds for graphs with no isolated vertices.*

3.2 Reduction to Theorem 5

By Corollary 10, one can restrict our study to graphs F with $\delta(F) \geq 1$. We shall now prove that for such an F, we have $\mathrm{np}(F) \leq +\infty$ as soon as there is some k for which $(\Delta \leq k)$-F-OVERLAY is NP-complete. To prove this, we introduce the notion of degree-gadget that will be useful in almost all the following proofs.

Let F be graph with $\delta(F) \geq 1$. For any integer $d \geq \lambda_1$, a ***d*-degree-gadget** (with respect to F) at vertex v, is the subgraph $D(d, v)$ defined as follows. Let $\alpha = \lfloor d/\lambda_1 \rfloor$ and $\beta = d - \alpha\lambda_1$. If $\beta = 0$, then $D(d, v)$ is the union of α pendant hyperedges at v. If $\beta \geq 1$, then $D(d, v)$ is the union of $\alpha - 1$ pendant hyperedges at v and two hyperedges which intersect in $I \cup \{v\}$ where I is a set of $\lambda_1 - \beta$ vertices. (See Figure 3).

Degree-gadgets are useful because of the following proposition whose easy proof is left to the reader.

Proposition 11. *Let F be graph with $\delta(F) \geq 1$. Then for any $d \geq \lambda_1$, we have the following.*

(i) *In any graph G that F-overlays $D(d, v)$, vertex v has degree at least d.*
(ii) *There is a graph G_v that F-overlays $D(d, v)$ in which v has degree exactly d, and every other vertex has degree at most $\Delta(F)$ if $\delta(F)$ divides d (i.e. $\beta = 0$) and at most $2\Delta(F) - 1$ otherwise.*

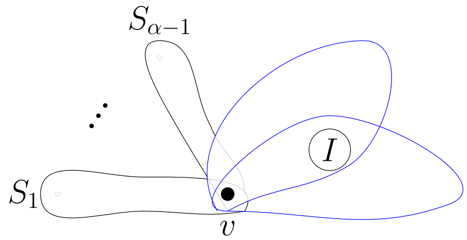

Figure 3: **A d-degree-gadget D at vertex v.** The set $I \neq \emptyset$ is the intersection of the two blue hyperedges. $\beta \neq 0$ when I is different from these two hyperedges; and $\beta = 0$ when I and the two blue hyperedges are equal.

Lemma 12. *Let F be a graph with $\delta(F) \geq 1$. Assume that $(\Delta \leq k_0)$-F-Overlay is* NP*-complete.*
(i) *If $\delta(F) = 1$, then* $\text{np}(F) \leq k_0$.
(ii) $\text{np}(F) \leq \max\{k_0 + \delta(F), 2\Delta(F) - 1\}$.

Proof. Observe that $k_0 \geq \Delta(F)$, because $(\Delta \leq k)$-F-Overlay is trivially polynomial-time solvable for every $k < \Delta(F)$.

(i) Let $k > k_0$. We shall prove that $(\Delta \leq k)$-F-Overlay is NP-complete. We give a reduction from $(\Delta \leq k_0)$-F-Overlay. Let H_0 be an $|F|$-uniform hypergraph. Let H be the hypergraph obtained from H_0 by adding a $(k - k_0)$-degree-gadget DG_v on every vertex v. Such a degree-gadget exists because $k - k_0 \geq 1 = \delta(F)$. Let us prove that there is an (F, H_0, k_0)-graph G_0 if and only if there exists an (F, H, k)-graph G.

Assume there is an (F, H_0, k_0)-graph G_0. By Proposition 11-(ii), for every $v \in V(H_0)$, there is a graph G_v that F-overlays DG_v, in which v has degree $k - k_0$, and every other vertex as degree at most $\Delta(F) \leq k_0$. Consider $G = G_0 \cup \bigcup_{v \in V(H_0)} G_v$. Clearly, G is an (F, H, k)-graph.

Conversely, assume that there is an (F, H, k)-graph G. By Proposition 11-(i), every vertex v of $V(H_0)$ has degree at least $k - k_0$ in DG_v. Thus it has degree at most k_0 in $G[H_0]$. Therefore, $G[H_0]$ is an (F, H_0, k_0)-graph.

(ii) The proof is identical to (i). Taking $k \geq \max\{k_0 + \delta(F), 2\Delta(F) - 1\}$ and using the same reduction as above we get that $(\Delta \leq k)$-F-Overlay is NP-complete. Note that $(k - k_0)$-degree-gadgets exist because $k - k_0 \geq \delta(F)$. ☐

By this lemma, in order to prove that $\text{np}(F)$ is bounded, it suffices to prove that there exists k_0 such that $(\Delta \leq k_0)$-F-Overlay is NP-complete.

4 Particular cases

In this section, we prove the NP-completeness of $(\Delta \leq k)$-F-Overlay for pairs (F, k) where F is either a **regular graph**, or a **complete graph minus an edge** K_p^- (*i.e.* it is obtained by removing an edge from K_p) or a **disjoint union of the complete bipartite graph** $K_{a,a+1}$, and k is an integer (depending on F).

4.1 Regular graphs

Theorem 13. *Let λ be a positive integer, and let F be a λ-regular graph which is not complete.*
Then $(\Delta \leq 6\lambda - 1)$-$F$-Overlay is NP*-complete.*

Proof. Set $p = |F|$. Since F is not complete, we have $p > \lambda + 1$.

We give a reduction from (3,4)-Sat to $(\Delta \leq 6\lambda - 1)$-$F$-Overlay.

Given a formula Φ of (3,4)-Sat with n variables $x_t, t \in [n]$, and m clauses $C_j, j \in [m]$, we construct a p-uniform hypergraph H as follows.

- For each variable x_t, we construct a *variable gadget* H_t as follows. We first create a *center vertex* w_t, a set of $4p - 4$ vertices $U_t = \{u_t^1, \ldots, u_t^{4p-4}\}$, and $4p - 4$ hyperedges $S_t^j = \{w_t, u_t^j, \ldots, u_t^{j+p-2}\}$ (superscripts are modulo 4) for $j \in [4p - 4]$. We then add a $(2\lambda - 1)$-degree-gadget at w_t and a 4λ-degree-gadget on each $u_t^{(p-1)i-j}$ for any $i \in [4]$ and $j \in [\lambda - 1]$. For $r \in [4]$ let $x_t^r = u_t^{r(p-1)-p+2}$ and

$\bar{x}_t^r = u_t^{r(p-1)-p+3}$. Set $X_t = \{x_t^1, x_t^2, x_t^3, x_t^4\}$ and $\overline{X}_t = \{\bar{x}_t^1, \bar{x}_t^2, \bar{x}_t^3, \bar{x}_t^4\}$. The vertices of X_t (resp. $\overline{X}_t$) are called the *non-negated* (resp. *negated) literal vertices* of H_t. See Figure 4.

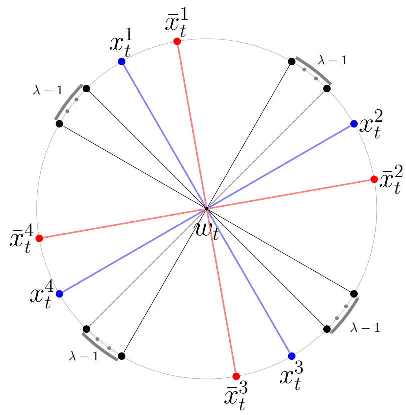

Figure 4: **The variable gadget H_t.** The center vertex w_t is in a $(2\lambda-1)$-degree-gadget. There are four sets of $\lambda-1$ vertices (in black), each of which is adjacent to the center vertex w_t and in a 4λ-degree-gadget. Blue and red vertices are respectively non-negated and negated literal vertices.

- For each clause $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$ we identify y_1, y_2, y_3 into a *clause vertex* c_j, where $y_i = x_t^r$ if $\ell_i = x_t$ and ℓ_i is the r-th occurrence of x_t, and $y_i = \bar{x}_t^r$ if $\ell_i = \bar{x}_t$ and is the r-th occurrence of x_t.

We will prove that there exists an assignment ϕ satisfying Φ if and only if there is an $(F, H, 6\lambda-1)$-graph G. The general idea is that a variable $x_t = true$ (resp. $false$) if and only if the vertices of X_t (resp. $\overline{X}_t$) have degree $2\lambda-1$ in $G[H_t]$ and so they are adjacent to the center vertex while the ones of the other set are not.

Assume that there exists a truth assignment ϕ satisfying Φ. Let G be the graph obtained as follows.

For each $t \in [n]$, let $(v_0, v_1, \dots, v_{p-1})$ be an ordering of $V(F)$ such that $N_F(v_0) = \{v_{p-\lambda+1}, \dots, v_{p-1}\} \cup \{v_1\}$ if $\phi(x_t) = true$ and $N_F(v_0) = \{v_{p-\lambda+1}, \dots, v_{p-1}\} \cup \{v_2\}$ if $\phi(x_t) = false$. For every $j \in [4p-4]$, we let $G[S_t^j]$ be the copy of F in which w_t corresponds to v_0 and u_t^i for $i \in \{j, \dots, j+p-1\}$ corresponds to the vertex $v_{i'}$ such that $i \equiv i' \mod (p-1)$. Observe that each u_t^i corresponds to the same vertex of F in all the $p-1$ copies of F induced by the S_t^j to which it belongs. Therefore either u_t^i is not adjacent to w_t and it has 2λ neighbors in $G[H_t]$ or u_t^i is adjacent to w_t and it has $2\lambda-1$ neighbors in $G[H_t]$. In particular, if $\phi(x_t) = true$ (resp. $\phi(x_t) = false$), then all vertices of X_t (resp. $\overline{X}_t$) have degree $2\lambda-1$ in $G[H_t]$ In addition, for every d-degree-gadget D at some vertex v, we let $G[D]$ be an $(F, D, 6\lambda-1)$-graph in which v has degree d.

Let us check that every vertex has degree at most $6\lambda-1$ in G.

- Each center vertex w_t has degree $2\lambda-1$ in its $(2\lambda-1)$-degree-gadget and it is adjacent to 4λ vertices in H_t, so $6\lambda-1$ in total.
- Every vertex in $\{u_t^{(p-1)i-j} \mid i \in [4] \text{ and } j \in [\lambda-1]\}$ has $2\lambda-1$ neighbors in H_t and 4λ other in its 4λ-degree-gadget. Hence its total degree is $6\lambda-1$.
- Every vertex in $U_t \setminus \{u_t^{(p-1)i-j} \mid i \in [4] \text{ and } j \in [\lambda-1]\}$ which is not identified in a clause vertex has only neighbors in H_t and thus degree at most $2\lambda < 6\lambda-1$.
- Each clause vertex is the identification of three literal vertices which have degree 2λ or $2\lambda-1$ in their variable gadgets. Moreover, at least one of the literals is true, so at least one of those vertices has only $2\lambda-1$ neighbors in its variable gadget. Hence its degree in G is at most $6\lambda-1$.

Hence, G is an $(F, H, 6\lambda-1)$-graph.

Conversely, assume that G is an $(F, H, 6\lambda-1)$-graph.
Consider a variable gadget H_t. The center vertex w_t has degree at least $2\lambda-1$ in its $(2\lambda-1)$-degree-gadget, so it has at most 4λ neighbors in $V(H_t)$. But w_t has degree at least λ in each of the S_t^j, and the hyperedges S_t^j, S_t^{j+p-1}, S_t^{j+2p-2}, S_t^{j+3p-3} pairwise intersect only in w_t. So this vertex has exactly λ neighbors in each of these sets, and so exactly λ neighbors in each S_t^j. Furthermore, if u_t^j is adjacent to w_t, then w_t has $\lambda-1$

neighbors in $\{u_t^{j+1}, \ldots, u_t^{j+p-2}\}$ and so u_t^{j+p-1} is adjacent to w_t because S_t^{j+1} contains λ neighbors of w_t. In particular, the vertices of X_t (resp. $\overline{X}_t$) are either all adjacent to w_t or all non-adjacent to w_t.

Now each of the $\lambda-1$ vertices in $\{u_t^{p-r+2}, \ldots, u_t^{p-1}\}$ is in a 4λ-gadget in which it has degree 4λ. Therefore, it has degree $2\lambda-1$ in $G[H_t]$ and must be adjacent to w_t. Hence at most one vertex in $\{x_t^1, \bar{x}_1^t\}$ is adjacent to w_t. Thus the vertices of X_t and those of $\overline{X}_t$ cannot be simultaneoulsy adjacent to w_t.

Let ϕ be the truth assignment defined by $\phi(x_t) = \mathit{true}$ if w_t is adjacent to X_t, and $\phi(x_t) = \mathit{false}$ otherwise. In any clause vertex c_j, we identified three literal vertices corresponding to the three literals. But c_j has degree at most $6\lambda-1$, so there is at least one literal vertex having degree $2\lambda-1$ in its variable gadget. This implies that this literal is true. Therefore, ϕ satisfies Φ. □

4.2 Complete graph minus an edge

Theorem 14. $(\Delta \leq 3p-1)$-K_p^--Overlay *is* NP-*complete for all* $p \geq 3$.

Proof. Reduction from (3,4)-Sat. Given a formula Φ of (3,4)-Sat with variables $x_t, t \in [n]$ and clauses $C_j, j \in [m]$, we build a hypergraph H as follows.

- For each variable x_t, we add a *variable gadget* H_t containing a *center set* C_t of size $p-2$, a set U_t of 8 vertices $U_t = \{u_t^1, \ldots, u_t^8\}$, and 8 hyperedges $S_t^i = C_t \cup \{u_t^i, u_t^{i+1}\}$ (superscripts are modulo 8) for $i \in [8]$. Set $X_t = \{u_t^{2i-1} \mid i \in [4]\}$ and $\overline{X}_t = \{u_t^{2i} \mid i \in [4]\}$. The vertices of X_t (resp. $\overline{X}_t$) are called the *non-negated literal vertices* (resp. *negated literal vertices*).
- For each clause $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$, we add a *clause vertex* c_j in which, for each literal ℓ_i which is the r-th occurrence of the variable x_t, we identify u_t^{2r-1} (resp. u_t^{2r}) if $\ell_i = x_t$ (resp. $\ell_i = \bar{x}_t$).
- In any center set C_t, if $p=3$, in which case $|C_t| = 1$, we add a $(2p-2)$-degree-gadget at the vertex of C_t; if $p \geq 4$, we add a $(2p-4)$-degree-gadget at $\max\{0, 6-p\}$ vertices of C_t and a $(2p-5)$-degree-gadget at $\min\{4, 2p-8\}$ vertices among the other ones.

We will show that there is an assignment ϕ satisfying Φ if and only there is a $(K_p^-, H, 3p-1)$-graph G.

Assume that ϕ satisfies Φ, then we construct G as follows.

In a variable gadget H_t, for every $i \in [8]$, we let $G[S_t^i]$ be a copy of K_p^- such that

- every vertex in C_t which is not in any degree-gadget is a $(p-1)$-vertex, and so is adjacent to all vertices of H_t;
- if $\phi(x_t) = \mathit{true}$ (resp. $\phi(x_t) = \mathit{false}$), then each vertex in X_t (resp. $\overline{X}_t$) is a $(p-2)$-vertex in every hyperedge containing it and each vertex in $\overline{X}_t$ (resp. X_t) is a $(p-1)$-vertex in every hyperedge containing it.
- any vertex in C_t which is in a d-degree-gadget is adjacent to all vertices in H_t except $p+5-(3p-1-d)$ literal vertices in exactly one of the two sets $X_t, \overline{X}_t$.

For any d-degree-gadget D at a vertex v, let $G[D]$ be a $(K_p^-, D, 3p-1)$-graph in which v has degree d.
Let us check that $\Delta(G) \leq 3p-1$.

- Each vertex in C_t which is not in any degree-gadget is adjacent to all vertices of H_t. So it has degree at most $p+5 \leq 3p-1$ in G.
- Each vertex in C_t which is in a d-degree-gadget has d neighbors in its degree-gadget and is adjacent to $3p-1-d$ vertices of H_t. So it has degree $3p-1$ in G.
- Any literal vertex which is not identified in any clause vertex has either $p-1$ or p neighbors in its variable gadget. Thus it has degree less than $3p-1$.
- Each clause vertex is the identification of three literal vertices. Each of those has degree either $p-1$ or p in its variable gadget. Moreover at least one of the literals is true so its corresponding literal vertex has degree $p-1$ in its variable gadget. Therefore the clause vertex has degree at most $3p-1$.
- Any vertex which is in a degree-gadget but in no variable gadget belongs to at most two hyperedges.Thus it has degree at most $2(p-1) < 3p-1$.

Hence, G is a $(K_p^-, H, 3p-1)$-graph.

Conversely, assume that G is a $(K_p^-, H, 3p-1)$-graph. For every hyperedge S of H, let F_S be a subgraph of $G[S]$ isomorphic to K_p^-.

Claim 14.1. *For every* $t \in [n]$*, we have the following:*

(i) $G[C_t]$ *is complete.*

(i) *There are exactly four non-edges between $X_t \cup \overline{X}_t$ and C_t. Moreover, either each vertex of X_t is incident to one of those non-edges, or each vertex of $\overline{X}_t$ is incident to one of those non-edges.*

Proof of Claim. Assume that $G[C_t]$ is not complete, then there is an edge $uv \notin G[C_t]$ for $u, v \in V(C_t)$. Since all hyperedges of H_t are K_p^--overlaid the edge uv is the only one missing in each subgraph $G[S_t^i]$, $i \in [8]$. Thus u_t^i has degree p in $G[H_t]$. Therefore every vertex of C_t is adjacent to all vertices of $X_t \cup \overline{X}_t$. A vertex z of C_t is in a d-degree-gadget with $d \geq 2p-5$ so it has at least $2p-5$ neighbors in this gadget. It is adjacent to at least $p-3$ vertices in C_t and the eight of $X_t \cup \overline{X}_t$. So it has degree at least $3p$, a contradiction.This proves (i)

(ii) Consider a vertex of C_t that is in a $(2p-6+i)$-degree-gadget. It has degree at least $(2p-6+i)$ in it its gadget and $p-1$ in C_t by (i). Hence it has at most $8-i$ neighbors in $X_t \cup \overline{X}_t$ and thus is non-adjacent to i vertices in $X_t \cup \overline{X}_t$; Hence if $p=3$ then the vertex of C_t is non-adjacent to four vertices in $X_t \cup \overline{X}_t$; if $p=4$, then two vertices of C_t are non-adjacent to two vertices in $X_t \cup \overline{X}_t$ each; if $p=5$, then the one vertex of C_t non-adjacent to two vertices in $X_t \cup \overline{X}_t$, and two other vertices are non-adjacent to one vertex in $X_t \cup \overline{X}_t$ each; if $p \geq 6$, then four vertices of C_t are non-adjacent to one vertex in $X_t \cup \overline{X}_t$ each. In all cases, there four non-edges between $X_t \cup \overline{X}_t$ and C_t. Now since every $G[S_t^i]$ has at most one non-edge, there are exactly four non-edges between $X_t \cup \overline{X}_t$ and C_t, and each vertex of X_t is incident to one of those non-edges, or each vertex of $\overline{X}_t$ is incident to one of those non-edges. This proves (ii). ♢

By Claim 14.1, we define a truth assignment ϕ by $\phi(x_t) = true$ (resp. $\phi(x_t) = false$) if th four non-edges between $X_t \cup \overline{X}_t$ and C_t are. incident to vertices of X_t (resp. $\overline{X}_t$). Observe that a literal vertex has degree $p-1$ (resp. p) in H_t if its corresponding literal is true (resp. false).

A clause vertex c_j is the identification of three literal vertices. Since it has degree at most $3p-1$, then at least one of those literal vertices has degree at most $p-1$ in its variable gadget. Thus this vertex corresponds to a true literal in the clause C_j. Therefore, ϕ satisfies Φ. □

4.3 Disjoint union of the complete bipartite graph $K_{a,a+1}$

In this section, we study on the family of disjoint union of the graph $K_{a,a+1}$. We aim to prove the following.

Theorem 15. *Let $rK_{a,a+1}$ be the disjoint union of r copies of $K_{a,a+1}$. Then* $\mathrm{np}(rK_{a,a+1}) \leq 3a+5$.

In order prove this theorem, we first prove Theorem 16 which show that $\mathrm{np}(K_{a,a+1}) \leq 3a+5$, and then deduce it using Lemma 17.

Theorem 16. $(\Delta \leq 3a+5)$-$K_{a,a+1}$-OVERLAY *is* NP*-complete.*

Proof. Reduction from (3,4)-SAT. Given a formula Φ of (3,4)-SAT with variables $x_t, t \in [n]$ and clauses $C_j, j \in [m]$, we build a hypergraph H as follows.

- For each variable x_t, we add a *variable gadget* H_t containing a set C_t^1 of size a, a set C_t^2 of size $a-1$ and a set U_t of eight vertices $U_t = \{u_t^1, \dots, u_t^8\}$, and eight hyperedges $S_t^i = C_t^1 \cup C_t^2 \cup \{u_t^i, u_t^{i+1}\}$ (superscripts are modulo 8) for $i \in [8]$. Set $X_t = \{u_t^1, u_t^3, u_t^5, u_t^7\}$ and $\overline{X}_t = \{u_t^2, u_t^4, u_t^6, u_t^8\}$. The vertices of X_t (resp. $\overline{X}_t$) are called the *non-negated literal vertices* (resp. *negated literal vertices*).
- For each clause $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$, we add a *clause vertex* c_j in which, for each literal ℓ_i which is the rth occurrence of the variable x_t, we identify u_t^{2r-1} (resp. u_t^{2r}) if $\ell_i = x_t$ (resp. $\ell_i = \bar{x}_t$).
- We add degree-gadgets on some vertices:
  - we add a $(2a+2)$-degree-gadget at each of vertices in C_t^1.
  - we add a $(2a+1)$-degree-gadget at each of vertices in C_t^2.

We will show that there is an assignment ϕ satisfying Φ if and only there is a $(K_{a,a+1}, H, 3a+5)$-graph G.

Assume that ϕ satisfies Φ, then we construct G as follows.
In a variable gadget H_t, for every $i \in [8]$, we let $G[S_t^i]$ be a copy of $K_{a,a+1}$ such that

- every vertex in C_t^1 is an a-vertex and each vertex in C_t^2 is an $(a+1)$-vertex (so $G[C_t]$ is $K_{a,a-1}$ with partition (C_t^1, C_t^2));

- if $\phi(x_t) = true$ (resp. $\phi(x_t) = false$, then each vertex in X_t (resp. $\overline{X}_t$) is an a-vertex in every hyperedge containing it and each vertex in $\overline{X}_t$ (resp. X_t) is an $(a+1)$-vertex in every hyperedge containing it.

For any d-degree-gadget D at a vertex v, let $G[D]$ be a $(K_{a,a+1}, D, 3a+5)$-graph in which v has degree d.

Let us check that $\Delta(G) \le 3a+5$.

- Each vertex in C_t^1 has degree $2a+2$ in its $(2a+2)$-degree-gadget. It is also adjacent the $a-1$ vertices in C_t^2, and to the four vertices of exactly one of the two sets $X_t, \overline{X}_t$. Thus, this vertex has degree $3a+5$ in G.
- Each vertex in C_t^2 has degree $2a+1$ in its $(2a+1)$-degree-gadget. It is also adjacent the a vertices in C_t^1 and to the four vertices in exactly one of sets $X_t, \overline{X}_t$. Hence, it has degree $3a+5$.
- Any literal vertex which is not identified in any clause vertex has degree at most $a+2$ in its variable gadget. So, it has degree $a+2 < 3a+5$ in G.
- Each clause vertex is the identification of three literal vertices. Each of those has degree either $a+1$ or $a+2$ in its variable gadget. Moreover at least one of the literals is true so its corresponding literal vertex has degree $a+1$ in its variable gadget. Therefore the clause vertex has degree at most $2(a+2)+a+1 = 3a+5$.
- Any vertex which is in a degree-gadget but in no variable gadget has degree at most $2(a+1) < 3a+5$ since it belongs to at most two hyperedges.

Hence, G is a $(K_{a,a+1}, H, 3a+5)$-graph.

Conversely, assume that G is a $(K_{a,a+1}, H, 3a+5)$-graph. For every hyperedge S of H, let F_S be a subgraph of $G[S]$ isomorphic to $K_{a,a+1}$. Free to remove some edges, we may assume that G is the union of the F_S over all hyperedges S of H. We have the following.

Claim 16.1. *For every $t \in [n]$, the following hold.*

(i) *In a hyperedge of H_t, the two literal vertices cannot be both a-vertices or both $(a+1)$-vertices.*

(ii) *In every hyperedge of H_t, the vertices in C_t^1 are a-vertices and the vertices in C_t^2 are $(a+1)$-vertices.*

(iii) *The vertices of one of the two sets $X_t, \overline{X}_t$ are a-vertices in all hyperedges of H_t to which they belong, and the vertices of the other of those sets are $(a+1)$-vertices in all hyperedges of H_t.*

Proof of Claim. Observe that any vertex in C_t^1 is in a $(2a+2)$-degree-gadget, so it has degree at most $a+3$ in $G[H_t]$. Similarly, any vertex in C_t^2 is in a $(2a+1)$-degree-gadget, so it has degree at most $a+4$ in $G[H_t]$.

(i) Assume for a contradiction that there is $i \in [8]$ such that u_t^i, u_t^{i+1} are both a-vertices in S_t^i. There are $a-1$ other a-vertices in S_t^i. Thus, at least one vertex v in C_t^1 is an $(a+1)$-vertex in S_t^i, and thus adjacent to u_t^i, u_t^{i+1} and the $a-1$ other a-vertices in S_t^i which are in $C_t^1 \cup C_t^2$.

Assume for a contradiction that v is adjacent to exactly $a-1$ vertices in $C_t^1 \cup C_t^2$. Then because v has degree at least a in every hyperedge, it must be adjacent to at least one literal vertex in each $S_t^{i'}$ for all $i' \in [8]$. In particular v is adjacent to at least one literal vertex in S_t^{i+2}, S_t^{i+4}, and S_t^{i+6}. Hence v has degree $a+4$ in $G[H_t]$, a contradiction to the above observation.

Consequently, v is adjacent to at least a and at most $a+1$ vertices in $C_t^1 \cup C_t^2$.

- If v is adjacent to exactly a vertices in $C_t^1 \cup C_t^2$, then there is a vertex u in $C_t^1 \setminus \{v\}$ which is adjacent to v since there are only $a-1$ vertices in C_t^2. Vertex u has degree at least a in S_t^i. Since v has degree $a+2$ in $S_t^i \cup C_t^1 \cup C_t^2$, it is adjacent to at most one vertex, among the six literal vertices u_t^{i+1+j}, $j \in [6]$. Hence there are two hyperedges S, S' in $\{S_t^{i+2}, S_t^{i+4}, S_t^{i+6}\}$ such that v is adjacent to no literal vertex of S and S'. Now in each of those two hyperedges, v has degree exactly a. Hence it must be an a-vertex, and each of its neighbors, including u, is an $(a+1)$-vertex and thus is adjacent to the two literal vertices. Hence u is adjacent to at least $a+4$ vertices in $G[H_t]$ (at least a in S_t^i plus the four literal vertices of S and S'). This is a contradiction.
- If v is adjacent to $a+1$ vertices in $C_t^1 \cup C_t^2$, then there are two vertices u, u' in $C_t^1 \setminus \{v\}$ which are adjacent to v and each of them has degree at least a in S_t^i. Since v has degree $a+2$ in $S_t^i \cup C_t^1 \cup C_t^2$, it is not adjacent to any of the six other literal vertices than u_t^i, u_t^{i+1}. Consider the three hyperedges $S_t^{i+2}, S_t^{i+4}, S_t^{i+6}$;
  - if v is an $(a+1)$-vertex in one of these hyperedges, then u, u' must be a-vertices and thus adjacent to the two literal vertices in this hyperedge.

– if v is an a-vertex in one of those hyperedges, then at least one of u, u' is an $(a+1)$-vertex in this hyperedge and is adjacent to its two literal vertex.

Thus at least one of u,u' is adjacent to at least four literal vertices in $S_t^{i+2} \cup S_t^{i+4} \cup S_t^{i+6}$, and so has degree at least $a+4$, a contradiction.

This proves that the two literal vertices of a hyperedge of H_t are not both a-vertices.

Let us now prove that the two literal vertices of a hyperedge of H_t cannot be both $(a+1)$-vertices. Assume for a contradiction that there is $i \in [8]$ such that u_t^i, u_t^{i+1} are both $(a+1)$-vertices. Any a-vertex x in S_t^i is adjacent to u_t^i, u_t^{i+1} and at least $a-2$ vertices in $C_t^1 \cup C_t^2$. If x is adjacent to exactly $a-2$ (resp. $a-1$) vertices in $C_t^1 \cup C_t^2$, then, since it has degree at least a in any hyperedge, it is adjacent to all six (resp. at least three) literal vertices in $S_t^{i+2} \cup S_t^{i+4} \cup S_t^{i+6}$. Thus x has degree $a+6$ (resp. $a+4$) in $G[H_t]$, a contradiction. Hence every a-vertex in S_t^i has at least a neighbors in $C_t^1 \cup C_t^2$.
There are $a+1$ a-vertices in S_t^i, so there must be one, say v, in C_t^1. It has degree at most $a+3$ in $G[H_t]$ and at least $a+2$ in S_t^i. Thus it is adjacent to at most one literal vertex in $S_t^{i+2} \cup S_t^{i+4} \cup S_t^{i+6}$. Hence v is not adjacent to the literal vertices of two hyperedges S, S' in $S_t^{i+2} \cup S_t^{i+4} \cup S_t^{i+6}$. Thus the literal vertices of the hyperedge S (resp. S') are both in a same part of F_S (resp. $F_{S'}$), and so they are $(a+1)$-vertices.
Now in each hyperedege of H_t, there are more a-vertices than $(a+1)$-vertices. Thus there is a vertex z which is an a-vertex in at least three hyperedges S_1, S_2, S_3 in $S_t^i \cup S_t^{i+2} \cup S_t^{i+4} \cup S_t^{i+6}$. In any of these hyperedges at least one of the literal vertices is an $(a+1)$-vertex, and in at least two of them the two literal vertices are $(a+1)$-vertices. Hence z is adjacent to at least five literal vertices. Moreover, as above, we can show that z has at least a neighours in $C_t^1 \cup C_t^2$. Thus z has degree at least $a+5$ in $G[H_t]$, a contradiction. This completes the proof of (i).

(ii) Assume for a contradiction that a vertex $w \in C_t^1$ is an $(a+1)$-vertex in S_t^i. By (i), w is adjacent to a literal vertex in S_t^i, and so it is adjacent to a other vertices in $C_t^1 \cup C_t^2$. Furthermore, by (i), in each hyperedge of H_t, w is adjacent to a literal vertex (either to an a-vertex or an $(a+1)$-vertex). Thus w is adjacent to four literal vertices in H_t, and so has degree at least $a+4$ in $G[H_t]$, a contradiction. Therefore the a vertices of C_t^1 are a-vertices. Moreover, by (i), one of the literal vertex of each S_t^i is an a-vertex. Therefore all vertices of C_t^2 must be $(a+1)$-vertices.

(iii) Let v be a vertex in C_t^1. It is an a-vertex in each S_t^i, so by (i) it is adjacent to one vertex in $\{u_t^i, u_t^{i+1}\}$ for all $i \in [8]$ and it is adjacent to the $a-1$ vertices of C_t^2. But v has degree at most $a+3$ in $G[H_t]$, so v is either adjacent to all vertices of X_t and non-adjacent to all vertices of $\overline{X}_t$, or non-adjacent to all vertices of X_t and adjacent to all vertices of $\overline{X}_t$. $\Diamond$

By Claim 16.1, we define a truth assignment ϕ by $\phi(x_t) = true$ (resp. $\phi(x_t) = false$) if all vertices in X_t are a-vertices (resp. $(a+1)$-vertices) in the hyperedges of H_t to which they belong.

Observe that, by Claim 16.1, if a literal vertex u_t^i is an $(a+1)$-vertex in the hyperedges of H_t to which it belongs then it has degree at least $a+2$ in $G[H_t]$ because it is adjacent to the a vertices of C_t^1 and the two literal vertices u_t^{i-1}, u_t^{i+1}.

A clause vertex c_j is the identification of three literal vertices. Since it has degree at most $3a+5$, then at least one of those literal vertices has degree at most $a+1$ in its variable gadget. By the above observation, this vertex is an a-vertex in the hyperedges of H_t to which it belongs. Thus this vertex corresponds to a true literal in the clause C_j. Therefore, ϕ satisfies Φ. $\square$

Lemma 17. *Let r be a positive integer. If $(\Delta \leq k)$-$K_{a,a+1}$-*Overlay *is* NP*-complete, then $(\Delta \leq k)$-$rK_{a,a+1}$-*Overlay *is* NP*-complete.*

Proof. $K_{a,a+1}$ has $a+1$ vertices of degree a and a vertices of degree $a+1$. Hence, in $rK_{a,a+1}$, there are $r(a+1)$ vertices of degree a and ra vertices of degree $a+1$.

We shall give a reduction from $(\Delta \leq k)$-$K_{a,a+1}$-Overlay to $(\Delta \leq k)$-$rK_{a,a+1}$-Overlay.
Let H be a $(2a+1)$-uniform hypergraph. We construct an $r(2a+1)$-uniform hypergraph H' from H as follows. We create a set A of $(r-1)(a+1)$ vertices, a set B of $(r-1)a$ vertices, and a set C of $2a+1$ vertices. We add the hyperedge $S_C = A \cup B \cup C$ to $E(H')$, and for every hyperedge S of H, we add the hyperedge $S' = S \cup A \cup B$ to $E(H')$. Finally, we add a $(k-a)$-degree-gadget at every vertex in A and a $(k-a-1)$-degree-gadget at every vertex in B.

Let us prove that there is a $(K_{a,a+1}, H, k)$-graph G if and only if there is an $(rK_{a,a+1}, H', k)$-graph G'.

Assume that G is a $(K_{a,a+1}, H, k)$-graph. We construct G' from G as follows. Let $G'[H] = G[H]$, so $G'[S] = G[S]$ for each $S \in E(H)$; let $G'[C]$ be a copy of $K_{a,a+1}$; let $G'[A \cup B]$ be a copy of $(r-1)K_{a,a+1}$ in which every vertex in A has degree a and every vertex in B has degree $a+1$; for each d-degree-gadget D at a vertex v, let $G'[D]$ be an $(rK_{a,a+1}, D, k)$-graph in which v has degree d. Clearly, for any $S' \in E(H')$, $G'[S']$ contains $rK_{a,a+1}$ and so does $G'[S_C]$. Moreover, one easily checks that every vertex of G' has degree at most k. Therefore, G' is an $(rK_{a,a+1}, H', k)$-graph.

Assume now that there is an $(rK_{a,a+1}, H', k)$-graph G'. Every vertex $v \in A$ is in a $(k-a)$-degree-gadget, so it has degree at most a in $G'[V(H) \cup A \cup B \cup C]$. Thus it must be an a-vertex in every hyperedge S' for all $S \in E(H)$.

Let v be a vertex in A. It is adjacent to $a-i$ vertices in B. Then v must be adjacent to at least i vertices in C and i vertices in $V(H)$. Thus the degree of v is at least $a+i$ in $G'[V(H) \cup A \cup B \cup C]$. Therefore $i = 0$, so v is adjacent to a vertices in B and no vertex in $V(H) \cup C$.

This implies that there are $(d-1)a(a+1)$ edges between A and B. But every vertex $u \in B$ is in a $(k-a-1)$-degree-gadget, and so has degree at most $a+1$ in $G'[V(H) \cup A \cup B \cup C]$. Thus, each vertex in B has $a+1$ neighbors in A, and is adjacent to vertex in $V(H) \cup C$.

Consider now a hyperedge $S' = S \cup A \cup B$. The graph $G'[S']$ contains $rK_{a,a+1}$. Since there is no edge between $A \cup B$ and $V(H)$, necessarily $G'[S]$ contains $K_{a,a+1}$. So S is $K_{a,a+1}$-overlaid by G'. Consequently, $G = G'[V(H)]$ is a $(K_{a,a+1}, H, k)$-graph. □

5 Proof of Theorem 5

The aim of this section is to prove Theorem 5. The proof divides into four cases, Theorem 13, Theorem 18, Theorem 19 and Theorem 20 as follows.

Proof of Theorem 5 (assuming Theorems 18, 19 and 20). Let F be a graph with degree values $1 \le \delta(F) = \lambda_1 < \cdots < \lambda_t = \Delta(F)$.

If $t = 1$, (i.e. F is regular), then we have the result by Theorem 13. Henceforth, we may assume that $t \ge 2$.

If there exists $i \in [t-1]$ such that $\lambda_{i+1} > \lambda_i + 1$, then Theorem 18 yields the result. Henceforth, we may assume that $\lambda_{i+1} = \lambda_i + 1$ for all $i \in [t-1]$.

If $t \ge 3$, then $\lambda_t + \lambda_1 \ge 2\lambda_2$ and Theorem 19 yields the result. Henceforth, we may assume that $t = 2$ which we then have the result by Theorem 20. □

It thus remains to prove Theorems 18, 19 and 20.

The proofs of the first two are reductions from 3-Colorability on 4-regular graphs which are similar to the one used to prove Theorem 9. Given a 4-regular graph G, we build a hypergraph H which includes, for each vertex $v \in V(G)$, a *vertex gadget* with three hyperedges which makes three choices of degrees on vertices c_v^1, c_v^2, c_v^3 (as three colors labeled 1, 2, 3 of vertex v) and a *color gadget* represented as a binary tree with 4 leaves which copies each choice to four (leaves) vertices in other hyperedges (with respect to four neighbors of $v \in V(G)$). For any edge uv, we simply identifies the two leaves of u, v. The idea is that for a proper coloring c of G, $c(v)$ corresponds to a vertex c_v^i having a certain degree d; then $c(v) = i$ if and only if c_v^i as degree d in its vertex gadget (see Figure 5). However, the set of hyperedges which are in a color gadget of the two theorems are different, see Figure 6 in Theorem 18 and Figure 7 in Theorem 19.

Theorem 18. *Let F be a graph on p vertices with degree values $1 \le \lambda_1 < \cdots < \lambda_t$. If there exists $i^* \in \{2, \ldots, t\}$ such that $\lambda_{i^*} \ge \lambda_{i^*-1} + 2$, then there is k such that $(\Delta \le k)$-F-Overlay is* NP*-complete.*

Proof. Set $k = \max\{2\lambda_t, 2\lambda_{i^*} + \lambda_{i^*-1} + \lambda_1\}$. We give a reduction from 3-Colorability on 4-regular graphs. Given a 4-regular graph G, we build a hypergraph H as follows.

- For each vertex $v \in V(G)$, we create a *vertex gadget* H_v with three hyperedges $S_v^i = \{c_v, c_v^i\} \cup X_v^i \cup Y_v^i$ for $i \in [3]$ where $|X_v^i| = \sum_{j=1}^{i^*-1} \alpha_j - 1$, $|Y_v^i| = p - |X_v^i| - 2$. We add a $(k - \lambda_{i^*} + 1)$-degree-gadget at each vertex $x \in X_v^i$ for $i \in [3]$, a $(k - 2\lambda_{i^*} - \lambda_{i^*-1})$-degree-gadget at c_v. We say that S_v^i is the *parent hyperedge* of c_v^i for each $i \in [3]$.

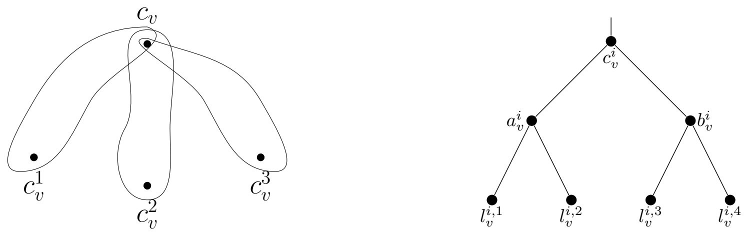

Figure 5: **The construction of the reduction.** The vertex gadget for vertex v (left) and the binary tree representing the color gagdet (right). In the construction, each edge of this tree is replaced by hyperedges such that the degree of the root c_v^i is transmitted to all its descendants.

- For each vertex v and each $i \in [3]$, we construct a *color gadget* H_v^i for $i \in [3]$ as follows.
  - We create a binary tree T_v^i with vertex set $\{c_v^i, a_v^i, b_v^i, \ell_v^{i,1}, \ell_v^{i,2}, \ell_v^{i,3}, \ell_v^{i,4}\}$ and edge set $\{c_v^i a_v^i, c_v^i b_v^i, a_v^i \ell_v^{i,1}, a_v^i \ell_v^{i,2}, b_v^i \ell_v^{i,3}, b_v^i \ell_v^{i,4}\}$, rooted at c_v^i. In this tree, a_v^i and b_v^i are the children of c_v^i, $\ell_v^{i,1}$ and $\ell_v^{i,2}$ are the children of a_v^i, and $\ell_v^{i,3}$ and $\ell_v^{i,4}$ are the children of b_v^i.
  - For any vertex $x \in \{c_v^i, a_v^i, b_v^i\}$, let y_1, y_2 be its children in T_v^i, and let $e_1 = xy_1, e_2 = xy_2$. We first add a $(k - 2\lambda_{i^*} + 1)$-degree-gadget at x. Then we construct an *x-edge-gadget* as follows: we add a set A_x of $\sum_{j=1}^{i^*-1} \alpha_j - 1$ new vertices and a set B_x of $p - |A_x| - 2$ new vertices, the hyperedges $S(e_1) = \{x, y_1\} \cup A_x \cup B_x$ and $S(e_2) = \{x, y_2\} \cup A_x \cup B_x$, and a $(k - \lambda_{i^*} + 1)$-degree-gadget at every vertex $a \in A_x$. For convenience, we say that $S(xy_1)$ (resp. $S(xy_2)$) is the parent hyperedge of y_1 (resp. y_2). Moreover, for any leaf $\ell_v^{i,j}$, we denote by $S_v^{i,j}$ the hyperedge containing the vertex $\ell_v^{i,j}$. See Figure 6.
- For every vertex $v \in V(G)$, let $e_v^1, e_v^2, e_v^3, e_v^4$, be an ordering of the edges incident to v. For each edge $uv \in E(G)$, let j_u and j_v be the indices such that $uv = e_u^{j_u} = e_v^{j_v}$. Then, for all $i \in [3]$, we identify the vertices ℓ_u^{i,j_u} and ℓ_v^{i,j_v} and we add a $(k - \lambda_{i^*} - \lambda_1)$-degree-vertex at this vertex.

Note that each of the d-degree-gadgets exists because we have $d \geq \lambda_1$ by our choice of k.

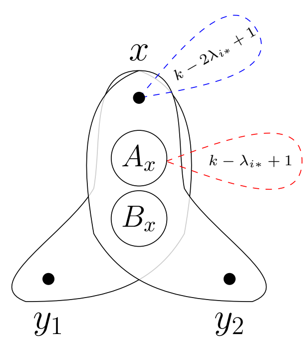

Figure 6: The x-edge-gadget with degree-gadgets at x and every vertex in A_x.

Let us now prove that there is a proper 3-coloring of G if and only if there is an (F, H, k)-graph G^*.

Assume first that there is an (F, H, k)-graph G^*.

Let $v \in V(G)$. The vertex c_v has degree at least $(k - 2\lambda_{i^*} - \lambda_{i^*-1})$ in its $(k - 2\lambda_{i^*} - \lambda_{i^*-1})$-degree-gadget. Hence c_v has degree at most $2\lambda_{i^*} + \lambda_{i^*-1}$ in $S_v^1 \cup S_v^2 \cup S_v^3$. But those hyperedges pairwise intersect in $\{c_v\}$. Thus there is $i \in [3]$ such that c_v has degree less than λ_{i^*} in S_v^i. Moreover, since any vertex $x \in X_v^i$ has degree at least $k - \lambda_{i^*} + 1$ in its $(k - \lambda_{i^*} + 1)$-degree-gadget, so it has degree less than λ_{i^*} in S_v^i. Thus c_v^i must have degree at least λ_{i^*} in S_v^i. Therefore, we can define a 3-coloring ϕ by $\phi(v) = i$ where i is an index such that c_v^i has degree at least λ_{i^*} in S_v^i.

Let us now prove that ϕ is proper. We need the following claim.

Claim 18.1. *Let $v \in V(G)$ and $i \in [3]$. If c_v^i has degree at least λ_{i^*} in S_v^i, then so does the leaf $\ell_v^{i,j}$ in $S_v^{i,j}$ for all $j \in [4]$.*

Proof of Claim. It suffices to prove that for any $x \in \{c_v^i, b_v^i, a_v^i\}$, if x has degree at least λ_{i^*} in its parent hyperedge, then both y_1, y_2 have degree at least λ_{i^*} in their parent hyperedges.

Assume that x has degree at least λ_{i^*} in its parent hyperedge. Since x has degree at least $k - 2\lambda_{i^*} + 1$ in its $(k - 2\lambda_{i^*} + 1)$-degree-gadget, it has degree at most $\lambda_{i^*} - 1$ in $S(xy_1) \cup S(xy_2)$. Moreover, any $a \in A_x$ has degree at least $k - \lambda_{i^*} + 1$ in its $(k - \lambda_{i^*} + 1)$-degree-gadget and so has degree less than λ_{i^*} in $S(xy_1) \cup S(xy_2)$ and so in each of $S(xy_1), S(xy_2)$. Since A_x is of size $\sum_{j=1}^{i^*-1} \alpha_1 - 1$, the vertex y_1 (resp. y_2) must have degree at least λ_{i^*} in $S(xy_1)$ (resp. $S(xy_2)$). $\Diamond$

Consider an edge $uv \in E(G)$, $i \in [3]$. The vertex $\ell = \ell_u^{i,j_u} = \ell_v^{i,j_v}$ has degree at least $k - \lambda_{i^*} - \lambda_1$ in its $(k - \lambda_{i^*} - \lambda_1)$-degree-gadget and is the unique common vertex of the hyperedges S_u^{i,j_u} and S_v^{i,j_v}. Therefore it has degree λ_{i^*} in at most one of S_u^{i,j_u}, S_v^{i,j_v}. Hence, by the Claim 18.1, at most one of c_u^i, c_v^i has degree λ_{i^*} in its parent hyperedge. Thus at most one of u, v is colored i by ϕ. Therefore, ϕ is a proper 3-coloring of G.

Assume now that ϕ is a proper 3-coloring of G. We construct a graph G^* as follows.

- For any vertex gadget H_v, $i \in [3]$, let $G^*[S_v^i]$ be a copy of F in which every vertex in X_v^i has degree at most λ_{i^*-1}, every vertex in Y_v^i has degree at least λ_{i^*}, and c_v has degree λ_{i^*-1} (resp. λ_{i^*}) and c_v^i has degree λ_1 (resp. λ_{i^*}) in S_v^i if $\phi(v) = i$ (resp. $\phi(v) \neq i$).
- In every color gadget H_v^i, for $x \in \{c_v^i, b_v^i, a_v^i\}$ with children y_1 and y_2, let $G^*[S(xy_1)]$ and $G^*[S(xy_2)])$ be two similar copies of F such that:
  - if $i \neq \phi(v)$, then x has degree λ_{i^*} in $G^*[S(xy_1)]$ and $G^*[S(xy_2)]$ (and so at most $\lambda_{i^*} + 1$ in $G^*[S(xy_1) \cup S(xy_2)]$) and y_1 and y_2 have degree λ_1 in $G^*[S(xy_1)]$ and $G^*[S(xy_2)]$ respectively.
  - if $i = \phi(v)$, then x has degree λ_{i^*-1} in $G^*[S(xy_1)]$ and $G^*[S(xy_2)]$ (and so at most $\lambda_{i^*-1} + 1$ in $G^*[S(xy_1) \cup S(xy_2)]$) and y_1 and y_2 have degree λ_{i^*} in $G^*[S(xy_1)]$ and $G^*[S(xy_2)]$ respectively.
  - every vertex in A_x has degree at most λ_{i^*-1} in both $G^*[S(xy_1)]$ and $G^*[S(xy_2)]$ and so at most $\lambda_{i^*} + 1$ in $G^[S(xy_1) \cup S(xy_2)]$;
  - every vertex in B_x is degree at least λ_{i^*} in both $G^*[S(xy_1)]$ and $G^*[S(xy_2)]$ and so at most $\lambda_t + 1$ in $G^*[S(xy_1) \cup S(xy_2)]$;
- For any d-degree-gadget D at vertex v, we let $G^*[D]$ be an (F, D, k)-graph in which v has degree d.

By construction, G^* F-overlays H. Let us check that $\Delta(G^*) \leq k$. Let u be a vertex of G^*.

- If u is in at most two hyperedges (in particular, if $=$ u is in Y_v^i or u is in B_x for x internal vertex in some T_v^i or u is only in a d-degree-gadget), then u has degree at most $2\lambda_t \leq k$.
- If $u \in X_v^i$ for $v \in V(G)$, then u has degree $k - \lambda_{i^*} + 1$ in its $(k - \lambda_{i^*} + 1)$-degree-gadget and at most λ_{i^*-1} in $S_v^i]$, thus u has degree at most $k - \lambda_{i^*} + \lambda_{i^*-1} + 1 \leq k$.
- If $u \in A_x$ for $v \in V(G)$ and x internal vertex in some tree T_v^i, then u has degree $k - \lambda_{i^*} + 1$ in its $(k - \lambda_{i^*} + 1)$-degree-gadget and at most $\lambda_{i^*-1} + 1$ in $G^*[S(xy_1) \cup S(xy_1)]$, thus u has degree at most $k - \lambda_{i^*} + \lambda_{i^*-1} + 2 \leq k$.
- For $u \in \{c_v^i, a_v^i, b_v^i\}$ for $i \in [3]$ with u parent of y_1, y_2, it has degree $k - 2\lambda_{i^*} + 1$ in its $(k - 2\lambda_{i^*} + 1)$-degree-gadget. And if $i = \phi(v)$ (resp. $i \neq \phi(v)$), then u has degree λ_{i^*} (resp. λ_1) in its parent hyperedge and $\lambda_{i^*-1} + 1$ (resp. $\lambda_{i^*} + 1$) in $G^*[S_{uy_1}^1 \cup S_{uy_2}^1]$. Hence u has degree at most $k - \lambda_{i^*} + \lambda_{i^*-1} + 2 \leq k$.
- Assume that u is the identification of ℓ_v^{i,j_v} and ℓ_w^{i,i_w} for an edge $vw \in E(G)$. First, u has degree $k - \lambda_{i^*} - \lambda_1$ in its $(k - \lambda_{i^*} - \lambda_1)$-degree-gadget. Moreover, since either $\phi(v) \neq i$ or $\phi(w) \neq i$, then u has degree λ_1n one of S_v^{i,j_v}, S_w^{i,j_w} and at most λ_{i^*} in the other. Therefore, u has degree at most k in G^*.

Consequently, G^* is an (F, H, k)-graph. $\square$

Theorem 19. *Let a graph F on p vertices with degree sequence $\mathbf{d} = (d_1, \ldots, d_p)$ such that $\lambda_t + \lambda_1 \geq 2\lambda_2$. Then there exists k such that $(\Delta \leq k)$-F-Overlay is NP-complete.*

Proof. Observe that the condition $\lambda_t + \lambda_1 \geq 2\lambda_2$ implies $t \geq 3$. Set $k = 2\lambda_t + \lambda_{t-1}$.
We give a reduction from 3-Colorability on 4-regular graphs.

Given a 4-regular graph G, we build a p-uniform hypergraph H as follows.

- For each vertex $v \in V(G)$, we create a *vertex gadget* H_v with three hyperedges $S_v^i = \{c_v, c_v^i\} \cup X_v^i \cup Y_v^i$ for $i \in [3]$ where $|X_v^i| = \sum_{i=1}^{t-1} \alpha_i - 1$, $|Y_v^i| = p - \alpha_t - 1$. For $i \in [3]$, we add a $(k - \lambda_{t-1})$-degree-gadget at each vertex $x \in X_v^i$
  We say that S_v^i is the parent hyperedge of each c_v^i, $i \in [3]$.
- For each vertex $v \in V(G)$ and each $i \in [3]$, we construct a *color gadget* H_v^i for $i \in [3]$ as follows.
  - We create a binary tree T_v^i with vertex set $\{c_v^i, a_v^i, b_v^i, \ell_v^{i,1}, \ell_v^{i,2}, \ell_v^{i,3}, \ell_v^{i,4}\}$ and edge set $\{c_v^i a_v^i, c_v^i b_v^i, a_v^i \ell_v^{i,1}, a_v^i \ell_v^{i,2}, b_v^i \ell_v^{i,3}, b_v^i \ell_v^{i,4}\}$, rooted at c_v^i. In this tree, a_v^i and b_v^i are the children of c_v^i, $\ell_v^{i,1}$ and $\ell_v^{i,2}$ are the children of a_v^i, and $\ell_v^{i,3}$ and $\ell_v^{i,4}$ are the children of b_v^i.
  - For each edge $e = xy$ of T_v^i with x the parent of y in T_v^i, we construct an *edge-gadget* containing x, y, a new vertex z_e, and four disjoint sets $U_e^1 W_e^1, U_e^2, W_e^2$ of new vertices, U_e^1 of size $\alpha_1 - 1$, W_e^1 of size $p - |U_e^1| - 1$, U_e^2 of size $p - \alpha_t - 1$, W_e^2 of size $\alpha_t - 1$. We add the hyperedges $S_e^1 = \{x, z_e\} \cup U_e^1 \cup W_e^1$ and $S_e^2 = \{z_e, y\} \cup U_e^2 \cup W_e^2$. See Figure 7. We finally add a $(k - \lambda_t - 2\lambda_1)$-degree-gadget at x, a $(k - \lambda_1)$-degree-gadget at each vertex of U_e^1, a $(k - \lambda_t + 1)$-degree-gadget at each of U_e^2, and a $(k - \lambda_2 - \lambda_t + 1)$-degree-gadget pendant at z_e.
- For every vertex $v \in V(G)$, let $e_v^1, e_v^2, e_v^3, e_v^4$, be an ordering of the edges incident to v. For each edge $uv \in E(G)$, let j_u and j_v be the indices such that $uv = e_u^{j_u} = e_v^{j_v}$. Then, for all $i \in [3]$, we identify the vertices ℓ_u^{i,j_u} and ℓ_v^{i,j_v} and we add a $(k - 2\lambda_t + 1)$-degree-gadget at this vertex.

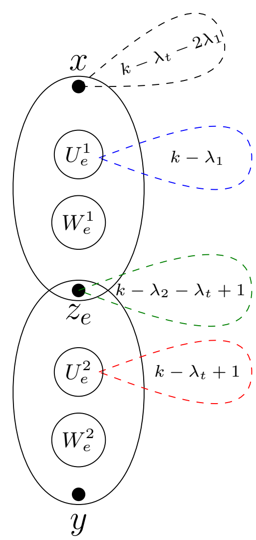

Figure 7: The edge-gadget for an edge $e = xy$ with degree-gadgets at x, z_e and every vertex in U_e^1, U_e^2.

Let us now prove that there is a proper 3-coloring of G if and only if there is an (F, H, k)-graph G^*.

Assume first that there is an (F, H, k)-graph G^*.

Let $v \in V(G)$. The vertex c_v has degree at most $2\lambda_t + \lambda_{t-1}$ in $S_v^1 \cup S_v^2 \cup S_v^3$. But those hyperedges pairwise intersect in $\{c_v\}$. Thus there is $i \in [3]$ such that c_v has degree less than λ_t in S_v^i.

Moreover, each vertex $x \in X_v^i$ has degree at least $k - \lambda_{t-1}$ in its $(k - \lambda_{t-1})$-degree-gadget, and so at most λ_{t-1} in S_v^i. Together with c_v, there are $\sum_{i=1}^{t-1} \alpha_t$ vertices of degree at most λ_{t-1} in S_v^i. Thus c_v^i have degree λ_t in its parent hyperedge S_v^i. Therefore, we can define a 3-coloring ϕ by $\phi(v) = i$ where i is an index such that c_v^i has degree λ_t in S_v^i.

Let us now prove that ϕ is proper. We need the following claim.

Claim 19.1. *Let $v \in V(G)$ and $i \in [3]$. If c_v^i has degree λ_t in S_v^i, then so does any leaf $\ell_v^{i,j}$ in $S_v^{i,j}$ for $j \in [4]$.*

Proof of Claim. It suffices to prove that for any $x \in \{c_v^i, b_v^i, a_v^i\}$, if x has degree λ_t in its parent hyperedge, then both y_1, y_2 have degree λ_t in their parent hyperedges.

Assume that x is a λ_t-vertex in its parent hyperedge. Since x has degree at least $k - \lambda_t - 2\lambda_1$ in its $(k - \lambda_t - 2\lambda_1)$-degree-gadget, and degree λ_t in its parent hyperedge, it has has degree at most $2\lambda_1$ in $S^1_{xy_1} \cup S^1_{xy_2}$, and so λ_1 in each of $S^1_{xy_1}, S^1_{xy_2}$. Let $e = xy$ be one of the two edges xy_1, xy_2. Any vertex in U^1_e has degree at least $k - \lambda_1$ in its $(k - \lambda_1)$-degree-gadget, and thus λ_1 in S^1_e. It implies that z_e has degree at least λ_2 in S^1_e. Since it is also in a $(k - \lambda_2 - \lambda_t + 1)$-degree-gadget, z_e has degree less than λ_t in S^2_e. Moreover, any vertex in U^2_e is in a $(k - \lambda_t + 1)$-degree-gadget, then none of them has degree λ_t in S^2_e except those in W^2_e which is of size $\alpha_t - 1$. Thus, y must have degree λ_t in S^2_e. ◊

Consider an edge $uv \in E(G)$, $i \in [3]$. The vertex $\ell = \ell^{i,j_u}_u = \ell^{i,j_v}_v$ has degree at least $k - 2\lambda_t + 1$ in its $(k - 2\lambda_t + 1)$-degree-gadget and is the unique common vertex of the hyperedges S^{i,j_u}_u and S^{i,j_v}_v. Therefore it has degree λ_t in at most one of S^{i,j_u}_u and $G^* S^{i,j_v}_v$. Hence, by Claim 19.1, at most one of c^i_u, c^i_v has degree λ_t in its parent hyperedge. Thus at most one of u, v is colored i by ϕ. Therefore, ϕ is a proper 3-coloring of G.

Assume now that ϕ is a proper 3-coloring of G. We construct a graph G^* as follows.

- For any vertex gadget H_v, $i \in [3]$, let $G^*[S^i_v]$ be a copy of F in which every vertex in X^i_v has degree at most λ_{t-1}, every vertex in Y^i_v has degree λ_t, and c_v has degree λ_{t-1} (resp. λ_t) and c^i_v has degree λ_t (resp. λ_1) if $\phi(v) = i$ (resp. $\phi(v) \neq i$).
- In every color gadget H^i_v, for each edge $e = xy$ of T^i_v with x parent of y, let $G^*[S^1_e], G^*[S^2_e]$ be copies of F such that:
  - every vertex in U^1_e has degree λ_1;
  - every vertex in U^2_e has degree at most λ_{t-1};
  - every vertex in W^2_e has degree λ_t;
  - if $i = \phi(v)$, then x has degree λ_1 in S^1_e, z_e has degree λ_2 in S^1_e and λ_{t-1} in S^2_e, and y has degree λ_t in S^2_e;
  - if $i \neq \phi(v)$, then x has degree λ_2 in S^1_e, z_e has degree λ_1 in S^1_e and λ_t in S^2_e, and y has degree λ_1 in S^2_e.
- For any d-degree-gadget D at vertex v, we let $G^*[D]$ be an (F, D, k)-graph in which v has degree d.

By construction, G^* F-overlays H. Let us check that $\Delta(G^*) \leq k$. Let u be a vertex of G^*.

- If u is in at most two hyperedges, (in particular if u is in Y^i_v, in $W^1_e \cup W^2_e$ in an edge-gadget or only in a degree-gadget), then u has degree at most $2\lambda_t < k$ in G^*.
- If $u = c_v$, then it has degree λ_{t-1} in S^i_v for the index $i = \phi(v)$, and λ_t in S^i_v for the two indices $i \neq \phi(v)$. Hence c_v has degree $2\lambda_1 + \lambda_{t-1} = k$.
- If $u \in X^i_v$ for $v \in V(G)$, then u has degree $k - \lambda_{t-1}$ in its $(k - \lambda_{t-1})$-degree-gadget and at most λ_{t-1} in S^i_v, thus u has degree at most k in G^*.
- If $u \in U^1_e$ for some edge e of T^i_v, then u has degree $k - \lambda_1$ in its degree-gadget and λ_1 in S^1_e, thus u has degree k in G^*.
- If $u \in U^2_e$, then u has degree $k - \lambda_t + 1$ in its degree-gadget and at most λ_{t-1} in S^2_e, thus u has degree at most k in G^*.
- If $u \in \{c^i_v, a^i_v, b^i_v\}$ for $i \in [3]$ with children y_1, y_2, then u has degree $k - \lambda_t - 2\lambda_1$ in its $(k - \lambda_t - 2\lambda_1)$-degree-gadget. Moreover, if $i = \phi(v)$ (resp. $i \neq \phi(v)$), then u has degree λ_t (resp. λ_1) in its parent hyperedge and λ_1 (resp. λ_2) in both $S^1_{uy_1}, S^1_{uy_2}$. Hence u has degree at most $k - \lambda_t - 2\lambda_1 + \lambda_t + 2\lambda_1 = k$ (resp. $k - \lambda_t - 2\lambda_1 + \lambda_1 + 2\lambda_2 \leq k$ by the assumption $\lambda_t + \lambda_1 \geq 2\lambda_2$) in G^*.
- If $u = z_e$ for some edge e of T^i_v, then u has degree $k - \lambda_2 - \lambda_t + 1$ in its $(k - \lambda_2 - \lambda_t + 1)$-degree-gadget. Moreover, if $i = \phi(v)$ (resp. $i \neq \phi(v)$), then u has degree λ_2 (resp. λ_1) in S^1_e and λ_{t-1} (resp. λ_t) in S^2_e. Hence, u has degree at most $k - \lambda_2 - \lambda_t + 1 + \lambda_2 + \lambda_{t-1} \leq k$ (resp. $k - \lambda_2 - \lambda_t + 1 + \lambda_1 + \lambda_t \leq k$) in G^*.
- Assume that u is the identification of ℓ^{i,j_v}_v and ℓ^{i,i_w}_w for an edge $vw \in E(G)$. First, u has degree $k - 2\lambda_t + 1$ in its $(k - 2\lambda_t + 1)$-degree-gadget. Moreover, since either $\phi(v) \neq i$ or $\phi(w) \neq i$, then u has degree less than λ_t in one of S^{i,j_v}_v, S^{i,j_w}_w. Therefore, u has degree at most k in G^*.

Consequently, G^* is an (F, H, k)-graph. □

Theorem 20. *Let F be a graph with α_1 vertices of positive degree λ_1 and $\alpha_2 = p - \alpha_1$ vertices of degree $\lambda_2 = \lambda_1 + 1$. Then $(\Delta \leq k)$-F-*Overlay *is* NP*-complete for some k.*

There are several cases in the proof, depending on the structure of graph F. In each case, we give a reduction from (3,4)-SAT problem, which follows the same general idea as the proof of Theorem 13 : we construct variable gadgets H_t containing some negated and non-negated *literal vertices* and identify some of them in such a way that for an assignment ϕ satisfying Φ, $\phi(x_t) = true$ (resp. $false$) if and only if non-negated (resp. negated) literal vertices in the variable gadget are adjacent to w_t in an (F, H, k)-graph.

Lemma 21. *Let F be a graph on p vertices with α_1 vertices of degree λ_1 and $\alpha_2 = p - \alpha_1$ vertices of degree $\lambda_2 > \lambda_1$ such that $F[V_{\lambda_2}]$ is μ-regular but neither complete nor anticomplete. Then there exists k such that $(\Delta \leq k)$-F-OVERLAY is NP-complete.*

Proof. Set $\gamma = \lambda_2 - \mu$ and $k = \max\{4\gamma(\alpha_2 - 1) + 4\mu + \lambda_1, 3\gamma(\alpha_2 - 1) + 6\mu - 1 + \lambda_1\}$.

We give a reduction from (3,4)-SAT. Given a formula Φ of (3,4)-SAT with variables $x_t, t \in [n]$ and clauses $C_j, j \in [m]$, we construct a hypergraph H as follows.

1. For each variable x_t, we construct a *variable gadget* H_t in the following way.
   We first create a *center vertex* w_t, a set of $4(\alpha_2 - 1)$ vertices $U_t = \{u_t^1, \ldots, u_t^{4(\alpha_2-1)}\}$, and for each $i \in [4(\alpha_2 - 1)]$, create a set of α_1 new vertices W_t^i, and a hyperedge $S_t^i = W_t^i \cup \{w_t, u_t^i, \ldots, u_t^{i+\alpha_2-2}\}$ (superscripts are modulo $4(\alpha_2 - 1)$).
   For $r \in [4]$, let $x_t^r = u_t^{r(\alpha_2-1)-\alpha_2+2}$ and $\bar{x}_t^r = u_t^{r(\alpha_2-1)-\alpha_2+3}$. Set $X_t = \{x_t^1, x_t^2, x_t^3, x_t^4\}$ and $\overline{X}_t = \{\bar{x}_t^1, \bar{x}_t^2, \bar{x}_t^3, \bar{x}_t^4\}$. The vertices of X_t (resp. $\overline{X}_t$) are called the *non-negated* (resp. *negated) literal vertices* of H_t.
2. For each clause $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$, we identify y_1, y_2, y_3 into a *clause vertex* c_j, where $y_i = x_t^r$ if $\ell_i = x_t$ and ℓ_i is the r-th occurrence of x_t, and $y_i = \bar{x}_t^r$ if $\ell_i = \bar{x}_t$ and is the r-th occurrence of x_t.
3. Finally, we add degree-gadgets on some vertices.
   - We add a $(k - 4\gamma(\alpha_2 - 1) - 4\mu)$-degree gadget on vertex w_t.
   - We add a $(k - \lambda_1)$-degree gadget at every vertex in W_t^i for all $i \in [4(\alpha_2 - 1)]$.
   - For $i \in [\mu - 1]$, we add a $(k - \gamma(\alpha_2 - 1) - 2\mu + 1)$-degree-gadget on each $u_t^{(\alpha_2-1)r-j}$ for $r \in [4]$.

Observe that every vertex in W_t^i, for $i \in [4(\alpha_2-1)]$, has degree at least $k-\lambda_1$ in its $(k-\lambda_1)$-degree-gadget, and so degree at most λ_1 in S_t^i. Thus, each of those vertices must have degree λ_1 in S_t^i. It implies that all the other vertices must have degree at least λ_2 in any hyperedge of H_t. In particular, w_t has degree λ_2 in any hyperedge of H_t. Since w_t is in a $(k - 4\gamma(\alpha_2 - 1) - 4\mu)$-degree-gadget and is adjacent to γ vertices in every W_t^i for $i \in [4(\alpha_2 - 1)]$, it has degree at most 4μ in $\bigcup_{i=1}^{4(\alpha_2-1)} S_t^i \setminus W_t^i$.

Moreover, each vertex $u_t^i \in U_t$ is a λ_2-vertex in every hyperedge $S_t^{i'}$ of H_t containing it, and so adjacent to γ vertices in $W_t^{i'}$. Since u_t^i belongs to $\alpha_2 - 1$ hyperedges of H_t, thus u_t^i is adjacent to $\gamma(\alpha_2 - 1)$ vertices in $\bigcup_{i'=1}^{4(\alpha_2-1)} W_t^{i'}$. For $i \in [\mu - 1]$, $u^{(\alpha_2-1)r-i}$ is in a $(k - \gamma(\alpha_2 - 1) - 2\mu + 1)$-degree-gadget, then it has degree at most $\gamma(\alpha_2 - 1) + 2\mu - 1$ in H_t, and so at most $2\mu - 1$ in $\bigcup_{i'=1}^{4(\alpha_2-1)} S_t^{i'}$. Moreover, $F[V_{\lambda_2}]$ is μ-regular (but not complete or anticomplete). The following is then similar to the proof of Theorem 13 for $F[V_{\lambda_2}]$. So we just sketch it.

Assume that there exists a truth assignment ϕ satisfying Φ. Let G be the graph obtained as follows.

We let $(v_0, v_1, \ldots, v_{\alpha_2-1})$ be an ordering of V_{λ_2} such that $N_F(v_0) = \{v_{\alpha_2-\mu+1}, \ldots, v_{\alpha_2-1}\} \cup \{v_1\}$ if $\phi(x_t) = true$ and such that $N_F(v_0) = \{v_{\alpha_2-\mu+1}, \ldots, v_{\alpha_2-1}\} \cup \{v_2\}$ if $\phi(x_t) = false$). For every $i \in [4\alpha_2 - 4]$, we let $G[S_t^i]$ be the copy of F in which every vertex in W_t^i is a λ_1-vertex, w_t corresponds to v_0 and $u_t^{i'}$ for $i' \in \{i, \ldots, i + \alpha_2 - 1\}$ corresponds to the vertex $v_{i''}$ such that $i' \equiv i'' \mod \alpha_2 - 1$. In addition, for every d-degree-gadget D at some vertex v, we let $G[D]$ be an (F, D, k)-graph in which v has degree d.

The graph G F-overlays H and one can check that $\Delta(F) \leq k$.

Conversely, assume that G is an (F, H, k)-graph. One can prove the following claim.

Claim 21.1. *For every $t \in [n]$ the following hold.*

(a) *Every vertex in W_t^i for $i \in [4\alpha_2 - 4]$ is a λ_1-vertex in S_t^i.*

(b) *w_t is a λ_2-vertex in every hyperedges of H_t. Furthermore, it is adjacent to γ vertices in each W_t^i and the vertices $u_t^{(\alpha_2-1)r-i}$ for $r \in [4]$, $i \in [\mu - 1]$.*

Therefore the truth assignment ϕ defined by $\phi(x_t) = true$ (resp. $false$) if w_t is adjacent to X_t (resp. $\overline{X}_t$), satisfies Φ. ☐

Proof of Theorem 20. Let V_d be the set of vertices of degree d in F and $\mathbf{d} = (d_1, \ldots, d_p)$ be the non-decreasing degree sequence of F. Let N_s be the set of vertices of V_{λ_2} having exactly s neighbors in V_{λ_1}, and let $N_{\geq s} = \bigcup_{s' \geq s} N_{s'}$.

For technical reasons, we distinguish several cases as follows.
If $F[V_{\lambda_1}]$ is not anticomplete, then see **Case A**. Otherwise, $F[V_{\lambda_1}]$ is anticomplete.

Assume first that $F[V_{\lambda_2}]$ is regular. If $F[V_{\lambda_2}]$ is neither complete nor anticomplete, then we have the result by Lemma 21. If $F[V_{\lambda_2}]$ is anticomplete, then F is a disjoint union of $K_{\lambda_1,\lambda_1+1}$ and we have the result by Theorem 15.

Hence we may assume that $F[V_{\lambda_2}]$ is complete. Observe $\alpha_2 \geq \lambda_1$ because a vertex of V_{λ_1} has all its neighbors in V_{λ_2} and $\alpha_2 \leq \lambda_1 + 1$ because every vertex of V_{λ_2} is adjacent to all other vertices of V_{λ_2} and at least one in V_{λ_1}. If $\alpha_2 = \lambda_1 + 1$, then every vertex of V_{λ_2} has exactly one neighbor in V_{λ_1}, and so $\alpha_2 = \lambda_1 \times \alpha_1$. Hence $\alpha_2 = 2 = \alpha_1$ and $\lambda_1 = 1$. Thus $F = K_3^-$ and we have the result by Theorem 14. If $\alpha_2 = \lambda_1$, then every vertex of V_{λ_1} is adjacent to all vertices of V_{λ_2}. Thus F is $K_{\lambda_1+2}^-$ and we have the result by Theorem 14.

Assume now that $F[V_{\lambda_2}]$ is not regular, that is $F[V_{\lambda_2}]$ has at least two degree values. In particular, $\alpha_2 \geq 2$.

If $N_{\geq 2}$ is empty, then $V_{\lambda_2} = N_0 \cup N_1$ and both N_0, N_1 are non-empty. See **Case B**-(i).

If there is a vertex in $N_{\geq 2}$ which is not adjacent to a vertex in V_{λ_1}, see **Case C**-(i).

Otherwise, every vertex in $N_{\geq 2}$ is adjacent to all vertices in V_{λ_1} (so here $N_{\geq 2} = N_{\alpha_1}$ with $\alpha_1 \geq 2$). If $N_1 = \emptyset$, then see **Case C**-(ii). Otherwise, $N_1 \neq \emptyset$ and any vertex in V_{λ_1} is not adjacent to all vertices in N_1, see **Case B**.

Case A: We set k depending on the subgraph $F[V_{\lambda_1}]$ of F.

(1) If $F[V_{\lambda_1}]$ is not complete, then $k = 6\lambda_1 - 1$.

(2) If $F[V_{\lambda_1}]$ is complete, then every vertex of V_{λ_1} is not adjacent to some vertex in V_{λ_2}. We set $k = 6\lambda_1 + 3$.

We give a reduction from (3,4)-Sat .

Given a formula Φ of (3,4)-Sat with variables $x_t, t \in [n]$ and clauses $C_j, j \in [m]$, we build a hypergraph H as follows.

1. For each variable x_t, we construct a *variable gadget* H_t in the following way.
   We first create a *center vertex* w_t, a set of $4p-4$ vertices $U_t = \{u_t^1, \ldots, u_t^{4p-4}\}$, and $4p-4$ hyperedges $S_t^i = \{w_t, u_t^i, \ldots, u_t^{i+p-2}\}$ (superscripts are modulo $4(p-1)$) for $i \in [4p-4]$.
   For $r \in [4]$, let $x_t^r = u_t^{r(p-1)-p+2}$ and $\bar{x}_t^r = u_t^{r(p-1)-p+3}$. Set $X_t = \{x_t^1, x_t^2, x_t^3, x_t^4\}$ and $\overline{X}_t = \{\bar{x}_t^1, \bar{x}_t^2, \bar{x}_t^3, \bar{x}_t^4\}$. The vertices of X_t (resp. $\overline{X}_t$) are called the *non-negated* (resp. *negated) literal vertices* of H_t.
2. For each variable x_t, we add a set of $p-\lambda_1$ vertices W_t, and a hyperedge $S'_t = W_t \cup \{u_t^{p-1}, \ldots, u_t^{p-\lambda_1+1}\}$ and we add a $(k - 4\lambda_1 - 1)$-degree-gadget at w_t.
3. For each clause $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$, we identify y_1, y_2, y_3 into a *clause vertex* c_j, where for all $i \in [3]$, $y_i = x_t^r$ if $\ell_i = x_t$ and ℓ_i is the r-th occurrence of x_t, and $y_i = \bar{x}_t^r$ if $\ell_i = \bar{x}_t$ and is the r-th occurrence of x_t.

Let z be a vertex in V_{λ_1} which is adjacent to the minimum number $a > 0$ of vertices in this set. Let $(z, z_1, \ldots, z_{p-1})$ be an ordering of F such that :

- z_j has degree λ_1 and is adjacent to z for all $j \in [a]$,
- z_j has degree λ_1 and is not adjacent to z for all $a+1 \leq j \leq \alpha_1 - 1$,
- z_j has degree λ_2 and is adjacent to z for all $\alpha_1 \leq j \leq \alpha_1 + \lambda_1 - a - 1$.
- z_j has degree λ_2 and is not adjacent to z for all $\alpha_1 + \lambda_1 - a \leq j \leq p-1$.

We will show that there is an assignment ϕ satisfying Φ if and only there is an (F, H, k)-graph G.

Assume that ϕ satisfies Φ, then we construct G as follows. For all $i \in [4p-4]$, let $G[S_t^i]$ be copies of F such that w_t corresponds to the vertex z and the following hold.

In **Case A**-(1),

- if $\phi(x_t) = true$ (resp. $\phi(x_t) = false$), then each vertex in X_t (resp. $\overline{X}_t$) corresponds to z_1, and each of $\overline{X}_t$ (resp. X_t) corresponds to $z_{\alpha_1 - 1}$.
- for all $r \in [4]$ and $2 \leq i \leq \alpha_1 - 2$, $u^{(p-1)r+1-i}$ corresponds to z_i.

- for all $r \in [4]$ and $\alpha_1 \leq i \leq p-1$, $u^{(p-1)r+2-i}$ corresponds to z_i.

In **Case A**-(2),

- if $\phi(x_t) = true$ (resp. $\phi(x_t) = false$), then each vertex in X_t (resp. $\overline{X}_t$) corresponds to z_1, and each of $\overline{X}_t$ (resp. X_t) corresponds to z_{p-1}.
- for all $r \in [4]$ and $2 \leq i \leq p-2]$, $u^{(p-1)r+1-i}$ corresponds to z_i.

For any d-degree-gadget D at a vertex v, let $G[D]$ be an (F, D, k)-graph in which v has degree d.

Let us check that $\Delta(G) \leq k$.

- w_t is adjacent to $4\lambda_1$ vertices in H_t and one more in $W_t \subset V(S'_t)$, and it has degree $k - 4\lambda_1 - 1$ in its degree-gadget. Thus w_t has degree k in total.
- Any literal vertex which is not identified to any clause vertex and is not in S'_t has degree has degree at most $2\lambda_2$ in its variable gadget. So, it has degree less than k.
- Any literal vertex which is in S'_t has degree has degree at most $2\lambda_2$ in its variable gadget and it is adjacent to at most λ_1 vertices in W_t. So, it has degree less than k.
- Each clause vertex c_j is in three literal variable gadgets. In **Case A**-(1) (resp. **Case A**-(2)), c_j has degree at most $2\lambda_1$ (resp. $2\lambda_2$) in each variable gadget. Moreover at least one of the literals is true so its corresponding literal vertex has degree $2\lambda_1 - 1$ (resp. $2\lambda_2 - 1$) Therefore c_j has degree at most $6\lambda_1 - 1$ (resp. $6\lambda_2 - 1$) in its variable gadget. Now it has degree $k - 6\lambda_1 + 1$ (resp. $k - 6\lambda_2 + 1$) in its degree-gadget, and so at most k in total.
- Any vertex which is in a degree-gadget but in no variable gadget has degree at most $2\lambda_t \leq k$ since it belongs to at most two hyperedges.
- Any vertex in W_t has degree at most $\lambda_2 < k$.

Hence, G is an (F, H, k)-graph.

Conversely, let G be an (F, H, k)-graph.

Claim 21.2. *For every $t \in [n]$ the following hold.*

(a) *w_t has degree $4\lambda_1$ in H_t and w_t has degree exactly λ_1 in every hyperedge containing it.*

(b) *There is $I \in [p-1]$ of size λ_1 such that for all $i \in I$ and $r \in [4]$, $u_t^{(p-1)r-i+1}$ is adjacent to w_t; and $[\lambda_1 - 1] \subset I$.*

Proof of Claim. Observe that w_t is in a $(k - 4\lambda_1 - 1)$-degree-gadget, so it has degree at most $4\lambda_1 + 1$ in $H_t \cup S'_t$. Since w_t is in S'_t which intersects S^1_t in $\lambda_1 - 1$ vertices and it is at least λ_1 in S'_t, then w_t is adjacent to at least one vertex in $W_t \subset V(S'_t)$. Thus, w_t has degree at most $4\lambda_1$ in H_t. Now for every $i \in [p-1]$, w_t belongs to the four hyperedges $S_t^{(p-1)r-i}$, $r \in [4]$, which pairwise intersect in $\{w_t\}$. Hence w_t has degree exactly λ_1 in each $S_t^{(p-1)r-i}$ and then $4\lambda_1$ in H_t. This proves (a).

Now, if a vertex u^i_t is adjacent to w_t, then so is u_t^{i+p-1} because w_t has degree exactly λ_1 in both S^i_t and S^{i+1}_t. Therefore there is $I \in [p-1]$ of size λ_1 such that w_t is adjacent to $u_t^{(p-1)r-i+1}$ for all $i \in I$ and $r \in [4]$. Since w_t has degree $4\lambda_1$ in H_t, then w_t is adjacent to exactly one vertex in W_t and so must be adjacent to $\lambda_1 - 1$ vertices in $S'_t \setminus W_t$ which are $u_t^{p-1}, \dots, u^{p-\lambda_1+1}$. It implies that $[\lambda_1 - 1] \subset I$. This proves (b). ◊

Claim 21.2 implies that the vertices of X_t (resp. $\overline{X}_t$) are either all adjacent to w_t or all non-adjacent to w_t. Moreover, w_t is adjacent to $\lambda_1 - 1$ vertices not in $X_t \cup \overline{X}_t$. Hence if the vertices of X_t are adjacent to w_t, the vertices of $\overline{X}_t$ are not (and vice-versa).

Let ϕ be the truth assignment defined by $\phi(x_t) = true$ if w_t is adjacent to H_t, and $\phi(x_t) = false$ otherwise. In any clause vertex c_j, we identified three literal vertices corresponding to the three literals.

- In **Case A**-(1), c_j has degree at most $k = 6\lambda_1 - 1$, so it has degree less than $2\lambda_1$ in one of its three variable gadgets H_t. Since any vertex u^i_t for $i \in [4p-4]$ belongs to two hyperedges S^i_t and S_t^{i-p+2} which intersect in $\{u^i_t, w_t\}$ and has degree at least λ_1 in each, then it has degree $2\lambda_1 - 1$ in H_t only if it is adjacent to w_t. Hence, c_j is adjacent to w_t.
- In **Case A**-(2), c_j has degree at most $k = 6\lambda_1 + 3 < 6\lambda_2$, so it has degree less than $2\lambda_2$ in one of its three variable gadgets H_t.

  Moreover, $F[V_{\lambda_1}]$ is complete, then w_t is adjacent to all λ_1-vertices in every hyperedges of H_t (because it is a λ_1-vertex in every hyperedge of H_t). If c_j is not adjacent to no center vertex of the three variable gadgets it belongs to, then it must be a λ_2-vertex in each hyperedge of those gadgets. Thus it has degree at least $2\lambda_2$ in each variable gadget and so at least $6\lambda_2$ in total, a contradiction. Thus c_j is adjacent to the center of at least one variable gadget w_t.

Hence, the corresponding literal to the literal vertex adjacent to w_t for variable x_t is true and clause C_j is satisfied.

Consequently, ϕ satisfies Φ.

Case B: Recall that in that case $N_1 \neq \emptyset$. Let $\gamma = \max\{|N(v) \cap V_{\lambda_1}| \mid v \in V_{\lambda_2}\}$. We have $V_{\lambda_2} = \bigcup_{s=0}^{\gamma} N_s$. Let k as follows.

(i) If $N_0 \neq \emptyset$, then set $k = \max\{6\lambda_2 - 1 + \lambda_1, \gamma\alpha_2 + 2(\lambda_2 - \gamma) + \lambda_1\}$.

(ii) If $N_0 = \emptyset$, $N_{\geq 2} \neq \emptyset$ and every vertex of $N_{\geq 2}$ is adjacent to all vertices of V_{λ_1}, then set $k = \max\{6\lambda_1 + 3\alpha_2 - 1 + \lambda_1, \gamma\alpha_2 + 2(\lambda_2 - \gamma) + \lambda_1\}$. Note that in that case every vertex in V_{λ_1} is adjacent to a vertex in N_1 but not all.

We give a reduction from (3,4)-SAT.

Given a formula Φ of (3,4)-SAT with variables $x_t, t \in [n]$ and clauses $C_j, j \in [m]$, we build a hypergraph as follows.

1. For each variable x_t, we construct a *variable gadget* H_t in the following way.
   We first create a *center vertex* w_t, $4\alpha_2$ sets of $\alpha_1 - 1$ vertices A_t^i for $i \in [4\alpha_2]$, a set of $4\alpha_2$ vertices $U_i = \{u_t^1, \dots, u_t^{4\alpha_2}\}$, and $4\alpha_2$ hyperedges $S_t^i = A_t^i \cup \{w_t, u_t^i, \dots, u_t^{i+\alpha_2-1}\}$ (superscripts are modulo $4\alpha_2$) for $i \in [4\alpha_2]$.
   For $r \in [4]$, let $x_t^r = u_t^{\alpha_2(r-1)+1}$ and $\bar{x}_t^r = u_t^{\alpha_2(r-1)+2}$. Set $X_t = \{x_t^1, x_t^2, x_t^3, x_t^4\}$ and $\overline{X}_t = \{\bar{x}_t^1, \bar{x}_t^2, \bar{x}_t^3, \bar{x}_t^4\}$. The vertices of X_t (resp. $\overline{X}_t$) are called the *non-negated* (resp. *negated) literal vertices* of H_t.
2. For each variable x_t,
   - we create a set of $p - \lambda_1$ vertices B_t and a hyperedge $S'_t = B_t \cup \{w_t, u_t^{\alpha_2}, \dots, u_t^{\alpha_2-\lambda_1+2}\}$.
   - add a $(k - 4\lambda_1 - 1)$-degree-gadget on w_t.
   - add a $(k - \lambda_1)$-degree-gadget on every vertex in A_t^i for $i \in [4\alpha_2]$.
3. For each clause $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$, we identify y_1, y_2, y_3 into a *clause vertex* c_j, where $y_i = x_t^r$ if $\ell_i = x_t$ and ℓ_i is the r-th occurrence of x_t, and $y_i = \bar{x}_t^r$ if $\ell_i = \bar{x}_t$ and is the r-th occurrence of x_t. We also add a $(k - 6\lambda_2 - 1)$-degree vertex at c_j in **Case B**-(i), and a $(k - 6\lambda_1 - 3\alpha_2 + 1)$-degree vertex at c_j in **Case B**-(ii).

We will show that there is an assignment ϕ satisfying Φ if and only there is an (F, H, k)-graph G.

Let z be a vertex in V_{λ_1} adjacent to a vertex $y$in N_1 and let $\bar{y}$ be a vertex in N_0 in in **Case B**-(i) or a vertex in N_1 not adjacent to z in **Case B**-(ii). Note that $\bar{y}$ and z are not adjacent. Let $(y_1, \dots, y_{\alpha_2-2})$ be an ordering of $V_{\lambda_2} \setminus \{y, \bar{y}\}$ such that $y_1, \dots, y_{\lambda_1-1}$ are adjacent to z and $y_{\lambda_1}, \dots, y_{\alpha_2-2}$ are not adjacent to z.

Assume that there is ϕ satisfying Φ, we construct a graph G as follows. Let $G[S'_t]$ be a copy of F such that w_t has degree λ_1 and is adjacent to the $\lambda_1 - 1$ vertices $u_t^{\alpha_2}, \dots, u_t^{\alpha_2-\lambda_1+2}$.
In a variable gadget H_t, for every $i \in [4\alpha_2]$, we let $G[S_t^i]$ be a copy of F such that w_t corresponds to the vertex z, and

- A_t^i corresponds to $V_{\lambda_1} \setminus \{z\}$.
- if $\phi(x_t) = \mathit{true}$ (resp. $\phi(x_t) = \mathit{false}$), then each vertex in X_t (resp. $\overline{X}_t$) corresponds to y, and each vertex in $\overline{X}_t$ (resp. X_t) to $\bar{y}$.
- for $i \in [\alpha_2 - 2]$, $u_t^{(p-1)r-i+1}$ corresponds to y_i.

For any d-degree-gadget D at a vertex v, let $G[D]$ be an (F, D, k)-graph in which v has degree d.

Let us check that $\Delta(G) \leq k$.

- w_t is adjacent to $4\lambda_1$ vertices in H_t and one more in $W_t \subset V(S'_t)$, and it has degree $k - 4\lambda_1 - 1$ in its degree-gadget, then w_t has degree k in total.
- Any literal vertex which is not identified to any clause vertex and not in S'_t has degree at most $\gamma\alpha_2 + 2(\lambda_2 - \gamma)$ in its variable gadget (it is adjacent to at most γ vertices in each A_t^i in a hyperedge to which it belongs and there are α_2 such hyperedges; and $f(x) = x\alpha_2 + 2(\lambda_2 - x)$ is increasing). So, it has degree less than k.
- Any literal vertex which is not identified to any clause vertex and in S'_t has degree at most $\gamma\alpha_2 + 2(\lambda_2 - \gamma)$ in its variable gadget and is adjacent to at most λ_1 vertices in B_t. So it has degree at most k.

- Each clause vertex c_j is in three variable gadget. In **Case B**-(i), c_j (resp. **Case B**-(ii)), in each of these gadgets, c_j has degree either $2\lambda_2 - 1$ if it is adjacent to w_t or $2\lambda_2$ (resp. $\alpha_2 + 2(\lambda_2 - 1)$ otherwise. Moreover at least one of the literals is true, its corresponding literal vertex has degree $2\lambda_2 - 1$ in its variable gadget. Therefore c_j at most $6\lambda_2 - 1$ neighbors (resp. $2\alpha_2 + 6(\lambda_2 - 1) + 1$) in variable gadgets. It also has $k - 6\lambda_2 + 1$ (resp. $k - 6\lambda_1 - 3\alpha_2 + 1$) neighbors in its degree-gadget. Hence, in G, it has degree at most k.
- Any vertex which is in a degree-gadget but in no variable gadget has degree at most $2\lambda_t \leq k$ since it belongs to at most two hyperedges.
- Any vertex in B_t has degree at most $\lambda_2 < k$.

Hence, G is an (F, H, k)-graph.

Conversely, let G be an (F, H, k)-graph.
Observe that any vertex in A_t^i for $i \in [4\alpha_2]$ is in a $(k - \lambda_1)$-degree-gadget, then it has degree at most λ_1 in S_t^i. Since any vertex has degree at least λ_1 in a hyperedge, then every vertex in $\bigcup_{i=1}^{4\alpha_2} A_t^i$ is a λ_1-vertex in any hyperedge to which it belongs.

Claim 21.3. *For every $t \in [n]$ the following hold.*
(a) *w_t has degree $4\lambda_1$ in H_t and w_t is a λ_1-vertex in every hyperedge of H_t.*
(b) *There is $I \in [\alpha_2]$ of size λ_1 such that for all $i \in I$ and $r \in [4]$, $u_t^{\alpha_2 r - i + 1}$ is adjacent to w_t; and $[\lambda_1 - 1] \subset I$.*

This claim can be proved in exactly the same way as Claim 21.2.

We have that every vertex in U_t is a λ_2-vertex in any hyperedge to which it belongs (since w_t and $\alpha_1 - 1$ vertices of A_t^i for $i \in [4\alpha_2]$ are λ_1-vertices).
Claim 21.3 implies that the vertices of X_t (resp. $\overline{X}_t$) are either all adjacent to w_t or all non-adjacent to w_t. Moreover, w_t is adjacent to $\lambda_1 - 1$ vertices in U_t but not in $X_t \cup \overline{X}_t$. Hence if the vertices of X_t are adjacent to w_t, the vertices of $\overline{X}_t$ are not (and vice-versa).

Let ϕ be the truth assignment defined by $\phi(x_t) = true$ if w_t is adjacent to all vertices of X_t in H_t, and $\phi(x_t) = false$ otherwise.
A clause vertex c_j has degree at most k. Because of its degree-gadget, in **Case B**-(i) (resp. **Case B**-(ii)), it has degree at most $6\lambda_2 - 1$ (resp. $6\lambda_1 + 3\alpha_2 - 1$) in H_t. Now, since it is the identification of three literal vertices, c_j has degree less than $2\lambda_2$ (resp. $2\lambda_1 + \alpha_2$) in one variable gadget H_t.

Claim 21.4. *Let $i \in [4\alpha_2]$. If u_t^i is not adjacent to w_t, then the following holds.*
(i) *u_t^i has degree at least $2\lambda_2$ in $G[H_t]$;*
(ii) *If $N_0 = \emptyset$, then u_t^i has degree at least $2\lambda_1 + \alpha_2$ in $G[H_t]$;*

Proof of Claim. u_t^i has at least λ_2 neighbors in each of S_t^i and $S_t^{i-\alpha_1+1}$. But the intersection of those hyperedges is $\{w_t, u_t^i\}$. As it is not adjacent to w_t, u_t^i has at least $2\lambda_2$ neighbors in $S_t^i \cup S_t^{i-\alpha_1+1}$. This proves (i).

If $N_0 = \emptyset$, then for all $i - \alpha_2 + 1 < i' < i$. u_t^i must be adjacent to at least one λ_1-vertex of $S_t^{i'}$ which is in $A_t^{i'}$. Hence u_i^t has at least $\alpha_2 - 2$ in $\bigcup_{i-\alpha_2+1<i'<i} A_t^{i'}$ which is disjoint from $S_t^i \cup S_t^{i-\alpha_1+1}$. Hence u_t^i has degree at least $2\lambda_2 + \alpha_2 - 2 = 2\lambda_1 + \alpha_2$ in $G[H_t]$. This proves (ii). $\lozenge$

This claim implies that there is at least one variable gadget H_t in which c_j is adjacent to w_t. It implies that the corresponding literal of this vertex in C_j is true, and so C_j is satisfied.
Consequently, ϕ satisfies Φ.

Case C: In this case, $F[V_{\lambda_1}]$ is anticomplete, $F[V_{\lambda_2}]$ is not regular, and V_{λ_2} satisfies one of the following.
(i) there is a vertex of $N_{\geq 2}$ that is not adjacent to all vertices in V_{λ_1} in F.
(ii) $V_{\lambda_2} = N_{\geq 2} \cup N_0$ and every vertex of $N_{\geq 2}$ is adjacent to all vertices of V_{λ_1}. Since $F[V_{\lambda_2}]$ is not complete, then there is a vertex in $N_{\geq 2}$ which is not adjacent to a vertex in either N_0 **Case C**-(ii)-a or $N_{\geq 2}$ **Case C**-(ii)-b.

We set $a = \max_{\substack{u \in V_{\lambda_1} \\ v \in N(u)}} \left| N(v) \cap N(u) \right|$, and let $k = 4(p-2)(2\lambda_1 - a) + 4\lambda_1 + 1$.

For conveniences, we denote some vertices of graph F as follows. Let z_0 be a vertex in $N_{\geq 2}$ such that there is $z_1 \in V_{\lambda_1}$ adjacent to z_0 with $a = |N(z_0) \cap N(z_1)|$. Let $z \in N_{\geq 2}$ which is adjacent to the minimum number of vertices in N_0, and $y, y' \in V_{\lambda_1}$ be vertices adjacent to z and

- in **Case C**-(i), let $\bar{y} \in V_{\lambda_1}$ be a vertex not adjacent to z.
- in **Case C**-(ii), let $\bar{y}$ be a vertex not adjacent to z such that $\bar{y} \in N_0$ if z is not adjacent to all vertices in N_0 and $\bar{y} \in N_{\geq 2}$ otherwise.

We give a reduction from (3,4)-Sat.

Given a formula Φ of (3,4)-Sat with variables $x_t, t \in [n]$ and clauses $C_j, j \in [m]$, we build a hypergraph as follows.

1. For each variable x_t, we construct a *variable gadget* H_t in the following way. We first create a *center vertex* w_t, a set of $4(p-2)$ vertices $D_t = \{d_t^1, \ldots, d_t^{4(p-2)}\}$, a set of $4(p-2)$ vertices $U_t = \{u_t^1, \ldots, u_t^{4(p-2)}\}$, and $4(p-2)$ hyperedges $S_t^i = \{w_t, d_t^i, u_t^i, \ldots, u_t^{i+p-3}\}$ (superscripts are modulo $4(p-2)$) for $i \in [4(p-2)]$.
   For $r \in [4]$, let $x_t^r = u_t^{r(p-2)-p+3}$ and $\bar{x}_t^r = u_t^{r(p-2)-p+4}$. Set $X_t = \{x_t^1, x_t^2, x_t^3, x_t^4\}$ and $\overline{X}_t = \{\bar{x}_t^1, \bar{x}_t^2, \bar{x}_t^3, \bar{x}_t^4\}$. The vertices of X_t (resp. $\overline{X}_t$) are called the *non-negated* (resp. *negated) literal vertices* of H_t.
2. For each variable x_t,
   - We create a set Y_t of $p - \lambda_1$ vertices and a hyperedge $S'_t = Y_t \cup \{w_t\} \cup \{u_t^{p-2}, \ldots, u_t^{p-\lambda_1}\}$.
   - For any $i \in [4(p-2)]$, we add two sets of $p - \lambda_1 - 1$ vertices A_t^i, B_t^i and a set of $\lambda_1 - 1$ vertices C_t^i, and two hyperedges $A_t^i \cup C_t^i \cup \{d_t^i, w_t\}$ and $B_t^i \cup C_t^i \cup \{d_t^i, w_t\}$. We call this a *fickle-gadget* F_t^i.
   - We add a $(k - 2\lambda_1 + 1)$-degree-gadget on every vertex in D_t.
3. For each clause $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$, we identify y_1, y_2, y_3 into a *clause vertex* c_j, where $y_i = x_t^r$ if $\ell_i = x_t$ and ℓ_i is the r-th occurrence of x_t, and $y_i = \bar{x}_t^r$ if $\ell_i = \bar{x}_t$ and is the r-th occurrence of x_t. We also add at c_j a $(k - 6\lambda_1 + 1)$-degree-gadget in **Case C**-(i), a $(k - 6\lambda_1 - 3)$-degree-gadget in **Case C**-(ii)-a, and a $(k - 6\lambda_1 - 3p + 6)$-degree-gadget in **Case C**-(ii)-b.

We will show that there is an assignment ϕ satisfying Φ if and only if there is an (F, H, k)-graph G.

Assume that there is ϕ satisfying Φ, we construct a graph G as follows.

Let $(y_1, \ldots, y_{p-4})$ be an ordering of $V(F) \setminus \{z, y, y', \bar{y}\}$ such that $y_1, \ldots, y_{\lambda_1 - 2}$ are adjacent to z and $y_{\lambda_1 - 1}, \ldots, y_{p-4}$ are not adjacent to z.

In both hyperedges of any fickle-gadget F_t^i, w_t corresponds to z_0 and d_t^i corresponds z_1. d_t^i is adjacent to w_t and all vertices in C_t^i, while w_t is adjacent to a vertices in C_t^i and $\lambda_2 - a$ other ones in each of A_t^i, B_t^i. Let $G[S'_t]$ be a copy of F such that w_t has degree λ_1 and is adjacent to the $\lambda_1 - 1$ vertices $u_t^{p-2}, \ldots, u_t^{p-\lambda_1}$.

For each variable gadget H_t, for every $i \in 4[\alpha_2]$, let $G[S_t^i]$ be a copy of F such that w_t corresponds to z, and

- if $\phi(x_t) = true$ (resp. $false$), then each vertex of X_t (resp. $\overline{X}_t$) corresponds to y and each vertex of $\overline{X}_t$ (resp. X_t) corresponds to $\bar{y}$.
- in any S_t^i, d_t^i corresponds to y'.
- for $i \in [p-4]$ and $r \in [4]$, $u_t^{(p-2)r+1-i}$ corresponds to y_i.

For any d-degree-gadget D at a vertex v, let $G[D]$ be an (F, D, k)-graph in which v has degree d.

Let us check that G has degree at most k.

- Any vertex $d_t^i \in D_t$ has degree $(k - 2\lambda_1 + 1)$ in its degree-gadget and λ_1 in the fickle-gadget F_t^i and $\lambda_1 - 1$ other vertices in $V(S_t^i \setminus \{w_t\})$, thus it has degree k.
- Any vertex in $Y_t \subset V(S'_t)$ has degree at most λ_2.
- w_t has $(2\lambda_1 - a)$ neighbor in each of the $4(p-2)$ fickle-gadgets and is adjacent to $4\lambda_1$ vertices in U_t and one more in Y_t. Thus it has degree k in G.
- Any vertex in a degree-gadget which is not in H_t has degree at most $2\lambda_2$.
- Any vertex in U_t but not in $X_t \cup \overline{X}_t \cup S'_t$ has degree at most $2\lambda_2$ if it is not adjacent to any vertex in D_t or at most $2\lambda_2 + p - 2$ if adjacent to a vertex in D_t for each hyperedge to which it belongs.
- Any vertex in $U_t \cap S'_t$ has degree at most $2\lambda_1 + p - 2$ in H_t and it is adjacent to at most λ_1 vertices in Y_t, so it has degree less than k.
- Any clause vertex c_j has degree d in its d-degree-gadget. Moreover, in each of its variable gadget, c_j has degree either $2\lambda_1 - 1$ if it adajcent to the center vertex or $2\lambda_1$ in **Case C**-(i), $2\lambda_2$ in **Case C**-(ii)-a,

and $2\lambda_1 + p - 2$ in **Case C**-(ii)-b otherwise. Since there at least one of three literals of the clause C_j is true, c_j has $2\lambda_1 - 1$ in one of its variable gadget, and thus degree at most k in total.

Hence, G is an (F, H, k)-graph.

Conversely, let G be an (F, H, k)-graph.

Claim 21.5. *For every $t \in [n]$ the following hold.*

(a) *For all $i \in [4(p-2)]$, d_t^i is adjacent to w_t and has degree λ_1 in any hyperedge of $F_t^i \cup S_t^i$.*

(b) *w_t is a λ_2-vertex in every hyperedge of H_t. Furthermore, there is $I \in [p-2]$ of size λ_1 such that for all $i \in I$ and $r \in [4]$, $u^{(p-2)r-i+1}$ is adjacent to w_t and $[\lambda_1 - 1] \subset I$.*

(c) *w_t has degree k in G.*

Proof of Claim. Observe that any vertex $d_t^i \in X_t$ is in a $(k - 2\lambda_1 + 1)$-degree-gadget, then it has degree at most $2\lambda_1 - 1$ in $G[F_t^i \cup S_t^i]$. Since S_t^i and F_t^i intersect only in $\{d_t^i, w_t\}$ and d_t^i has degree at least λ_1 in each, then d_t^i has degree at least $2\lambda_1 - 1$ in $G[F_t^i \cup S_t^i]$. The equality holds when d_t^i is adjacent to w_t and $\lambda_1 - 1$ vertices in C_t^i. Thus, d_t^i has degree λ_1 in all hyperedges of $F_t^i \cup S_t^i$ and is adjacent to w_t. This proves (a).

In any fickle-gadget F_t^i, from (a), every vertex in $\{w_t\} \cup C_t^i$ is adjacent to d_t^i and must be a λ_2-vertex in the two hyperedges of F_t^i. Thus, w_t is adjacent to at most a vertices in C_t^i, and so has degree at least $2\lambda_2 - a$ in $G[F_t^i]$. Since w_t is in $4(p-2)$ fickle-gadgets, then it has degree at least $4(p-2)(2\lambda_2 - a)$ in $G[\bigcup_{i=1}^{4(p-2)} F_t^i]$. Moreover, from (a), for $i \in [4(p-2)]$, w_t is adjacent to d_t^i which has degree λ_1 in S_t^i. Thus w_t must be a λ_2-vertex in S_t^i because F is anticomplete.

Since w_t is in S'_t which intersects H_t in $\lambda_1 - 1$ vertices and w_t must have degree at least λ_1 in $G[S'_t]$, then it is adjacent to at least one vertex in $Y_t \subset V(S'_t)$. Therefore, w_t is adjacent to at most $k - 4(p-2)(2\lambda_2 - a) - 1 = 4\lambda_1$ vertices in H_t.

Now for every $i \in [p-2]$, w_t belongs to four hyperedges $S_t^{(p-2)r-i}$, $r \in [4]$, which pairwise intersect in $\{w_t\}$. Hence w_t has degree exactly λ_1 in each $S^{(p-2)r-i} \setminus \{d_t^{(p-2)r-i}\}$ and then is adjacent to $4\lambda_1$ vertices in U_t.

If a vertex u_t^i is adjacent to w_t, then so is u_t^{i+p-2} because w_t has degree exactly λ_1 in both $S_t^i \setminus \{d_t^i\}$ and $S_t^{i+1} \setminus \{d_t^{i+1}\}$. Therefore there is $I \in [p-2]$ of size λ_1 such that w_t is adjacent to $u_t^{(p-2)r-i+1}$ for all $i \in I$ and $r \in [4]$.

Since w_t has degree $4\lambda_1$ in H_t, then w_t is adjacent to exactly one vertex in Y_t and so must be adjacent to $\lambda_1 - 1$ vertices in $S'_t \setminus Y_t$ which are $u_t^{p-2}, \dots, u_t^{p-\lambda_1}$. It implies that $[\lambda_1 - 1] \subset I$. This completes the proof of (b).

From (a), (b) we have that w_t has degree $4(p-2)(2\lambda_2 - a)$ in $G[\bigcup_{i=1}^{4(p-2)} F_t^i]$, it is adjacent to $4\lambda_1$ vertices in U_t and one in Y_t. Thus, w_t has degree k in total. This proves (c). $\Diamond$

Claim 21.5(b) implies that the vertices of X_t (resp. $\overline{X}_t$) are either all adjacent to w_t or all non-adjacent to w_t. Moreover, w_t is adjacent to $4(\lambda_1 - 1)$ vertices in $U_t \setminus (X_t \cup \overline{X}_t)$. Hence if the vertices of X_t are adjacent to w_t, the vertices of $\overline{X}_t$ are not (and vice-versa).

Let ϕ be the truth assignment defined by $\phi(x_t) = true$ if w_t is adjacent to all vertices of X_t in H_t, and $\phi(x_t) = false$ otherwise. Observe the following.

- In **Case C**-(i), any clause vertex c_j is in a $(k - 6\lambda_1 + 1)$-degree-gadget, so it has degree at most $6\lambda_1 - 1$ in the union of its three variable gadgets. Thus it has degree less than $2\lambda_1$ in one of its variable gadgets H_t. Since any vertex in U_t has degree at least $2\lambda_1 - 1$ in $G[H_t]$, with equality only if it is adjacent to w_t, the vertex c_j is adjacent to w_t. Hence, the corresponding literal to this literal vertex is true and so C_j is satisfied.
- In **Case C**-(ii), any clause vertex c_j is in a $k - d$-degree-gadget, then has degree at most d in the union of its three variable gadgets. Hence c_j has degree at most $\lfloor d/3 \rfloor$ neighbors in one of those variable gagdget, say H_t. Let i be the index such that $c_j = u_t^i$.

  Suppose for a contradiction that c_j is not adjacent to w_t. Then it is a λ_2-vertex in every hyperedge of H_t.

  Vertex c_j has at least λ_2 neighours in each of S_t^i and S_t^{i-p-3} which intersect in $\{c_j, w_t\}$. Hence c_j has at least $2\lambda_2$ neighours in $S_t^i \cup S^{i-p-3}$. In **Case C**-(ii)-a, $\lfloor d/3 \rfloor = 2\lambda_1 + 1 < 2\lambda_2$, so we get a contradiction.

In **Case C**-(ii)-b, since w_t is adjacent to all d_t^i by Claim 21.5 (b), c_j corresponds to a vertex in $N_{\geq 2}$ in every hyperedge of H_t to which it belongs. Therefore it is adjacent to all λ_1-vertices in these hyperedges and thus in particular to all $d_t^{i'}$ for all $i-p+3 < i' < i$. Hence c_j has degree at least $2\lambda_2 + p - 4$ in $V(H_t)$. But $\lfloor d/3 \rfloor = 2\lambda_1 + p - 3 = 2\lambda_2 + p - 5$, a contradiction.

In both subcases, the vertex c_j is adjacent to w_t. Hence, the corresponding literal to this literal vertex is true and so C_j is satisfied.

Consequently, ϕ satisfies Φ. □

6 Further research

Problem 1 asks for a characterization of the pairs (F, k) for which $(\Delta \leq k)$-F-Overlay is polynomial-time solvable and those for which it is NP-complete. As a partial answer, we proved that $\mathrm{np}(F) < +\infty$ if and only if F is **standard**, that is neither a complete graph nor an anticomplete graph. We believe that the following holds.

Conjecture 22. For every graph F, $(\Delta \leq k)$-F-Overlay is polynomial-time solvable when $k < \mathrm{np}(F)$ (and NP-complete otherwise).

Thus answering Problem 1 is equivalent to determining $\mathrm{np}(F)$. However, it would already be interesting to prove that for any pair (F, k), $(\Delta \leq k)$-F-Overlay is either polynomial-time solvable or NP-complete. A first step to prove this is to prove Conjecture 2.

Furthermore, we made no attempt to minimize the upper bound on $\mathrm{np}(F)$, our goal was just to prove such a bound exists. In fact, our proof in Section 5 shows the general upper bound $\mathrm{np}(F) \leq 8|F|\delta(F)$ for every standard graph F. However, there are many graphs for which the proof shows $\mathrm{np}(F) \leq 6\Delta(F)$. It motivates the following questions.

Problem 23. Does there exist a constant c such that $\mathrm{np}(F) \leq c \cdot \Delta(F)$ for every standard graph F ?

Moreover, better upper bounds can certainly be obtained for certain classes of graph F. For example, for every path P_p with $p \geq 4$, Theorem 20 **Case B** (i) yields $\mathrm{np}(P_p) \leq p+1$, and if F is a disjoint union of paths, then Theorem 20 yields $\mathrm{np}(F) \leq 8|F| - 11$ (the worst case is given by **Case C** (i) when F has a component which is a P_3). In Appendix A, we show $\mathrm{np}(P_p) \leq 4$. We also obtain a better upper bound for disjoint union of paths: we prove that $\mathrm{np}(F) \leq 5$ for such a graph F.

Getting lower bounds would also be interesting. The trivial lower bound is $\mathrm{np}(F) \geq \Delta(F)$. There are graphs for which this lower bound is attained (the graphs with one edge of order at least 4 for example), and other for which it is not t(he paths for example, see Havet et al. [10]). It would be nice to characterize the graphs such that $\mathrm{np}(F) = \Delta(F)$. It would also be nice to find graphs F such that $\mathrm{np}(F) - \Delta(F)$ is large. The largest known difference is 2 for C_4, the cycle on four vertices. Indeed Havet et al. [10] proved $\mathrm{np}(C_4) \geq 4 = \Delta(C_4) + 2$.

There are very few standard graphs F for which $\mathrm{np}(F)$ is known. The only ones are the graphs with one edge. Havet et al. [10] proved $\mathrm{np}(O_3) = 2$ and $\mathrm{np}(O_p) = 1$ for all $p \geq 4$. It would be nice to determine $\mathrm{np}(F)$ for other graphs. A first problem is to do it for paths. Havet et al. [10] proved that $\mathrm{np}(P_p) > 2$ for all p. In Appendix A of this paper, we prove $\mathrm{np}(P_p) \leq 4$ for $p \geq 3$. Hence it is open to answer whether $\mathrm{np}(P_p) = 3$. Note that paths have minimum degree 1, so by Lemma 12, Conjecture 2 holds for such graphs. A second natural step is to determine $\mathrm{np}(F)$ when F is a cycle. Indeed, Conjecture 2 is not yet proved for such graphs and there are non-trivial polynomial-time algorithms when F is a cycle, as shown by the example of C_4.

In this paper, we only considered the case when the family $\mathcal{F}$ of admissible graphs has size 1. It is natural and interesting to study the more general case when $\mathcal{F}$ can have an arbitrary size, finite or infinite.

Problem 24. Characterize the pairs $(\mathcal{F}, k)$ for which $(\Delta \leq k)$-$\mathcal{F}$-Overlay is polynomial-time solvable and those for which it is NP-complete.

We believe that Conjectures 25 and 2 extends to any family $\mathcal{F}$.

Conjecture 25. For every family of graphs $\mathcal{F}$, there exists an integer k_0 such that $(\Delta \leq k)$-$\mathcal{F}$-Overlay is polynomial-time solvable when $k < k_0$ and NP-complete otherwise.

Conjecture 26. If $(\Delta \leq k)$-$\mathcal{F}$-Overlay is NP-complete, then $(\Delta \leq k{+}1)$-$\mathcal{F}$-Overlay is also NP-complete.

We also strongly believe that Theorem 3 can be extended to any family $\mathcal{F}$. Defining $\mathrm{np}(\mathcal{F})$ as the minimum integer k_0 such that $(\Delta \leq k)$-$\mathcal{F}$-Overlay is NP-complete for all $k \geq k_0$ or $+\infty$ if no such k_0 exists, we conjecture the following.

Conjecture 27. $\mathrm{np}(\mathcal{F}) = +\infty$ if and only if all elements of $\mathcal{F}$ are complete or anticomplete.

References

[1] D. Agarwal, C. Caillouet, D. Coudert, and F. Cazals. Unveiling Contacts within Macro-molecular assemblies by solving Minimum Weight Connectivity Inference Problems. *Molecular and Cellular Proteomics*, 14:2274–2284, 2015.

[2] D. Angluin, J Aspnes, and L. Reyzin. Inferring social networks from outbreaks. In *ALT '10*, pages 104–118, 2010.

[3] U. Brandes, S. Cornelsen, B. Pampel, and A. Sallaberry. Blocks of hypergraphs - applied to hypergraphs and outerplanarity. In *IWOCA '10*, pages 201–211, 2010.

[4] G. Chockler, R. Melamed, Y. Tock, and R. Vitenberg. Constructing scalable overlays for pub-sub with many topics. In *PODC '07*, 2007.

[5] V. Conitzer, J. Derryberry, and T. Sandholm. Combinatorial auctions with structured item graphs. In *Proceeding of AAAI'2004*, pages 212–218, 2004.

[6] D.-Z. Du and D. F. Kelley. On complexity of subset interconnection designs. *Journal of Global Optimization*, 6(2):193–205, 1995.

[7] D.-Z. Du and Z. Miller. Matroids and subset interconnection design. *SIAM Journal on Discrete Mathematics*, 1(4):416–424, 1988.

[8] H. Fan, C. Hundt, Y.-L. Wu, and J. Ernst. Algorithms and implementation for interconnection graph problem. In *Combinatorial Optimization and Applications*, pages 201–210, 2008.

[9] H. Fan and Y.-L. Wu. Interconnection graph problem. In *Proceedings of FCS'2008*, pages 51–55, 2008.

[10] F. Havet, D. Mazauric, V.-H. Nguyen, and R. Watrigant. Overlaying a hypergraph with a graph with bounded maximum degree. In *Proceedings of CALDAM'2020*, volume 12016 of *LNCS*, pages 403–414, 2020.

[11] I. Holyer. The NP-completeness of edge-coloring. *SIAM J. Comput.*, 10(4):718–720, 1981.

[12] D. S. Johnson and H. O. Pollak. Hypergraph planarity and the complexity of drawing Venn diagrams. *Journal of Graph Theory*, 11(3):309–325, 1987.

[13] M. Onus and A. Richa. Minimum maximum-degree publish-subscribe overlay network design. *IEEE/ACM Trans. Netw.*, 19(5):1331–1343, 2011.

[14] A. C. Tovey. A simplified NP-complete satisfiability problem. *Discrete Applied Mathematics*, 8(1):85 – 89, 1984.

A Disjoint union of paths

In this appendix, we study on the family of disjoint union of paths and aim to prove the following theorem.

Theorem 28. *Let F be a disjoint union of paths with $\delta(F) \geq 1$. If $F \neq K_2$, then* $\mathrm{np}(F) \leq 5$.

A disjoint union of paths contains several paths with their lengths. The following result allows us to consider only a longest path in a disjoint of paths which is simpler.

Lemma 29. *Let F be a disjoint union of paths among which P is a shortest one. If $(\Delta \leq k)$-$(F-P)$-*Overlay *is* NP*-complete, then $(\Delta \leq k)$-F-*Overlay *is* NP*-complete. Hence* $\mathrm{np}(F) \leq \mathrm{np}(F-P)$.

Proof. Set $|F| = p$, $|P| = q$, and $F' = F - P$. So $|F'| = p - q$.

We give a reduction from $(\Delta \leq k)$-F'-Overlay to $(\Delta \leq k)$-F-Overlay.

Let H' be a $(p-q)$-uniform hypergraph. Let us build a p-uniform hypergraph H from H'. For every hyperedge S of H', we do the following: we create a set $U_S = \{u_S^1, \dots, u_S^q\}$ of q vertices, and a set W_S of $p-q$ vertices; we add the hyperedges $S \cup U_S$ and $U_S \cup W_S$ to $E(H)$; we add $k-1$ pendant hyperedges at u_S^1 and u_S^q, and $k-2$ pendant hyperedges at each u_S^i for $2 \leq i \leq q-1$.

We shall prove that there is an (F, H, k)-graph G if and only if there is an (F', H', k)-graph G'.

Assume first that there is an (F, H, k)-graph G. Note that every vertex in a hyperedge S of H has degree at least 1 in $G[S]$. For each hyperedge S of H', the vertex u_S^1 is in $k-1$ pendant hyperedges, in each of which it has degree 1. Therefore u_S^1 has degree at most 1 in $G[S \cup U_S \cup W_S]$. Now u_S^1 has degree 1 in $G[S \cup U_S]$ and $G[U_S \cup W_S]$, so necessarily u_S^1 has a unique neighbor in U_S and no neighbor in $S \cup W_S$. Similarly, u_S^q has a unique neighbor in U_S and no neighbor in $S \cup W_S$. If $q = 2$, then $G[U_S] = P$ and there is no edge between U_S an S in G. If $q > 2$, then the neighbor of u_S^1 in U_S cannot be u_S^q for otherwise $G[\{u_S^1, u_S^q\}]$ would be a connected component of $G[S \cup U_S]$ on two vertices, a contradiction to the fact that P is the smallest component of F. Hence, without loss of generality, we may assume that the unique neighbor of u_S^1 in U_S is u_S^2. But now since $q > 2$, the vertex u_S^2 has at least two neighbors in both $G[S \cup U_S]$ and $G[U_S \cup W_S]$. Moreover, u_S^2 is in $k-2$ pendant hyperedges. Therefore u_S^2 has exactly two neighbors in U_S and no neighbor in $S \cup W_S$. If $q = 3$, then $G[U_S] = P$ and there is no edge between U_S an S in G. If $q > 3$, then the neighbor of u_S^2 distinct from u_S^1 in U_S cannot be u_S^q for otherwise $G[\{u_S^1, u_S^2, u_S^q\}]$ would be a connected component of $G[S \cup U_S$ of order 3, a contradiction to the fact that P is the smallest component of F. Hence, without loss of generality, we may assume that the neighbor of u_S^2 distinct from u_S^1 in U_S is u_S^3. And so on, by induction on $i \leq q-1$, one can show that the neighbors of u_S^i in $G[S \cup U_S \cup W_S]$ are u_S^{i-1} and u_S^i. Therefore, $G[U_S] = P$ and there is no edge between U_S and S in G, and so $G[S] = F - P = F'$. Consequently, $G' = G[H']$ F'-overlays H' and so is an (F', H', k)-graph G'.

Assume now that there is an (F', H', k)-graph G'. Let G be the graph built from G' as follows. For each hyperedge S, we let $G[U_s \cup W_S]$ be a copy of F such that $G[U_S] = P$, and the subgraph of G induced by every pendant hyperedge at some vertex x is any copy of F in which x as degree 1. Observe that $G[S \cup U_s] = G'[S] \cup P$. Thus $G[S \cup U_s]$ contains F because $G'[S]$ contains F'. Therefore G F-overlays H. One easily checks that $\Delta(G) \leq k$, so G is an (F, H, k)-graph. ☐

By Lemma 29, it is sufficient to prove Theorem 28 for paths, and $2K_2$ – the 1-regular graph on four vertices. By Theorem 13, we have $\mathrm{np}(2K_2) \leq 5$, so it suffices to prove the result for paths.

Theorem 30. $\mathrm{np}(P_p) \leq 4$ *for all* $p \geq 4$.

Proof. By Lemma 12, it suffices to prove that $(\Delta \leq 4)$-P_p-Overlay is NP-complete for $p \geq 4$. We give a reduction from (3,4)-Sat.

Given a formula Φ of (3,4)-Sat with n variables $x_t, t \in [n]$, and m clauses $C_j, j \in [m]$, we construct a hypergraph H as follows.

- For each variable x_t, we add a *variable gadget* $H(x_t)$ containing a *center vertex* c_t, four pairs of *literal vertices* $x_t^1, \bar{x}_t^1, \dots, x_t^4, \bar{x}_t^4$, and eight sets of $p-3$ vertices $U_t^1, \dots, U_t^4$ and $T_t^1, \dots, T_t^4$. Set $X_t = \{x_t^i \mid i \in [4]\}$ and $\overline{X}_t = \{\bar{x}_t^i \mid i \in [4]\}$. We add the four hyperedges $\{c_t, x_t^i, \bar{x}^i\} \cup U_t^i$ for $i \in [4]$, and the four hyperedges $\{c_t, \bar{x}_t^i, x_t^{i+1}\} \cup T_t^i$ for $i \in [4]$ (superscripts are modulo 4).
- For each clause $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$, we add a *clause gadget* $H(C_j)$ containing two *clause vertices* y_j, z_j, one set of $p-2$ vertices V_j, three sets of $p-3$ vertices W_j^1, W_j^2, W_j^3, and we distinguish a vertex $w_j^i \in W_j^i$ for each $i \in [3]$. We add one hyperedge $S_j = \{y_j, z_j\} \cup V_j$, and for each literal ℓ_i which is the r_i-th occurrence of this variable, $r_i \in [4]$, we add one hyperedge $S_j^{\ell_i} = \{y_j, z_j, \ell_i^{r_i}\} \cup W_j^i$. Finally, for $i \in [3]$ we add three pendant hyperedges at each w_j^i (with new vertices).

We will prove that there exists a truth assignment ϕ satisfying Φ if and only if there is a $(P_p, H, 4)$-graph G. The general idea is that a variable $x_t = true$ (resp. $false$) if and only if the center vertex c_t is adjacent to all vertices of X_t (resp. $\overline{X}_t$) (so each x_t^i has degree 2 (resp. 3) in $G[H(x_t)]$ for all $i \in [4]$).

Assume that there exists a truth assignment ϕ satisfying Φ. Let G be the graph obtained as follows.

- We first consider each variable x_t, $t \in [n]$. If $\phi(x_t) = true$ (resp. $false$), then any subgraph induced by a hyperedge S of $H(x_t)$ is a copy of P_p whose two endpoints are c_t and the vertex of $S \cap X_t$ (resp. $S \cap \bar{X}_t$), and in which c_t is adjacent to the vertex of $S \cap \bar{X}_t$ (resp. $S \cap X_t$). Then $G[H(x_t)]$ is called a *true variable subgraph* (resp. *false variable subgraph*) on x_t, if in this subgraph the literal vertices in X_t have degree 2 (resp. 3) and the literal vertices in $\overline{X}_t$ have degree 3 (respectively 2).
- We then consider each clause $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$, $j \in [m]$. The induced subgraph $G[S_j]$ is a copy of P_p where y_j is an endpoint and y_j is adjacent to z_j (therefore y_j has degree 1 and z_j has degree 2 in $G[S_j]$. Let $i \in [3]$. If $\phi(\ell_i) = true$, then let $G[S_j^{\ell_i}]$ be a *true literal subgraph*, that is it is a copy of P_p starting with $(z_j, y_j, \ell_i^{\alpha_i})$ and ending at w_j^i; it increases the degree of y_j by 1 in G and does not increase the degree of z_j in G. If $\phi(\ell_i) = false$, then let $G[S_j^{\ell_i}]$ be a *false literal subgraph*, that is a copy of P_p starting with (w_j^i, z_j, y_j) and ending at $\ell_i^{\alpha_i}$; it increases the degree of both y_j and z_j by 1 in G.

  Finally for any hyperedge pendant at w_j^i, the subgraph induced by this hyperedge is a copy of P_p in which w_j^i as degree 1.

Observe that all the vertices of G have degree at most 4. In particular, for each clause C_j, the vertex z_j has degree 4 because there is a literal such that $\phi(\ell_i) = true$. Hence, G is a $(P_p, H, 4)$-graph.

Conversely, assume that G is a $(P_p, H, 4)$-graph.

Observe that the subgraph induced by the hyperedges of each variable gadget is either a true or a false variable subgraph. Indeed, c_t has degree at most 4, and degree 1 in each of the eight hyperedges of the variable gadget. In order to have all hyperedges P_p-overlaid it must contain either the four edges $c_t x_t^i$, $i \in [4]$ (true vertex subgraph), or the four edges $c_t \bar{x}_t^i$ for $i \in [4]$ (false vertex subgraph). Thus, we define a truth assignment ϕ by setting $\phi(x_t) = true$ (resp. $false$) if $H(x_t)$ is a true (resp. false) variable subgraph. We shall prove that ϕ satisfies Φ. We need the following claim.

Claim 30.1. *For any clause gadget $H(C_j)$, $y_j z_j \in E(G)$ and at least one of three literal vertices in $H(C_j)$ has at least two neighbors in $V(H(C_j))$.*

Proof of Claim. Observe that in each $G[S_j^{l^i}]$, the vertex w_j^i must have degree 1, because it has three neighbors in its three pendant hyperedges.

Assume for contradiction that $y_j z_j \notin E(G)$, then both y_j and z_j have a neighbor in V_j. Moreover, in each $G[S_j^{l^i}]$ at least one of y_j, z_j has degree at least 2 because w_j^i has degree 1. Consequently, at least one of y_j, z_j has degree more than 4 in G, a contradiction. Therefore, $y_j z_j \in E(G)$.

Assume for a contradiction that the three literal vertices have only one neighbor in $V(H(C_j))$. Then both y_j and z_j have at least two neighours in each $S_j^{\ell_i}$. Moreover, one of them, say z_j, has two neighbors in S_j. Thus z_j is adjacent to y_j and at least one vertex in each of the four disjoints sets $S_j^{\ell_i} \setminus \{y_j, z_j\}$, $i \in [3]$ and V_j. Hence z_j has degree 5 in G, a contradiction. $\Diamond$

From this claim, in any $G[H(C_j)]$, at least one of the three literal vertices in $H(C_j)$, say $\ell_i^{\alpha_i}$, has at least two neighbors in $V(H(C_j))$. But then $\ell_i^{\alpha_i}$ must have degree 2 in its variable gadget. Therefore, by definition of ϕ, we have $\phi(\ell_i) = true$. We conclude that ϕ satisfies Φ. $\square$

Theorem 31. $\mathrm{np}(P_3) \leq 4$.

Proof. By Lemma 12, it suffices to prove that $(\Delta \leq 4)$-P_3-Overlay is NP-complete. We give a reduction from (3,4)-Sat problem. The proof is very similar to that of Theorem 30, the differences lying in the construction of the clause gadgets. Therefore, we just give a sketch of the proof and leave the details to the reader.

Given a formula Φ of (3,4)-Sat with n variables $x_t, t \in [n]$, and m clauses $C_j, j \in [m]$, we construct a 3-uniform hypergraph H as follows.

- For each variable x_t, we add a variable gadget $H(x_t)$ containing c_t, four couples $(x_t^i, \bar{x}_t^i)$ and the hyperedges are of the form $\{c_t, x_t^i, \bar{x}_t^i\}$ and $\{c_t, \bar{x}_t^i, x_t^{i+1}\}$ for $i \in [4]$ (superscript are modulo 4).
- For each clause $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$, we add a clause gadget $H(C_j)$ six new vertices $y_j^1, y_j^2, y_j^3, w_j^1, w_j^2, w_j^3$. We add the hyperedge $S_j = \{y_j^1, y_j^2, y_j^3\}$. For each literal ℓ_i for $i \in [3]$ which is the r_i-th occurrence of its variable, we add $S_j^{\ell_i} = \{y_j^i, \ell_i^{r_i}, w_j^i\}$. For each $i \in [3]$, we add three pendant hyperedges at w_j^i and a pendant hyperedge at y_j^i.

One that then can easily prove the following analogue of Claim 30.1.

Claim 31.1. *For any clause gadget $H(C_j)$, in any $(P_3, H, 4)$-graph, at least one of three literal vertices in $H(C_j)$ has at least two neighbors in $V(H(C_j))$.*

This allows to show that there exists a truth assignment ϕ satisfying Φ if and only if there is a $(P_3, H, 4)$-graph G. □

Lemma 29 and Theorems 30, 31, and 13 directly imply Theorem 28.