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***Abstract*—In this paper, we study a class of generalized Persidskii systems with external disturbances and establish conditions, in the form of linear matrix inequalities, for input-to-output stability (IOS) and robust synchronization for these systems. We apply the obtained results to the robust control design for synchronizing linear systems and to the synchronization of Hindmarsh-Rose models of neurons.**

***Index Terms*—Input-to-output stability, robust synchronization, generalized Persidskii systems.**

I. Introduction

Stability analysis for dynamical systems is a complicated problem, especially in the nonlinear case and in the presence of external perturbations [2]–[4]. One of the most general frameworks in this domain is given by the input-to-output stability (IOS) concept [5], [6], which quantifies the boundedness and the convergence of an output signal for a nonlinear dynamical system in the presence of essentially bounded exogenous inputs. This theory also provides necessary and sufficient conditions for IOS in terms of the existence of corresponding Lyapunov functions [6]. The well-known input-to-state stability (ISS) property [7] is a particular case of IOS when the whole state is considered as the system output. The main drawback of applying these concepts is the lack of constructive methods to design the related Lyapunov functions.

Most existing approaches for synthesizing Lyapunov functions for nonlinear dynamics involve various canonical forms such as Lur'e systems [8], Lipschitz dynamics, Persidskii systems [9], and homogeneous models. In the present paper, we focus our attention on Persidskii systems, which have been extensively studied in the context of neural networks [10] and power systems [11]. This class of models was introduced in [9], [12], and recently the ISS conditions have been established in [13] for its generalized form. The properties of the Lyapunov function proposed in [13] and its time derivative are analyzed using linear matrix inequalities (LMI), which is a rare case for nonlinear systems taking into account a possible highly nonlinear nature of generalized Persidskii systems. The main goal of our work is to extend those results to input-to-output stability (IOS) and apply them to study synchronization conditions for a family of generalized Persidskii systems.

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A preliminary conference version of this work [1] mainly focuses on the synchronization application, and the IOS conditions are only briefly discussed (solely a disturbance-free case is studied).

Synchronization is a complex phenomenon that is frequently observed in networked and interconnected systems. It has been extensively investigated in various fields, *e.g.*, robotics, communication security, autonomous driving [14]–[17]. The principal methods to achieve synchronization for nonlinear systems are based on the passivity theory [18], [19], output regulation [20], incremental stability [21], Lyapunov approach [22], to mention a few recent results. Lur'e systems constitute a popular benchmark for testing these theories [23]–[26]. Here we employ the IOS theory to derive conditions for robust synchronization (in the presence of external inputs).

The outline of this paper is as follows. In Section II, the stability definitions are presented. The class of considered systems is defined in Section III, and in Section IV, the conditions for IOS and the analysis of the selected Lyapunov function are given. The synchronization measure and an approach to study synchronization of a family of generalized Persidskii systems are introduced in Section V-A. Robust control design for synchronization of linear systems subject to highly nonlinear perturbations is presented in Section V-B. The Hindmarsh-Rose model is considered as an example in Section VI to examine the efficiency of our proposed results.

Notation

- $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_+$ represent the sets of natural numbers, real numbers, and nonnegative real numbers, respectively. The symbols $|\cdot|$ and $\|\cdot\|$ denote the absolute value in $\mathbb{R}$ and the Euclidean norm on $\mathbb{R}^n$ (and the induced matrix norm $\|A\|$ for a matrix $A \in \mathbb{R}^{m\times n}$), correspondingly.
- The identity matrix of dimension n is denoted by I_n and the n-dimensional all-ones vector by $\mathbf{1}_n$. The set of diagonal matrices with nonnegative elements on the main diagonal is denoted by $\mathbb{D}_+^n \subset \mathbb{R}_+^{n\times n}$. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, $\lambda_{\max}(A)$ denotes its maximal eigenvalue. For a matrix $B \in \mathbb{R}^{m\times n}$, let $\ker(B)$ denote its kernel.
- For p, $n \in \mathbb{N}$ with $p \le n$, the notation $\overline{p,n}$ is used to represent the set $\{p,\dots,n\}$.
- For a Lebesgue measurable function $u\colon \mathbb{R}_+ \to \mathbb{R}^m$, define the norm $\|u\|_{[t_1,t_2)} = \operatorname{ess\,sup}_{t\in[t_1,t_2)}\|u(t)\|$ for $[t_1,t_2) \subset \mathbb{R}_+$. Let $\mathscr{L}_\infty^m$ be the space of functions u with $\|u\|_\infty := \|u\|_{[0,\infty)} < +\infty$ and $\tilde{\mathscr{L}}_\Theta^m \subset \mathscr{L}_\infty^m$ be the space of functions taking values in a compact subset $\Theta \subset \mathbb{R}^m$.
- A continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathscr{K}$ if it is strictly increasing and $\sigma(0) = 0$; it belongs to class $\mathscr{K}_\infty$ if it also satisfies $\lim_{r\to\infty}\sigma(r) = \infty$. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathscr{KL}$ if for fixed $s \in \mathbb{R}_+$, $\beta(\cdot, s) \in \mathscr{K}$ and for fixed $r \in \mathbb{R}_+$, $\beta(r,\cdot)$ is a decreasing function with $\lim_{s\to\infty}\beta(r,s) = 0$.

- For a continuously differentiable function $V\colon \mathbb{R}^n \to \mathbb{R}$, denote by $\nabla V(v) f(v)$ the Lie derivative of V along the vector field f evaluated at point $v \in \mathbb{R}^n$.

II. Preliminaries

Consider a class of nonlinear systems:

$$\begin{aligned} &\dot{x}(t) = f(x(t), d(t)),\ \forall t \geq 0,\ \text{with } f(0,0) = 0,\ x(0) = x_0, \\ &y(t) = h(x(t)), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $d(t) \in \mathbb{R}^m$ is the external perturbation, $d \in \mathscr{L}_\infty^m$, $y(t) \in \mathbb{R}^p$ is the output vector. Moreover, $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a locally Lipschitz continuous function and $h\colon \mathbb{R}^n \to \mathbb{R}^p$ is a continuously differentiable function. For an initial state $x_0 \in \mathbb{R}^n$ and $d \in \mathscr{L}_\infty^m$, the corresponding solution of (1) is denoted by $x(t, x_0, d)$ for the values of $t \geq 0$ the solution exists, so the corresponding output is $y(t, x_0, d) = h(x(t, x_0, d))$.

The system (1) is *forward complete* if for all $x_0 \in \mathbb{R}^n$ and $d \in \mathscr{L}_\infty^m$, the solution $x(t, x_0, d)$ is uniquely defined for all $t \geq 0$.

In the rest of the paper, to lighten the notation, the time-dependency of variables might remain implicitly understood, for instance we write x for $x(t)$. Let us give some definitions that are used in the sequel.

Definition 1. [6] *A forward complete system* (1) *is said to be:*

1) practical input-to-output stable *(pIOS) if there exist $\beta \in \mathscr{KL}$, $\gamma \in \mathscr{K}$ and $c \in \mathbb{R}_+$ such that*
$$\|y(t, x_0, d)\| \leq \beta(\|x_0\|, t) + \gamma(\|d\|_\infty) + c,\ \forall t \geq 0$$
for any $x_0 \in \mathbb{R}^n$ and $d \in \mathscr{L}_\infty^m$. The system is input-to-output stable *(IOS) if $c = 0$. When $y = x$, the IOS property is called* input-to-state stability *(ISS).*
2) output-Lagrange input-to-output stable *(OLIOS) if it is IOS and there exist $\sigma_1, \sigma_2 \in \mathscr{K}$ such that*
$$\|y(t, x_0, d)\| \leq \max\{\sigma_1(\|h(x_0)\|), \sigma_2(\|d\|_\infty)\},\ \forall t \geq 0$$
for any $x_0 \in \mathbb{R}^n$ and $d \in \mathscr{L}_\infty^m$.
3) state-independent input-to-output stable *(SIIOS) if there exist $\beta \in \mathscr{KL}, \gamma \in \mathscr{K}$ such that*
$$\|y(t, x_0, d)\| \leq \beta(\|h(x_0)\|, t) + \gamma(\|d\|_\infty),\ \forall t \geq 0$$
for any $x_0 \in \mathbb{R}^n$ and $d \in \mathscr{L}_\infty^m$.
4) robustly output stable *(ROS) if there exist a smooth function $\alpha \in \mathscr{K}_\infty$ and $\beta \in \mathscr{KL}$ such that the system*
$$\dot{x} = \tilde{f}(x, \varsigma) := f(x, \varsigma\alpha(\|h(x)\|)) \tag{2}$$
is forward complete, and the estimate
$$\|y_\alpha(t, x_0, \varsigma)\| \leq \beta(\|x_0\|, t),\ \forall t \geq 0$$
is satisfied for all $\varsigma \in \tilde{\mathscr{L}}_{\mathscr{C}}^m$, where $\mathscr{C} := \{\mu \in \mathbb{R}^m : \|\mu\| \leq 1\}$, and $y_\alpha(t, x_0, \varsigma) = h(x(t, x_0, \varsigma))$ denotes the output function of the system (2).

Definition 2. [6] *A forward complete system* (1) *is said to be* uniformly bounded-input-bounded-state stable *(UBIBS) if there exists $\sigma \in \mathscr{K}$ such that*
$$\|x(t, x_0, d)\| \leq \max\{\sigma(\|x_0\|), \sigma(\|d\|_\infty)\},\ \forall t \geq 0$$
for all $x_0 \in \mathbb{R}^n$ and $d \in \mathscr{L}_\infty^m$.

Definition 3. [6] *For the system* (1)*, a smooth function $V\colon \mathbb{R}^n \to \mathbb{R}_+$ is:*

1) *an IOS*-Lyapunov function *if there exist $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$, $\chi \in \mathscr{K}$ and $\alpha_3 \in \mathscr{KL}$ such that*
$$\begin{aligned} \alpha_1(\|h(x)\|) \leq V(x) \leq \alpha_2(\|x\|), \\ V(x) \geq \chi(\|d\|) \Rightarrow \nabla V(x) f(x, d) \leq -\alpha_3(V(x), \|x\|) \end{aligned} \tag{3}$$
for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$.
2) *an OLIOS*-Lyapunov function *if there exist $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$, $\chi \in \mathscr{K}$ and $\alpha_3 \in \mathscr{KL}$ such that*
$$\begin{aligned} \alpha_1(\|h(x)\|) \leq V(x) \leq \alpha_2(\|h(x)\|), \\ V(x) \geq \chi(\|d\|) \Rightarrow \nabla V(x) f(x, d) \leq -\alpha_3(V(x), \|x\|) \end{aligned} \tag{4}$$
for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$.
3) *an SIIOS*-Lyapunov function *if there exist $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$ and $\chi, \alpha_3 \in \mathscr{K}$ such that*
$$\begin{aligned} \alpha_1(\|h(x)\|) \leq V(x) \leq \alpha_2(\|h(x)\|), \\ V(x) \geq \chi(\|d\|) \Rightarrow \nabla V(x) f(x, d) \leq -\alpha_3(V(x)) \end{aligned}$$
for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$.
4) *an ROS*-Lyapunov function *if there exist $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$, $\chi \in \mathscr{K}$ and $\alpha_3 \in \mathscr{KL}$ such that*
$$\begin{aligned} \alpha_1(\|h(x)\|) \leq V(x) \leq \alpha_2(\|x\|), \\ \|h(x)\| \geq \chi(\|d\|) \Rightarrow \nabla V(x) f(x, d) \leq -\alpha_3(V(x), \|x\|) \end{aligned}$$
for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$.

Theorem 1. [6] *A UBIBS system* (1) *is IOS (OLIOS, SIIOS, ROS) if and only if it admits an IOS (OLIOS, SIIOS, ROS)-Lyapunov function.*

Remark 1. *Note that for a sufficient condition of IOS, SIIOS, or ROS, the UBIBS requirement can be discarded provided that the system* (1) *is forward complete (or it possesses the unboundedness observability property [27]) and an IOS/SIIOS-Lyapunov function or a ROS-Lyapunov function satisfies* (3) *((4) in SIIOS case) and respectively,*
$$V(x) \geq \chi(\|d\|) \Rightarrow \nabla V(x) f(x, d) \leq -\alpha_3(V(x)) \tag{5}$$
or
$$\|h(x)\| \geq \chi(\|d\|) \Rightarrow \nabla V(x) f(x, d) \leq -\alpha_3(V(x))$$
for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$, some $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$ and $\chi, \alpha_3 \in \mathscr{K}$.

III. Problem statement

Consider the following class of systems:

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{j=1}^{M} A_j f^j(x(t)) + d(t),\ \forall\ t \geq 0, \\ y(t) &= Cx(t), \end{aligned} \tag{6}$$

where $x(t) = [x_1(t) \dots x_n(t)]^\top \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^p$ is the output signal, $C \in \mathbb{R}^{p \times n}$ with $C \neq 0$, $d \in \mathscr{L}_\infty^n$ is the external perturbation, $f^j : \mathbb{R}^n \to \mathbb{R}^n$ with $f^j(x) = [f_1^j(x_1) \dots f_n^j(x_n)]^\top$, $j \in \overline{1, M}$ $(M \in \mathbb{N} \backslash \{0\})$ are continuous functions ensuring the existence of solutions of (6) in the forward time at least locally and $A_k \in \mathbb{R}^{n\times n},\ k \in \overline{0,M}$.

In this paper, it is assumed that if the upper limit of a summation or a sequence is smaller than the lower one, then the corresponding terms (or conditions) are omitted.

Assumption 1. *For any $i \in \overline{1,n},\ j \in \overline{1,M}$:*

$$\tau f_i^j(\tau) > 0, \quad \forall \tau \in \mathbb{R}\setminus\{0\}.$$

This assumption states that all nonlinearities belong to a sector and may take zero values at zero only, and it is the main restriction on the class of systems given in (6).

For further use, we denote by the index $m \in \overline{0,M}$, a positive integer such that for all $i \in \overline{1,n},\ a \in \overline{1,m}$:

$$\lim_{\tau\to\pm\infty} f_i^a(\tau) = \pm\infty$$

and by $\mu \in \overline{m,M}$, a positive integer such that for all $i \in \overline{1,n}$, $b \in \overline{1,\mu}$:

$$\lim_{\nu\to\pm\infty} \int_0^\nu f_i^b(\tau)d\tau = +\infty.$$

The index $m > 0$ characterizes the radially unbounded nonlinearities, and $m = 0$ corresponds to the case when all nonlinearities are bounded (at least for negative or positive arguments). The index $\mu > 0$ selects the nonlinearities having unbounded integrals. Clearly, if $m > 0$, then all radially unbounded nonlinearities also have unbounded integrals, thus $\mu \geq m$ due to the introduced sector condition. Indexes m and μ can be obtained after a proper re-indexing and decomposition of the f^j, and the featured restriction of (6) is formulated in Assumption 1 (the sector condition).

Remark 2. *The Lur'e models under the sector conditions [28], [29] may be presented in the form* (6) *under Assumption 1. The advantage of* (6) *over Lur'e dynamics is that all cross-terms between x_i and $f_i^j(x_i)$ appearing in the expressions of V and $\dot V$ can be accurately treated, rather than be considered as perturbations (see [13] or the next section). The same analysis in the conventional form of Lur'e model can be less straightforward (especially for $M > 1$ and for $f_i^j(x_i)f_i^k(x_i)$ with $j \neq k \in \overline{1,M}$).*

The principal goals of this work are to propose an approach to check IOS, ROS, and SIIOS properties for the generalized Persidskii system in (6), to apply the obtained conditions for the synchronization analysis in a family of systems as (6), and to design a nonlinear robust synchronization control in this framework.

IV. IOS CONDITIONS

The main result of this paper is as follows:

Theorem 2. *Let Assumption 1 be satisfied. If there exist $0 \leq P_1 = P_1^\top \in \mathbb{R}^{p\times p}$, $0 \leq P_2 = P_2^\top \in \mathbb{R}^{n\times n}$, $\Lambda^j = \mathrm{diag}(\Lambda_1^j, \ldots, \Lambda_n^j) \in \mathbb{D}_+^n$ for $j \in \overline{1,M}$, $\Theta \in \mathbb{D}_+^n$, $\Psi \in \mathbb{D}_+^p$, $\Xi^k \in \mathbb{D}_+^n$ for $k \in \overline{0,M}$, $\Upsilon_{s,z} \in \mathbb{D}_+^n$ for $s \in \overline{0,M-1},\ z \in \overline{s+1,M}$ and $0 < \Phi = \Phi^\top \in \mathbb{R}^{n\times n}$ such that*

$$P_1 > 0 \text{ or } P_2 > 0 \text{ or } \sum_{j=1}^{\mu}\Lambda^j > 0; \tag{7}$$

$$P_2 \leq \Theta,\ Q \leq 0,$$

where

$$Q = \begin{bmatrix} Q_{1,1} & Q_{1,2} & Q_{1,3} & \cdots & Q_{1,M+1} & P_C \\ Q_{1,2}^\top & Q_{2,2} & Q_{2,3} & \cdots & Q_{2,M+1} & \Lambda^1 \\ Q_{1,3}^\top & Q_{2,3}^\top & Q_{3,3} & \cdots & Q_{3,M+1} & \Lambda^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Q_{1,M+1}^\top & Q_{2,M+1}^\top & Q_{3,M+1}^\top & \cdots & Q_{M+1,M+1} & \Lambda^M \\ P_C & \Lambda^1 & \Lambda^2 & \cdots & \Lambda^M & -\Phi \end{bmatrix};$$

$$P_C = C^\top P_1 C + P_2;\quad Q_{1,1} = A_0^\top P_C + P_C A_0 + \Xi^0 + C^\top \Psi C;$$

$$Q_{j+1,j+1} = A_j^\top \Lambda^j + \Lambda^j A_j + \Xi^j,\ j \in \overline{1,M};\quad Q_{1,j+1} = P_C A_j + A_0^\top \Lambda^j + \Upsilon_{0,j},\ j \in \overline{1,M};$$

$$Q_{s+1,z+1} = A_s^\top \Lambda^z + \Lambda^s A_z + \Upsilon_{s,z},\ s \in \overline{1,M-1},\ z \in \overline{s+1,M},$$

then a forward complete system (6) *is ROS if*

$$\Psi > 0,\ \Theta + \sum_{j=1}^{M}\Lambda^j \leq \xi\left(\sum_{k=0}^{M}\Xi^k + 2\sum_{s=0}^{M-1}\sum_{z=s+1}^{M}\Upsilon_{s,z}\right),$$

or IOS if

$$P_1 \leq \xi\Psi,\ \Theta + \sum_{j=1}^{M}\Lambda^j \leq \xi\left(\sum_{k=0}^{m}\Xi^k + 2\sum_{s=0}^{m-1}\sum_{z=s+1}^{m}\Upsilon_{s,z}\right), \tag{8}$$

for some $\xi > 0$.

Proof. Consider a candidate Lyapunov function

$$V(x) = x^\top P_C x + 2\sum_{j=1}^{M}\sum_{i=1}^{n}\Lambda_i^j \int_0^{x_i} f_i^j(\tau)d\tau. \tag{9}$$

If $P_1 > 0$, then

$$y^\top P_1 y \leq V(x) \leq \alpha_2(\|x\|), \tag{10}$$

with $\alpha_2(\tau) \leq \lambda_{\max}(P_C)\tau^2 + 2nM \max_{i\in\overline{1,n},j\in\overline{1,M}}\left\{\Lambda_i^j \int_0^\tau f_i^j(\gamma)d\gamma\right\}$

a function from class $\mathcal{K}_\infty$, so (3) is verified. If instead, $P_2 > 0$ or $\sum_{j=1}^{\mu}\Lambda^j > 0$ (see (7)), then $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$ for a function $\alpha_1 \in \mathcal{K}_\infty$ (due to Assumption 1) and the definition of μ. Since $\|y\| \leq \|C\|\|x\|$ with $C \neq 0$, then (3) is again satisfied. Consider the derivative of V calculated for (6):

$$\begin{aligned} \dot V &= \dot x^\top P_C x + x^\top P_C \dot x + 2\sum_{j=1}^{M}\sum_{i=1}^{n}\Lambda_i^j f_i^j(x_i)\dot x_i \\ &= x^\top\left(A_0^\top P_C + P_C A_0\right)x + \left(\sum_{j=1}^{M} f^j(x)^\top A_j^\top\right)P_C x \\ &\quad + x^\top P_C \sum_{j=1}^{M} A_j f^j(x) + 2x^\top P_C d \\ &\quad + 2\sum_{j=1}^{M}\left(x^\top A_0^\top + d^\top + \left(\sum_{s=1}^{M} f^s(x)^\top A_s^\top\right)\right)\Lambda^j f^j(x). \end{aligned}$$

Therefore, under (7) we obtain

$$\begin{aligned}\dot V &= \begin{bmatrix} x \\ f^1(x) \\ \vdots \\ f^M(x) \\ d \end{bmatrix}^\top Q \begin{bmatrix} x \\ f^1(x) \\ \vdots \\ f^M(x) \\ d \end{bmatrix} - x^\top(C^\top \Psi C + \Xi^0)x \\ &\quad - \sum_{j=1}^M f^j(x)^\top \Xi^j f^j(x) - 2\sum_{j=1}^M x^\top \Upsilon_{0,j} f^j(x) \\ &\quad -2\sum_{s=1}^{M-1}\sum_{z=s+1}^{M} f^s(x)^\top \Upsilon_{s,z} f^z(x) + d^\top \Phi d \\ &\le -x^\top(C^\top\Psi C+\Xi^0)x - 2\sum_{s=1}^{M-1}\sum_{z=s+1}^{M} f^s(x)^\top \Upsilon_{s,z} f^z(x) \\ &\quad - \sum_{j=1}^M f^j(x)^\top \Xi^j f^j(x) - 2\sum_{j=1}^M x^\top \Upsilon_{0,j} f^j(x) + d^\top\Phi d.\end{aligned}$$

If $\Psi > 0$, then under the restriction $\frac{1}{2}y^\top \Psi y \ge d^\top \Phi d$ we conclude that

$$\begin{aligned}\dot V &\le -x^\top(\frac{1}{2}C^\top\Psi C+\Xi^0)x - \sum_{j=1}^M f^j(x)^\top \Xi^j f^j(x) \\ &\quad -2\sum_{j=1}^M x^\top \Upsilon_{0,j} f^j(x) - 2\sum_{s=1}^{M-1}\sum_{z=s+1}^{M} f^s(x)^\top \Upsilon_{s,z} f^z(x).\end{aligned}$$

Now we have to show that there exists $\alpha \in \mathcal{K}$ such that

$$\begin{aligned}\alpha(V(x)) &\le x^\top(\frac{1}{2}C^\top\Psi C+\Xi^0)x + \sum_{j=1}^M f^j(x)^\top \Xi^j f^j(x) \\ &+2\sum_{j=1}^M x^\top \Upsilon_{0,j} f^j(x) + 2\sum_{s=1}^{M-1}\sum_{z=s+1}^{M} f^s(x)^\top \Upsilon_{s,z} f^z(x), \end{aligned} \quad (11)$$

which is true taking into account the form of V and if

$$P_1 \le \xi\Psi,\ \Theta + \sum_{j=1}^M \Lambda^j \le \xi\left(\sum_{k=0}^M \Xi^k + 2\sum_{s=0}^{M-1}\sum_{z=s+1}^{M}\Upsilon_{s,z}\right)$$

for some $\xi > 0$. The latter properties are imposed in the theorem (the first inequality can be verified since $\Psi > 0$). Hence,

$$\frac{1}{2}y^\top\Psi y \ge d^\top\Phi d \Rightarrow \dot V \le -\alpha(V).$$

Recalling Remark 1, by Theorem 1, we conclude that the system is ROS. To ensure the IOS property, if the function $\alpha \in \mathcal{K}_\infty$ in (11), then the property (5) can be guaranteed:

$$V \ge \alpha^{-1}(2d^\top\Phi d) \Rightarrow \dot V \le -\frac{1}{2}\alpha(V),$$

which according to Theorem 1 and Remark 1 is necessary to substantiate (the condition (3) has been already verified). The function α can be selected in the required class under the introduced conditions (8) since only the first m nonlinearities and the quadratic term are radially unbounded. □

Remark 3. *In the case that IOS conditions are verified with $P_2 > 0$ or $\sum_{j=1}^{\mu}\Lambda^j > 0$, the system is UBIBS, and the requirement on forward completeness stated in Theorem 2 can be dropped.*

Remark 4. *The Lyapunov function* (9) *was frequently used in the absolute stability theory [28]–[30].*

To formulate the conditions of OLIOS or SIIOS for the system (6), note that according to Definition 3, the difference between the corresponding Lyapunov functions is in the form of the function α_3 only (it can belong to the class $\mathcal{KL}$ or $\mathcal{K}$). As we can conclude from the proof of Theorem 2, within the applied framework, only $\alpha_3 \in \mathcal{K}$ can be obtained. Hence, we have to restrict our analysis to the SIIOS case and the following additional hypothesis is needed:

Assumption 2. *For any $j \in \overline{1,m}$:*

$$x^\top C^\top C f^j(x) > 0,\ \forall x \in \mathbb{R}^n \setminus \{x \in \mathbb{R}^n : Cx = 0\}.$$

The last assumption assumes that all unbounded nonlinearities possess a kind of symmetry that Cf^j takes zero on the set where $y = 0$ only. Such a restriction is satisfied if, for example, $f_i^j(s) = f_1^j(s)$ for all $i \in \overline{2,n}$ and $j \in \overline{1,m}$, and $C = \Gamma$ as in (15), under which Assumption 2 is essentially an incremental passivity condition for all nonlinearities [31], [32].

Theorem 3. *Let assumptions 1 and 2 be satisfied and $C^\top C \in \mathbb{D}_+^n$. If there exist $0 < P_1 = P_1^\top \in \mathbb{R}^{p\times p}$; $\Lambda^j = \mathrm{diag}(\Lambda_1^j,\dots,\Lambda_n^j) \in \mathbb{D}_+^n$ with $\ker(\Lambda^j) = \ker(C)$ for $j \in \overline{1,M}$; $\Xi^k \in \mathbb{D}_+^p$ for $k \in \overline{0,M}$, $\Upsilon_{s,z} \in \mathbb{D}_+^p$ for $s \in \overline{0,M-1}$, $z \in \overline{s+1,M}$; and $0 < \Phi = \Phi^\top \in \mathbb{R}^{n\times n}$ such that*

$$Q \le 0,$$

where the matrix Q is given in Theorem 2 under substitutions $\Psi \to 0$, $\Upsilon_{s,z} \to C^\top\Upsilon_{s,z}C$ for $s \in \overline{0,M-1}$ and $z \in \overline{s+1,M}$, $\Xi^k \to C^\top\Xi^k C$ for $k \in \overline{0,M}$ with $P_2 = 0$, then a forward complete system (6) *is SIIOS if for some $\xi > 0$:*

$$P_1 \le \xi\Xi^0,\ \sum_{j=1}^M \Lambda^j \le \xi C^\top\left(\sum_{k=1}^m \Xi^k + 2\sum_{s=0}^{m-1}\sum_{z=s+1}^{m}\Upsilon_{s,z}\right)C. \quad (12)$$

Proof. Consider a candidate Lyapunov function

$$V(x) = x^\top C^\top P_1 C x + 2\sum_{j=1}^M\sum_{i=1}^n \Lambda_i^j \int_0^{x_i} f_i^j(\tau)d\tau,$$

then it is straightforward that (4) is verified for any $P_1 > 0$ and due to the imposed conditions on the kernels of Λ^j and C. The derivative of V calculated for (6) under the assumptions of the theorem can be upper estimated as follows for $Q \le 0$:

$$\begin{aligned}\dot V &\le -x^\top C^\top \Xi^0 C x - \sum_{j=1}^M f^j(x)^\top C^\top \Xi^j C f^j(x) \\ &\quad -2\sum_{j=1}^M x^\top C^\top \Upsilon_{0,j} C f^j(x) \\ &\quad -2\sum_{s=1}^{M-1}\sum_{z=s+1}^{M} f^s(x)^\top C^\top \Upsilon_{s,z} C f^z(x) + d^\top\Phi d\end{aligned}$$

and for the condition (12) there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\dot V \le -\alpha(V) + d^\top \Phi d,$$

which according to Theorem 1 and Remark 1 implies in our case SIIOS. □

The conditions of both theorems, 2 and 3, can be combined and also used for stability analysis (with $d(t) = 0$ for all $t \geq 0$):

Corollary 1. *Let assumptions 1 and 2 be satisfied and there exist* $0 \leq P_1 = P_1^\top \in \mathbb{R}^{p\times p}$, $0 \leq P_2 = P_2^\top \in \mathbb{R}^{n\times n}$; $\Xi^k \in \mathbb{D}_+^p$ *for* $k \in \overline{0,M}$; $\Lambda^j \in \mathbb{D}_+^n$ *for* $j \in \overline{1,M}$; $\Upsilon_{s,z} \in \mathbb{D}_+^p$ *for* $s \in \overline{0,M-1}$ *and* $z \in \overline{s+1,M}$ *such that*

$$P_2 > 0 \text{ or } \sum_{q=1}^{\mu} \Lambda^q > 0;\ Q \leq 0,$$

where

$$Q = \begin{bmatrix} Q_{1,1} & Q_{1,2} & Q_{1,3} & \cdots & Q_{1,M+1} \\ Q_{1,2}^\top & Q_{2,2} & Q_{2,3} & \cdots & Q_{2,M+1} \\ Q_{1,3}^\top & Q_{2,3}^\top & Q_{3,3} & \cdots & Q_{3,M+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Q_{1,M+1}^\top & Q_{2,M+1}^\top & Q_{3,M+1}^\top & \cdots & Q_{M+1,M+1} \end{bmatrix};$$

$$Q_{1,1} = A_0^\top P_C + P_C A_0 + C^\top \Xi^0 C; \quad P_C = C^\top P_1 C + P_2;$$

$$Q_{j+1,j+1} = A_j^\top \Lambda^j + \Lambda^j A_j + C^\top \Xi^j C,\ j \in \overline{1,M};$$

$$Q_{1,j+1} = P_C A_j + A_0^\top \Lambda^j + C^\top \Upsilon_{0,j} C,\ j \in \overline{1,M};$$

$$Q_{s+1,z+1} = A_s^\top \Lambda^z + \Lambda^s A_z + C^\top \Upsilon_{s,z} C,\ s \in \overline{1,M-1},\ z \in \overline{s+1,M}.$$

Then the system (6) *is UBIBS and* $\lim_{t\to+\infty} \|y(t,x_0,0)\| = 0$ *for all* $x_0 \in \mathbb{R}^n$ *if*

$$\sum_{k=0}^{M} \Xi^k + 2\sum_{s=0}^{M-1}\sum_{z=s+1}^{M} \Upsilon_{s,z} > 0.$$

Proof. Consider the Lyapunov function (9) with $P_C = C^\top P_1 C + P_2$. If $P_2 > 0$ or $\sum_{z=1}^{\mu} \Lambda^z > 0$, then

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

for some functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, due to Assumption 1 and the definition of μ. Hence, such a Lyapunov function is positive definite and radially unbounded. The derivative of V calculated for (6) with $d = 0$ can be upper estimated as follows for $Q \leq 0$:

$$\dot{V} \leq -x^\top C^\top \Xi^0 C x - \sum_{j=1}^{M} f^j(x)^\top C^\top \Xi^j C f^j(x)$$
$$-2\sum_{j=1}^{M} x^\top C^\top \Upsilon_{0,j} C f^j(x) - 2\sum_{s=1}^{M-1}\sum_{z=s+1}^{M} f^s(x)^\top C^\top \Upsilon_{s,z} C f^z(x).$$

Since $\sum_{k=0}^{M} \Xi^k + 2\sum_{s=0}^{M-1}\sum_{z=s+1}^{M} \Upsilon_{s,z} > 0$ and due to assumptions 1 and 2, there exists a function $\alpha \in \mathcal{K}$ such that

$$\dot{V} \leq -\alpha(\|y\|).$$

The proven properties of V and the fact that $\dot{V} \leq 0$ implies that all solutions of (6) are bounded: $\|x(t,x_0,0)\| \leq \alpha_1 \circ \alpha_2^{-1}(\|x_0\|)$ for all $x_0 \in \mathbb{R}^n$ and all $t \geq 0$. Applying standard LaSalle arguments [4], we obtain $\lim_{t\to+\infty} \|y(t,x_0,0)\| = 0$, for all $x_0 \in \mathbb{R}^n$. ◻

Remark 5. *A minor modification of the conditions given in this section is needed if*

$$y(t) = \begin{bmatrix} x(t)^\top C_0^\top & (f^1(x(t)))^\top C_1^\top & \ldots & (f^M(x(t)))^\top C_M^\top \end{bmatrix}^\top.$$

The concepts of IOS and SIIOS can be used for many analysis and design problems, e.g.*, for synchronization or estimation, and the former issue is considered below.*

V. Synchronization

A. Synchronization of a family of generalized Persidskii systems

In this subsection, we consider an application of the previously proposed theory.

1) Family of generalized Persidskii systems: Consider a family of $N \geq 2$ systems of the following form:

$$\dot{x}_z(t) = A_{z,0} x_z(t) + \sum_{j=1}^{M} A_{z,j} f^j(x_z(t)) + B_z u_z(t) \tag{13}$$
$$+ d_z(t),\ z \in \overline{1,N},\ \forall t \geq 0,$$

where $x_z(t) = [x_{z,1}(t) \ldots x_{z,n}(t)]^\top \in \mathbb{R}^n$ is the state vector of a subsystem, $A_{z,s} \in \mathbb{R}^{n\times n}$ for $s \in \overline{0,M}$, $B_z \in \mathbb{R}^{n\times r}$, $u_z(t) = [u_{z,1}(t) \ldots\ u_{z,r}(t)]^\top \in \mathbb{R}^r$ is the controlled input, and $d_z(t) \in \mathbb{R}^n$ is the external perturbation, $d_z \in \mathcal{L}_\infty^n$; $f^j(x_z(t)) = [f_1^j(x_{z,1}(t)) \ldots f_n^j(x_{z,n}(t))]^\top$ for $j \in \overline{1,M}$ are the functions ensuring existence of the solutions of the system (13) in the forward time at least locally. The sector restrictions on f^j, $j \in \overline{1,M}$ are imposed as in Assumption 1.

In this study, we consider the synchronization of the common dynamics of (13), *i.e.*, a system in the following form:

$$\dot{X}(t) = A_0 X(t) + \sum_{j=1}^{M} A_j F^j(X(t)) + BU(t) + d(t), \tag{14}$$

where $X(t) = [x_1(t)^\top \ldots x_N(t)^\top]^\top \in \mathbb{R}^{Nn}$ is the state vector, $A_s = \mathrm{diag}(A_{1,s} \ldots A_{N,s}) \in \mathbb{R}^{Nn\times Nn}$ for $s \in \overline{0,M}$, $B = \mathrm{diag}(B_1 \ldots B_N) \in \mathbb{R}^{Nn\times Nr}$, $U(t) = [u_1(t)^\top \ldots u_N(t)^\top]^\top \in \mathbb{R}^{Nr}$ is the controlled input, $d(t) = [d_1^\top(t) \cdots d_N^\top(t)]^\top \in \mathbb{R}^{Nn}$ is the common perturbation, $d \in \mathcal{L}_\infty^{Nn}$; $F^j(X(t)) = [f^j(x_1(t))^\top \ldots f^j(x_N(t))^\top]^\top \in \mathbb{R}^{Nn}$ for $j \in \overline{1,M}$. Clearly, the functions F^j, $j \in \overline{1,M}$ also satisfy the sector condition. We denote the *consensus set* of (13) as

$$\mathcal{W} := \left\{X \in \mathbb{R}^{Nn} \mid x_i = x_1 \quad \text{for} \quad i \in \overline{2,N}\right\}$$

and we say that (14) is in the *synchronous mode* if $X(t) \in \mathcal{W}$, for all $t \geq 0$. To quantify the closeness of the system to the synchronous regime, we use a synchronization measure: a continuously differentiable function $\rho : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ such that $\rho(X) = 0$ implies that $X \in \mathcal{W}$. The presence of the disturbances $d \neq 0$ does not allow the system to be in the synchronous mode.

Then the robust synchronization problem can be set: to design a feedback $U = U(X)$ that renders the system (14) to be IOS with respect to the output ρ and the input d. If $d = 0$, then such a control U pushes (14) to the synchronous mode.

2) Conditions of synchronization: In this study, the synchronization measure $\rho(X)$ is defined as

$$\rho(X) = \Gamma X, \tag{15}$$

$$\text{where } \Gamma = \begin{bmatrix} -I_n & I_n & 0 & \cdots & 0 \\ 0 & -I_n & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -I_n & I_n \\ I_n & 0 & \cdots & 0 & -I_n \end{bmatrix} \in \mathbb{R}^{Nn\times Nn}.$$

Note that due to properties of F^j, in the synchronization mode $\Gamma F^j(X) = 0$, for all $j \in \overline{1,M}$ and $X \in \mathcal{W}$, *i.e.*, an analog of Assumption 2 is satisfied for F^j, $j \in \overline{1,M}$.

The feedback to robustly synchronize the system (14) (to stabilize the system (14) in IOS sense) is selected in the form of diffusive coupling:

$$U = K_0 \Gamma X + \sum_{j=1}^{M} K_j \Gamma F^j(X) \tag{16}$$

with $K_k \in \mathbb{R}^{rN \times nN}$ for $k \in \overline{0, M}$ designed below.

Remark 6. *The control* (16) *can also be selected in the form of direct coupling [33]:*

$$U = K_0 X + \sum_{j=1}^{M} K_j F^j(X),$$

i.e.*, the coupling is diffusive if it is proportional to the synchronization measure* ρ *as in* (16)*, and it is direct if it is given in the form of a generic state feedback. Both types of coupling can be analyzed in the proposed framework, but for brevity the synchronization conditions are formulated below for the diffusive case only.*

Substituting the control (16) into the equations of the system (14) we obtain the following closed-loop dynamics:

$$\dot{X}(t) = \tilde{A}_0 X(t) + \sum_{j=1}^{M} \tilde{A}_j F^j(X(t)) + d(t),\ Y(t) = \Gamma X(t), \tag{17}$$

where $\tilde{A}_k = A_k + BK_k\Gamma$ for $k \in \overline{0, M}$.

Clearly, the system (17) is in the form (6) and assumptions 1 and 2 are satisfied, then theorems 2 and 3 or Corollary 1 can be directly applied.

Corollary 2. *If the IOS conditions of Theorems 2 are satisfied under the substitution of* $p \to nN, n \to nN, C \to \Gamma, A_k \to \tilde{A}_k$*,* $k \in \overline{0, M}$*, then a forward complete system* (17) *is robustly synchronized.*

Proof. As we remarked above, assumptions 1 and 2 are verified by the system (17), and it is forward complete due to hypotheses of the corollary. Then, the IOS property guarantees boundedness of the synchronization error ρ in the presence of essentially bounded perturbations $d \neq 0$, and asymptotic convergence of the synchronization error to zero for $d = 0$ (that corresponds to the achievement of the synchronous mode). □

Corollary 3. *If the conditions of Corollary 1 are satisfied under the substitution of* $p \to nN, n \to nN, C \to \Gamma, A_k \to \tilde{A}_k$*,* $k \in \overline{0, M}$*, then the system* (17) *with* $d(t) = 0, \forall t \in \mathbb{R}_+$ *reaches the synchronous mode.*

Proof. It is a direct consequence of Corollary 1 since assumptions 1 and 2 hold. □

B. Robust synchronization of linear systems

Let us consider how the control gains $K_k \in \mathbb{R}^{rN \times nN}$ for $k \in \overline{0, M}$ can be designed to ensure synchronization.

For brevity, in this subsection, we consider the robust synchronization of two linear systems

$$\dot{x} = \begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \end{bmatrix} = Ax + Bu + d \tag{18}$$

where $x^1, x^2 \in \mathbb{R}^n$ are the states, $A \in \mathbb{R}^{2n \times 2n}$, $B \in \mathbb{R}^{2n \times m}$, $u \in \mathbb{R}^m$ is the controlled input, $d \in \mathbb{R}^{2n}$ is the external perturbation, and we assume two scenarios: either $d \in \mathscr{L}_\infty^{2n}$ or d is a nonlinear function of the state x admitting an upper bound

$$\|d\|^2 \le \sum_{i=1}^{2n} R_i^0 |x_i|^2 + R_i^1 |x_i|^{1+\zeta} + R_i^2 |x_i|^{1+\pi}, \tag{19}$$

where $\zeta \in (0,1)$, $\pi > 1$ are growth parameters and $R^s = \mathrm{diag}(R_1^s \dots R_{2n}^s) \in \mathbb{D}_+^{2n}$ are given matrices for $s \in \overline{0, 2}$. In the latter case (18) is a nonlinear system, and if $R^1 \neq 0$ or $R^2 \neq 0$, then a linear feedback cannot ensure robust synchronization of this system (in the sense of IOS), while Corollary 2 provides a tool for synchronization of the system (18) with such a disturbance.

For (18) we propose to use a feedback control in the form

$$u = K_0 \Gamma x + K_1 \Gamma f^1(x) + K_2 \Gamma f^2(x),$$

where $K_0, K_1, K_2 \in \mathbb{R}^{m \times 2n}$ are the tuning gains,

$$\Gamma = \begin{bmatrix} -I_n & I_n \\ I_n & -I_n \end{bmatrix}$$

is the matrix defining synchronization measure for $N = 2$ (in such case we may take $\Gamma = [I_n \ -I_n]$ without losing generality), and f^1, f^2 are the functions following the imposed conditions of the system (6) and Assumption 1:

$$f_i^1(x_i) = |x_i|^\zeta \operatorname{sign}(x_i),\ f_i^2(x_i) = |x_i|^\pi \operatorname{sign}(x_i),$$
$$f^j(x) = [f_1^j(x_1), \dots, f_{2n}^j(x_{2n})],\ \forall i \in \overline{1, 2n},\ j \in \overline{1, 2}.$$

Then the resulting closed-loop system is

$$\dot{x} = A_0 x + A_1 f^1(x) + A_2 f^2(x) + d, \tag{20}$$

where $A_0 = A + BK_0\Gamma$, $A_1 = BK_1\Gamma$ and $A_2 = BK_2\Gamma$. Using the same arguments as in subsection V-A2, we define the output function, or the synchronization measure, of (20) as

$$y(t) = \Gamma x(t).$$

Applying the Lyapunov function from Theorem 2:

$$V(x) = x^\top P_\Gamma x + 2 \sum_{z=1}^{2} \sum_{j=1}^{2} \sum_{i=1}^{n} \Lambda_i^j \int_0^{x_i^z} f_i^j(\tau) d\tau, \tag{21}$$

where $P_\Gamma = \Gamma^\top P_1 \Gamma + P_2$, for the system (20) its derivative is calculated as

$$\dot{V}(x) = \begin{bmatrix} x \\ f^1(x) \\ f^2(x) \\ d \end{bmatrix}^\top \mathcal{Q} \begin{bmatrix} x \\ f^1(x) \\ f^2(x) \\ d \end{bmatrix} - x^\top (\Xi^0 + \Gamma^\top \Psi \Gamma) x + \phi d^\top d$$
$$- 2 \sum_{j=1}^{2} x^\top \Upsilon_{0,j} f^j(x) + 2 \sum_{j=1}^{2} x^\top (P_\Gamma A_j + A_0^\top \Lambda^j + \Upsilon_{0,j}) f^j(x),$$

where $\Xi^0, \Upsilon_{0,j}, \Lambda^j$ are given in the formulation of Theorem 2, $\Phi = \phi I_{2n}$ and

$$\mathcal{Q} = \begin{bmatrix} A_0^\top P_\Gamma + P_\Gamma A_0 + \Xi^0 + \Gamma^\top \Psi \Gamma & 0 & 0 & P_\Gamma \\ 0 & A_1^\top \Lambda^1 + \Lambda^1 A_1 & A_1^\top \Lambda^2 + \Lambda^1 A_2 & \Lambda^1 \\ 0 & A_2^\top \Lambda^1 + \Lambda^2 A_1 & A_2^\top \Lambda^2 + \Lambda^2 A_2 & \Lambda^2 \\ P_\Gamma & \Lambda^1 & \Lambda^2 & -\phi I_{2n} \end{bmatrix}$$

for some $\phi > 0$. For the last term in $\dot{V}$, applying Young's inequality for all cross-terms out the main diagonal:

$$x_i |x_k|^{\zeta} \operatorname{sign}(x_k) \le \frac{|x_i|^{1+\zeta}}{1+\zeta} + \frac{\zeta |x_k|^{1+\zeta}}{1+\zeta},$$
$$x_i |x_k|^{\pi} \operatorname{sign}(x_k) \le \frac{|x_i|^{1+\pi}}{1+\pi} + \frac{\pi |x_k|^{1+\pi}}{1+\pi}$$

for any $i \neq k \in \overline{1,2n}$, we obtain that if

$$\mathcal{Q} \le 0, \tag{22}$$
$$\mathbf{1}_{2n}^{\top}[(1+\zeta)\delta(P_\Gamma A_1 + A_0^{\top}\Lambda^1) + \zeta\omega(P_\Gamma A_1 + A_0^{\top}\Lambda^1) + \omega^{\top}(P_\Gamma A_1 + A_0^{\top}\Lambda^1) + \Upsilon_{0,1}] \le 0, \tag{23}$$
$$\mathbf{1}_{2n}^{\top}[(1+\pi)\delta(P_\Gamma A_2 + A_0^{\top}\Lambda^2) + \pi\omega(P_\Gamma A_2 + A_0^{\top}\Lambda^2) + \omega^{\top}(P_\Gamma A_2 + A_0^{\top}\Lambda^2) + \Upsilon_{0,2}] \le 0, \tag{24}$$

where $\delta(\mathscr{A})$ denotes the diagonal matrix having the diagonal elements of $\mathscr{A}$, and $\omega(\mathscr{A})$ has zero diagonal elements and absolute values of other elements of $\mathscr{A}$, then

$$x^{\top}(P_\Gamma A_j + A_0^{\top}\Lambda^j + \Upsilon_{0,j}) f^j(x) \le 0, \ \forall j \in \overline{1,2}$$

hence, $\dot{V} \le -x^{\top}(\Xi^0 + \Gamma^{\top}\Psi\Gamma)x - 2\sum_{j=1}^{2} x^{\top}\Upsilon_{0,j} f^j(x) + \phi d^{\top} d.$

This allows us to present the main result of this section:

Theorem 4. *Given $K_0, K_1, K_2 \in \mathbb{R}^{m\times 2n}, \zeta \in (0,1)$ and $\pi > 1$, if there exist $0 \le P_1 = P_1^{\top} \in \mathbb{R}^{2n\times 2n}$, $0 \le P_2 = P_2^{\top} \in \mathbb{R}^{2n\times 2n}$, $\Lambda^j = \operatorname{diag}(\Lambda_1^j, \dots, \Lambda_{2n}^j) \in \mathbb{D}_+^{2n}$ for $j \in \overline{1,2}$, $\Theta \in \mathbb{D}_+^{2n}$, $\Psi \in \mathbb{D}_+^{2n}$, $\Xi^k \in \mathbb{D}_+^{2n}$ for $k \in \overline{0,2}$, $\Upsilon_{s,z} \in \mathbb{D}_+^{2n}$ for $s \in \overline{0,1}$, $z \in \overline{s+1,2}$ and $\phi > 0$ such that*

$$P_1 > 0 \text{ or } P_2 > 0 \text{ or } \sum_{j=1}^{2}\Lambda^j > 0; \ P_2 \le \Theta,$$

(22), (23) and (24) are satisfied, then a forward complete system (20) is IOS (robustly synchronized) if

$$P_1 \le \xi\Psi, \ \Theta + \sum_{j=1}^{2}\Lambda^j \le \xi\left(\Xi^0 + 2\sum_{j=1}^{2}\Upsilon_{0,j}\right)$$

for some $\xi > 0$. If, additionally,

$$\Xi^0 + \Gamma^{\top}\Psi\Gamma > \phi R^0, \ 2\Upsilon_{0,j} > \phi R^j, \ j = 1,2, \tag{25}$$

then for (19) the system is asymptotically reaching the synchronous mode.

Proof. Assume that there exists a function $\alpha \in \mathscr{K}_\infty$ such that

$$2\alpha(V) \le x^{\top}(\Xi^0 + \Gamma^{\top}\Psi\Gamma)x + 2\sum_{j=1}^{2} x^{\top}\Upsilon_{0,j} f^j(x),$$

then under the restriction $V(x) \ge \alpha^{-1}(\phi d^{\top} d)$, we get $\dot{V} \le -\alpha(V)$.

The selection of $\alpha \in \mathscr{K}_\infty$ follows the conditions:

$$P_1 \le \xi\Psi, \ \Theta + \sum_{j=1}^{2}\Lambda^j \le \xi\left(\Xi^0 + 2\sum_{j=1}^{2}\Upsilon_{0,j}\right)$$

for some $\xi > 0$. The remaining steps repeat the proof of Theorem 2. If the perturbation d satisfies (19), *i.e.*, $d^{\top} d \le x^{\top} R^0 x + x^{\top} R^1 f^1(x) + x^{\top} R^2 f^2(x)$, then for (25) under the same conditions we get that $\dot{V} \le -\epsilon\alpha(V)$ for some $\epsilon \in (0,1)$, implying global stability and convergence of the output Γx to zero. ◻

VI. Application

The Hindmarsh-Rose (HR) model [34] is widely used to investigate chaotic behavior in isolated biological cells and neuronal dynamics (being a compact version of the general case [35]):

$$\begin{aligned} \dot{x}_1 &= a x_1^2 - x_1^3 - x_2 + x_3 + d, \\ \dot{x}_2 &= (a+\alpha)x_1^2 - x_2, \\ \dot{x}_3 &= \mu(b x_1 - x_3) + u, \end{aligned} \tag{26}$$

where $x = [x_1\, x_2\, x_3]^{\top} \in \mathbb{R}^3$ is the state, $d \in \mathbb{R}$ is the disturbance (equivalently applied current in experiments), $u \in \mathbb{R}$ is the control and $a, \alpha, \mu, b \in \mathbb{R}$. Let $\theta > \frac{1}{4}$ be an auxiliary parameter. Then the system (26) can be rewritten as

$$\dot{x} = \alpha_0 x + \alpha_1 f^1(x) + \alpha_2 f^2(x) + \tilde{b}u + \tilde{d}, \tag{27}$$

where $\tilde{b} = [0\ 0\ 1]^{\top}$, $\tilde{d} = [d\ 0\ 0]^{\top}$,

$$\alpha_0 = \begin{bmatrix} -a\theta & -1 & 1 \\ -(a+\alpha)\theta & -1 & 0 \\ \mu b & 0 & -\mu \end{bmatrix}, \ \alpha_1 = \begin{bmatrix} -1-a & 0 & 0 \\ -a-\alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$\alpha_2 = \begin{bmatrix} a & 0 & 0 \\ a+\alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ f^1(x) = \begin{bmatrix} x_1^3 \\ x_2^3 \\ x_3^3 \end{bmatrix}, \ f^2(x) = \begin{bmatrix} x_1(x_1^2 + x_1 + \theta) \\ x_2(x_2^2 + x_2 + \theta) \\ x_3(x_3^2 + x_3 + \theta) \end{bmatrix},$$

the new nonlinearities f^1 and f^2 satisfy the sector condition given in Assumption 1. Let us set the number of systems in the family $N = 2$, $a = 2.8$, $d = 3.1$, $\alpha = 1.6$, $\mu = 10^{-3}$, $b = 9$ and $\theta = 0.3$. Therefore, the common dynamics of models (27) is

$$\dot{X} = A_0 X + A_1 F^1(X) + A_2 F^2(X) + BU + D, \tag{28}$$

where

$$X = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \in \mathbb{R}^6, \ U = \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \in \mathbb{R}^2, \ F^j(X) = \begin{bmatrix} f^j(x^1) \\ f^j(x^2) \end{bmatrix}, \forall j \in \overline{1,2},$$
$$B = \begin{bmatrix} \tilde{b} & 0 \\ 0 & \tilde{b} \end{bmatrix}, \ A_s = \begin{bmatrix} \alpha_s & 0 \\ 0 & \alpha_s \end{bmatrix}, \forall s \in \overline{0,2}, \ D = \begin{bmatrix} \tilde{d}^1 \\ \tilde{d}^2 \end{bmatrix}$$

and $x^1, x^2 \in \mathbb{R}^3$ are the solutions of each of the couples HR models (27). Evidently, the system (28) is in the form (14). Consider a feedback control in the form (16), then U is a vector of scalar controls affecting the HR model to synchronize the system (28), we obtain the closed-loop system in the form (17)

$$\dot{X} = (A_0 + BK_0\Gamma)X + (A_1 + BK_1\Gamma)F^1(X) + (A_2 + BK_2\Gamma)F^2(X) + D.$$

The synchronization measure is selected as (15) with $\Gamma = \begin{bmatrix} I_3 & -I_3 \end{bmatrix}$. Let

$$K_0 = \begin{bmatrix} 0.4283 & 0.4820 & 0.1206 \\ 0.5895 & 0.2262 & 0.3846 \end{bmatrix},$$
$$K_1 = \begin{bmatrix} 0 & 0 & 0.5830 \\ 0 & 0 & 0.2518 \end{bmatrix}, \ K_2 = \begin{bmatrix} 0 & 0 & 0.2904 \\ 0 & 0 & 0.6171 \end{bmatrix},$$

then following Corollary 2, there exist matrices solving the proposed LMIs in Theorem 2. The norm of the difference $e := x^1 - x^2$ and the state trajectories x^1, x^2 of the closed-loop system with distinct initial states $x^1(0) = \begin{bmatrix} 0.12 & -0.21 & 0.80 \end{bmatrix}^{\top}$,

$x^2(0) = \begin{bmatrix} 0.41 & 0.91 & 0.88 \end{bmatrix}^\top$ are shown in Fig. 1 and Fig. 2, respectively. The simulation results imply that the system (28) is synchronized by the feedback controller, while each separate subsystem remains oscillating.

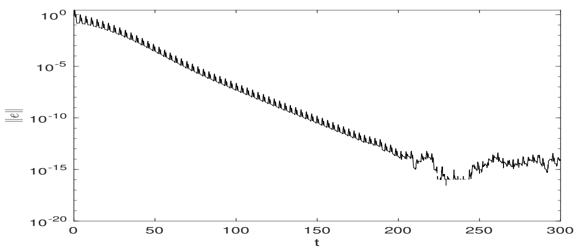

Fig. 1. The norm of the synchronization error e versus the time t

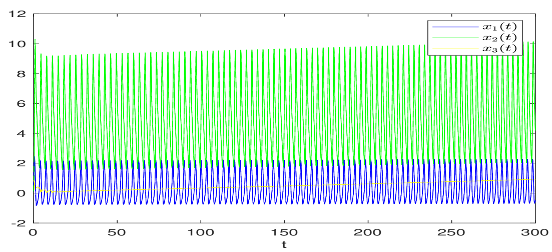

Fig. 2. The state trajectories x^1 and x^2 versus the time t

VII. Conclusion

In this paper, we proposed IOS, SIIOS, and ROS conditions for generalized Persidskii systems. The conditions were obtained in the form of LMIs. As an application of the developed approach, the problem of the robust synchronization of a family of generalized Persidskii systems was solved. Furthermore, a robust synchronization method was proposed for linear systems with essentially bounded perturbations also admitting a hard upper bound in terms of the state components. The efficiency of the proposed results was illustrated through the synchronization of Hindmarsh-Rose models.

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