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Received 26 April 2016; | Revised 6 June 2016; | Accepted 6 June 2016

DOI: xxx/xxxx

ARTICLE TYPE

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Summary

This work addresses the problem of robust output feedback model predictive control (MPC) of constrained, discrete-time, time-delayed linear systems subject to (bounded) additive disturbances. The proposed predictive controller incorporates an interval observer, that exploits the available measurement to update the set-membership of the system states, and an interval predictor, used in the prediction step of the MPC. The proposed algorithm is similar to conventional MPC for linear systems, offering guarantees on recursive feasibility and robust constraint satisfaction, with low computational complexity. Furthermore, the delay is explicitly taken into account in the design, and the obtained scheme easily encompasses delays on the state, on the measurement or the control input.

KEYWORDS:
Predictive control, time delay, robust control

1 | INTRODUCTION

Many real-world applications, such as bio-chemical processes, mechanical, thermal or transportation systems, present delayed dynamics. For this reason, control and observation of time-delayed systems (TDS) have been an active field of research over the last decades[1]. Indeed, the presence of delay imposes serious challenges, since it often leads to a deterioration of performance and stability of control systems[2].

Besides, if the dynamical system to be controlled is subject to constraints (on the states, inputs or outputs), the control design is known to be a difficult task – or even impossible – to be tackled using classical feedback solutions[3]. However, constraints are recurrent in many industrial systems and are often related to physical limitations, safety or performance requirements.

A popular tool to deal with constrained control is Model Predictive Control (MPC)[4]. The MPC algorithm solves, at each decision instant, an online (constrained) optimal control problem (OCP). The MPC algorithm exploits the available information of the system's states and predicts, over a time window of finite length, its future behaviour through a dynamical model. Then, in a receding-horizon fashion, the first control input from the obtained optimal sequence is applied to the real plant.

Although intuitive, the MPC algorithm faces some challenges if the system to be controlled is uncertain: sources of error such as unmodelled dynamics, disturbances, measurement noises, and unknown or unaccounted delays, cause a mismatch between the prediction (obtained by the model) and the real system. Furthermore, MPC requires a full-state measurement, which is not always available. If estimated variables are used, even more uncertainty is added to the problem due to inherent estimation errors and leads to the problem of *output feedback* MPC. In this sense, an MPC scheme is *robust*[5] if it accomplishes the control task while reliably guaranteeing constraint satisfaction for *any* realization of these uncertainties in a given range.

†This work is partially supported by IPL CoSY, and by the Ministry of Science and Higher Education of Russian Federation, passport of goszadanie no. 2019-0898.

Over the last decades, several results concerning MPC algorithms for TDS have been proposed. Although considering delays on state and/or control inputs, most of these works require full-state measurement and impose, most commonly, constraints *only* on the input signal. This is not surprising, since one of the main challenges of state-feedback control for time-delayed systems relates to input saturation (caused by *after-effects*[1]). These algorithms define the optimization problem (similarly as for non-delayed systems) in terms of Linear Matrix Inequalities (LMIs): assuming known, as in Kothare et al.[6], and uncertain but bounded delays, as in Park et al.[7]. The case of delayed control input has also been reported by Park et al.[8] and Shi et al.[9]. In a different setting, the nonlinear case (considering no terminal constraints) has been extensively investigated by Reble[10]. Only Ding et al.[11] proposed a robust MPC scheme considering both input and state constraints, also accounting for parametric uncertainties.

However, to the best of our knowledge, no MPC algorithm has been proposed for constrained TDS considering the output-feedback case (*i.e.*, the states are not directly available for measurement). In this sense, the contribution of this work is twofold: propose new interval estimators (an observer and a predictor) and develop an algorithm solving the output-feedback MPC problem for linear, discrete-time, time-delayed systems with both state and control constraints and additive perturbations. The proposed MPC algorithm follows a recent idea applied to linear time-invariant (LTI) systems[12], in which the novelty relies on the use of interval observers (IO)[13]: an estimator that offers a computationally inexpensive way to compute the set-membership of the system states (in the form of *intervals*) with guaranteed stability.

Indeed, by incorporating an interval predictor (IP) (an open-loop interval estimator, *i.e.*, independent of measurements) in the prediction step of the MPC, we can easily guarantee that the constraints will be respected for all possible trajectories of the real perturbed system, including those related to the states which are not directly measured. We show that the proposed algorithm explicitly takes the (known) delay into account, and encompasses scenarios in which the delays appear on the state, on the measurement signal or the control input.

This paper is organized as follows: Section 2 states the problem to be solved, the design of the interval estimators is discussed in Section 3, where offline conditions based on LMIs are given. Section 4 presents the novel MPC algorithm to be solved online, guaranteeing stability and robust constraint satisfaction, while Section 5 discusses its complexity and the performance. Finally, a numerical example illustrates the methodology in Section 6, and conclusions and future directions are given in Section 7.

NOTATION:

- The sets of real and integer numbers are defined by $\mathbb{R}$ and $\mathbb{Z}$, respectively, then $|\cdot|$ represents the absolute value for an element of these sets; $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$ and $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $\|x\|$.

- A matrix M is said to be non-negative if all of its elements are non-negative. A matrix M is said to be Schur stable if all of its eigenvalues have absolute value less than one. The identity matrix of dimension n is defined by $\mathcal{I}_n$. For a symmetric matrix A, the symmetric entry (*i.e.,* $A_{i,j} = A_{j,i}$) is denoted by $\star$. The transpose (resp. the inverse transpose) of the matrix A is denoted by $A^\top$ (resp. $A^{-\top}$). We denote $A = \text{diag}(a_1, \dots, a_n)$ and $V = \text{vec}(v_1, \dots, v_v) \in \mathbb{R}^n$ as, respectively, the diagonal matrix with block entries $A_{ii} = a_i$, and the vector composed by the concatenation of each vector $v_i \in \mathbb{R}^{n_i}, n = \sum_{i=1}^{v} n_i$. The eigenvalues of a matrix A are denoted by $\Lambda(A)$.

- For a function $x : \mathbb{Z}_+ \to \mathbb{R}^n$, we use the convention $x_k = x(k)$ and denote $|x|_\infty = \sup_{k \in \mathbb{Z}_+} \|x_k\|$. Furthermore, we define as ℓ_∞^n the set of all sequences such that $|x|_\infty < \infty$.

- Let $x_1, x_2 \in \mathbb{R}^n$ be two vectors and $A_1, A_2 \in \mathbb{R}^{n\times n}$ be two matrices, then the relations $x_1 \leq x_2$ and $A_1 \leq A_2$ are to be understood component-wise. For a matrix A we define $A^+ = \max\{0, A\}$ (also understood component-wise), $A^- = A^+ - A$ (similarly for vectors), and also denote the matrix of absolute values of all elements by $|A| = A^+ + A^-$. Furthermore, for a symmetric matrix $A \in \mathbb{R}^{n\times n}$ the relation $A \prec 0$ (resp. $A \succeq 0$) means that $A \in \mathbb{R}^{n\times n}$ is negative (resp. positive semi-) definite.

2 | PROBLEM STATEMENT

Consider the following uncertain, linear, discrete-time, delayed system:

$$\begin{aligned} x_{k+1} &= A_0 x_k + A_1 x_{k-h} + B u_k + w_k, \quad k \in \mathbb{Z}_+ \\ x_k &= \phi_k, \quad k \in [-h, \dots, 0] \\ y_k &= C x_k + v_k \end{aligned} \tag{1}$$

where $x_k, x_{k-h} \in \mathbb{R}^n$ are, respectively, the current and the delayed values of the state with a known delay $h > 0$, $u_k \in \mathbb{R}^m$ is the control input, $y_k \in \mathbb{R}^p$ is the measured output, $v \in \ell_\infty^p$ is the measurement noise, $w \in \ell_\infty^n$ is the process disturbance. The matrices $A_0, A_1 \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are known. The following hypotheses are imposed:

Assumption 1. Initial conditions of (1) are bounded such as $\underline{\phi}_k \le \phi_k \le \overline{\phi}_k$, $k \in [-h, \dots, 0]$, for some known vectors $\underline{\phi}_k, \overline{\phi}_k \in \mathbb{R}^n$. Furthermore, the additive perturbations $w_k \in [\underline{w}_k, \overline{w}_k]$ and $v_k \in [\underline{v}_k, \overline{v}_k]$ for all $k \in \mathbb{Z}_+$, where $\underline{w}, \overline{w} \in \ell_\infty^n$ and $\underline{v}, \overline{v} \in \ell_\infty^p$ are known signals.

Assumption 1 is usual in the design of IOs and means that the sources of uncertainty in (1) (*i.e.*, initial conditions, w_k and v_k) are enclosed in (known) bounded intervals. The following hypothesis (which can always be verified by means of a change of coordinates) is imposed:

Assumption 2. Let $C \ge 0$.

Let $\mathbb{X} \subset \mathbb{R}^n$ and $\mathbb{U} \subset \mathbb{R}^m$ be given bounded convex sets of admissible values for the state x_k and the input u_k, respectively.

Problem 1. Let $[\underline{\phi}_k, \overline{\phi}_k] \subset \mathbb{X}$, $k \in [-h, \dots, 0]$, and Assumption 1 be satisfied. The objective is to design an output feedback controller that stabilizes the delayed system (1) in a vicinity of the origin, while also guaranteeing the satisfaction of state and input constraints:

$$x_k \in \mathbb{X}, \quad u_k \in \mathbb{U} \quad \forall k \in \mathbb{Z}_+,$$

for a given time-delay h and any realization (under Assumption 1) of disturbances w_k, v_k.

3 | DESIGN OF INTERVAL ESTIMATORS

Interval observers (IO) are estimators that, while having the form of an observer, provide information of the set-membership of the system's states in the form of intervals. The main feature of such estimators is the guaranteed satisfaction (under Assumption 1) of the following relation:

$$\underline{x}_k \le x_k \le \overline{x}_k, \quad \forall k \in \mathbb{Z}_+ \tag{2}$$

where $\underline{x}_k$ and $\overline{x}_k$ are, respectively, the upper and lower estimates of x_k. Furthermore, in the same sense, an interval predictor (IP) (also called *framer*, see [14]) can be introduced as an open-loop interval observer, *i.e.,* independent of measurements y_k.

Therefore, the main ideas of this work are *(i)* to use an IO to update the set-membership of the states at every instant k, and *(ii)* to incorporate an IP into an MPC algorithm, and use (2) to check constraint satisfaction, since

$$[\underline{x}_k, \overline{x}_k] \subset \mathbb{X} \Rightarrow x_k \subset \mathbb{X}.$$

In this light, Section 3.1 and 3.2 are devoted to provide conditions, in the form of LMIs, for the design of an IO and an IP, respectively. Section 3.3 then discusses the static feedback control design for the IP, which will be then used to derive stabilizing features for the MPC algorithm in the subsequent section. Notice that all LMI conditions presented in this section are verified offline.

Preliminaries

In the sequel, we will use the convention that $\underline{v}_{k-h} = \underline{v}_0$ and $\overline{v}_{k-h} = \overline{v}_0$ for $k < h$, and the following results:

Lemma 1. [15] Let A be a matrix of proper dimensions and $x, \underline{x}, \overline{x} \in \mathbb{R}^n$ be such that $\underline{x} \le x \le \overline{x}$, then

$$A^+\underline{x} - A^-\overline{x} \le Ax \le A^+\overline{x} - A^-\underline{x}. \tag{3}$$

Lemma 2. [16] For $A, A_d \in \mathbb{R}_+^{n\times n}$ the system

$$x_{k+1} = Ax_k + A_d x_{k-h} + \omega_k, \tag{4}$$

$$\omega : \mathbb{Z}_+ \to \mathbb{R}_+^n,\ \omega \in \ell_\infty^n,\ k \in \mathbb{Z}_+,$$

has a non-negative solution $x_k \in \mathbb{R}_+^n$ for all $k \in \mathbb{Z}_+$ provided that $x_k \geq 0$ for all $k \in [-h, \dots, 0]$.

A system as in above lemma is called cooperative (monotone) or non-negative.

Lemma 3. [17] A matrix $A \in \mathbb{R}_+^{n\times n}$ is Schur stable iff there exists a diagonal matrix $P \in \mathbb{R}^{n\times n}$, $P > 0$ such that $A^\top PA - P \prec 0$.

Lemma 4. [2] Let $V : \mathbb{Z}_+ \to \mathbb{R}_+$ and $\omega_k \in \mathbb{R}$. If there exist scalars $\lambda \in (0,1)$ and $b > 0$ such that

$$W_k \triangleq V_{k+1} - \lambda V_k - b|w_k|^2 \leq 0, \quad k \in \mathbb{Z}_+,$$

then $V_k \leq \lambda^k V(0) + \frac{b}{1-\lambda}\max\{|w_0|^2, \dots, |w_k|^2\}$.

3.1 | Interval observer

In this section, we will present conditions for designing an IO for system (1), by exploiting the available measurement y_k. In this sense, let us rewrite the equation for x_k in (1) as

$$x_{k+1} = (A_0 - L_0C)x_k + (A_1 - L_1C)x_{k-h} + Bu_k + w_k + L_0 y_k - L_0 v_k + L_1 y_{k-h} - L_1 v_{k-h} \tag{5}$$

for any gains $L_0, L_1 \in \mathbb{R}^{n\times p}$. From (5), the following IO can be proposed under Assumption 1:

$$\begin{aligned}
\overline{x}_{k+1} &= (A_0 - L_0C)\overline{x}_k + (A_1 - L_1C)\overline{x}_{k-h} + Bu_k + \overline{w}_k + L_0y_k + L_1y_{k-h} - L_0^+\underline{v}_k + L_0^-\overline{v}_k - L_1^+\underline{v}_{k-h} + L_1^-\overline{v}_{k-h},\\
\underline{x}_{k+1} &= (A_0 - L_0C)\underline{x}_k + (A_1 - L_1C)\underline{x}_{k-h} + Bu_k + \underline{w}_k + L_0y_k + L_1y_{k-h} - L_0^+\overline{v}_k + L_0^-\underline{v}_k - L_1^+\overline{v}_{k-h} + L_1^-\underline{v}_{k-h},\\
\overline{x}_k &= \overline{\phi}_k, \qquad \underline{x}_k = \underline{\phi}_k \quad \text{for } k \in [-h, \dots, 0].
\end{aligned} \tag{6}$$

Proposition 1. Let Assumption 1 hold and there exist matrices L_0 and L_1 such that $A_0 - L_0C$ and $A_1 - L_1C$ are non-negative. Then, the relation (2) holds for all $k \in \mathbb{Z}_+$.

Proof. The increments of the estimation errors $\underline{e}_k = x_k - \underline{x}_k$ and $\overline{e}_k = \overline{x}_k - x_k$ are given by

$$\begin{aligned}
\overline{e}_{k+1} &= (A_0 - L_0C)\overline{e}_k + (A_1 - L_1C)\overline{e}_{k-h} + \overline{\rho}_k,\\
\underline{e}_{k+1} &= (A_0 - L_0C)\underline{e}_k + (A_1 - L_1C)\underline{e}_{k-h} + \underline{\rho}_k,
\end{aligned} \tag{7}$$

where $\overline{\rho}_k = \overline{w}_k - w_k + L_0v_k - L_0^+\underline{v}_k + L_0^-\overline{v}_k + L_1v_{k-h} - L_1^+\underline{v}_{k-h} + L_1^-\overline{v}_{k-h}$ and $\underline{\rho}_k = w_k - \underline{w}_k - L_0v_k + L_0^+\overline{v}_k - L_0^-\underline{v}_k - L_1v_{k-h} + L_1^+\overline{v}_{k-h} - L_1^-\underline{v}_{k-h}$. Note that these inputs, as well as $\overline{e}_{k-h}, \underline{e}_{k-h} \geq 0$ for $k \in [-h, \dots, 0]$, are non-negative under Assumption 1. Therefore, provided that $A_0 - L_0C$ and $A_1 - L_1C$ are non-negative, we have that $\underline{e}_k, \overline{e}_k \geq 0$ for all $k \in \mathbb{Z}_+$ under Lemma 2. □

Due to the presence of delayed components, the stability of (6) will be addressed using the Lyapunov-Krasovskii framework. For brevity in the following, let us denote $D_0 = A_0 - L_0C$ and $D_1 = A_1 - L_1C$. This brings us to the following result:

Theorem 1. Let Assumption 1 hold. Given constant scalars $\lambda \in (0,1)$ and $\varepsilon > 0$, if there exist a diagonal matrix $P_2 \in \mathbb{R}^{n\times n}$, matrices $S, P \in \mathbb{R}^{n\times n}$, $U_0, U_1 \in \mathbb{R}^{n\times p}$ and a constant b such that the following inequalities are verified:

$$\begin{gathered}
P_2A_0 - U_0C \geq 0, \quad P_2A_1 - U_1C \geq 0\\
\begin{bmatrix}
(1-\lambda)P + P_2A_0 - U_0C - P_2 + A_0^\top P_2 - C^\top U_0^\top - P_2 + S & P + \varepsilon(A_0^\top P_2 - C^\top U_0^\top - P_2) - P_2 & P_2A_1 - U_1C & P_2\\
\star & P - \varepsilon(P_2 + P_2) & \varepsilon(P_2A_1 - U_1C) & \varepsilon P_2\\
\star & \star & -\lambda^h S & 0\\
\star & \star & \star & -b\mathcal{I}_n
\end{bmatrix} \prec 0\\
b > 0, P \succ 0, S \succ 0, P_2 > 0
\end{gathered} \tag{8}$$

then system (6) with gains $L_i = P_2^{-1}U_i$ $(i = 0, 1)$ is an IO for system (1): relation (2) holds for all $k \in \mathbb{Z}_+$, and the estimation errors $\underline{e}, \overline{e} \in \ell_\infty^n$.

Proof. The two first constraints imply that $A_0 - L_0C \geq 0$ and $A_1 - L_1C \geq 0$. Then, according to Proposition 1, this implies cooperativity of the estimation errors (7). Since the dynamics of (7) is composed by two similar and independent subsystems, the stability analysis can be performed independently for $\underline{e}_k$ and $\overline{e}_k$. Hence, below the variable e_k represents $\underline{e}_k$ or $\overline{e}_k$, while ρ_k

relates to $\underline{\rho}_k$ or $\overline{\rho}_k$. Consider a candidate Lyapunov-Krasovskii functional (LKF) given by $V_k = e_k^\top P e_k + \sum_{i=k-h}^{k-1} \lambda^{k-i-1} e_i^\top S e_i$, and $\tilde{y}_k = e_{k+1} - e_k$. Then, following Lemma 4, we compute

$$
\begin{aligned}
W_k &= V_{k+1} - \lambda V_k - b|\rho_k|^2 \\
&= (\tilde{y}_k + e_k)^\top P(\tilde{y}_k + e_k) - \lambda e_k^\top P e_k + \sum_{i=k-h+1}^{k} \lambda^{k-i} e_i^\top S e_i - \lambda \sum_{i=k-h}^{k-1} \lambda^{k-i-1} e_i^\top S e_i - b|\rho_k|^2 \\
&= \tilde{y}_k^\top P \tilde{y}_k + 2 e_k^\top P \tilde{y}_k + e_k^\top (1-\lambda) P e_k + e_k^\top S e_k - \lambda^h e_{k-h}^\top S e_{k-h} - b|\rho_k|^2 .
\end{aligned}
\tag{9}
$$

Using the *descriptor method*[18], we add the following expression to the right-hand side of the relation above

$$
2\left(e_k^\top P_2 + \tilde{y}_k^\top P_3^\top\right)\left[(D_0 - \mathcal{I}_n)e_k + D_1 e_{k-h} + \rho_k - \tilde{y}_k\right] = 0,
$$

where $P_3 \in \mathbb{R}^{n\times n}$ is an auxiliary matrix variable, and rewrite W_k in a matrix form, which yields

$$
W_k = \begin{bmatrix} e_k \\ \tilde{y}_k \\ e_{k-h} \\ \rho_k \end{bmatrix}^\top
\underbrace{\begin{bmatrix}
(1-\lambda)P + P_2(D_0 - \mathcal{I}_n) + (D_0 - \mathcal{I}_n)^\top P_2 + S & P + (D_0 - \mathcal{I}_n)^\top P_3 - P_2 & P_2 D_1 & P_2 \\
\star & P - P_3^\top - P_3 & P_3^\top D_1 & P_3^\top \\
\star & \star & -\lambda^h S & 0 \\
\star & \star & \star & -b\mathcal{I}_n
\end{bmatrix}}_{\Pi_1}
\begin{bmatrix} e_k \\ \tilde{y}_k \\ e_{k-h} \\ \rho_k \end{bmatrix}
$$

and, thus, the condition $W_k < 0$ amounts to $\Pi_1 \prec 0$, that leads to (8) by substituting $P_3 = \varepsilon P_2$ and introducing new decision variables $U_0 = P_2 L_0$ and $U_1 = P_2 L_1$. This implies that (7) is input-to-state stable with respect to ρ_k according to Lemma 4. □

Remark 1. If a solution for the conditions imposed in Theorem 1 cannot be found, then a change of coordinates can be used to derive cooperativity conditions when designing L_0, L_1, see Efimov et al.[15].

Remark 2. The case of delayed measurements: If the measurements are delayed, *i.e.*, $y_k = Cx_{k-h}$, then it is obvious that $L_0 = 0$ in (6). The existence of an IO for (1) can be verified if either the matrix A_0 is non-negative, or a cooperative change of coordinates can be found, as discussed above.

Remark 3. An optimizing criterion can be also introduced when solving the conditions of Theorem 1, aiming to minimize the impact of the disturbances in the accuracy of the observer by an optimal selection of gains.

3.2 | Interval predictor

Since the IO (6) depends on y_k – which is obviously unknown in future time steps – it is not suitable for prediction. Hence, we need to further investigate (5) so only known terms are used in the predictor design.

In order to distinguish the development of the IP from the IO presented previously, we will replace in (5) the variable x_k by z_k, and the gains L_0, L_1 by F_0, F_1, respectively, throughout this subsection. Then, under Assumption 2 and Lemma 1, the terms that are unavailable in future steps in (5), *i.e.*, $F_0 y_k + F_1 y_{k-h} - F_0 v_k - F_1 v_{k-h} = F_0 C x_k + F_1 C x_{k-h}$, can be replaced by their interval estimates:

$$
\begin{aligned}
F_0^+ C \underline{z}_k - F_0^- C \overline{z}_k \le F_0 C z_k \le F_0^+ C \overline{z}_k - F_0^- C \underline{z}_k, \\
F_1^+ C \underline{z}_{k-h} - F_1^- C \overline{z}_{k-h} \le F_1 C z_{k-h} \le F_1^+ C \overline{z}_{k-h} - F_1^- C \underline{z}_{k-h},
\end{aligned}
\tag{10}
$$

for some gains $F_0, F_1 \in \mathbb{R}^{n\times p}$ to be designed, and where $\underline{z}_k$ and $\overline{z}_k$ are, respectively, the predicted upper and lower bounds of z_k such that

$$
\underline{z}_k \le x_k \le \overline{z}_k, \quad \forall z \in \mathbb{Z}_+ . \tag{11}
$$

Then, using relation (10), we are in position to propose the following IP:

$$
\begin{aligned}
\overline{z}_{k+1} &= (A_0 - F_0 C)\overline{z}_k + (A_1 - F_1 C)\overline{z}_{k-h} + Bu_k + \overline{w}_k + F_0^+ C \overline{z}_k - F_0^- C \underline{z}_k + F_1^+ C \overline{z}_{k-h} - F_1^- C \underline{z}_{k-h}, \\
\underline{z}_{k+1} &= (A_0 - F_0 C)\underline{z}_k + (A_1 - F_1 C)\underline{z}_{k-h} + Bu_k + \underline{w}_k + F_0^+ C \underline{z}_k - F_0^- C \overline{z}_k + F_1^+ C \underline{z}_{k-h} - F_1^- C \overline{z}_{k-h}, \\
\overline{z}_k &= \overline{\phi}_k, \qquad \underline{z}_k = \underline{\phi}_k \quad \text{for } k \in [-h, \dots, 0].
\end{aligned}
\tag{12}
$$

Then, to derive stability conditions for (12) while also ensuring satisfaction of relation (11), let us consider the dynamics of the interval width, obtained by introducing a change of coordinates $\delta z_k = \overline{z}_k - \underline{z}_k$:

$$\delta z_{k+1} = (A_0 + 2F_0^- C)\delta z_k + (A_1 + 2F_1^- C)\delta z_{k-h} + \delta w_k, \tag{13}$$

where, analogously, $\delta w_k = \overline{w}_k - \underline{w}_k$. This brings us to the following result:

Theorem 2. Let assumptions 1–2 be satisfied. Given constant scalars $\lambda \in (0,1)$ and $\varepsilon > 0$, if there exist a diagonal matrix $P_2 \in \mathbb{R}^{n\times n}$, matrices $S, P \in \mathbb{R}^{n\times n}$, and $Y_i^+, Y_i^- \in \mathbb{R}_+^{n\times p}$ $(i = 0, 1)$, and a constant $b_2 > 0$ such that the following inequalities are verified:

$$P_2 A_0 - (Y_0^+ - Y_0^-)C \geq 0, \quad P_2 A_1 - (Y_1^+ - Y_1^-)C \geq 0$$

$$\begin{bmatrix} (1-\lambda)P + P_2 A_0 + 2Y_0^- C - P_2 + A_0^\top P_2 + 2C^\top (Y_0^-)^\top - P_2 + S & P + \varepsilon(A_0^\top P_2 + 2C^\top (Y_0^-)^\top - P_2) - P_2 & P_2 A_1 + 2Y_1^- C & P_2 \\ \star & P - 2\varepsilon P_2 & \varepsilon(P_2 A_1 + 2Y_1^- C) & \varepsilon P_2 \\ \star & \star & -\lambda^h S & 0 \\ \star & \star & \star & -b_2 \mathcal{I}_n \end{bmatrix} \prec 0 \tag{14}$$

$$P \succ 0, S \succ 0, P_2 > 0$$

then, selecting gains $F_i^- = P_2^{-1} Y_i^-$ and $F_i^+ = P_2^{-1} Y_i^+$ $(i = 0, 1)$, system (12) is an IP for system (1): the relation (11) holds, and the interval width $\delta z \in \ell_\infty^n$.

Proof. Let us denote the estimation errors as $\overline{\epsilon}_k = \overline{z}_k - x_k$ and $\underline{\epsilon}_k = x_k - \underline{z}_k$. Computing the increments of the estimation errors yields

$$\overline{\epsilon}_{k+1} = (A_0 - F_0 C)\overline{\epsilon}_k + (A_1 - F_1 C)\overline{\epsilon}_{k-h} + \left(F_0^+ C\overline{z}_k - F_0^- C\underline{z}_k - F_0 C z_k\right) + \left(F_1^+ C\overline{z}_{k-h} - F_1^- C\underline{z}_{k-h} - F_1 C z_{k-h}\right) + \overline{w}_k - w_k,$$
$$\underline{\epsilon}_{k+1} = (A_0 - F_0 C)\underline{\epsilon}_k + (A_1 - F_1 C)\underline{\epsilon}_{k-h} + \left(F_0 C z_k - F_0^+ C\underline{z}_k + F_0^- C\overline{z}_k\right) + \left(F_1 C z_{k-h} - F_1^+ C\underline{z}_{k-h} + F_1^- C\overline{z}_{k-h}\right) + w_k - \underline{w}_k,$$

where, according to Assumption 1 and (10), all terms independent of ϵ_k are non-negative, as well as $\underline{\epsilon}_{k-h}, \overline{\epsilon}_{k-h}$ for $k \in [-h, \dots, 0]$. Therefore, similarly as in Proposition 1, the cooperativity conditions to satisfy are $A_0 - F_0 C > 0$ and $A_1 - F_1 C > 0$. Hence, the first two constraints ensure that $\underline{\epsilon}_k, \overline{\epsilon}_k \geq 0$, implying that relation (11) holds.

Furthermore, notice that

$$A_0 - F_0 C = A_0 - F_0^+ C + F_0^- C \leq A_0 + 2F_0^- C, \quad A_1 - F_1 C = A_1 - F_1^+ C + F_1^- C \leq A_1 + 2F_1^- C.$$

Thus, one readily has that $A_0 + 2F_0^- C$ and $A_1 + 2F_1^- C$ are also non-negative provided that the first two inequalities are verified. In this light, we can use Lemma 4 and derive the last inequality for the stability of (13), similarly as done in Theorem 1. The feasibility of such conditions implies input-to-state stability of δz_k with respect to δw_k, and concludes the proof. □

Remark 4. Note that, if the control input contains components with a known delay (*e.g.*, $u_k = v_{k-h}$, for some signal $v_k \in \mathbb{R}^m$), both IO (6) and IP (12) will preserve the same structure. Furthermore, as for IO (6), if the output is available for delayed measurements, then the IP (12) can be designed with $F_0 = 0$.

Remark 5. Similarly as for Theorem 1, an optimizing criterion can be also introduced when solving the conditions of Theorem 2, aiming to minimize the impact of the disturbances in the accuracy of the predictor.

3.3 | Feedback control design

In this section, we will address the design of a static feedback controller for IP (12). This feedback will be used to derive the terminal ingredients for the MPC design, as we will discuss in the following. Let us introduce a new change of coordinates, $z_k^\star = \frac{\overline{z}_k + \underline{z}_k}{2}$, which describes the center of the interval obtained by (12) and whose increments are given by:

$$z_{k+1}^\star = A_0 z_k^\star + A_1 z_{k-h}^\star + Bu_k + w_k^\star \tag{15}$$

where, analogously, $w_k^\star = \frac{\overline{w}_k + \underline{w}_k}{2}$. Note that the dynamics (15) is completely known and controlled by u_k, meaning that it can be used to steer the center of the interval, *i.e.*, vec$(\underline{z}_k^\star, \overline{z}_k^\star)$, over the state-space. In this sense, defining a control law such as

$$u_k = K_0 z_k^\star + K_1 z_{k-h}^\star + Dw_k^\star \tag{16}$$

where $K_0, K_1, D \in \mathbb{R}^{n\times m}$ are gains to be designed, the following closed-loop system is obtained:

$$z_{k+1}^\star = (A_0 + BK_0)z_k^\star + (A_1 + BK_1)z_{k-h}^\star + (\mathcal{I}_n + BD)w_k^\star. \tag{17}$$

Theorem 3. Let assumptions 1–2 be satisfied. Given constant scalars $\lambda \in (0,1)$ and $\varepsilon > 0$, if the following inequality is verified for a scalar $b_3 > 0$, matrices $P, S, P_2 \in \mathbb{R}^{n\times n}$, and $\Psi_0, \Psi_1, \Psi_d \in \mathbb{R}^{n\times m}$:

$$\begin{bmatrix} (1-\lambda)P + S + P_2^\top A_0^\top + \Psi_0^\top B^\top - P_2^\top + A_0 P_2 + B\Psi_0 - P_2 & \varepsilon(P_2^\top A_0^\top + \Psi_0^\top B^\top - P_2^\top) - P_2 + P & A_1 P_2 + B\Psi_1 & P_2 + B\Psi_d \\ \star & P - \varepsilon(P_2^\top + P_2) & \varepsilon(A_1 P_2 + B\Psi_1) & \varepsilon(P_2 + B\Psi_d) \\ \star & \star & -\lambda^h S & 0 \\ \star & \star & \star & -b_3 \mathcal{I}_n \end{bmatrix} \prec 0 \tag{18}$$

$$P \succ 0,\ P_2 \succ 0,\ S \succ 0,$$

then, selecting gains $K_0 = \Psi_0 P_2^{-1}$, $K_1 = \Psi_1 P_2^{-1}$ and $D = \Psi_d P_2^{-1}$, system (15) under control (16) is input-to-state stable with respect to the input $w_k^\star$.

Proof. Let us consider a LKF candidate given by $V_k = z_k^{\star\top}\bar{P}z_k^\star + \sum_{i=k-h}^{k-1}\lambda^{k-1-i}z_i^{\star\top}\bar{S}z_i^\star$ and let $\bar{y}_k = z_{k+1}^\star - z_k^\star$. Then, following Lemma 4, we compute

$$\begin{aligned} W_k &= V_{k+1} - \lambda V_k - b_3|w_k^\star|^2 \\ &= (\bar{y}_k + z_k^\star)^\top \bar{P}(\bar{y}_k + z_k^\star) - \lambda z_k^{\star\top}\bar{P}z_k^\star + \sum_{i=k-h+1}^{k}\lambda^{k-i}z_i^{\star\top}\bar{S}z_i^\star - \lambda\sum_{i=k-h}^{k-1}\lambda^{k-i-1}z_i^{\star\top}\bar{S}z_i^\star - b_3|w_k^\star|^2 \\ &= \bar{y}_k^\top\bar{P}\bar{y}_k + 2z_k^{\star\top}\bar{P}\bar{y}_k + z_k^{\star\top}(1-\lambda)\bar{P}z_k^\star + z_k^{\star\top}\bar{S}z_k^\star - \lambda^h z_{k-h}^{\star}{}^\top\bar{S}z_{k-h}^\star - b_3|w_k^\star|^2. \end{aligned}$$

Next, similarly as in Theorem 1, we add the following expression to the right-hand side of the relation above:

$$2\left[z_k^{\star\top}\bar{P}_2^\top + \bar{y}_k^\top\bar{P}_3^\top\right]\left[(A_0 + BK_0 - \mathcal{I}_n)z_k^\star + (A_1 + BK_1)z_{k-h}^\star + (\mathcal{I}_n + DB)w_k^\star - \bar{y}_k\right] = 0$$

for some slack variables $\bar{P}_2, \bar{P}_3 \in \mathbb{R}^{n\times n}$. Denoting $\eta_k = \mathrm{vec}(z_k^\star, \bar{y}_k, z_{k-h}^\star, w_k^\star)$, the expression for W_k can be rewritten as $W_k = \eta_k^\top\Pi_3\eta_k$, where Π_3 is given by :

$$\Pi_3 = \begin{bmatrix} (1-\lambda)\bar{P} + \bar{S} + (A_0 + BK_0 - \mathcal{I}_n)^\top\bar{P}_2 + \bar{P}_2^\top(A_0 + BK_0 - \mathcal{I}_n) & (A_0 + BK_0 - \mathcal{I}_n)^\top\bar{P}_3 - \bar{P}_2^\top + \bar{P} & \bar{P}_2^\top(A_1 + BK_1) & \bar{P}_2^\top(\mathcal{I}_n + BD) \\ \star & \bar{P} - \bar{P}_3 - \bar{P}_3^\top & \bar{P}_3^\top(A_1 + BK_1) & \bar{P}_3^\top(\mathcal{I}_n + BD) \\ \star & \star & -\lambda^h\bar{S} & 0 \\ \star & \star & \star & -b_3\mathcal{I}_n \end{bmatrix}$$

and thus, the conditions $W_k < 0$ is equivalent to $\Pi_3 \prec 0$. Denoting the slack variables as $\bar{P}_3 = \varepsilon\bar{P}_2$ and $P_2 = \bar{P}_2^{-1}$ and multiplying Π_3 by $\mathrm{diag}(P_2^\top, P_2^\top, P_2^\top, \mathcal{I}_n)$ and $\mathrm{diag}(P_2, P_2, P_2, \mathcal{I}_n)$, on the left and on the right, respectively, yields

$$\begin{bmatrix} (1-\lambda)P + S + P_2^\top A_0 + P_2^\top K_0^\top B^\top - P_2^\top + A_0 P_2 + BK_0 P_2 - P_2 & \varepsilon(P_2^\top A_0^\top + P_2^\top K_0^\top B^\top - P_2^\top) - P_2 + P & A_1 P_2 + BK_1 P_2 & (\mathcal{I}_n + BD)P_2 \\ \star & P - \varepsilon(P_2^\top - P_2) & \varepsilon(A_1 P_2 + BK_1 P_2) & \varepsilon(\mathcal{I}_n + BD)P_2 \\ \star & \star & -\lambda^h S & 0 \\ \star & \star & \star & -b_3\mathcal{I}_n \end{bmatrix} \prec 0$$

where $P = P_2^\top\bar{P}P_2$, $S = P_2^\top\bar{S}P_2$. Finally, (18) is obtained by introducing decision variables $\Psi_0 = K_0 P_2$, $\Psi_1 = K_1 P_2$ and $\Psi_d = DP_2$ on the inequality above. □

4 | MPC DESIGN

In this section, we will present the design of the predictive controller with guaranteed stability and constraint satisfaction, based on the interval estimators proposed previously. Since the IP (12) is completely known (under Assumption 1), we will rely on classic axioms of Mayne et al.[3] to show stability. Section 4.1 then introduces the *stabilizing ingredients* required, and Section 4.2 presents the proposed algorithm.

4.1 | Stabilizing ingredients

If the conditions presented in Theorem 3 can be verified, we can employ the Lyapunov function used therein (*i.e.*, the matrix $\bar{P}$) to obtain the following set:

$$\bar{\mathbb{X}} = \left\{ z^\star \in \mathbb{R}^n \,:\, z^{\star\top}\bar{P}z^\star \le \frac{b_3}{1-\lambda}\sup_{k\in\mathbb{Z}_+}|w_k^\star|^2 \right\}, \tag{19}$$

which is an exponentially attractive ellipsoid for any initial functions[2] of (15) as a direct consequence of Lemma 4. Therefore, this set can be used as the terminal set $\mathbb{X}_f$ in the MPC algorithm and the associated Lyapunov function as terminal cost.

However, for well-posedness of the MPC, as it will be discussed in the following, the static feedback control (16) must satisfy the constraint on the control input. In this sense, a conventional assumption in MPC design is imposed:

Assumption 3. Let $\mathbb{X}_f \subseteq \bar{\mathbb{X}} \subseteq \mathbb{X}$ and the control input computed by (16) satisfy $u_k \in \mathbb{U}$ for any $z^\star_k, z^\star_{k-h} \in \mathbb{X}_f$.

The assumption above states that the terminal set used in the MPC algorithm should be contained in the ellipsoid defined by (19), which, in turn, must also be contained in the state constraint set. Furthermore, any control input computed in this terminal set should also satisfy the control constraint. If the input constraint set is ellipsoidal, then Assumption 3 may be relaxed by imposing additional conditions in Theorem 3:

Corollary 1. Let there exist positive definite and symmetric matrices $\mathcal{U} \in \mathbb{R}^{m\times m}$ and $\Gamma \in \mathbb{R}^{n\times n}$ such that $\mathbb{U} = \{u \in \mathbb{R}^m : u^\top \mathcal{U} u \leq 1\}$ and $w^\star_k \in \{w^\star_k \in \mathbb{R}^n : w^{\star\top}_k \Gamma w^\star_k \leq 1\}$. If there exists a matrix $\tilde{\Gamma} \in \mathbb{R}^{n\times n}$ such that the conditions of Theorem 3 are satisfied with the following additional inequalities:

$$\begin{bmatrix} P & 0 & 0 & \Psi_0^\top \\ \star & P & 0 & \Psi_1^\top \\ \star & \star & \tilde{\Gamma} & \Psi_d^\top \\ \star & \star & \star & \mathcal{U}^{-1} \end{bmatrix} \geq 0, \qquad \begin{bmatrix} \tilde{\Gamma} & P_2^\top \\ \star & \Gamma^{-1} \end{bmatrix} \geq 0, \tag{20}$$

then control (16) satisfies the constraint $u_k \in \mathbb{U}$ for all $z^\star_\tau \in \eta^{-1}\bar{\mathbb{X}}$, $\tau \in [k-h, \dots, k]$, where $\eta = 4\frac{b_3}{1-\lambda}\Lambda^{-1}_{\min}(\Gamma)$.

Proof. First, note that the condition $u_k \in \mathbb{U}$ is equivalent to

$$\begin{bmatrix} z^\star_k & z^\star_{k-h} & w^\star_k \end{bmatrix} \begin{bmatrix} K_0 \\ K_1 \\ D \end{bmatrix} \mathcal{U} \begin{bmatrix} K_0 & K_1 & D \end{bmatrix} \begin{bmatrix} z^\star_k \\ z^\star_{k-h} \\ w^\star_k \end{bmatrix} \leq 1.$$

From this fact, it is clear that the relation

$$\begin{bmatrix} K_0 \\ K_1 \\ D \end{bmatrix} \mathcal{U} \begin{bmatrix} K_0 & K_1 & D \end{bmatrix} \leq \begin{bmatrix} \bar{P} & 0 & 0 \\ 0 & \bar{P} & 0 \\ 0 & 0 & 0.5\Gamma \end{bmatrix} \tag{21}$$

implies that the control constraint is satisfied in a set that verifies

$$z^{\star\top}_k \bar{P} z^\star_k + z^\star_{k-h}{}^\top \bar{P} z^\star_{k-h} + 0.5 w^{\star\top}_k \Gamma w^\star_k \leq 1. \tag{22}$$

However, this set must also be invariant under control (16). Using the description of the attractive ellipsoid (19), one can deduce that $z^{\star\top}_k \bar{P} z^\star_k + z^\star_{k-h}{}^\top \bar{P} z^\star_{k-h} + w^{\star\top}_k \Gamma w^\star_k \leq 2\frac{b_3}{1-\lambda}\Lambda^{-1}_{\min}(\Gamma) + 0.5$. In this sense, if $z^\star_\tau \in \eta^{-1}\bar{\mathbb{X}}$ for $\tau = [k-h, \dots, k]$, then (22) is satisfied and, consequently, the control constraint follows under (21).

The conditions given in this corollary are obtained as follows. Applying the Schur complement in (21) and multiplying the resulting inequality by $\mathrm{diag}(P_2^\top, P_2^\top, P_2^\top, \mathcal{I}_m)$ and $\mathrm{diag}(P_2, P_2, P_2, \mathcal{I}_m)$, by the left and the right sides, respectively, yields

$$\begin{bmatrix} P & 0 & 0 & \Psi_0^\top \\ \star & P & 0 & \Psi_1^\top \\ \star & \star & P_2^\top \Gamma P_2 & \Psi_d^\top \\ \star & \star & \star & \mathcal{U}^{-1} \end{bmatrix} \geq 0.$$

Since the $P_2^\top \Gamma P_2$ is nonlinear in the decision variable P_2, the first LMI of this corollary is obtained by substituting it for a slack variable $\tilde{\Gamma}$ under the following additional constraint

$$\tilde{\Gamma} - P_2^\top \Gamma P_2 \geq 0,$$

which, by applying the Schur complement, yields the second LMI. This concludes the proof. □

Remark 6. Note that it would be possible to obtain similar results if we multiplied the right-hand side of (21) by η^{-1}. However, this is not admissible since b_3 and P are both decision variables on Theorem 3, whose multiplication would lead to a nonlinear condition.

In the next subsection, the IP (12) will be reinitialized regularly. To avoid confusion, we will denote $\underline{z}_{k,i}, \overline{z}_{k,i}$ as the predictions obtained on the i-th step at the decision instant k. In the previous section, we simply used the notation $\underline{z}_k, \overline{z}_k$ since the IP was initialized only once, at $k = 0$.

4.2 | Design of the predictive controller

The core idea of the proposed predictive controller is as follows: since the IP (12) depends solely on known variables, we can use it to predict the behaviour of an envelope containing all trajectories of system (1). Furthermore, by exploiting the available measurement y_k, the IO updates the set-membership of the states of (1) at every instant k.

Let us define the smallest envelope computed using the estimates of both IO and IP:

$$\underline{\hat{x}}_k = \max\{\underline{x}_k, \underline{z}_{k-1,1}\}, \qquad \hat{\overline{x}}_k = \min\{\overline{x}_k, \overline{z}_{k-1,1}\}$$

and, thus, at each decision instant k, the IP (12) is initialized with $\underline{z}_{k,j} = \underline{\hat{x}}_{k+j}$ and $\overline{z}_{k,j} = \hat{\overline{x}}_{k+j}$, $j \in [-h, \dots, 0]$. If an input sequence $\mathcal{S}_N = \{s_0, \dots, s_{N-1}\}$ is available, one can compute the values of $\underline{z}_{k,i+1}, \overline{z}_{k,i+1}$ over a time window of length N (under substitution of $u_{k+i} = s_i$, $i \in [0, \dots, N-1]$, and recalling that $\underline{w}_k, \overline{w}_k$ are known for all $k \in \mathbb{Z}_+$, in accordance with Assumption 1). Following this rationale, the OCP to be solved online by the MPC is stated as follows:

$$\mathcal{S}_N^k := \arg\min_{\mathcal{S}_N} V_f(z^\star_{k,N}) + \sum_{i=0}^{N-1} \ell(z^\star_{k,i}, s_i), \tag{23}$$

where $V_f(z^\star_{k,N}) = {z^\star_{k,N}}^\top \Phi_1 z^\star_{k,N}$ is the terminal cost, and $\ell(z^\star_{k,i}, s_i) = {z^\star_{k,i}}^\top \Phi_2 z^\star_{k,i} + s_i^\top \Phi_3 s_i$ is the stage cost, for $\Phi_1, \Phi_2 \in \mathbb{R}_+^{n\times n}$ and $\Phi_3 \in \mathbb{R}_+^{m\times m}$ being symmetric weighting matrices. The OCP (23) must be solved under the following constraints:

$$\underline{z}_{k,j} = \underline{\hat{x}}_k \text{ and } \overline{z}_{k,j} = \hat{\overline{x}}_k, \text{ for } j \in [-h, \dots, 0], \tag{24a}$$

$$\underline{z}_{k,i+1}, \overline{z}_{k,i+1} \text{ are computed by (12)}, \tag{24b}$$

$$\underline{z}_{k,i}, \overline{z}_{k,i} \subset \mathbb{X}, \quad s_i \in \mathbb{U} \text{ for } i \in [0, \dots, N-1], \tag{24c}$$

$$z^\star_{k,N} \in \mathbb{X}_f \tag{24d}$$

Algorithm 1 summarizes the proposed MPC scheme. We are now in position to state the main result concerning the proposed MPC:

Algorithm 1 IO-MPC

Offline: Solve LMIs (8), (14), (18)–(20), estimate $\mathbb{X}_f$, select $\Phi_1 = \bar{P}$, $\Phi_2 = \frac{(1-\lambda)}{2}\bar{P}$ and $\Phi_3 = \frac{(1-\lambda)}{8}\bar{P}$.
Input: Initial conditions $\underline{\phi}_k, \overline{\phi}_k$, $k \in [-h, \dots, 0]$, constraint sets $\mathbb{X}, \mathbb{U}$, and prediction horizon N.
Online:

1: **for** each decision instant $k \in \mathbb{Z}_+$ **do**
2: Measure y_k and update IO (6).
3: Initialize IP (12) such as $[\underline{z}_{k,0}, \overline{z}_{k,0}] = [\underline{\hat{x}}_k, \hat{\overline{x}}_k]$.
4: Solve OCP (23) under constraints (24a)–(24d), and assign $u_k = s_0^k$.
5: Apply u_k to system (1).
6: **end for**

Remark 7. Not that if the control input signal is delayed (as pointed out in Remark 4), then an extra constraint is needed to initialize its previous values and take their effect into account in the prediction (similarly as for the states in (24a)), *i.e.*, $s_j = u_{k+j}$ for $j \in [-h, \dots, -1]$.

Theorem 4. Let $[\underline{\phi}_k, \overline{\phi}_k] \subset \mathbb{X}$ for $k \in [-h, \dots, 0]$ and assumptions 1–3 be satisfied with $[\underline{w}_{k+1}, \overline{w}_{k+1}] \subseteq [\underline{w}_k, \overline{w}_k]$. Then, following Algorithm 1, the closed-loop system composed by (1), (6), (12) has the following features:

1. Recursive feasibility of reaching the terminal set in N steps;
2. ISS of dynamics (15) in $\mathbb{X}_f$ and practical ISS for (1);
3. Constraint satisfaction.

Proof. Since the proposed scheme is based on standard *stabilizing ingredients*[3], this proof benefits of conventional arguments for MPC algorithms. Suppose that for any $[\underline{\phi}_k, \overline{\phi}_k] \subset \mathbb{X}$ for $k \in [-h, \dots, 0]$ a solution of OCP (23) exists, *i.e.*, a sequence $\mathcal{S}_N$

exists and steers the center of the trajectories of the IP (12) to terminal set $\mathbb{X}_f$. Thus, if $u_k = s_0^k$ is applied, then $[\underline{z}_{k,i+1}, \overline{z}_{k,i+1}] \subset [\underline{z}_{k+1,i}, \overline{z}_{k+1,i}] \subset \mathbb{X}$ is guaranteed (at least for $i = 0$) and, also, $z^\star_{k,N} \subset \mathbb{X}_f$ thanks to constraint (24d). Then, following Algorithm 1, this procedure can be iteratively repeated for $k \in \mathbb{Z}_+$ since $[\underline{w}_{k+1}, \overline{w}_{k+1}] \subseteq [\underline{w}_k, \overline{w}_k]$. As a direct consequence of relation (2), the control sequence $\mathcal{S}_N$ that steers $z^\star_{k,0}$ to $\mathbb{X}_f$, also steers x_k to a vicinity of the origin, implying point *(1)*.

For point *(2)*, note that the dynamics of $z^\star_k$ is nominal (*i.e.*, completely known) under Assumption 1, and that a solution of OCP (23) implies that $z^\star_{k,N} \subset \mathbb{X}_f$. The ISS in $\mathbb{X}_f$ follows directly from the selection of the terminal ingredients Φ_i ($i \in \{1, 2, 3\}$) which implies that

$$V(z^\star_{k+1,N}) - V(z^\star_{k,N}) \leq \frac{b_3}{1-\lambda}|w^\star_k|^2 - \ell(z^\star_{k,N}, s_N)$$

due to Lemma 4. Furthermore, the practical ISS property of (1) follows from

$$|x_k| \leq |\varkappa_k|, \quad \text{and} \quad |\varkappa_0| \leq |\underline{x}_0| + |\overline{x}_0| \leq |x_0| + C_0,$$

where $\varkappa_k = \mathrm{vec}(\underline{z}_{k,0}, \overline{z}_{k,0})$ and $C_0 = |\underline{x}_0| + |\overline{x}_0|$.

Finally, due to constraint (24c), relation (11) and under Assumption 3, a solution of OCP (23) implies that $x_k \in [\underline{z}_{k,i}, \overline{z}_{k,i}] \subset \mathbb{X}$ and $u_k \in \mathbb{U}$. This implies point *(3)* and concludes the proof. □

5 | COMPLEXITY AND PERFORMANCE

The OCP (23) under constraints (24a)–(24d) is a quadratic programming (QP). As it can be seen, the complexity of solving such a problem is of order $\mathcal{O}(Nn)$, *i.e.*, the conditions to be met scale linearly with the length of the prediction horizon and the dimension of the system. This shows that the proposed method is interesting, since its complexity is similar to any conventional MPC algorithm, and it can be efficiently solved by the existing optimization routines. Furthermore, it is worth noticing that, due to the structure of the pair IO/IP, this complexity is fixed (*i.e.*, does not increase with time).

Since the presented approach does not require constraint tightening to ensure robust constraint satisfaction (which is a common approach for LTI systems, *e.g.*, using Tube-MPC [19]). Hence, two interesting advantages are potentially obtained: a larger feasible region and the possibility of using the full range of the control signal.

Finally, the conservativeness of the approach is directly related to how large the predictive intervals obtained by the IP are. However, since all terms in (12) are known, it is possible to evaluate the predicted interval width in the steady-state (*i.e.*, as $k \to \infty$ and any error due to initial conditions have vanished). The steady-state width of the IP (12) also allows us to infer how close to the origin the trajectories of the real (perturbed) system will be.

6 | NUMERICAL EXAMPLE

In this section, a numerical example illustrates the usefulness of the proposed methodology. Consider the following time-delayed system:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 0.5 & -0.1 \\ 0.5 & 0.2 \end{bmatrix} x_k + \begin{bmatrix} 0.1 & -0.3 \\ 0 & -0.1 \end{bmatrix} x_{k-h} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k + w_k \\ y_k &= \begin{bmatrix} 0 & 1 \end{bmatrix} x_k + v_k \end{aligned} \tag{25}$$

where $h = 10$, $w_k \in [-0.2, 0.2]^2$, $v_k \in [-0.5, 0.5]$ and the constraint sets are taken as $\mathbb{X} = [-9, 3] \times [-7, 4]$ and $\mathbb{U} = [-1, 1]$. By selecting $\lambda = 0.99$ and $\varepsilon = 1$ and solving LMIs (8) and (14), the following gains were obtained:

$$L_0 = [-0.1435, 0.0807],\ L_1 = [-0.3252, -0.1691],\ F_0 = [-0.1025, 0.0982] \text{ and } F_1 = [-0.3024, -0.1036],$$

while solving the LMIs (18)–(20) yielded the following Lyapunov matrix:

$$\bar{P} = \begin{bmatrix} 7.550 & -0.8982 \\ -0.8982 & 11.445 \end{bmatrix}.$$

As described in Algorithm 1, we selected the prediction horizon as $N = 20$ and the weighting matrices Φ_j ($j \in \{1, 2, 3\}$), and estimated $\mathbb{X}_f$ according to (19). The simulations shown in the following considered a time span of $T = 60$ steps, the initial conditions for the estimators as $\underline{\phi}_k = [-9, -7]$ and $\overline{\phi}_k = [-8, -6]$, for all $k \in [-h, \ldots, 0]$, while the initial conditions for (25) are randomly selected such as $x_j \in [\underline{\phi}_j, \overline{\phi}_j]$, for $j \in [-h, \dots, 0]$. This scenario is simulated 100 times accounting for several realizations of the additive disturbances w_k, v_k.

Figure 1 shows the evolution of the states of (25). We have also launched the IP at $k = 0$ to make a contrast between its trajectories and the constraint set. Note that all constraints were satisfied and the IP even reaches the boundaries of the constraint set, indicating low conservativeness. It is worth noticing that the IP is not diverging, but slowly converging to its final value (which can be conservative bounds). However, this is not a problem thanks to the re-initialization of the IP, repeated at every decision instant k, and the fact that the MPC aims to steer $z_k^\star$ to $\mathbb{X}_f$, and not the whole envelope $[\underline{z}_k, \overline{z}_k]$.

Analogously, Figure 2 shows the computed control inputs. As it can be seen, since the OCP (23) does not tighten the constraint sets, a low cost on the control moves allowed the MPC to use its full range – but still respecting the given constraints.

The average computation time for the solution of OCP (23) was 0.0042 ± 0.0021 second/step, with a maximum time of 0.1358 second. All simulations were performed using MATLAB 2017a, using an Intel i7-8565U processor (1.8GHz) and 16GB RAM. Also, we used YALMIP[20] to solve the offline LMIs (for the design of the IO/IP) and to set-up the optimization problem (using the solver *quadprog*).

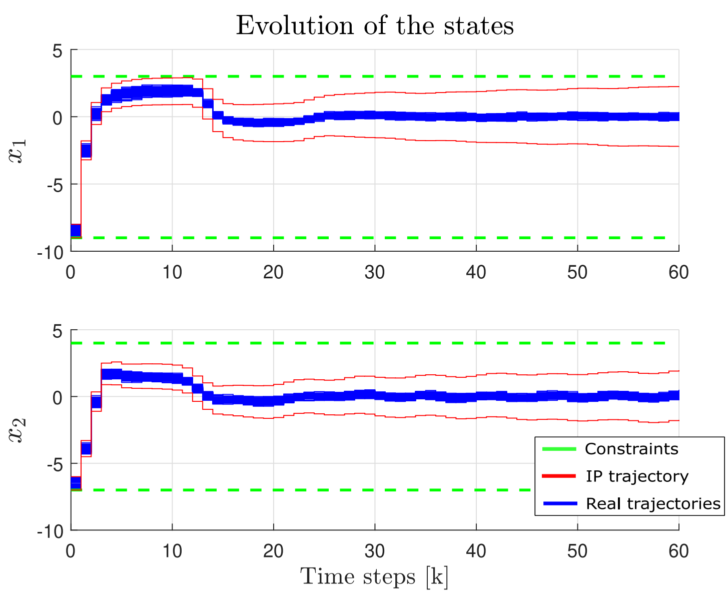

FIGURE 1 Trajectories of the states of the real system, the IP and the constraint set.

7 | CONCLUSION

In this work, a novel robust output feedback MPC scheme has been presented for linear time-delayed systems subject to additive disturbances. The novelty relies on the incorporating new interval estimators on the MPC algorithm, ensuring constraint satisfaction and recursive feasibility. The resulting algorithm is similar to the conventional MPC, requiring similar stabilizing ingredients and having low computational complexity. The interval estimators, as well as these ingredients, are obtained by the solution of (offline) LMIs. A numerical example illustrates the proposed methodology.

Future directions of research include the practical case in which the time-delay is unknown and/or possibly time-varying. Although the MPC algorithm would be similar as presented in this work, such a scenario would require revisiting the presented interval estimators and their design conditions.

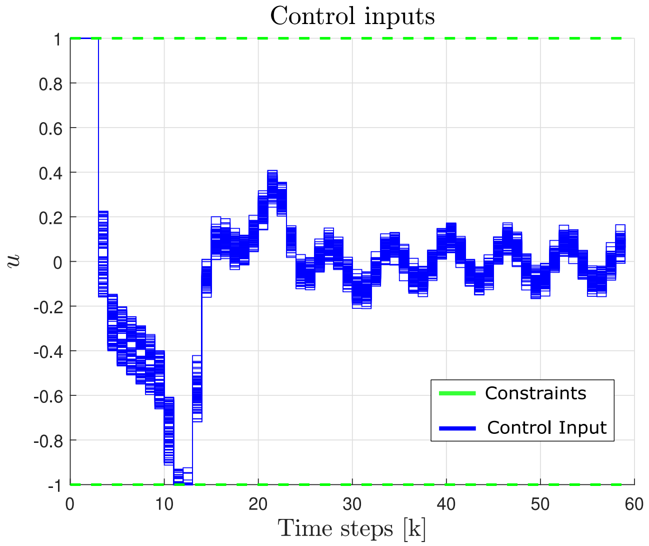

FIGURE 2 Evolution of the control inputs in constrast to the constraints.

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