Bridging the multiscale hybrid-mixed and multiscale hybrid high-order methods

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SIAM GS21, June 2021

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Outline

1. What are multiscale methods?
2. The multiscale hybrid-mixed (MHM) method
3. The multiscale hybrid high-order (MsHHO) method
4. MHM $=$ HHO?

What are multiscale methods?

Multiscale problems

Consider $\Omega \subset \mathbb{R}^d$. Given $f : \Omega \to \mathbb{R}$, find $u : \Omega \to \mathbb{R}$ such that

$$\left\{ \begin{array}{rcll} -\boldsymbol{\nabla} \cdot (\boldsymbol{A}_\varepsilon \boldsymbol{\nabla} u) & = & f & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{array} \right.$$

where the diffusion tensor $\boldsymbol{A}_\varepsilon$ exhibits “multiscale” features.

A key applications is Darcy equations, modeling flows in porous rocks.

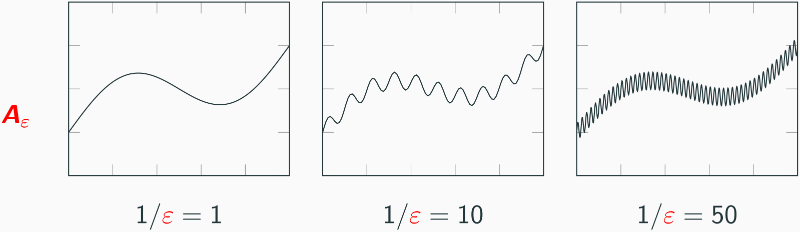
What are these mutliscale features?

For the sake, of simplicity, we will consider here the case where

$$\boldsymbol{A}_{\varepsilon}(\boldsymbol{x}) := \widehat{\boldsymbol{A}}\left(\boldsymbol{x}, \left\{\frac{\boldsymbol{x}}{\varepsilon}\right\}\right),$$

where $\widehat{\boldsymbol{A}} : \Omega \times (0,1)^d \to \mathbb{R}^{d\times d}$ is smooth, and

$$\widehat{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y}) \text{ is } \boldsymbol{y}\text{-periodic for all } \boldsymbol{x} \in \Omega.$$



Goal

Resolving the fine scale with a mesh size $h \simeq \varepsilon$ is to expensive.

We want to capture the “macroscopic behavior” of u, with a mesh size H.

We would like error estimates in $\mathcal{O}(H)$, with $H \gg \varepsilon$.

Limitations of standard FEM

Standard FEM fails. Indeed, we have

$$-f = \nabla \cdot (\boldsymbol{A}_\varepsilon \nabla u) = (\nabla \cdot \boldsymbol{A}_\varepsilon) \cdot \nabla u + \boldsymbol{A}_\varepsilon : \nabla^2 u$$

with $|\nabla \cdot \boldsymbol{A}_\varepsilon| \simeq 1/\varepsilon$, leading to

$$\|\nabla^2 u\|_\Omega \lesssim \frac{1}{\varepsilon}\|f\|_\Omega,$$

and

$$\|\nabla(u - u_H)\|_\Omega \lesssim \frac{H}{\varepsilon}\|f\|_\Omega,$$

with $\mathcal{P}_1$ elements.

This is sharp: standard FEM needs to resolve the small scales ($H \leq \varepsilon$).

Key ideas in multiscale methods

We would like to keep the same number of dofs as the standard $\mathcal{P}_1$ FEM.

Hence, we preserve the mesh size H, but we modify the shape functions.

These modified basis functions will “upscale” fine scale information.

New shape functions via local problems

Consider a simplex $K \in \mathcal{T}_H$, one if its vertices $\boldsymbol{a}$ and

$$\psi^{\boldsymbol{a}} \in \mathcal{P}_1(K), \qquad \psi^{\boldsymbol{a}}(\boldsymbol{a}') = \delta_{\boldsymbol{a}\boldsymbol{a}'} \quad \forall \boldsymbol{a}' \in \mathcal{V}_K$$

the "hat function" associated with $\boldsymbol{a}$.

An alternate definition of $\psi^{\boldsymbol{a}}$ is that it satisfies

$$\begin{cases} -\Delta \psi^{\boldsymbol{a}} &= 0 \quad \text{in } K, \\ \psi^{\boldsymbol{a}} &= \psi^{\boldsymbol{a}} \quad \text{on } \partial K. \end{cases}$$

This motivates the definition

$$\begin{cases} -\boldsymbol{\nabla} \cdot (\boldsymbol{A}_\varepsilon \boldsymbol{\nabla} {\psi_\varepsilon}^{\boldsymbol{a}}) &= 0 \quad \text{in } K, \\ \psi_\varepsilon^{\boldsymbol{a}} &= \psi^{\boldsymbol{a}} \quad \text{on } \partial K, \end{cases}$$

for the multiscale shape functions.

Multiscale shape functions

Recall that:

$$\begin{cases} -\nabla \cdot (\boldsymbol{A}_\varepsilon \psi_\varepsilon{}^{\boldsymbol{a}}) &= 0 \quad \text{in } K, \\ \psi_\varepsilon^{\boldsymbol{a}} &= \psi^{\boldsymbol{a}} \quad \text{on } \partial K. \end{cases} \tag{1}$$

Since we preserved the boundary conditions at each element, we automatically construct an $H_0^1(\Omega)$-conforming basis.

The $\psi_\varepsilon^{\boldsymbol{a}}$ will incorporate some ε-scale oscilations.

In practice, (1) is approximated numerically on a fine mesh, which is possible because the problem is local to the element K.

Multiscale methods

Step 1: compute for each vertex the hat function $\psi_\varepsilon^{\boldsymbol{a}}$.
This is done by uncoupled solving fine scale problems on small domains.

Step 2: compute the multiscale approxiation $u_{\varepsilon,H} \in \operatorname*{span}_{\boldsymbol{a}\in\mathcal{V}_H}\{\psi_\varepsilon^{\boldsymbol{a}}\}$

$$(\boldsymbol{A}_\varepsilon \nabla u_{\varepsilon,H}, \nabla\psi_\varepsilon^{\boldsymbol{a}'})_\Omega = (f, \nabla\psi_\varepsilon^{\boldsymbol{a}'})_\Omega \quad \forall \boldsymbol{a}' \in \mathcal{V}_H.$$

Multiscale error estimates

Standard FEM

$$\|\nabla(u - u_H)\|_\Omega \lesssim \frac{H}{\varepsilon}\|f\|_\Omega$$

Multiscale FEM

$$\|\nabla(u - u_{\varepsilon,H})\|_\Omega \lesssim \left(H + \sqrt{\frac{\varepsilon}{H}}\right)\|f\|_\Omega$$

T.Y. Hou, X.H Wu, Z. Cai, 1999.

Summary on multiscale methods

Multiscale methods are robust when $H \gg \varepsilon$.

Modified shape functions are computed via local fine scale problems. Here, we will assume these problems are exatcly solved for simplicity.

The MHM and MsHHO are two multiscale methods recently introduced and able to handle polytopal meshes.

The multiscale hybrid-mixed (MHM) method

Some history

P.A. Raviart and J.M. Thomas, 1977: A source of inspiration.

R. Araya, C. Harder, D. Paredes and F. Valentin, 2013: Darcy equations.

D. Paredes, F. Valentin and H. Versieux, 2017: Multiscale error analysis.

R. Araya, C. Harder, A.H. Pozas and F. Valentin, 2017: Stoke's equations.

T. Chaumont-Frelet and F. Valentin, 2020: Helmholtz problems.

The multiscale hybrid-mixed (MHM) method

The primal hybrid formulation

The primal hybrid formulation

Consider the energy functional

$$J(v) := \frac{1}{2}(\boldsymbol{A}\boldsymbol{\nabla} v, v) - (f, v)_\Omega.$$

Then, we have the characterization

$$u = \arg \min_{v \in H_0^1(\Omega)} J(v).$$

The key idea of the primal hybrid formulation is to relax continuity through the use of a Lagrange multiplier.

P.A. Raviart and J.M. Thomas, 1977.

Piecewise smooth functions

Consider a polytopal mesh $\mathcal{T}_H$ of Ω and the "broken" space

$$V := \left\{ v \in L^2(\Omega) \mid v|_K \in H^1(K) \, \forall K \in \mathcal{T}_H \right\}.$$

We would like to characterize $H_0^1(\Omega) \subset V$.

Let $v \in V$. Then $v \in H_0^1(\Omega)$ iff

$$\sum_{K \in \mathcal{T}_H} \int_{\partial K} \boldsymbol{\sigma} \cdot \boldsymbol{n}_K v = 0$$

for all $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$.

Lagrange multiplier

We see that traces of $\boldsymbol{H}(\mathrm{div}, \Omega)$ are involved. We thus set

$$\Lambda := \left\{ \lambda \in \prod_{K \in \mathcal{T}_H} H^{-1/2}(\partial K) \mid \exists \boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega);\ \lambda|_{\partial K} = \boldsymbol{\sigma} \cdot \boldsymbol{n}_K \ \forall K \in \mathcal{T}_H \right\},$$

and

$$\langle \lambda, v \rangle := \sum_{K \in \mathcal{T}_H} \int_{\partial K} \lambda v.$$

Hence $v \in V$ belongs to $H_0^1(\Omega)$ iff

$$\langle \lambda, v \rangle = 0 \quad \forall \lambda \in \Lambda.$$

The primal hybrid formulation

We have

$$u = \arg\min_{v \in H_0^1(\Omega)} J(v) = \arg\min_{\substack{v \in V \\ \langle \mu, v \rangle = 0 \ \forall \mu \in \Lambda}} J(v).$$

The primal hybrid formulation is the Euler-Lagrange equations:
Find $(u, \lambda) \in V \times \Lambda$ such that

$$\begin{cases} (\boldsymbol{A}\nabla u, \nabla v)_{\mathcal{T}_H} + \langle \lambda, v \rangle = (f, v) & \forall v \in V, \\ \langle \mu, u \rangle = 0 & \forall \mu \in \Lambda. \end{cases}$$

u is the original solution, and $\lambda|_{\partial K} = \nabla u \cdot \boldsymbol{n}_K$.

The multiscale hybrid-mixed (MHM) method

The MHM formulation

Local problems

We can rewrite the first equation of the primal hybrid formulation as

$$(\boldsymbol{A}\boldsymbol{\nabla} u, \boldsymbol{\nabla} v)_{\mathcal{T}_H} = (f, v)_\Omega - \langle \lambda, v \rangle \quad \forall v \in V,$$

which we may rewrite element-wise as

$$\begin{cases} -\boldsymbol{\nabla} \cdot (\boldsymbol{A}\boldsymbol{\nabla} u) & = & f & \text{in } K, \\ \boldsymbol{A}\boldsymbol{\nabla} u \cdot \boldsymbol{n}_K & = & \lambda & \text{on } \partial K, \end{cases}$$

for all $K \in \mathcal{T}_H$.

If λ is known, we obtain u via Neumann problems.
This idea is behind the definition of multiscale shape functions.

Kernel extraction

Neumann problems have a kernel: we need to extract constants.

Let $V = \mathcal{P}_0(\mathcal{T}_H) \overset{\perp}{\oplus} V^\perp$ and $u = u_0 + u^\perp$. We have

$$(\boldsymbol{A}\boldsymbol{\nabla} u^\perp, \boldsymbol{\nabla} v)_{\mathcal{T}_H} + \langle \lambda, v\rangle = (\boldsymbol{\nabla} u, \boldsymbol{\nabla} v)_{\mathcal{T}_H} + \langle \lambda, v\rangle = \langle f, v\rangle$$

Hence, $u^\perp = T\lambda + \widehat{T}f$, where the operators

$$(\boldsymbol{A}\boldsymbol{\nabla}(T\mu), v^\perp) = -\langle \mu, v^\perp\rangle \quad (\boldsymbol{A}\boldsymbol{\nabla}(\widehat{T}k), v^\perp) = (k, v^\perp)_\Omega \quad \forall v^\perp \in V^\perp$$

for $\mu \in \Lambda$ and $k \in L^2(\Omega)$ are defined via local Neumann problem.

The MHM formulation

Recall the primal hybrid formulaion

$$\begin{cases} (\boldsymbol{A}\boldsymbol{\nabla} u, \boldsymbol{\nabla} v)_{\mathcal{T}_H} + \langle \lambda, v \rangle & = & (f, v) & \forall v \in V, \\ \langle \mu, u \rangle & = & 0 & \forall \mu \in \Lambda. \end{cases}$$

the decomposition $u = u_0 + u^\perp$, and $u^\perp = T\lambda + \widehat{T} f$.

It gives: Find $(u_0, \lambda) \in \mathcal{P}_0(\mathcal{T}_H) \times \Lambda$ such that

$$\begin{cases} \langle \mu, u_0 \rangle + \langle \mu, T\lambda \rangle & = & -\langle \mu, \widehat{T} f \rangle & \forall \mu \in \Lambda, \\ \langle \lambda, v_0 \rangle & = & (f, v_0)_\Omega & \forall v_0 \in \mathcal{P}_0(\mathcal{T}_H). \end{cases}$$

This global saddle point problem is the MHM formulation.

The multiscale hybrid-mixed (MHM) method

The MHM method

Discrete spaces

Recall that

$$\begin{cases} \langle \mu, u_0 \rangle + \langle \mu, T\lambda \rangle & = & -\langle \mu, \widehat{T} f \rangle & \forall \mu \in \Lambda, \\ \langle \lambda, v_0 \rangle & = & (f, v_0)_\Omega & \forall v_0 \in \mathcal{P}_0(\mathcal{T}_H). \end{cases}$$

The space $\mathcal{P}_0(\mathcal{T}_H)$ is already discrete. We thus need $\Lambda_H \subset \Lambda$.

Fixing $k \geq 0$, we consider

$$\Lambda_H := \{ \mu_H \in \Lambda \mid \mu|_F \in \mathcal{P}_k(F) \, \forall F \in \mathcal{F}_H \} \, .$$

The MHM method

The MHM method consists in finding $(u_{0,H}, \lambda_H) \in \mathcal{P}_0(\mathcal{T}_H) \times \Lambda_H$ s.t.

$$\begin{cases} \langle \mu_H, u_{0,H} \rangle + \langle \mu_H, T\lambda \rangle &= -\langle \mu_H, \widehat{T} f \rangle \quad \forall \mu_H \in \Lambda_H, \\ \langle \lambda_H, v_0 \rangle &= (f, v_0)_\Omega \qquad \forall v_0 \in \mathcal{P}_0(\mathcal{T}_H). \end{cases}$$

and then setting

$$u_H := u_{0,H} + T\lambda_H + \widehat{T} f.$$

Implementation

Step 1: consider a basis $(\mu^\ell)_{\ell=1}^N$ of Λ_H.

Step 2: solve the local Neumann problems: find $\eta^\ell|_K \in H^1_\star(K)$

$$(\boldsymbol{A}\boldsymbol{\nabla}\eta^\ell, \boldsymbol{\nabla} v_\star)_K = -\int_{\partial K} \mu^\ell v_\star \quad \forall v_\star \in H^1_\star(K)$$

for all $K \in \mathcal{T}_H$. Compute $\widehat{T}f$ similarly.

Step 3: $T\lambda = \eta = \sum_{\ell=1}^N R_\ell \eta^\ell$, $u_{0,H} = \sum_{K\in\mathcal{T}_H} V_K \mathbf{1}_K$, and

$$\begin{cases} \langle \mu_H, u_{0,H}\rangle + \langle \mu_H, \eta\rangle &= \langle \mu_H, \widehat{T}f\rangle \quad \forall \mu_H \in \Lambda_H \\ \langle \lambda_H, v_{0,H}\rangle &= (f, v_{0,H}) \quad \forall v_{0,H} \in \mathcal{P}_0(\mathcal{T}_H) \end{cases}$$

Step 4: Assemble the solution

$$u_H = u_{0,H} + T\lambda_H + \widehat{T}f = \sum_{K\in\mathcal{T}_H} V_K \mathbf{1}_K + \sum_{\ell=1}^N R_\ell \eta^\ell + \widehat{T}f.$$

The multiscale hybrid high-order (MsHHO) method

Background

- D. Di Pietro, A. Ern and S. Lemaire, 2014: original HHO article.
- D. Di Pietro and A. Ern, 2015: Elasticity.
- D. Di Pietro, and J. Droniou, 2017: Leray-Lions equations.
- M. Cicuttin, A. Ern and S. Lemaire, 2019: multiscale extension.

Degrees of freedom and reconstruction operator

The HHO method relies on the following set of discrete uknowns

$$\widehat{U}_H := \mathcal{P}_k(\mathcal{T}_H) \times \mathcal{P}_k(\mathcal{F}_H)$$

These unkowns are respectively meant to reprent $u|_K$ and $u|_F$.

The method "normally" hinges on a discrete reconstruction operator.
$\mathscr{G} : \widehat{U}_H \to \boldsymbol{\nabla}\mathcal{P}_{k+1}(\mathcal{T}_H)$, defined for $u := (u_\mathcal{T}, u_\mathcal{F}) \in \widehat{U}_H$ by

$$(\mathscr{G} u, \boldsymbol{\nabla} v)_K = (\boldsymbol{\nabla} u_K, \boldsymbol{\nabla} v)_K + \sum_{F \subset \partial K} (u_F - u_K, \boldsymbol{\nabla} v \cdot \boldsymbol{n}_K)_F$$

for all $v \in \mathcal{P}_{k+1}(K)$ and all $K \in \mathcal{T}_H$.

The multiscale variant

In the MsHHO, the “discrete” operator is replaced by local PDEs.
This enables to upscale fine scale details onto the coarse mesh.

Consider the finite-dimensional space

$$\mathscr{U}_k(K) := \left\{ v \in H^1(K) \mid \nabla \cdot (\boldsymbol{A}\nabla v) \in \mathcal{P}_k(K),\ \boldsymbol{A}\nabla v \cdot \boldsymbol{n}_K|_F \in \mathcal{P}_k(F) \right\}$$

and $\mathscr{U}_k(\mathcal{T}_H) := \prod_{K \in \mathcal{T}_H} \mathscr{U}_k(K)$.

For all $\widehat{u} := (u_{\mathcal{T}}, u_{\mathcal{F}}) \in \widehat{U}_H$ and $K \in \mathcal{T}_H$, $\exists ! r_K(u) \in \mathscr{U}_k(K)$ such that

$$\begin{aligned} (\boldsymbol{A}\nabla r_K(\widehat{u}), \nabla v)_K &= -(v_K, \nabla \cdot (\boldsymbol{A}\nabla v))_K + \textstyle\sum_{F \subset \partial K} (u_F, \boldsymbol{A}\nabla v \cdot \boldsymbol{n}_K)_F \\ (r_K(\widehat{u}), \mathbf{1}_K)_K &= \textstyle\sum_{F \subset \partial K} (u_F, \mathbf{1}_F)_F \end{aligned}$$

for all $v \in \mathscr{U}_k(K)$.

The MsHHO method

Step 1: Assemble the $\mathscr{U}_k(K)$ spaces by solving local Neumann problems.

Step 2: Find dofs $\widehat{u} \in \widehat{U}_H$ such that

$$\sum_{K \in \mathcal{T}_H} (\boldsymbol{A} \nabla r_K(\widehat{u}), \nabla r_K(\widehat{v}))_K = \sum_{K \in \mathcal{T}_H} (f, \widehat{v}_K)_K$$

for all $\widehat{v} \in \widehat{U}_H$.

Step 3: Assemble the solution

$$u_H = \sum_{K \in \mathcal{T}_H} r_K(\widehat{u}).$$

MHM = HHO?

A first naive look

Cons:

The number of dofs does not match!
The dofs of MHM correspond $u_{0,K} := \int_K u$ and $\lambda|_{\partial K} = \boldsymbol{A}\nabla u \cdot \boldsymbol{n}_K$.
The dofs of MsHHO correspond to $u|_K$ and $u|_{\partial K}$.

The global linear systems have different features:
Saddle-point for MHM, SPD for MsHHO.

Pros:

Both methods employ Neumann problems to construct the multiscale basis functions.

The central result

Theorem

For all $f \in L^2(\Omega)$, there exists a unique $u \in H^1(\mathcal{T}_H)$ s.t.

$$-\boldsymbol{\nabla} \cdot (\boldsymbol{A} \boldsymbol{\nabla}_H u) = f \text{ in } \Omega$$

and

$$\boldsymbol{A} \boldsymbol{\nabla}_H u \cdot \boldsymbol{n}_F \in \mathcal{P}_k(F) \quad ([\![u]\!], q)_F = 0 \quad \forall q \in \mathcal{P}_k(F)$$

for all $F \in \mathcal{T}_H$.

This u is linked to the solutions of MHM and MsHHO methods.

The MHM solution (1/2)

Recall that for the MHM method

$$u_H := u_{0,H} + T\lambda_H + \widehat{T}f.$$

Hence,

$$-\nabla \cdot (\boldsymbol{A}\nabla_H u_H) = -\nabla \cdot (\boldsymbol{A}\nabla_H T\lambda_H) - \nabla \cdot (\boldsymbol{A}\nabla_H \widehat{T}f) = f$$

and

$$(\boldsymbol{A}\nabla_H u_H) \cdot \boldsymbol{n}_F = (\boldsymbol{A}\nabla_H T\lambda_H) \cdot \boldsymbol{n}_F + (\boldsymbol{A}\nabla_H \widehat{T}f) \cdot \boldsymbol{n}_F = \lambda_H|_F \in \mathcal{P}_k(F)$$

for all $F \in \mathcal{F}_H$.

By construction, the MHM solution automatically fullfils two conditions.

The MHM solution (2/2)

Recall the global MHM saddle point problem

$$\begin{cases} \langle \mu_H, u_{0,H} \rangle + \langle \mu_H, T\lambda_H \rangle & = & -\langle \mu_H, \widehat{T} f \rangle & \forall \mu_H \in \Lambda_H, \\ \langle \lambda_H, v_0 \rangle & = & (f, v_0)_\Omega & \forall v_0 \in \mathcal{P}_0(\mathcal{T}_H). \end{cases}$$

We may rewrite the first line as

$$\langle \mu_H, u_H \rangle = 0 \quad \forall \mu_H \in \Lambda_H,$$

or after breaking the elementwise integrals, as

$$\sum_{F \in \mathcal{F}_H} (\mu_H, [\![u_H]\!])_F = 0 \quad \forall \mu_H \in \Lambda_H.$$

The MHM solution is the unique function of the main theorem!

The MsHHO solution (1/2)

One easily shows that

$$(r_K(\widehat{v}), q_K)_K = (\widehat{v}_K, q_K)_K \quad \forall q_K \in \mathcal{P}_k(K)$$
$$(r_K(\widehat{v}), q_F)_F = (\widehat{v}_F, q_F)_F \quad \forall q_F \in \mathcal{P}_k(F)$$

for all $\widehat{v} \in \widehat{U}_H$

Since the $\widehat{v}_F$ are univalued on faces, we see that

$$(\llbracket r_K(\widehat{v}) \rrbracket, q)_F = 0 \quad \forall q \in \mathcal{P}_k(F)$$

for all $F \in \mathcal{F}_H$.

The MsHHO solution satisfies the third condition by construction.

The MsHHO solution (2/2)

Recall global MsHHO problem

$$\sum_{K\in\mathcal{T}_H} (\boldsymbol{A}\boldsymbol{\nabla} r_K(\widehat{u}), \boldsymbol{\nabla} r_K(\widehat{v}))_K = \sum_{K\in\mathcal{T}_H} (f, v_K)_K,$$

which we may rewrite as

$$-\sum_{K\in\mathcal{T}_H} (\boldsymbol{\nabla}\cdot(\boldsymbol{A}\boldsymbol{\nabla} r_K(\widehat{u})), \widehat{v}_K)_K + \sum_{F\in\mathcal{T}_H} ([\![\boldsymbol{A}\boldsymbol{\nabla} r_K(\widehat{u})]\!]\cdot\boldsymbol{n}_F, \widehat{v}_F)_F = \sum_{K\in\mathcal{T}_H} (\Pi_k f, v_K)_K,$$

since $\boldsymbol{\nabla}\cdot(\boldsymbol{A}\boldsymbol{\nabla} r_K(\widehat{u})) \in \mathcal{P}_k(K)$ and $\boldsymbol{A}\boldsymbol{\nabla} r_K(\widehat{u})\cdot\boldsymbol{n}_F \in \mathcal{P}_k(F)$.

It follows that

$$-\boldsymbol{\nabla}\cdot(\boldsymbol{A}\boldsymbol{\nabla}_H u_H) = \Pi_k f \qquad \boldsymbol{A}\boldsymbol{\nabla} u_H\cdot\boldsymbol{n}_F \in \mathcal{P}_k(F).$$

What have we learnt?

MHM = HHO?

Whenever $f \in \mathcal{P}_k(\mathcal{T}_H)$, the MHM and MsHHO methods produce the same solution u_H.

This solution can be clearly indentified by conditions on the flux, the normal traces and the jumps.

More insights on the two methods

The two methods may be seen as “dual” to each other.

The MHM method spans a space of functions such that

$$-\nabla \cdot (\boldsymbol{A} \nabla_H v_H) = f \quad (\boldsymbol{A} \nabla_H v_H) \cdot \boldsymbol{n}_F \in \mathcal{P}_k(F)$$

for all $F \in \mathcal{F}_H$. Weak continuity is enforced by the global system

$$(\llbracket u_H \rrbracket, q)_F = 0 \quad \forall q \in \mathcal{P}_k(F).$$

The MsHHO method works exactly the other way around.

Perspective for improvements

We hope this work will shed a new light on the design of skeletal based multiscale methods.

We believe that several opportunities of improvement will arise in the future. For example, saddle-point MHM problems can be replaced by MsHHO SPD problems.

Thanks for you attention!