An equilibrated a posteriori error estimator for the curl-curl problem

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Outline

The Poisson problem

Poisson problem

Consider a Lipschitz polyhedral domain Ω and $f \in L^2(\Omega)$.

Our first model problem is to find $u \in H^1_0(\Omega)$ such that

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega$$

for all $v \in H^1_0(\Omega)$.

u_h is the Lagrange FEM approximation of u with degree $p+1$.

For the sake of simplicity, I will assume that $f = f_h \in \mathcal{P}_p(\mathcal{T}_h)$.

The idea of flux equilibration

Assume that we have a field $\boldsymbol{\sigma}_h \in \boldsymbol{H}(\mathrm{div}, \Omega)$ such that

$$\nabla \cdot \boldsymbol{\sigma}_h = f_h \text{ in } \Omega$$

at our disposal.

Then, we have

$$\begin{aligned}
(\nabla(u - u_h), \nabla v)_\Omega &= (f_h, v)_\Omega - (\nabla u_h, \nabla v) \\
&= (\nabla \cdot \boldsymbol{\sigma}_h, v)_\Omega - (\nabla u_h, \nabla v) \\
&= -(\boldsymbol{\sigma}_h + \nabla u_h, \nabla v),
\end{aligned}$$

for all $v \in H^1_0(\Omega)$ and in particular

$$\|\nabla(u - u_h)\|_\Omega \leq \|\boldsymbol{\sigma}_h + \nabla u_h\|_\Omega.$$

Prager-Synge theorem

Equilibrated flux

$$\boldsymbol{\sigma}_h \in \boldsymbol{H}(\mathrm{div}, \Omega); \quad \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_h = f_h \text{ in } \Omega$$

Error estimate

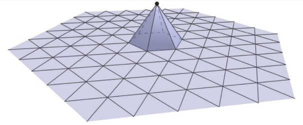
$$\|\boldsymbol{\nabla}(u - u_h)\|_\Omega \leq \|\boldsymbol{\sigma}_h + \boldsymbol{\nabla} u_h\|_\Omega.$$

W. Prager and J.L. Synge, 1947

The particular choice $\boldsymbol{\sigma} := -\boldsymbol{\nabla} u$ saturates the bound.

Local flux constructions

The idea flux would be $\boldsymbol{\sigma} := -\nabla u$.
Let's characterize it locally and cook up a discrete computable version.



Let us set

$$\boldsymbol{\sigma}^{\boldsymbol{a}} := \psi_{\boldsymbol{a}} \boldsymbol{\sigma}$$

where $\psi_{\boldsymbol{a}}$ is the "hat function" associated with the vertex $\boldsymbol{a}$, so that

$$\boldsymbol{\sigma} = \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}^{\boldsymbol{a}}.$$

Characterization of the local contributions

We know that $\boldsymbol{\sigma}^{\boldsymbol{a}} := -\psi_{\boldsymbol{a}} \boldsymbol{\nabla} u$. So that in particular

$$\boldsymbol{\sigma}^{\boldsymbol{a}} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$$

where $\omega_{\boldsymbol{a}}$ is the set of tetrahedra $K \in \mathcal{T}_h$ sharing the vertex $\boldsymbol{a}$, and

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^{\boldsymbol{a}} = \psi_{\boldsymbol{a}} f_h - \boldsymbol{\nabla} \psi_{\boldsymbol{a}} \cdot \boldsymbol{\nabla} u.$$

As a result, we have the characterization

$$\boldsymbol{\sigma}^{\boldsymbol{a}} = \arg \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \cdot \boldsymbol{v} = \psi_{\boldsymbol{a}} f_h - \boldsymbol{\nabla} \psi_{\boldsymbol{a}} \cdot \boldsymbol{\nabla} u}} \|\boldsymbol{v} + \psi_{\boldsymbol{a}} \boldsymbol{\nabla} u\|_{\omega_{\boldsymbol{a}}},$$

since the minimum is zero and achieved when $\boldsymbol{v} = -\psi_{\boldsymbol{a}} \boldsymbol{\nabla} u$.

Discrete construction

Recall that

$$\boldsymbol{\sigma}^{\boldsymbol{a}} = \arg \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v} = \psi_{\boldsymbol{a}} f_h - \nabla \psi_{\boldsymbol{a}} \cdot \nabla u}} \|\boldsymbol{v} + \psi_{\boldsymbol{a}} \nabla u\|_{\omega_{\boldsymbol{a}}}.$$

As a discrete counterpart, we set

$$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RT}_{p+1}(\mathcal{T}_h^{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v}_h = \psi_{\boldsymbol{a}} f_h - \nabla \psi_{\boldsymbol{a}} \cdot \nabla u_h}} \|\boldsymbol{v}_h + \psi_{\boldsymbol{a}} \nabla u_h\|_{\omega_{\boldsymbol{a}}}.$$

This is indeed well-defined since

$$(\psi_{\boldsymbol{a}} f - \nabla \psi_{\boldsymbol{a}} \cdot \nabla u_h, 1)_{\omega_{\boldsymbol{a}}} = (f, \psi_{\boldsymbol{a}})_\Omega - (\nabla u_h, \nabla \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = 0$$

whenever $\boldsymbol{a} \notin \partial\Omega$.

Reliability and efficiency

Summation provides an equilibrated flux:

$$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}},$$

and reliability follows from the Prager–Synge theorem.

Efficiency

$$\|\boldsymbol{\sigma}_h + \boldsymbol{\nabla} u_h\|_K \lesssim \|\boldsymbol{\nabla}(u - u_h)\|_{\widetilde{K}}$$

The hidden constant does not depend on p.

The curl–curl problem

The curl–curl problem

For the sake of simplicity, assume that Ω is simply connected, and consider a divergence-free right-hand side $\boldsymbol{J}_h \in \boldsymbol{RT}_p(\mathcal{T}_h)$.

Our model problem is then to find $\boldsymbol{A} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega)$ such that

$$(\boldsymbol{\nabla} \times \boldsymbol{A}, \boldsymbol{\nabla} \times \boldsymbol{v})_\Omega = (\boldsymbol{J}_h, \boldsymbol{v})_\Omega, \qquad (\boldsymbol{A}, \boldsymbol{\nabla} q)_\Omega = 0,$$

for all $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega)$ and $q \in H^1_0(\Omega)$.

$\boldsymbol{A}_h \in \boldsymbol{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_0(\mathbf{curl}, \Omega)$ is the Nédélec approximation of $\boldsymbol{A}$.

Prager-Synge theorem

Equilibrated flux

$$\boldsymbol{B}_h \in \boldsymbol{H}(\mathbf{curl}, \Omega); \quad \nabla \times \boldsymbol{B}_h = \boldsymbol{J}_h$$

Error estimate

$$\|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\|_\Omega \leq \|\boldsymbol{B}_h - \nabla \times \boldsymbol{A}_h\|_\Omega$$

The "ideal" flux $\boldsymbol{B} := \nabla \times \boldsymbol{A}$ saturates the bound.

The issue with localization (1/2)

We follow the same steps than for the Poisson problem.

The ideal flux is $\boldsymbol{B} := \boldsymbol{\nabla} \times \boldsymbol{A}$. If we set $\boldsymbol{B}^{\boldsymbol{a}} := \psi_{\boldsymbol{a}} \boldsymbol{\nabla} \times \boldsymbol{A}$, then

$$\boldsymbol{\nabla} \times \boldsymbol{B}^{\boldsymbol{a}} = \psi_{\boldsymbol{a}} \boldsymbol{J}_h + \boldsymbol{\nabla} \psi_{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A},$$

so that

$$\boldsymbol{B}^{\boldsymbol{a}} = \arg \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \times \boldsymbol{v} = \psi_{\boldsymbol{a}} \boldsymbol{J} + \boldsymbol{\nabla} \psi_{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}}} \|\boldsymbol{v} - \psi_{\boldsymbol{a}} \boldsymbol{\nabla} \times \boldsymbol{A}\|_{\omega_{\boldsymbol{a}}}.$$

The issue with localization (2/2)

Recall that

$$\boldsymbol{B}^{\boldsymbol{a}} = \arg \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \times \boldsymbol{v} = \psi_{\boldsymbol{a}} \boldsymbol{J}_h + \boldsymbol{\nabla} \psi_{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}}} \|\boldsymbol{v} - \psi_{\boldsymbol{a}} \boldsymbol{\nabla} \times \boldsymbol{A}\|_{\omega_{\boldsymbol{a}}}.$$

Unfortunately, we can not set

$$\boldsymbol{B}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{N}_{p+1}(\mathcal{T}_h^{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathbf{curl}, \omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \times \boldsymbol{v}_h = \psi_{\boldsymbol{a}} \boldsymbol{J}_h + \boldsymbol{\nabla} \psi_{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h}} \|\boldsymbol{v}_h - \psi_{\boldsymbol{a}} \boldsymbol{\nabla} \times \boldsymbol{A}_h\|_{\omega_{\boldsymbol{a}}}$$

as the minimization set is empty: the field

$$\psi_{\boldsymbol{a}} \boldsymbol{J}_h + \boldsymbol{\nabla} \psi_{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h$$

is not divergence-free!

Possible solutions

An initial idea for lowest-order elements:

- D. Braess and J. Schöberl, 2008

Recently developped extensions:

- J. Gedicke, S. Geevers and I. Perugia, 2019
- J. Gedicke, S. Geevers, I. Perugia and J. Schöberl, 2020
- T. Chaumont-Frelet and M. Vohralík, 2021

Here, I will detail the last construction.

The idea

Let $\boldsymbol{\theta}^{\boldsymbol{a}} := \nabla\psi_{\boldsymbol{a}} \times \nabla \times \boldsymbol{A}$. There are two important properties:

$$\sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\theta}^{\boldsymbol{a}} = \mathbf{0}, \qquad \nabla\cdot\boldsymbol{\theta}^{\boldsymbol{a}} = -\nabla\psi_{\boldsymbol{a}}\cdot\boldsymbol{J}_h.$$

At the discrete level, we have

$$\sum_{\boldsymbol{a}\in\mathcal{V}_h} (\nabla\psi_{\boldsymbol{a}} \times \nabla \times \boldsymbol{A}_h) = \mathbf{0}, \qquad \nabla\cdot(\nabla\psi_{\boldsymbol{a}} \times \nabla \times \boldsymbol{A}_h) \neq -\nabla\psi_{\boldsymbol{a}}\cdot\boldsymbol{J}_h.$$

It is tempting to set

$$\boldsymbol{\theta}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}\in\boldsymbol{RT}_p(\mathcal{T}_h^{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}=-\nabla\psi_{\boldsymbol{a}}\cdot\boldsymbol{J}_h}} \|\boldsymbol{v} - \nabla\psi_{\boldsymbol{a}} \times \nabla \times \boldsymbol{A}_h\|_{\omega_{\boldsymbol{a}}},$$

but it does not sum up to zero.

Trick #1: Over-constrained minimization

We instead set

$$\widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v} \in \boldsymbol{RT}_p(\mathcal{T}_h^{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ (\boldsymbol{v} - \boldsymbol{\nabla}\psi_{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h, \boldsymbol{r})_{\omega_{\boldsymbol{a}}} = 0 \ \forall \boldsymbol{r} \in \boldsymbol{\mathcal{P}}_0(\mathcal{T}_h^{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \cdot \boldsymbol{v} = -\boldsymbol{\nabla}\psi_{\boldsymbol{a}} \cdot \boldsymbol{J}_h}} \|\boldsymbol{v} - \boldsymbol{\nabla}\psi_{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h\|_{\omega_{\boldsymbol{a}}},$$

where the mean-value constrain works because $\boldsymbol{A}_h$ solves the discrete problem. We then set

$$\widehat{\boldsymbol{\theta}}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}}.$$

Importantly, we have

$$(\widehat{\boldsymbol{\theta}}_h, \boldsymbol{r})_\Omega = 0 \quad \forall \boldsymbol{r} \in \boldsymbol{\mathcal{P}}_0(\mathcal{T}_h).$$

Trick #2: Divergence-free decomposition of $\widehat{\theta}$ (1/2)

The $\widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}}$ have the correct divergence, but does not sum up to zero.
We need a divergence-free decomposition of $\widehat{\boldsymbol{\theta}}_h$ into contributions $\widetilde{\boldsymbol{\theta}}_h^{\boldsymbol{a}}$.

We set for each $\boldsymbol{a} \in \mathcal{V}_h$ and $K \in \mathcal{T}_h^{\boldsymbol{a}}$

$$\widetilde{\boldsymbol{\theta}}_h^{\boldsymbol{a}}|_K := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RT}_{p+1}(K) \\ \boldsymbol{\nabla}\cdot \boldsymbol{v}_h = 0 \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = \psi_{\boldsymbol{a}} \widehat{\boldsymbol{\theta}}_h \cdot \boldsymbol{n}_K \text{ on } \partial K}} \|\boldsymbol{v}_h - \psi_{\boldsymbol{a}} \widehat{\boldsymbol{\theta}}_h\|_K$$

This is indeed well-posed since

$$\begin{aligned}
(\psi_{\boldsymbol{a}} \widehat{\boldsymbol{\theta}}_h \cdot \boldsymbol{n}_K, 1)_{\partial K} &= (\psi_{\boldsymbol{a}}, \widehat{\boldsymbol{\theta}}_h \cdot \boldsymbol{n}_K)_{\partial K} \\
&= (\boldsymbol{\nabla} \psi_{\boldsymbol{a}}, \widehat{\boldsymbol{\theta}}_h)_K + (\boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\theta}}_h, \psi_{\boldsymbol{a}})_K \\
&= 0.
\end{aligned}$$

Trick #2: Divergence-free decomposition of $\widehat{\theta}$ (2/2)

We can then actually show that

$$\sum_{\boldsymbol{a}\in\mathcal{V}_h} \widetilde{\boldsymbol{\theta}}_h^{\boldsymbol{a}} = \widehat{\boldsymbol{\theta}}_h,$$

so that setting

$$\boldsymbol{\theta}_h^{\boldsymbol{a}} := \widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}} - \widetilde{\boldsymbol{\theta}}_h^{\boldsymbol{a}} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}),$$

we have

$$\boldsymbol{\nabla}\cdot\boldsymbol{\theta}_h^{\boldsymbol{a}} = -\boldsymbol{\nabla}\psi_{\boldsymbol{a}}\cdot\boldsymbol{J}_h, \qquad \sum_{\boldsymbol{a}\in\mathcal{V}_h}\boldsymbol{\theta}_h^{\boldsymbol{a}} = \boldsymbol{0}.$$

Local flux contruction and efficiency

Since the $\boldsymbol{\theta}_h^{\boldsymbol{a}}$ have all the required properties, we can now set

$$\boldsymbol{B}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{N}_{p+1}(\mathcal{T}_h^{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathbf{curl}, \omega_{\boldsymbol{a}}) \\ \nabla \times \boldsymbol{v}_h = \psi_{\boldsymbol{a}} \boldsymbol{J}_h + \boldsymbol{\theta}_h^{\boldsymbol{a}}}} \| \boldsymbol{v}_h - \psi_{\boldsymbol{a}} \nabla \times \boldsymbol{A}_h \|_{\omega_{\boldsymbol{a}}},$$

and we construct an equilibrated flux as

$$\boldsymbol{B}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{B}_h^{\boldsymbol{a}}.$$

Efficiency

$$\| \boldsymbol{B}_h - \nabla \times \boldsymbol{A}_h \|_K \lesssim \| \nabla \times (\boldsymbol{A} - \boldsymbol{A}_h) \|_{\widetilde{K}}$$

The hidden constant does not depend on p.

T. Chaumont-Frelet, A. Ern and M. Vohralík, 2021