# On the number of essential arguments of homomorphisms between products of median algebras

Miguel Couceiro, Gerasimos C Meletiou

# On the number of essential arguments of homomorphisms between products of median algebras

Miguel Couceiro and Gerasimos C. Meletiou

**Abstract.** In this paper we characterize classes of median-homomorphisms between products of median algebras, that depend on a given number of arguments, by means of necessary and sufficent conditions that rely on the underlying algebraic and on the underlying order structure of median algebras. In particular, we show that a median-homomorphism that take values in a median algebra that does not contain a subalgebra isomorphic to the $m$-dimensional Boolean algebra as a subalgebra cannot depend on more than $m-1$ arguments. In view of this result, we also characterize the latter class of median algebras. We also discuss extensions of our framework on homomorphisms over median algebras to wider classes of algebras.

**Mathematics Subject Classification.** 06A12, 06D99, 06F99.

**Keywords.** Median algebra, median-homomorphism, essential argument, hypercube-freeness, congruence distributivity.

## 1. Introduction

A *median algebra* is a structure $\mathbf{M} = \langle M, \mathbf{m} \rangle$ for a nonempty set $M$, called the *universe*, and a ternary operation $\mathbf{m}\colon M^3 \to M$, called *median*, such that

$$\begin{aligned} \mathbf{m}(x,x,y) &= x, \\ \mathbf{m}(x,y,z) &= \mathbf{m}(y,x,z) = \mathbf{m}(y,z,x), \\ \mathbf{m}(\mathbf{m}(x,y,z),t,u) &= \mathbf{m}(x,\mathbf{m}(y,t,u),\mathbf{m}(z,t,u)), \end{aligned} \tag{1.1}$$

for all $x,y,z,t,u \in M$. Note that condition (1.1) can be thought of as a distributive law. For general background see, e.g., [2, 14].

The first named author acknowleges the financial support from the CNRS Mastodons project QCM-BioChem.

It is well known [16, 17] that each element $a$ of a median algebra $\mathbf{M}$ gives rise to a meet-semilattice $\langle M, \leq_a \rangle$ where $\leq_a$ is defined by

$$x \leq_a y \quad \Longleftrightarrow \quad \mathbf{m}(a, x, y) = x. \tag{1.2}$$

Clearly, $a$ is the least element of $\langle M, \leq_a \rangle$ and it induces a meet operation $\wedge_a$ on $\langle M, \leq_a \rangle$, which is given by $x \wedge_a y = \mathbf{m}(a, x, y)$. Semilattices $\langle M, \leq_a \rangle$ thus constructed are *median semilattices* with a least element (namely, $a$) and they are characterized by the fact that their principal ideals

$$\downarrow x := \{y \in M : y \leq_a x\}$$

are distributive lattices, and by the fact that for any $a, b, c \in M$, the 3-element set $\{a, b, c\}$ has an upper bound whenever each of its 2-element subsets $\{a, b\}$, $\{b, c\}$, $\{c, a\}$ has an upper bound. In this case, a median operation $\mathbf{m} \colon M^3 \to M$ can be defined by

$$\mathbf{m}(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (z \wedge y), \tag{1.3}$$

for every $x, y, z \in M$.

A semilattice operation on $\mathbf{M}$ will be called *compatible* if the median operation $\mathbf{m}$ can be written in terms of $\wedge$ and the partial $\vee$ as in 1.3. Note that all meets of the form $\wedge_a$, $a \in M$, are compatible semilattice operations (see [5]). However there exist semilattice operations that are not of the form $\wedge_a$ but they give rise to semilattice orders that are preserved by the median operation. For instance, take $M = ]0, 1[ \subseteq \mathbb{R}$: the min and max operations are compatible semilattice operations.

A subset $C \subseteq M$ is said to be a *convex subset of* $\mathbf{M}$ if for every $a, b, c \in M$ so that $a, c \in C$, we have $\mathbf{m}(a, b, c) \in C$; see, e.g., [2, 18]. It is easy to verify that $C$ is then a median subalgebra of $M$. Moreover, $C$ is a convex subset of $\mathbf{M}$ if and only if it is a subsemilattice of $\langle M, \leq_a \rangle$, for every $a \in M$. For $a, b \in M$, the *convex hull* of $\{a, b\}$ or the *interval* from $a$ to $b$, denoted by $[a, b]$, is defined by

$$[a, b] := \{t \in M : t = \mathbf{m}(a, t, b)\} = \{\mathbf{m}(a, t, b) : t \in M\}.$$

It is well-known [6] that every interval $[a, b]$ in a median algebra can be endowed with a distributive lattice structure, and it is easy to verify that $\langle [a, b], \wedge_a, \wedge_b \rangle$ is a distributive lattice with $a$ and $b$ as the least and greatest elements, respectively.

Let $\mathbf{M} = \langle M, \mathbf{m} \rangle$ be a median algebra and let $\xi(\mathbf{M})$ be the set of all compatible semilattice operations on $\mathbf{M}$. The set $\xi(\mathbf{M})$ endowed with the median $\mathbf{m}_\xi$ defined by $\mathbf{m}_\xi(\wedge_1, \wedge_2, \wedge_3) = \wedge$, where $\wedge$ is given by

$$x \wedge y = \mathbf{m}(x \wedge_1 y, x \wedge_2 y, x \wedge_3 y)$$

for all $x, y \in M$, is a median algebra. In fact, we have the following result from [5].

**Theorem 1.1.** $\mathbf{M} = \langle M, \mathbf{m} \rangle$ *be a median algebra and let* $\xi(\mathbf{M})$ *be the set of all compatible semilattice operations on* $\mathbf{M}$ *endowed with the median* $\mathbf{m}_\xi$ *defined above. Then* $\xi(\mathbf{M})$ *is, up to isomorphism, the unique median algebra* $\mathbf{N}$ *such that*

- $\mathbf{M}$ *is a convex subalgebra of* $\mathbf{N}$*;*
- *every compatible semilattice operation on* $\mathbf{M}$ *uniquely extends to a compatible semilattice operation on* $\mathbf{N}$*;*
- *every compatible semilattice operation on* $\mathbf{N}$ *has a zero (annihilator).*

In [10, 11] median-homomorphisms, i.e., median-preserving mappings, between products of median algebras were investigated and Arrow type [1] impossibility results were derived. In particular, it was shown in [11] that if $\mathbf{A}_1, \cdots, \mathbf{A}_n$ are median algebras and if $\mathbf{B}$ is a tree (i.e., a $\wedge$-semilattice where no pair of incomparable elements have an upper bound), then a mapping

$$f\colon \mathbf{A}_1 \times \cdots \times \mathbf{A}_n \to \mathbf{B}$$

is a median-homomorphism if and only if $f$ is an essentially at most unary median homomorphism, i.e., there exists $i \in [n] = \{1, \ldots, n\}$ and a median homomorphism $h\colon \mathbf{A}_i \to \mathbf{B}$ such that

$$f(\mathbf{a}) = h(a_i), \quad \text{for all} \quad \mathbf{a} = (a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n.$$

In addition to this, the inverse problem was also tackled, namely: Given arbitrary median algebras $\mathbf{A}_1, \cdots, \mathbf{A}_n$, characterize those median algebras $\mathbf{B}$ for which all median-homomorphisms $f\colon \mathbf{A}_1 \times \cdots \times \mathbf{A}_n \to \mathbf{B}$ are essentially at most unary. As it turned out, the answer is that this is the case if and only if $\mathbf{B}$ is a tree. To locate trees among median algebras many characterizations were proposed in the literature; see [18] for a general reference. In [11], trees were characterized in terms of a relaxation of "conservativeness", namely, the simple condition on $\mathbf{m}$ that for every $x, y, z \in M$

$$\mathbf{m}(x, y, z) := (x \wedge y) \vee (x \wedge z) \vee (z \wedge y) \in \{x \wedge y, y \wedge z, z \wedge x\}.$$

In view of these results, one naturally asks for the characterization of median-homomorphisms $f\colon \mathbf{A}_1 \times \cdots \times \mathbf{A}_n \to \mathbf{B}$ that are essentially at most binary, ternary, etc. A median-homomorphism is said to be *essentially at most $k$-ary*, for $1 \leq k \leq n$, if there is $K \subseteq [n]$ with $|K| \leq k$, and a median-homomorphism $g\colon \Pi_{i\in K}\mathbf{A}_i \to \mathbf{B}$ such that

$$f(\mathbf{a}) = g(\mathbf{a}_K), \quad \text{for all} \quad \mathbf{a} = (a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n,$$

and where $\mathbf{a}_K = (a_i)_{i\in K} \in \Pi_{i\in K}\mathbf{A}_i$.

In this paper we provide characterizations of essentially at most $k$-ary median-homomorphisms, and we show that those median algebras $\mathbf{B}$ for which all median-homomorphisms $f\colon \mathbf{A}_1 \times \cdots \times \mathbf{A}_n \to \mathbf{B}$ are forcingly essentially at most $k$-ary do not contain a subalgebra isomorphic to the $k+1$-dimensional Boolean algebra; in fact, the latter condition is both necessary and sufficient. Moreover, for each $k \geq 1$, we present a simple condition on median algebras $\mathbf{B}$ that guarantees that the latter do not contain a subalgebra isomorphic to the $k+1$-dimensional Boolean algebra.

## 2. Essential arity of median-homomorphisms

In this section we will give necessary and sufficient conditions on a median algebra $\mathbf{B}$ that force any median-homomorphism of the form $f\colon \mathbf{A}_1\times\cdots\times\mathbf{A}_n \to \mathbf{B}$ to have at most $k$ essential arguments. To do so, we make use of the fact that median algebras are congruence distributive and thus that their finite products have no skew congruences, to decompose the image of such a median-homomorphism into a direct product of quotient median algebras. From this representation it will follow in particular that if $\mathbf{B}$ does not contain a subalgebra isomorphic to the $k+1$-dimensional Boolean algebra, then $f\colon \mathbf{A}_1 \times \cdots \times \mathbf{A}_n \to \mathbf{B}$ cannot have more than $k$ essential arguments.

To facilitate the presentation, we will adopt the following notation. Let $\mathbf{A}_1, \cdots, \mathbf{A}_n$ be median algebras and $\mathbf{x} = (x_i)_{i\in[n]} \in \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$. For $K \subseteq [n] = \{1, \ldots, n\}$ and $\mathbf{a} = (a_i)_{i\in K} \in \Pi_{i\in K} A_i$, we denote by $\mathbf{x}_K^{\mathbf{a}}$ the tuple in $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ whose $i$-th component is $a_i$ if $i \in K$ and $x_i$ otherwise. When $K = I \cup J$ with $I \cap J = \varnothing$, we extend this notation to $\mathbf{x}_{I|J}^{\mathbf{a}|\mathbf{b}} = (\mathbf{x}_I^{\mathbf{a}})_J^{\mathbf{b}} = (\mathbf{x}_J^{\mathbf{b}})_I^{\mathbf{a}}$.

By the classical homomorphism theorem, the image of a median-homomorphism $f\colon \mathbf{A}_1\times\cdots\times\mathbf{A}_n \to \mathbf{B}$ is isomorphic to $\mathbf{A}_1\times\cdots\times\mathbf{A}_n/\mathrm{Ker}(f)$. From Jónsson's characterization [15] it follows that the variety of median algebras is congruence distributive, and thus that $\mathrm{Ker}(f)$ is a product congruence; see, e.g., [8]. As an immediate consequence, we have the following equivalences.

**Fact 2.1.** Let $f\colon \mathbf{A}_1 \times \cdots \times \mathbf{A}_n \to \mathbf{B}$ be a median-homomorphism, and let $j \in [n]$ and $a, b \in \mathbf{A}_j$. Then the following assertions are equivalent:

(1) For every $\mathbf{c} \in A_1 \times \cdots \times A_n$, $f(\mathbf{c}_{\{j\}}^a) = f(\mathbf{c}_{\{j\}}^b)$;
(2) There exists $\mathbf{c} \in A_1 \times \cdots \times A_n$ such that $f(\mathbf{c}_{\{j\}}^a) = f(\mathbf{c}_{\{j\}}^b)$;
(3) There exists $\mathbf{c}, \mathbf{d} \in A_1 \times \cdots \times A_n$ such that $f(\mathbf{c}_{\{j\}}^a) = f(\mathbf{d}_{\{j\}}^b)$;
(4) $\{f(\mathbf{c}_{\{j\}}^a) : \mathbf{c} \in A_1 \times \cdots \times A_n\} \cap \{f(\mathbf{c}_{\{j\}}^b) : \mathbf{c} \in A_1 \times \cdots \times A_n\} \neq \varnothing$;
(5) $\{f(\mathbf{c}_{\{j\}}^a) : \mathbf{c} \in A_1 \times \cdots \times A_n\} = \{f(\mathbf{c}_{\{j\}}^b) : \mathbf{c} \in A_1 \times \cdots \times A_n\}$.

Consider binary relations $\sim_j^f$, $j \in [n]$, on $\mathbf{A}_j$ defined as follows. For $a, b \in A_j$, set $a \sim_j^f b$ if assertion (1) of Fact 2.1 (or, equivalently, all) holds. It is not difficult to see that $\sim_j^f$, $j \in [n]$, is an invariant of $\mathbf{m}$, i.e., $\sim_j^f$ is a *congruence* on $\mathbf{A}_j$.

**Proposition 2.2.** *For $j \in [n]$, consider the congruence $\sim_j^f$ on $\mathbf{A}_j$. The congruence classes of $\sim_j^f$ are convex subsets of $\mathbf{A}_j$. Moreover, for every $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots, b_n) \in A_1 \times \cdots \times A_n$, we have $f(\mathbf{a}) = f(\mathbf{b})$ if and only if $a_j \sim_j^f b_j$, for all $j \in [n]$.*

*Proof.* To show that the first statement holds, let $C \subseteq A_j$ be an equivalence class of $\sim_j^f$, and let $a, b \in C$ and $t \in A_j$. As $a \sim_j^f b$, $a \sim_j^f a$ and $t \sim_j^f t$, it follows from (1) that $a = \mathbf{m}(a, a, t) \sim_j^f \mathbf{m}(b, a, t)$ and thus that $\mathbf{m}(b, a, t) \in C$. Therefore, $C$ is a convex subset of $\mathbf{A}_j$.

Necessity in the last statement follows from (3) of Fact 2.1, whereas sufficiency can be established by observing that

$$f(\mathbf{a}) \overset{a_1 \sim_1^f b_1}{=} f(\mathbf{a}_{\{1\}}^{b_1}) \overset{a_2 \sim_2^f b_2}{=} f(\mathbf{a}_{\{1,2\}}^{b_1 b_2}) \overset{a_3 \sim_3^f b_3}{=} \dots \overset{a_n \sim_n^f b_n}{=} f(\mathbf{a}_{[n]}^{\mathbf{b}}) = f(\mathbf{b}). \quad \square$$

From the last statement in Proposition 2.2 we immediately obtain the following representation result.

**Corollary 2.3.** *Let $f\colon \mathbf{A}_1 \times \cdots \times \mathbf{A}_n \to \mathbf{B}$ be a median-homomorphism. Then*

$$f(\mathbf{A}_1 \times \cdots \times \mathbf{A}_n) = \{f(\mathbf{x}) : \mathbf{x} \in A_1 \times \cdots \times A_n\}$$

*is isomorphic to $\mathbf{A}_1/\sim_1^f \times \cdots \times \mathbf{A}_n/\sim_n^f$.*

Observe that the $i$-th argument of a median-homomorphism

$$f\colon \mathbf{A}_1 \times \cdots \times \mathbf{A}_n \to \mathbf{B}$$

is inessential if and only if $\sim_i^f$ is the trivial congruence (with a single congruence class). From this observation and Corollary 2.3 we thus obtain the following characterization of median-homomorphisms with at most $k$ essential arguments.

**Theorem 2.4.** *Let $\mathbf{A}_1, \cdots, \mathbf{A}_n$ be median algebras and let $\mathbf{B}$ be a median algebra not containing a subalgebra isomorphic to the $k+1$-dimensional Boolean algebra. Then a mapping*

$$f\colon \mathbf{A}_1 \times \cdots \times \mathbf{A}_n \to \mathbf{B}$$

*is a median-homomorphism if and only if there is a $K \subseteq [n]$ with $1 \leq |K| \leq k$ and a median-homomorphism $g\colon \Pi_{i \in K} \mathbf{A}_i \to \mathbf{B}$ such that*

$$f(x_1, \dots, x_n) = g((x_i)_{i \in K}), \qquad \textit{for all} \quad (x_1, \dots, x_n) \in A_1 \times \cdots \times A_n.$$

*In particular, $f$ is an essentially at most $k$-ary.*

**Remark 2.5.** Note that for any median algebras $\mathbf{A}_1, \cdots, \mathbf{A}_n$ such that at least $k$ of them are non-trivial (i.e., with at least two distinct elements), if $\mathbf{B}$ contains a subalgebra isomorphic to the $k$-dimensional Boolean algebra, then we can easily construct a median homomorphism $f\colon \mathbf{A}_1 \times \cdots \times \mathbf{A}_n \to \mathbf{B}$ with $k$ essential arguments.

Indeed, suppose that there is $K \subseteq [n]$, with $|K| = k$, such that for each $i \in K$, $\mathbf{A}_i$ is non trivial (i.e., $|A_i| \geq 2$). According to [2] (see Subsection 1.5), $\mathbf{A}_i$ is the subdirect power of the 2-element median algebra $\mathbf{2}$ with universe $\{0, 1\}$. Since $|A_i| \geq 2$, we can always find a surjective homomorphism $h_i\colon \mathbf{A}_i \to \mathbf{2}$.

If $\mathbf{B}$ contains a subalgebra isomorphic to the $k$-dimensional Boolean algebra $\mathbf{B}_k$, then we may assume that $B_k = \{0, 1\}^k$. It is not difficult to verify that the homomorphism $H\colon \mathbf{A}_1 \times \cdots \times \mathbf{A}_n \to \mathbf{B}_k$ given by $H(x_1, \dots, x_n) = \big(h_1(x_1), \dots, h_k(x_k)\big)$ depends on all arguments $i \in [k]$. Furthermore, $H$ is surjective.

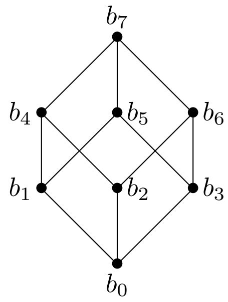


FIGURE 1. The Boolean algebra $\mathbf{B}_k$ for $k = 3$.

Let $k \geq 2$. Suppose that $\mathbf{B}$ is a median algebra containing a $k$-dimensional Boolean algebra $B_k = \{b_0, b_1, \dots, b_{2^k-1}\}$, that we may suppose equal to $\{0,1\}^k$. Assume that $b_1, \dots, b_k$ are the $k$ tuples at Hamming distance 1 from $b_0$ and that $b_{2^k-1}$ is the complement of $b_0$ (see Figure 2 with $k = 3$). For $i \in [k]$, let $\mathbf{A}_i$ be the median algebra with universe $A_i = [b_0, b_i]$ and let $\mathbf{C}$ be the median algebra whose universe $C$ is the convex hull of $B_k$. Note that $A_i \subseteq C$, for $i \in [k]$, and that

$$C = [b_0, b_{2^k-1}] = \{t \in M \,:\, t = b_0 \wedge_t \cdots \wedge_t b_{2^k-1}\}$$
$$= \{b_0 \wedge_t \cdots \wedge_t b_{2^k-1} \,:\, t \in M\}.$$

The following lemma is the median algebra variant of the well-known decomposition of bounded distributive lattices into direct products of principal ideals [7, 13].

**Lemma 2.6.** *The median algebras* $\mathbf{A}_1 \times \cdots \times \mathbf{A}_k$ *and* $\mathbf{C}$ *are isomorphic.*

*Proof.* Consider the function $f\colon A_1 \times \cdots \times A_k \to C$ given by

$$f(x_1, \dots, x_k) = x_1 \wedge_{b_{2^k-1}} \cdots \wedge_{b_{2^k-1}} x_k. \tag{2.1}$$

Note that $f$ is well-defined and that indeed $f(x_1, \dots, x_k) \in C$. Also, for each $i \in [k]$, consider $g_i\colon C \to A_i$ given by $g_i(t) = \mathbf{m}(b_0, t, b_i)$. It is not difficult to see that each $g_i$ is a median-homomorphism and that $g_i(x) = x$ whenever $x \in A_i$, and that $g_i(x) = b_0$ whenever $x \in A_j$ for $j \neq i$. Moreover, it is easy to see that $f$ and $(g_1, \dots, g_k)$ are inverses of one another. Since the function $(g_1, \dots, g_k)\colon \mathbf{C} \to \mathbf{A}_1 \times \cdots \times \mathbf{A}_k$ is a median isomorphism (i.e., bijective median homomorphism), its inverse $f$ is also a median isomorphism. □

In view of Lemma 2.6, each $t \in C$ can be thought of as a tuple

$$(x_1, \dots, x_k) \in A_1 \times \cdots \times A_k.$$

Moreover, if $t, t' \in C$, then $t = t'$ if and only if $x_i = x'_i$, for all $i \in [k]$. As an immediate consequence we obtain the following corollary.

**Corollary 2.7.** *For the isomorphism* $f\colon \mathbf{A}_1 \times \cdots \times \mathbf{A}_k \to \mathbf{C}$ *given by* (2.1), *we have that each equivalence* $\sim_i^f$, $i \in [k]$, *is in fact the equality relation.*

Using this result, we can now show that if a median algebra contains a subalgebra isomorphic to the $k$-dimensional Boolean algebra, then the same holds for any of its compatible semilattice orderings.

**Proposition 2.8.** *Let* $\mathbf{M}$ *be a median algebra. For any* $k \geq 2$*, the following assertions are equivalent.*

(1) *For every* $t \in M$*,* $\mathbf{M} = \langle M, \leq_t \rangle$ *contains a subsemilattice isomorphic to the* $k$*-dimensional Boolean algebra.*
(2) *There exists* $t \in M$ *such that* $\mathbf{M} = \langle M, \leq_t \rangle$ *contains a subsemilattice isomorphic to the* $k$*-dimensional Boolean algebra.*
(3) $\mathbf{M}$ *contains a subalgebra isomorphic to the* $k$*-dimensional Boolean algebra.*

*Proof.* Clearly, (1) implies (2), and (2) implies (3). We now show that (3) implies (1). As for Lemma 2.6, let $\mathbf{B}_k$ $k$-dimensional Boolean algebra with universe $B_k = \{b_0, b_1, \ldots, b_{2^k-1}\}$ and let $C$ be the convex hull of $B_k$. Recall that each $A_i = [b_0, b_i]$ has at least 2 elements, for instance, $b_0$ and $b_i$. We consider the two possible cases.
**Case 1:** $t \in C$. As in the proof of Lemma 2.6, $t = f(\mathbf{a})$ for some $\mathbf{a} = (a_1, \ldots, a_k) \in A_1 \times \cdots \times A_k$. Now, for each $i \in [k]$, consider $a'_i \in A_i$ such that $a'_i \neq a_i$, and let $t' = f(\mathbf{a}')$ for $\mathbf{a}' = (a'_1, \ldots, a'_k)$. Define

$$D = \{\mathbf{x}_{I|J}^{\mathbf{a}|\mathbf{a}'} \,:\, I \cup J = [k] \,\text{and}\, I \cap J = \varnothing\}.$$

It is not difficult to see that $|D| = 2^k$ and that $\mathbf{M} = \langle M, \leq_t \rangle$ contains a subsemilattice isomorphic to the $k$-dimensional Boolean algebra.
**Case 2:** $t \notin C$. Consider $t' = b_0 \wedge_t b_1 \wedge_t \cdots \wedge_t b_{2^k-1}$, and the remainder of the proof is similar to that of Case 1, since $\leq_t$ and $\leq_{t'}$ coincide on $C$. □

Following the same steps as in the proof of Proposition 2.8 and using the fact that $\mathbf{M}$ can be embedded into $\xi(\mathbf{M})$ as a convex subalgebra (Theorem 1.1 taken from [5]), we can generalize Proposition 2.8 to the following result.

**Proposition 2.9.** *Let* $\mathbf{M}$ *be a median algebra. For any* $k \geq 2$*, the following assertions are equivalent.*

(1) *For every compatible semilattice order* $\leq$*,* $\mathbf{M} = \langle M, \leq \rangle$ *contains a subsemilattice isomorphic to the* $k$*-dimensional Boolean algebra.*
(2) *There exists compatible semilattice order* $\leq$ *such that* $\mathbf{M} = \langle M, \leq \rangle$ *contains a subsemilattice isomorphic to the* $k$*-dimensional Boolean algebra.*
(3) $\mathbf{M}$ *contains a subalgebra isomorphic to the* $k$*-dimensional Boolean algebra.*

## 3. Characterizations of hypercube-free median algebras

In [11] we showed that the median semilattices (or, equivalently, median algebras) $\mathbf{B}$ that do not contain a substructure isomorphic to the 2-dimensional

Boolean algebra $\mathcal{P}(\{0,1\}^2)$ are exactly the $(2:3)$-*median semilattices*, i.e., those median semilattices $\mathbf{B} = \langle B, \leq \rangle$ such that for every $x, y, z \in B$ we have

$$\mathbf{m}(x,y,z) := (x \wedge y) \vee (x \wedge z) \vee (z \wedge y) \in \{x \wedge y, y \wedge z, z \wedge x\}. \tag{3.1}$$

As shown in [11], they coincide exactly with trees thought of as median semilattices.

In this section we will strengthen this result by characterizing, for any $k \geq 2$, those median semilattices that do not contain a subsemilattice isomorphic to the $k$-dimensional Boolean algebra and, using Proposition 2.8, showing that the same holds for any compatible semilattice order.

Let $\mathbf{B}$ be a median algebra thought of as meet semilattice, and consider the $k$-ary operation, $k \geq 3$,

$$\mathbf{m}_k(x_1, \ldots, x_k) = \bigvee_{\substack{I \subseteq [k] \\ |I| = k-1}} \bigwedge_{i \in I} x_i.$$

According to [3, 4], if $\mathbf{B}$ is closed under $\mathbf{m}_3 = \mathbf{m}$, then it is closed under $\mathbf{m}_k$ for all $k \geq 3$. Since we assume throughout that $\mathbf{B}$ is closed under $\mathbf{m}$, it is also closed under $\mathbf{m}_k$ for all $k \geq 3$.

In [3, 4], median semilattices are called 3-*median semilattices*, whereas semilattices closed under $\mathbf{m}_k$ are called $k$-*median semilattices*. Moreover, it is clear from [3, 4] that $\mathbf{m}_k$ is self-dual (i.e., the function remains the same when interchanging meets with joins) if and only if $k = 3$.

**Proposition 3.1.** *Let* $\mathbf{B}$ *be a median algebra thought of as a meet semilattice. The following assertions are equivalent:*

(1) *For every* $x_1, \ldots, x_k \in B$,

$$\bigwedge_{i \in [k]} x_i \in \{\bigwedge_{i \in I} x_i \,:\, I \subseteq [k],\, |I| = k-1\};$$

(2) *For every* $x_1, \ldots, x_k \in B$,

$$\mathbf{m}_k(x_1, \ldots, x_k) \in \{ \bigvee_{\substack{I \subseteq [k],\, j \in I \\ |I| = k-1}} \bigwedge_{i \in I} x_i \,:\, j \in [k]\};$$

(3) *For every* $x_1, \ldots, x_k \in B$, *if* $\bigvee_{i \in [k]} x_i$ *exists in* $\mathbf{B}$, *then*

$$\bigvee_{i \in [k]} x_i \in \{\bigvee_{i \in I} x_i \,:\, I \subseteq [k],\, |I| = k-1\}.$$

*Proof.* To show that (1) implies (3), let $x_1, \ldots, x_k \in B$ and suppose that $\bigvee_{i \in [k]} x_i$ exists in $\mathbf{B}$. Consider the dual of $\mathbf{m}_k$, i.e.,

$$\mathbf{m}_k^d(x_1, \ldots, x_k) = \bigwedge_{\substack{I \subseteq [k] \\ |I| = k-1}} \bigvee_{i \in I} x_i.$$

Note that the existence of $\bigvee_{i\in[k]} x_i$ in $\mathbf{B}$ guarantees that $\mathbf{m}_k^d$ is well defined. Moreover, from (1) it follows that

$$\mathbf{m}_k^d(x_1,\ldots,x_k) \in \{ \bigwedge_{\substack{I\subseteq[k],\, j\in I\\ |I|=k-1}} \bigvee_{i\in I} x_i \,:\, j\in[k]\}.$$

Without loss of generality, assume that the latter holds for $j=k$, that is,

$$\mathbf{m}_k^d(x_1,\ldots,x_k) = \bigwedge_{\substack{I\subseteq[k],\, k\in I\\ |I|=k-1}} \bigvee_{i\in I} x_i. \tag{3.2}$$

By taking the join with $\bigvee_{i\in[k-1]} x_i$ on both sides of (3.2), we obtain by distributivity and absorption

$$(\bigwedge_{\substack{I\subseteq[k]\\ |I|=k-1}} \bigvee_{i\in I} x_i) \vee \bigvee_{i\in[k-1]} x_i = \bigvee_{i\in[k-1]} x_i$$

on the left-hand side and

$$(\bigwedge_{\substack{I\subseteq[k],\, k\in I\\ |I|=k-1}} \bigvee_{i\in I} x_i) \vee \bigvee_{i\in[k-1]} x_i = \bigvee_{i\in[k]} x_i$$

on the right-hand side, thus showing that $\bigvee_{i\in[k-1]} x_i = \bigvee_{i\in[k]} x_i$, i.e., that (3) holds.

Clearly, (3) implies (2), and thus it remains to show that (2) implies (1). So consider the $k$-ary operation $\mathbf{m}_k$, $k\geq 3$. If (2) holds, then

$$\mathbf{m}_k(x_1,\ldots,x_k) \in \{ \bigvee_{\substack{I\subseteq[k],\, j\in I\\ |I|=k-1}} \bigwedge_{i\in I} x_i \,:\, j\in[k]\}.$$

By employing a reasoning dual to that in the proof of the implication (1) ⇒ (3), it then follows that (1) holds. □

**Theorem 3.2.** *Let* $\mathbf{B}$ *be a median algebra thought of as a meet semilattice, and let* $k\geq 2$. *Then the following assertions are equivalent:*

(1) $\mathbf{B}$ *does not contain a subalgebra isomorphic to the* $k$*-dimensional Boolean algebra.*

(2) *For all* $t\in B$, $\mathbf{B}=\langle B,\leq_t\rangle$ *does not contain a subsemilattice isomorphic to the* $k$*-dimensional Boolean algebra.*

(3) *There exists* $t\in B$ *such that* $\mathbf{B}=\langle B,\leq_t\rangle$ *does not contain a subsemilattice isomorphic to the* $k$*-dimensional Boolean algebra.*

(4) *For every* $x_1,\ldots,x_k\in B$,

$$\bigwedge_{i\in[k]} x_i \in \{\bigwedge_{i\in I} x_i \,:\, I\subseteq[k],\, |I|=k-1\}.$$

(5) *For every* $x_1,\ldots,x_k\in B$,

$$\mathbf{m}_k(x_1,\ldots,x_k) \in \{ \bigvee_{\substack{I\subseteq[k],\, j\in I\\ |I|=k-1}} \bigwedge_{i\in I} x_i \,:\, j\in[k]\}.$$

(6) *For every* $x_1, \ldots, x_k \in B$, *if* $\bigvee_{i\in[k]} x_i$ *exists in* $\mathbf{B}$, *then*

$$\bigvee_{i\in[k]} x_i \in \{\bigvee_{i\in I} x_i \,:\, I \subseteq [k],\, |I| = k-1\}.$$

*Proof.* By Proposition 2.8, assertions (1), (2) and (3) are pairwise equivalent. Also, from Proposition 3.1 it follows that (4), (5) and (6) are also pairwise equivalent. Also, it is not difficult to see that if (1) does not hold, then none of (4), (5) and (6) hold.

Hence, to complete the proof, it suffices to show that (1) implies (5). Suppose, on the contrary, that (5) does not hold, that is, there are $x_1, \ldots, x_k \in B$ such that

$$\mathbf{m}_k(x_1, \ldots, x_k) \notin \{\bigvee_{\substack{I\subseteq[k],\, j\in I\\ |I|=k-1}} \bigwedge_{i\in I} x_i \,:\, j \in [k]\}.$$

For each $i \in [k]$, set $x'_i := \mathbf{m}_k(x_1, \ldots, x_k) \wedge x_i$. Note that

$$\mathbf{m}_k(x_1, \ldots, x_k) = \mathbf{m}_k(x'_1, \ldots, x'_k),$$

$$\bigwedge_{i\in[k]} x'_i = \bigwedge_{i\in[k]} (\mathbf{m}_k(x_1, \ldots, x_k) \wedge x_i) = \mathbf{m}_k(x_1, \ldots, x_k) \wedge \bigwedge_{i\in[k]} x_i = \bigwedge_{i\in[k]} x_i$$

and that, for every $I \subseteq [k]$ such that $|I| = k-1$, we have

$$\bigwedge_{i\in I} x'_i = \bigwedge_{i\in I} x_i.$$

It is not difficult to see that the $2^k$ elements in

$$D = \{\mathbf{m}_k(x'_1, \ldots, x'_k)\} \cup \{\bigwedge_{i\in I} x'_i \,:\, 1 \leq |I| \leq k\}$$

are pairwise distinct. Hence, $D$ with $\wedge$ and $\vee$ given by

$$\begin{aligned} \bigwedge_{i\in I} x'_i \wedge \bigwedge_{i\in J} x'_i &= \bigwedge_{i\in I\cup J} x'_i \\ \bigwedge_{i\in I} x'_i \vee \bigwedge_{i\in J} x'_i &= \bigwedge_{i\in I\cap J} x'_i, \end{aligned}$$

constitutes a $k$-dimensional Boolean subalgebra of $\mathbf{B}$. Therefore, (4) does not hold, and the proof is now complete. □

Using Theorem 1.1, we can strengthen Theorem 3.2 to all compatible semilattice orders.

**Corollary 3.3.** *Assertions* 2 *and* 3 *(and thus all assertions) of Theorem* 3.2 *are equivalent to the following ones:*

(2') *For all compatible semilattice orders* $\leq$ *of* $\mathbf{B}$, $\mathbf{B} = \langle B, \leq\rangle$ *does not contain a subsemilattice isomorphic to the* $k$*-dimensional Boolean algebra.*

(3') *There exists a compatible semilattice order* $\leq$ *of* $\mathbf{B}$ *such that* $\mathbf{B} = \langle B, \leq\rangle$ *does not contain a subsemilattice isomorphic to the* $k$*-dimensional Boolean algebra.*

## 4. Concluding remarks

In this paper we extended the framework of [11] on essentially unary homomorphisms, and showed that all median homomorphisms $f : \mathbf{A}_1 \times \cdots \times \mathbf{A}_n \to \mathbf{B}$ from a product of median algebras to a median algebra $\mathbf{B}$ not containing a subalgebra isomorphic to the $k+1$-dimensional Boolean algebra or, equivalently, not containing a subsemilattice (independently of the ordering considered) isomorphic to the $k+1$-dimensional Boolean algebra, have at most $k$ essential arguments. Moreover, we provided descriptions of the latter median algebras $\mathbf{B}$ by means of necessary and sufficient conditions.

Now it seems to us that this type of results does not rely on the particular structure of median algebras, but rather on some general characteristics common to a wider family of algebraic structures. For instance, Chajda et al. [9] have recently extended our previous results [10, 11] dealing with essentially unary homomorphisms over median algebras to the wider class of hereditarily directly irreducible algebras.

In our current setting dealing with homomorphisms with at most $k$ essential arguments, the key observation for proving one of the main results of this paper (Theorem 2.4) was the fact that the variety of median algebras is congruence distributive and thus that it contains no skew congruences. Following the exact same steps, we can extend these results to other congruence distributive varieties such as that of majority algebras.

Hence, as future work we intend to investigate other variety classes with the hope of completely describing those that induce such impossibility results. In this direction, our first step will be to consider congruence modular varieties and, using commutator theory tools, investigate those homomorphisms whose kernels are skew.

## Acknowlegements

The authors would like to thank the anonymous referee for his/her thorough review and insightful remarks that improved the current paper and that led to the extension of our results to wider classes of algebras.

## References

[1] Arrow, K.J.: A difficulty in the concept of social welfare. J. Political Economy **58**:4, 328–346 (1950)

[2] Bandelt, H. J., Hedlíková, J.: Median algebras. Discrete math. **45**, 1–30 (1983)

[3] Bandelt, H. J., Janowitz, M., Meletiou, G. C.: $n$-Median semilattices. Order **8**, 185–195 (1991)

[4] Bandelt, H. J., Meletiou, G. C.: An algebraic setting for near-unanimity consensus. Order **7**, 169–178 (1990)

[5] Bandelt, H. J., Meletiou, G. C.: The zero-completion of a median algebra. Czech. Math. J. **43**(118):3, 409–417 (1993)

[6] Birkhoff, G., Kiss, S. A.: A ternary operation in distributive lattices. Bull. Amer. Math. Soc. **53**, 749–752 (1947)

[7] Birkhoff, G.: Lattice Theory, volume 25 of *American Mathematical Society Colloquium Publications*, revised edition. American Mathematical Society, New York (1948)

[8] Burris, S., Sankappanavar, H. P.: A course in universal algebra, Millenium edition. www.math.uwaterloo.ca/~snburris

[9] Chajda, I., Goldstern, M., Länger, H.: A note on homomorphisms between products of algebras. Algebra Universalis **79**:25 (2018) https://doi.org/10.1007/s00012-018-0517-9

[10] Couceiro, M., Marichal, J.-L., Teheux, B.: Conservative median algebras and semilattices. Order **33**:1, 121–132 (2016)

[11] Couceiro, M., Foldes, S., Meletiou, G. C.: On homomorphisms between products of median algebras. Algebra Universalis **78**:4, 545–553 (2017)

[12] Couceiro, M., Meletiou, G. C.: On a special class of median algebras. Miskolc Math. Notes **18**:1, 167–171 (2017)

[13] Grätzer, G.: General lattice theory. Birkhäuser Verlag, Basel, 2nd edition (1998)

[14] Isbell, J. R.: Median algebra. Trans. Amer. Math. Soc. **260**:2, 319–362 (1980)

[15] Jónsson, B.: Algebras whose congruence lattices are distributive. Math. Scand. **21**, 110–121 (1967)

[16] Sholander, M.: Trees, lattices, order, and betweenness. Proc. Amer. Math. Soc. **3**:3, 369–381 (1952)

[17] Sholander, M.: Medians, lattices, and trees. Proc. Amer. Math. Soc. **5**:5, 808–812 (1954)

[18] Van De Vel, M.: Theory of Convex Structures. North Holland, 1st edition (1993)

Miguel Couceiro
University of Lorraine, CNRS, Inria, LORIA, F-54000 Nancy, France
e-mail: miguel.couceiro[at]{inria, loria}.fr

Gerasimos C. Meletiou
TEI of Epirus, PO Box 110, 47100 Arta, Greece
e-mail: gmelet[at]teiep.gr