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ABSTRACT

We show that every perturbation $A(\lambda, \epsilon)$ of an $n \times n$ matrix polynomial $A(\lambda)$ such that $\det A(\lambda) = \lambda^m$ with $m \leq n$ can be reduced by equivalence transforms to a perturbed matrix polynomial whose leading matrix has maximal Smith form. This yields a reduced form for square perturbed matrix polynomials from which one can easily recover all the eigenvalue leading terms of the form $\mu\epsilon^\beta$ with $\beta^{-1} \in \mathbb{N}^*$.

Keywords

Matrix polynomials, perturbed eigenvalues, Smith normal form, Newton diagram.

1. INTRODUCTION

Let $A(\lambda) = \sum_{i=0}^{d} A_i \lambda^i$ be an $n \times n$ matrix polynomial over $\mathbb{Q}$ with $\det A(\lambda) \neq 0$. Following [20, 11, 13, 12, 22, 23] we consider the problem of the sensitivity of the eigenvalues of $A(\lambda)$ (i.e. the roots of $\det A(\lambda)$) to small perturbations of the matrix coefficients A_i. Such perturbations can be represented by *perturbed matrix polynomials* $A(\lambda, \epsilon) = \sum_{i=0}^{d} A_i(\epsilon)\lambda^i$ where $A_i(\epsilon) = \sum_{j \geq 0} A_{ij}\epsilon^j$ is an $n \times n$ formal power series matrix over $\mathbb{Q}$ such that $A_i(0) = A_i$ for $0 \leq i \leq d$. Regarding $A(\lambda, \epsilon)$ as a perturbation of its leading matrix $A(\lambda) = A(\lambda, 0)$, we may also write

$$A(\lambda, \epsilon) = A(\lambda) + \epsilon(A_{01} + \sum_{i=1}^{d} A_{i1}\lambda^i) + O(\epsilon^2). \quad (1)$$

The sensitivity of the eigenvalues of $A(\lambda)$ is described by the leading terms $\mu\epsilon^\beta$ of the formal Puiseux expansions $\lambda(\epsilon) = \mu\epsilon^\beta + \cdots$ at $\epsilon = 0$ of the solutions of $\det A(\lambda, \epsilon) = 0$. (See e.g. [2].) Such a $\lambda(\epsilon)$ will be referred to as an *ϵ^β-eigenvalue* of $A(\lambda, \epsilon)$.

Assuming that $\det A(\lambda) = \lambda^m$, we focus on how the zero eigenvalue of $A(\lambda)$ "splits" into m perturbed eigenvalues of $A(\lambda, \epsilon)$: when $A(\lambda)$ is in Smith form, some sufficient conditions on some entries of A_{01} were given in [11] for $A(\lambda, \epsilon)$ to have ri $\epsilon^{1/i}$-eigenvalues when $A(\lambda)$ has r invariant polynomials of order i; it is also known how to recover the leading coefficients μ from these particular entries. However, these conditions are not necessary. Consider for example the perturbed matrix polynomials

$$A(\lambda, \epsilon) = \begin{bmatrix} 1 & \epsilon \\ \epsilon & \lambda^2 \end{bmatrix} \quad \text{and} \quad \tilde{A}(\lambda, \epsilon) = \begin{bmatrix} \lambda & \epsilon \\ \epsilon & \lambda \end{bmatrix}.$$

Although the eigenvalues are in both cases ϵ and $-\epsilon$, the structure $A(\lambda) = \mathrm{diag}[1, \lambda^2]$ does not indicate a priori a splitting into two ϵ-eigenvalues and, in that sense, is misleading. On the other hand, one may easily verify that $\tilde{A}(\lambda, \epsilon) = U(\lambda, \epsilon)A(\lambda, \epsilon)V(\lambda, \epsilon)$ with

$$U(\lambda, \epsilon) = \begin{bmatrix} 1 & 0 \\ \frac{\lambda - \lambda^2}{\epsilon} & 1 \end{bmatrix} \quad \text{and} \quad V(\lambda, \epsilon) = \begin{bmatrix} 1 & 0 \\ \frac{\lambda - 1}{\epsilon} & 1 \end{bmatrix}.$$

We thus have found a matrix $\tilde{A}(\lambda, \epsilon)$ equivalent to $A(\lambda, \epsilon)$ whose leading Smith form $\tilde{A}(\lambda) = \mathrm{diag}[\lambda, \lambda]$ is non misleading: we can tell from $\tilde{A}(\lambda)$ that $\tilde{A}(\lambda, \epsilon)$ admits two ϵ-eigenvalues.

The two main contributions of this paper are as follows. First, we establish for $m \leq n$ that "non misleading" actually means that the leading Smith form is maximal for lexicographic ordering. Second, we prove that the observation made on the above example holds more generally: every perturbation of an $n \times n$ matrix polynomial $A(\lambda)$ such that $\det A(\lambda) = \lambda^m$ with $m \leq n$ can be reduced by equivalence transforms to a perturbed matrix polynomial whose leading matrix has maximal Smith form. This yields a reduced form for square perturbed matrix polynomials from which one can easily recover all the eigenvalue leading terms of the form $\mu\epsilon^\beta$ with $\beta^{-1} \in \mathbb{N}^*$.

These results represent a first step towards the effective formal reduction of matrix polynomials depending on a parameter. There are two main motivations. First, many problems in robust control [3, 1] can be described in terms of parameterized matrix polynomials. On the other hand, using matrix polynomials allows to extend some results obtained for the perturbation $A(\lambda, \epsilon) = \lambda I - J - B(\epsilon)$ of a nilpotent Jordan structure J. (See e.g. [14, 16, 15, 10].)

Previous studies about parameterized matrix polynomials were concerned by either determinant degree and Smith

*This work was done while the author was a postdoctoral fellow at the Symbolic Computation Group, University of Waterloo.

form computations [17, 18, 19] or perturbation theory for matrix pencils [7, 21], quadratic matrix polynomials [13, 22, 23] and matrix functions [20, 21, 11, 12]. But the approach that consists in searching for the "best" leading Smith form in a set of equivalent perturbed matrix polynomials seems to be new.

For our analysis, we first need to refine a perturbation result by Langer and Najman [11]. This is done in Section 2.1. Following the approach of [16], we further interpret this refinement in terms of the Newton diagram in Section 2.2. We then introduce perturbed matrix polynomials with maximal leading Smith form in Section 3, together with the notion of highest Newton envelope. Section 4.1 presents our reduced form for $m \leq n$ whereas Section 4.2 discusses the case $m > n$. We conclude in Section 5 with two extensions.

Notation. Throughout the paper, l denotes the degree in λ of $p(\lambda, \epsilon) = \det A(\lambda, \epsilon)$ and Π_m^n denotes the matrix set

$$\Pi_m^n = \left\{ A(\lambda, \epsilon) \in \mathbb{Q}[[\epsilon]][\lambda]^{n \times n} \ : \ \det A(\lambda) = \lambda^m \right\}$$

where $A(\lambda) = A(\lambda, 0)$.

2. PERTURBATION OF A SMITH FORM AND PERTURBED EIGENVALUES

Let $A(\lambda, \epsilon) \in \Pi_m^n$ and assume that its leading matrix $A(\lambda)$ is equal to the Smith form

$$S(\lambda) = \mathrm{diag}[1, \ldots, 1, (\lambda^{m_1})^{r_1} \cdots (\lambda^{m_q})^{r_q}] \qquad (2)$$

with $0 < m_1 < \cdots < m_q$. Here we wrote $(\lambda^a)^b$ for the $b \times b$ diagonal matrix whose all diagonal entries are equal to the monomial λ^a. One can think of m_j as the jth partial multiplicity of the zero eigenvalue of $A(\lambda)$ repeated r_j times, and note that $r_1 m_1 + \cdots + r_q m_q = m$ and $r_1 + \cdots + r_q \leq n$.

2.1 A perturbation theorem for eigenvalues

In order to investigate the link bewteen the structure (2) of $A(\lambda)$ and the leading terms of the eigenvalues of $A(\lambda, \epsilon)$, introduce the partial sums

$$s_j = r_j + \cdots + r_q \qquad \text{for } 1 \leq j \leq q,$$

and define L_j as the last principal submatrix of A_{01} in (1) of order s_j. For $1 \leq j < q$, one has $s_{j+1} = s_j - r_j$ and L_{j+1} lies in the lower right corner of L_j. The values of the determinants of such submatrices of A_{01} — together with the additional "intermediate" quantities recalled below — were already considered by Langer and Najman [11, 12].

Definition 1. *For $1 \leq j \leq q$, let Δ_j be the determinant of L_j. For $1 \leq j \leq q$ and $0 \leq k \leq r_j$, we further denote by $\Delta_j^{(k)}$ the sum of the principal minors of L_j with order $s_j - k$ that contain L_{j+1}.*

In this definition, L_{q+1} has been assumed to be the empty matrix, of determinant $\Delta_{q+1} = 1$. Note that $\Delta_j = \Delta_j^{(0)}$ and that $\Delta_{j+1} = \Delta_j^{(r_j)}$ for $1 \leq j \leq q$.

The theorem below provides some sufficient conditions on the sums $\Delta_j^{(k)}$ for the multiple zero eigenvalue of $A(\lambda)$ to split under perturbation according to the number and the order of its partial mutiplicities. A proof based on the Newton diagram will be given in Section 2.2.

Theorem 1. *Let $j \in \{1, \ldots, q\}$. Assuming that at least one of the sums $\Delta_j^{(k)}$ for $0 \leq k \leq r_j$ is nonzero, let k_1, k_2 be respectively minimal and maximal so that $\Delta_j^{(k_1)} \neq 0$ and $\Delta_j^{(k_2)} \neq 0$. Then*

i) $A(\lambda, \epsilon)$ admits $(k_2 - k_1) m_j$ ϵ^{1/m_j}-eigenvalues;

ii) writing $w_j = e^{2i\pi/m_j}$, the leading terms of these eigenvalues are

$$(\mu_{jk})^{1/m_j} w_j^l \epsilon^{1/m_j}\ , \quad k = 1, \ldots, k_2 - k_1\ , \quad l = 1, \ldots, m_j\ ,$$

where the complex numbers μ_{jk} denote the roots of the polynomial $\sum_{k=k_1}^{k_2} \Delta_j^{(k)} \lambda^{k-k_1}$ and where $(\mu_{jk})^{1/m_j}$ is one of the m_j distinct m_jth roots of μ_{jk}.

In the particular case where $(k_1, k_2) = (0, r_j)$, this result was obtained by Langer and Najman in [11], describing the typical eigenvalue splitting when A_{01} is dense. Note also that in this case the μ_{jk}'s are up to the sign the eigenvalues of the Schur complement of L_{j+1} in L_j. This follows from $\sum_{k=0}^{r_j} \Delta_j^{(k)} \lambda^k = \det(L_j + \lambda E_j)$ with $E_j = \mathrm{diag}[I_{r_j}, O_{s_{j+1}}]$, and from the determinantal equality (see e.g. [5, p. 190])

$$\begin{aligned} \det(L_j + \lambda E_j) \ &:= \ \det \begin{bmatrix} X + \lambda I & Y \\ Z & L_{j+1} \end{bmatrix} \\ &= \ \det L_{j+1} \det(X - Y L_{j+1}^{-1} Z + \lambda I). \end{aligned}$$

However, though simpler to check, the conditions "$\Delta_j \neq 0$" will prove to be not enough for our purposes.

As an illustration of the above Theorem 1, consider the 6×6 perturbed matrix polynomial of the form

$$A(\lambda, \epsilon) = \mathrm{diag}[1, 1, (\lambda)^3, \lambda^3] + \epsilon(A_{01} + A_{11}\lambda) \qquad (3a)$$

with A_{01} and A_{11} given by

$$A_{01} = \left[\begin{array}{cc|ccc|c} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ \hline 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \qquad (3b)$$

and

$$A_{11} = \left[\begin{array}{cc|ccc|c} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 & 1 & 1 \end{array}\right]. \qquad (3c)$$

Here $q = 2$, $(r_1, m_1) = (3, 1)$, $(r_2, m_2) = (1, 3)$ and

$$L_1 = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \end{array}\right], \quad L_2 = [\ 0\ ].$$

Hence $\Delta_1^{(0)} = 1$, $\Delta_1^{(1)} = 1$, $\Delta_1^{(2)} = 0$, $\Delta_1^{(3)} = 0$ and it follows from Theorem 1 (applied with $j = 1$) that $A(\lambda, \epsilon)$ has one eigenvalue of the form $\lambda(\epsilon) = -\epsilon + o(\epsilon)$.

2.2 Newton diagram-based characterization

Following Moro, Burke and Overton [16], let us interpret Theorem 1 in terms of the Newton diagram of $A(\lambda, \epsilon)$. Let $p(\lambda, \epsilon) = \det A(\lambda, \epsilon)$ of degree l in λ be written as

$$p(\lambda, \epsilon) = \sum_{i=0}^{l} a_i(\epsilon)\lambda^i.$$

For $0 \le i \le l$, there exists $(\hat{a}_i, \alpha_i) \in \mathbb{Q}^* \times \mathbb{N}$ such that $a_i(\epsilon) = \hat{a}_i \epsilon^{\alpha_i} + O(\epsilon^{\alpha_i+1})$ providing that $a_i(\epsilon)$ is not identically zero; we further set $\alpha_i = +\infty$ when $a_i(\epsilon)$ is zero. The lower boundary of the convex hull of $\{(i, \alpha_i) : 0 \le i \le l\}$ in the cartesian plane is the *Newton diagram* $\mathcal{N}$ associated with $A(\lambda, \epsilon)$. (See e.g. [16],[4],[2],[24].) Defining the length of a line segment as the length of the projection of this segment onto the horizontal axis, the Newton diagram thus consists of a finite number of segments with rational, possibly infinite $(-\infty)$ slopes and nonzero lengths. In particular, the length of the segment with slope $-\infty$ can be considered as equal to the valuation of $p(\lambda, \epsilon)$ in λ. We will denote by $\mathcal{N}^-$ the negative part of $\mathcal{N}$, i.e. the subdiagram of $\mathcal{N}$ which consists of negative (finite and infinite) slopes only. Note that $\mathcal{N}$ and $\mathcal{N}^-$ have lengths l, m respectively.

Let $(\rho, \nu, \delta) \in \mathbb{N} \times \mathbb{Z} \times \mathbb{N}^*$ be such that $\gcd(\nu, \delta) = 1$. The first main property of the Newton diagram is that $A(\lambda, \epsilon)$ has $\rho\delta$ $\epsilon^{\nu/\delta}$-eigenvalues if and only if its Newton diagram $\mathcal{N}$ has a segment $\mathcal{S}$ of slope $-\nu/\delta$ and length $\rho\delta$. Now, letting $x_{\mathcal{S}}$ be the minimum index i so that (i, α_i) belongs to this particular segment $\mathcal{S}$, we see that δ divides $i - x_{\mathcal{S}}$ for all $(i, \alpha_i) \in \mathcal{S}$. This allows to associate with each segment $\mathcal{S}$ of $\mathcal{N}$ the polynomial

$$p_{\mathcal{S}}(\lambda) = \sum_{(i,\alpha_i)\in\mathcal{S}} \hat{a}_i \lambda^{(i-x_{\mathcal{S}})/\delta} \quad (4a)$$

of degree ρ. (See e.g. [6, p. 137].) The second fundamental property of the Newton diagram is as follows: the leading terms of the $\rho\delta$ $\epsilon^{\nu/\delta}$-eigenvalues of $A(\lambda, \epsilon)$ are

$$(\mu_k)^{1/\delta} w^l \epsilon^{\nu/\delta}, \qquad k = 1, \ldots, \rho, \qquad l = 1, \ldots, \delta, \quad (4b)$$

where $w = e^{2i\pi/\delta}$ and where the μ_k's are the roots of $p_{\mathcal{S}}(\lambda)$.

For example, $A(\lambda, \epsilon)$ in (3) has determinant

$$\begin{aligned} p(\lambda, \epsilon) = \; & \left(-\epsilon^3 + 2\epsilon^4 + \boxed{\epsilon}\right)\lambda^7 + \\ & \left(3\epsilon^2 + \epsilon^5 + 3\epsilon^4 + \boxed{1} - 3\epsilon^3 + 4\epsilon\right)\lambda^6 + \\ & \left(-\epsilon^5 + 2\epsilon - 2\epsilon^6 + 4\epsilon^2 + 2\epsilon^4 + 3\epsilon^3\right)\lambda^5 + \\ & \left(7\epsilon^5 + 3\epsilon^4 + 2\epsilon^3 + \boxed{\epsilon} + 4\epsilon^6 + 3\epsilon^2\right)\lambda^4 + \\ & \left(-\epsilon^4 + 2\epsilon^3 + 9\epsilon^6 + \epsilon^2\right)\lambda^3 + \\ & \left(2\epsilon^4 - 2\epsilon^5\right)\lambda^2 + \left(\boxed{\epsilon^3} + \epsilon^4 + \epsilon^5\right)\lambda + \boxed{\epsilon^4}. \end{aligned}$$

Hence the Newton diagram $\mathcal{N}$ shown in Fig. 2. It follows that $A(\lambda, \epsilon)$ has one ϵ-eigenvalue, three $\epsilon^{2/3}$-eigenvalues, two $\epsilon^{1/2}$-eigenvalues and one ϵ^{-1}-eigenvalue. The corresponding polynomials of type (4a) — built from the boxed terms of $p(\lambda, \epsilon)$ only — are $1 + \lambda$, $1 + \lambda^3$, $1 + \lambda^2$, $1 + \lambda$ respectively.

Theorem 2 below expresses in terms of the characteristics $(r_1, m_1), \ldots, (r_q, m_q)$ of the Smith form $A(\lambda)$ what the rightmost possible vertices for the Newton diagram of $A(\lambda, \epsilon)$ are. This will yield a graphical characterization of the sufficient conditions of Theorem 1 and also allow for a simple proof of this perturbation result. The structure of $A(\lambda)$ and an ordinate y being given, consider all the perturbations of $A(\lambda)$ of the form (1) and denote by $x(y)$ the smallest possible abscissa for a vertex $(x(y), y)$ of the associated Newton diagrams. Additionally, recall that $s_j = r_j + \cdots + r_q$ and define

$$t_j = r_j m_j + \cdots + r_q m_q \quad \text{for } 1 \le j \le q.$$

THEOREM 2. *Let $j \in \{1, \ldots, q\}$ and $k \in \{0, \ldots, r_j\}$. If $y = s_j - k$ then $x(y) = m - t_j + k m_j$ and the coefficient of $\epsilon^y \lambda^{x(y)}$ in $p(\lambda, \epsilon)$ is $\Delta_j^{(k)}$.*

PROOF. By definition, $p(\lambda, \epsilon)$ is the sum of all the products (up to the sign) of n entries of the matrix $A(\lambda, \epsilon) = A(\lambda) + \epsilon(A_{01} + O(\lambda)) + O(\epsilon^2)$ such that no two entries belong to the same column or row. The entries involved in a product can be ϵ-terms from $A(\lambda, \epsilon) - A(\lambda)$ or λ-terms from $A(\lambda)$. Let us examine how to form a product of order ϵ^y in ϵ and of minimal order in λ. For ϵ-terms, we pick $y \le s_1 \le n$ entries in ϵA_{01}. It then remains to choose $n - y$ λ-terms. In order to exhaust the largest powers of λ from $A(\lambda)$, these ϵ-terms further lie on the last s_{j+1} columns and on any group of $r_j - k$ columns with index between $n - s_j + 1$ and $n - s_{j+1}$. The row indices for the ϵ-terms must vary in exactly the same ranges, for otherwise the diagonal structure of $A(\lambda)$ leads to a product equal to zero. Finally, we complete the product by taking the first $n - s_j$ diagonal entries of $A(\lambda)$ together with a group of k invariant polynomials of the form λ^{m_j}. This product thus has order $\lambda^{m-t_j+km_j}$ in λ, from which $x(y)$ follows. Deleting from $A(\lambda, \epsilon)$ the rows and columns corresponding to each of the above choices of λ-terms yields the sum $\Delta_j^{(k)} \epsilon^y$. $\square$

Now, for $1 \le j \le q$ and $0 \le k \le r_j$, let $\mathcal{P}_j^{(k)}$ be the point given by $(x(y), y)$ as in Theorem 2 and write $\mathcal{P}_j = \mathcal{P}_j^{(0)}$ and $\mathcal{P}_{j+1} = \mathcal{P}_j^{(r_j)}$ for $1 \le j \le q$. In particular, one has $\mathcal{P}_{q+1} = (m, 0)$. Using the terminology of Moro, Burke and Overton [16, p. 808], we call *Newton envelope associated with the Smith form of* $A(\lambda)$ the diagram $\mathcal{E}(A(\lambda))$ obtained by successively connecting the points $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{q+1}$. (See Fig. 2 for an example.) It then follows from Theorem 2 that for any perturbation $A(\lambda, \epsilon)$ of an $A(\lambda)$ with given structure (2), the negative part $\mathcal{N}^-$ has no point lying below the envelope. Furthermore, the sufficient regularity conditions of Theorem 1 can be characterized graphically as shown below.

COROLLARY 1. *Let $j \in \{1, \ldots, q\}$ and $k \in \{0, \ldots, r_j\}$. The sum $\Delta_j^{(k)}$ is nonzero if and only if the point $\mathcal{P}_j^{(k)}$ of the envelope is also a point of the Newton diagram.*

We end this section by providing a Newton diagram-based proof of Theorem 1.

PROOF. [Proof of Theorem 1] It follows from Corollary 1 that the Newton diagram of $A(\lambda, \epsilon)$ has a segment $\mathcal{S}_j =$

$[\mathcal{P}_j^{(k_1)}, \mathcal{P}_j^{(k_2)}]$ with length $(k_2-k_1)m_j$ and slope $-1/m_j$. The first property of the Newton diagram thus yields (i). In order to prove (ii), recall that the leading constants of the ϵ^{1/m_j}-eigenvalues are given as in (4) by the roots of

$$p_{\mathcal{S}_j}(\lambda) = \sum_{(x(y),y)\in\mathcal{S}_j} \hat{a}_{x(y)} \lambda^{(x(y)-x_{\mathcal{S}_j})/m_j}.$$

Now, let $k \in \{k_1, \ldots, k_2\}$ be such that $\mathcal{P}_j^{(k)} = (x(y), y)$ lies on $\mathcal{S}_j$ — or equivalently, $\Delta_j^{(k)} \neq 0$. One has $\alpha_{x(y)} = y$ and it follows from Theorem 2 that the coefficient of $p(\lambda, \epsilon)$ in $\epsilon^y \lambda^{x(y)}$ is $\hat{a}_{x(y)} = \Delta_j^{(k)}$. On the other hand, one may easily verify that $x(y) - x_{\mathcal{S}_j} = (k-k_1)m_j$. Hence $p_{\mathcal{S}_j}(\lambda) = \sum_{k=k_1}^{k_2} \Delta_j^{(k)} \lambda^{k-k_1}$. $\square$

With the approach above, the Smith form $A(\lambda)$ and thus $\mathcal{E}(A(\lambda))$ are given, and we identified among all the perturbations $A(\lambda, \epsilon)$ compatible with $A(\lambda)$ the ones that allow to recover as much of $\mathcal{N}^-$ as possible. In the next section, the question we consider is the opposite: $A(\lambda, \epsilon)$ and thus $\mathcal{N}^-$ being given, what is the "best" possible Smith form for $A(\lambda)$ compatible with $\mathcal{N}^-$?

3. PERTURBED MATRIX POLYNOMIALS WITH MAXIMAL LEADING SMITH FORM

Let $A(\lambda, \epsilon) \in \Pi_m^n$. In this section $\mathcal{N}$ is assumed to be known from the outset whereas the Smith form of $A(\lambda)$ is not. We denote by $\Pi_m^n(\mathcal{N}^-)$ the subset of Π_m^n that consists of the perturbed matrix polynomials whose Newton diagram has negative part $\mathcal{N}^-$. Let further $\Sigma_m^n(\mathcal{N}^-)$ be the set of the Smith forms of the corresponding leading matrices. This set is lexicographically ordered by identifying the Smith form (2) with

$$(0, \ldots, 0, \underbrace{m_1, \ldots, m_1}_{r_1}, \ldots, \underbrace{m_q, \ldots, m_q}_{r_q}) \in \mathbb{N}^n.$$

Additionally, $\Sigma_m^n(\mathcal{N}^-)$ is finite, for every element has determinant equal to λ^m and m is the length of $\mathcal{N}^-$. Denote by $S(\lambda)_{max}$ the maximum of $\Sigma_m^n(\mathcal{N}^-)$.

Definition 2. *We will refer to* $S(\lambda)_{max}$ *as the* maximal leading Smith form *associated with* $\Pi_m^n(\mathcal{N}^-)$ *and we will say that* $A(\lambda, \epsilon)$ *is a* perturbed matrix polynomial with maximal leading Smith form *when* $A(\lambda) = S(\lambda)_{max}$.

When $m \leq n$, perturbed matrix polynomials with maximal leading Smith form enjoy the property that the sufficient conditions of Theorem 1 are also necessary. In order to derive this result (Corollary 2), we first introduce a refinement of the Newton envelope of Section 2.2; this tool will then be used to characterize $S(\lambda)_{max}$ graphically.

3.1 Highest Newton envelope

We define the *highest Newton envelope* $\mathcal{E}_{max}$ *compatible with* $\mathcal{N}^-$ as the best lower approximation of $\mathcal{N}^-$ by a convex diagram whose length is m and whose every segment has slope equal to the inverse of a negative integer: the segment of $\mathcal{N}^-$ with slope $-\infty$ is lower bounded by a segment of $\mathcal{E}_{max}$, say $\mathcal{S}_0$, with the same length and the same rightmost point, but slope -1; each segment of $\mathcal{N}^-$ with slope of the form $-1/N$ with $N \in \mathbb{N}^*$ is also a segment of $\mathcal{E}_{max}$; the segments of $\mathcal{N}^-$ whose slope is strictly comprised between $-1/N$ and $-1/(N+1)$ are best lower bounded by two segments of slopes $-1/N$, $-1/(N+1)$ and respective lengths Nx, $(N+1)y$. More precisely, assume that $\mathcal{N}^-$ has s segments with slopes given by $-1/N < \beta_1 < \cdots < \beta_s < -1/(N+1)$ for some positive integer N. For $1 \leq j \leq s$, let us write $\beta_j = -\nu_j/\delta_j$ with ν_j, δ_j positive and coprime, and let the corresponding segment have length $\rho_j\delta_j$. Then, the pair $(x, y) \in \mathbb{N}^* \times \mathbb{N}^*$ is the unique solution of the linear system

$$\begin{bmatrix} 1 & 1 \\ N & N+1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \sum_{j=1}^{s} \rho_j \begin{bmatrix} \nu_j \\ \delta_j \end{bmatrix}. \quad (5)$$

This is illustrated in Fig. 1.

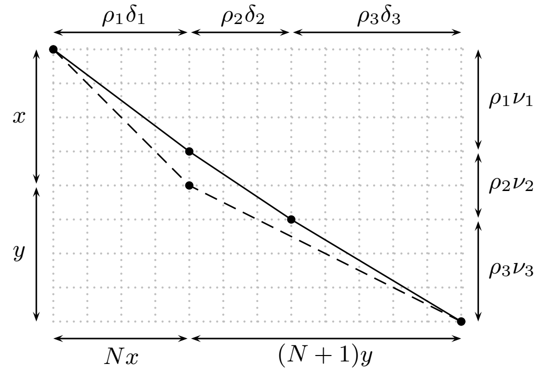

Figure 1: Best approximation of a group of segments of $\mathcal{N}^-$ by a pair of segments with slope $-1/N$ and $-1/(N+1)$ for $N \in \mathbb{N}^*$. Illustration with $N = 1$ and $(\beta_1, \beta_2, \beta_3) = (-3/4, -2/3, -3/5)$.

Now, let $v(\mathcal{N}^-)$ be the set of vertices of $\mathcal{N}^-$ with finite ordinate. (If it exists, the point $(0, +\infty)$ is thus disregarded.) In order to provide an explicit formula for $\mathcal{E}_{max}$ we introduce the integer sequence $(\sigma_i)_{i\in\mathbb{N}^*}$ given by

$$\sigma_i = 2\nu_i - \nu_{i+1} - \nu_{i-1} \quad (6a)$$

and

$$\nu_i = \min\{k + i\alpha_k \ : \ (k, \alpha_k) \in v(\mathcal{N}^-)\}. \quad (6b)$$

Graphically, the ν_i's can be interpreted as follows. For $i \geq 0$ let $\mathcal{N}_i^-$ be the diagram deduced from $\mathcal{N}^-$ by transforming $(k, \alpha_k) \in v(\mathcal{N}^-)$ into $(k, k + i\alpha_k) \in v(\mathcal{N}_i^-)$. Applying the same transform to the vertices of $\mathcal{E}_{max}\backslash\{\mathcal{S}_0\}$, we obtain the highest Newton envelope $\mathcal{E}_i$ compatible with $\mathcal{N}_i^-$. A segment of $\mathcal{N}^-$ or $\mathcal{E}_{max}\backslash\{\mathcal{S}_0\}$ with finite slope s is thus transformed into a segment of $\mathcal{N}_i^-$ or $\mathcal{E}_i$ with the same length but slope $is+1$. By definition of the highest envelope, ν_i is thus the smallest ordinate for the vertices of both $\mathcal{N}_i^-$ and $\mathcal{E}_i$.

Consider now the σ_i's. They are nonnegative, for the sequence $(\nu_{i+1} - \nu_i)_{i\in\mathbb{N}}$ is nonincreasing. Additionally, it follows from Lemma 1 below that $(i\sigma_i)_{i\in\mathbb{N}^*}$ defines an integer partition of $m - \nu_0$ and that $(\sigma_i)_{i\in\mathbb{N}^*}$ is zero almost everywhere. Theorem 3 also relies on this lemma.

Lemma 1. *The sequence $(\sigma_i)_{i \in \mathbb{N}^*}$ given by (6) satisfies*

$$\sum_{i=j+1}^{\infty} (i-j)\sigma_i = m - \nu_j \quad \textit{for all } j \in \mathbb{N}.$$

Proof. Let $s < 0$ be the largest slope of $\mathcal{N}^-$ and let $N = \lceil -s^{-1} \rceil$. For $i \geq N$, the diagram $\mathcal{N}_i^-$ therefore has slopes in $\mathbb{Q}^-$ only. Since $(m, 0)$ is a point of $\mathcal{N}^-$, it follows that $\nu_i = m$ for all $i \geq N$. Hence $\sigma_i = 0$ for $i > N$ and intermediate cancellations in the sum yield $\sum_{i=j+1}^{N}(i-j)\sigma_i = -\nu_j + (N-j+1)\nu_N - (N-j)\nu_{N+1}$. This is $m - \nu_j$. □

Theorem 3. *Let $\sigma_{m_1}, \ldots, \sigma_{m_q}$ denote the nonzero values of $(\sigma_i)_{i \in \mathbb{N}^*}$ numbered so that $m_1 < \cdots < m_q$. If additionally one defines $\mathcal{Q}_0 = (0, \nu_0 + \sum_{k=1}^{q} \sigma_{m_k})$ and, for $1 \leq j \leq q+1$, $\mathcal{Q}_j = (\nu_0 + \sum_{k=1}^{j-1} m_k \sigma_{m_k}, \sum_{k=j}^{q} \sigma_{m_k})$, then $\mathcal{E}_{max}$ is the diagram obtained by successively connecting the points $\mathcal{Q}_0, \mathcal{Q}_1, \ldots, \mathcal{Q}_{q+1}$.*

Proof. $\mathcal{E}_{max}$ consists of $p+1$ segments $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_p$ where $\mathcal{S}_0$ is the segment of slope -1 corresponding to the slope $-\infty$ of $\mathcal{N}^-$ and where, for $1 \leq j \leq p$, $\mathcal{S}_j$ has slope $-1/n_j$ and length $r_j n_j$ with $n_1 < \cdots < n_p$. For $j \in \{1, \ldots, p\}$ we prove that $\sigma_{n_j} = r_j$ as follows. Denote by (x_1, y_1), (x_2, y_2) the cartesian coordinates of the extreme points of $\mathcal{S}_j$. When successively looking at the transformed envelopes $\mathcal{E}_i$ for $i \in \{n_j - 1, n_j, n_j + 1\}$, we see that

$$\begin{aligned} \nu_{n_j - 1} &= x_1 + (n_j - 1)y_1; \\ \nu_{n_j} &= x_i + n_j y_i \quad \text{for } i = 1, 2; \\ \nu_{n_j+1} &= x_2 + (n_j + 1)y_2. \end{aligned}$$

Writing $(x_2, y_2) = (x_1 + r_j n_j, y_1 - r_j)$ then yields $\sigma_{n_j} = y_1 - y_2 = r_j$. On the other hand, it follows from the r_j's being positive that the σ_{n_j}'s define p out of the q nonzero values of $(\sigma_i)_{i \in \mathbb{N}^*}$. For $i = 0$, it follows from (6b) that the length of $\mathcal{S}_0$ is equal to ν_0. Hence $m = \nu_0 + \sum_{i \in \{n_1, \ldots, n_p\}} i\sigma_i$. Since $n_1 < \cdots < n_p$, Lemma 1 then implies that $p = q$ and $n_j = m_j$ for all $j \in \{1, \ldots, q\}$. Hence the coordinates of the $\mathcal{Q}_j$'s, noticing that $(m, 0)$ is the rightmost point of $\mathcal{E}_{max}$. □

For example, $\mathcal{N}^-$ in Fig. 2 has vertices $(0,4)$, $(1,3)$, $(4,1)$, $(6,0)$ and it follows from (6) that $\sigma_1 = 2$, $\sigma_2 = 2$, $\sigma_3 = \sigma_4 = \cdots = 0$. The corresponding highest envelope $\mathcal{E}_{max}$ is represented on the second graphic.

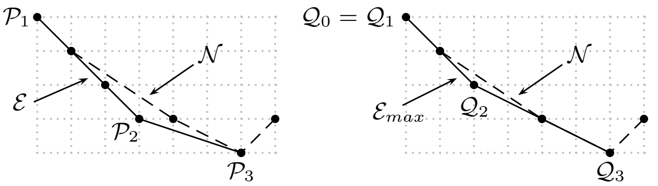

Figure 2: Newton diagram and Newton envelopes for the matrix $A(\lambda, \epsilon)$ of (3).

3.2 Recovery of some eigenvalue leading terms

When $m \leq n$, recovering some eigenvalue leading terms is made possible by the following relation between the highest envelope and the maximal leading Smith form. (We discuss the case where $m > n$ in Section 4.2.)

Theorem 4. *Let $m \leq n$ and let $A(\lambda, \epsilon) \in \Pi_m^n$ with Newton diagram $\mathcal{N}$. Then $\mathcal{E}_{max}$ is the Newton envelope associated with $S(\lambda)_{max}$.*

Proof. Let $S(\lambda) \in \Sigma_m^n(\mathcal{N}^-)$. If $S(\lambda) < S(\lambda)_{max}$ then $\mathcal{E}(S(\lambda)_{max})$ has at least one vertex strictly above $\mathcal{E}(S(\lambda))$, and thus $\mathcal{E}(S(\lambda)) \neq \mathcal{E}_{max}$.

In order to prove that the converse holds true, let us first construct $T(\lambda, \epsilon) \in \Pi_m^n(\mathcal{N}^-)$ such that $T(\lambda)$ is in Smith form and $\mathcal{E}(T(\lambda)) = \mathcal{E}_{max}$. Let $T(\lambda, \epsilon) = \mathrm{diag}[1, \ldots, 1, B(\lambda, \epsilon)]$ where we choose for $B(\lambda, \epsilon)$ the block diagonal matrix whose kth block corresponds to a segment (or a group of segments) of $\mathcal{N}^-$ as follows. To the infinite slope of $\mathcal{N}^-$ of length, say ρ, we associate the block $B^{(1)}(\lambda, \epsilon) = \lambda I_\rho$. To a segment of $\mathcal{N}^-$ with slope $-1/N$ and length ρN, we associate the $\rho \times \rho$ matrix $B^{(k)}(\lambda, \epsilon) = \mathrm{diag}[\lambda^N - \epsilon]_{1 \leq i \leq \rho}$. With the same notation as before, consider now a group of s segments with slopes $-1/N < \beta_1 < \cdots < \beta_s < -1/(N+1)$. For $1 \leq j \leq s$, let $(x_j, y_j) \in \mathbb{N}^* \times \mathbb{N}^*$ be the solution of the linear system deduced from (5) by replacing the right hand side with $\rho_j[\nu_j, \delta_j]^T$. Additionally, let $C_j^{(k)}(\lambda, \epsilon) = (\lambda^N)^{x_j}(\lambda^{N+1})^{y_j} + C_j^{(k)}(\epsilon)$ with $C_j^{(k)}(\epsilon)$ the $\rho_j \nu_j \times \rho_j \nu_j$ matrix of the form

$$\begin{bmatrix} 0 & \epsilon & & \\ & \ddots & \ddots & \\ & & 0 & \epsilon \\ \epsilon & & & 0 \end{bmatrix}.$$

Clearly, $C^{(k)}(\lambda, \epsilon) = \mathrm{diag}[C_1^{(k)}(\lambda, \epsilon), \ldots, C_s^{(k)}(\lambda, \epsilon)]$ has $\rho_j \delta_j$ ϵ^{β_j}-eigenvalues for $1 \leq j \leq s$. Since $x = \sum_{j=1}^{s} x_j$ and $y = \sum_{j=1}^{s} y_j$, we take for $B^{(k)}(\lambda, \epsilon)$ the matrix deduced from $C^{(k)}(\lambda, \epsilon)$ by permuting some rows and columns in such a way that the new leading matrix is $B^{(k)}(\lambda) = (\lambda^N)^x (\lambda^{N+1})^y$. Note that the size of $B(\lambda, \epsilon)$ is at most n by assumption on the length m of $\mathcal{N}^-$.

Now, assume that $\mathcal{E}(S(\lambda)) \neq \mathcal{E}_{max}$. Considering the first vertex of $\mathcal{E}(S(\lambda))$ that lies strictly below $\mathcal{E}_{max}$ and since $T(\lambda)$ is an element of $\Sigma_m^n(\mathcal{N}^-)$ whose envelope is $\mathcal{E}_{max}$, we obtain $S(\lambda) < T(\lambda)$. Hence $S(\lambda) < S(\lambda)_{max}$. □

Theorem 4 has two consequences: first, it can be combined with Theorem 3 in order to yield the formula

$$S(\lambda)_{max} = \mathrm{diag}[1, \ldots, 1, (\lambda)^{\nu_0} (\lambda^{m_1})^{\sigma_{m_1}} \cdots (\lambda^{m_q})^{\sigma_{m_q}}] \quad (7)$$

when $m \leq n$. For example, the matrix of (3) satisfies $m = n = 6$ and we deduce from $\mathcal{E}_{max}$ in Fig. 2 that $\nu_0 = 0$, $q = 2$, $(m_1, \sigma_{m_1}) = (1, 2)$, $(m_2, \sigma_{m_2}) = (2, 2)$ and, therefore, that $S(\lambda)_{max} = \mathrm{diag}[1, 1, (\lambda)^2, (\lambda^2)^2]$. Second, using Corollary 1, it follows that the sufficient conditions of Theorem 1 can be necessary too:

Corollary 2. *[Converse of Theorem 1 i)] Let $m \leq n$ and let $A(\lambda, \epsilon) \in \Pi_m^n$ be a perturbed matrix polynomial with* *maximal leading Smith form (2). Then, if $A(\lambda, \epsilon)$ has ri $\epsilon^{1/i}$-eigenvalues, there exists $j \in \{1, \ldots, q\}$ such that $i = m_j$ and $r = k_2 - k_1$ with k_1 minimal and k_2 maximal so that $\Delta_j^{(k_1)} \neq 0$ and $\Delta_j^{(k_2)} \neq 0$.*

This means that such perturbed matrix polynomials are precisely those for which all the eigenvalue leading terms of the form $\mu\epsilon^\beta$ with $\beta^{-1} \in \mathbb{N}^*$ can be recovered via Theorem 1. For example, $A(\lambda)$ in (3) is not maximal, for otherwise applying Theorem 1 to $A(\lambda, \epsilon)$ would have revealed not only its ϵ-eigenvalue but also its two $\epsilon^{1/2}$-eigenvalues. Nevertheless, as shown in the next section, $A(\lambda, \epsilon)$ can be reduced to an equivalent perturbed matrix polynomial whose leading Smith form is maximal.

4. MAXIMIZATION BY EQUIVALENCE TRANSFORMS

Let $\mathsf{F} = \mathbb{Q}[[\epsilon]][\epsilon^{-1}]$. Regarded as elements of $\mathsf{F}[\lambda]^{\mathsf{n}\times\mathsf{n}}$, two perturbed matrix polynomials $A(\lambda, \epsilon)$, $\tilde{A}(\lambda, \epsilon)$ are *equivalent* when $\tilde{A}(\lambda, \epsilon) = U(\lambda, \epsilon)A(\lambda, \epsilon)V(\lambda, \epsilon)$ for some unimodular $U(\lambda, \epsilon)$, $V(\lambda, \epsilon)$ in $\mathsf{F}[\lambda]^{\mathsf{n}\times\mathsf{n}}$. (See e.g. [8, vol. 1, p. 135].) It follows that two equivalent perturbed matrix polynomials have the same eigenvalue leading terms. Since their Newton diagrams coincide up to a translation along the y-axis, they also have the same sequence $(\sigma_i)_{i\in\mathbb{N}^*}$. More precisely, for any perturbed matrix polynomial equivalent to $A(\lambda, \epsilon)$, the invariant sequence of (6) is given by

$$\sigma_i = 2\nu_i - \nu_{i+1} - \nu_{i-1}, \quad \nu_i = \mathrm{val}_\epsilon\, p(\lambda\epsilon, \epsilon^i), \tag{8}$$

where val_ϵ is the valuation in ϵ and $p(\lambda, \epsilon) = \det A(\lambda, \epsilon)$.

4.1 Reduction theorem

THEOREM 5. *Let $m \leq n$. Then for every $A(\lambda, \epsilon) \in \Pi_m^n$ one can define a pair of equivalence transforms $U(\lambda, \epsilon)$ and $V(\lambda, \epsilon)$ such that $U(\lambda, \epsilon)A(\lambda, \epsilon)V(\lambda, \epsilon)$ is a perturbed matrix polynomial with maximal leading Smith form.*

PROOF. Over $\mathsf{F}[\lambda]$, the matrix polynomial $A(\lambda, \epsilon)$ is equivalent to its Smith form $\mathrm{diag}[I_{n-x}, s_1(\lambda, \epsilon), \ldots, s_x(\lambda, \epsilon)]$ with s_k monic in λ and of positive degree for $1 \leq k \leq x$. Multiplying the last x rows by suitable constants of F yields an equivalent matrix

$$P(\lambda, \epsilon) = \mathrm{diag}[I_{n-x}, p_1(\lambda, \epsilon), \ldots, p_x(\lambda, \epsilon)]$$

with no negative power of ϵ and such that

$$p(\lambda, \epsilon) = \prod_{k=1}^{x} p_k(\lambda, \epsilon).$$

Let $m_k = \deg p_k(\lambda, 0)$. Since $m \leq n$, one can partition n as $n_1 + \cdots + n_x$ so that $n_k \geq m_k$ for $1 \leq k \leq x$. The matrix $A(\lambda, \epsilon)$ is therefore equivalent to $\mathrm{diag}[P_k(\lambda, \epsilon)]_{1\leq k\leq x}$ with $P_k(\lambda, \epsilon) = \mathrm{diag}[I_{n_k-1}, p_k(\lambda, \epsilon)]$.

Now let $\mathcal{N}_k$, $\mathcal{E}_{k,max} = \mathcal{E}_{k,max}(\mathcal{N}_k^-)$ and $S_{k,max}(\lambda)$ be respectively the Newton diagram, the Newton envelope and the maximal leading Smith form associated with $P_k(\lambda, \epsilon)$. It follows from the equations of (5) being linear that the equality $\mathcal{E}_{max}(\cup\mathcal{N}_k^-) = \cup\mathcal{E}_{k,max}(\mathcal{N}_k^-)$ holds. Here, the union denotes the convex diagram obtained by merging convex diagrams for $1 \leq k \leq x$. For example $\cup\mathcal{N}_k^- = \mathcal{N}^-$. Since $m_k \leq n_k$ for $1 \leq k \leq x$, we deduce $S(\lambda)_{max}$ from $\mathrm{diag}[S_{k,max}(\lambda)]_{1\leq k\leq x}$ by row and column permutations. The P_k's can thus be handled independently and we may assume without loss of generality that $A(\lambda, \epsilon) = \mathrm{diag}[I_{n-1}, p(\lambda, \epsilon)]$.

Let us now reduce $A(\lambda, \epsilon)$ to an equivalent $\tilde{A}(\lambda, \epsilon)$ such that $\tilde{A}(\lambda)$ is equal to the maximal Smith form of (7). Let $\tau = \nu_0 + \sum_{j=1}^q \sigma_{m_j}$. It suffices to find a pair of $\tau \times \tau$ unimodular matrices $U(\lambda, \epsilon)$, $V(\lambda, \epsilon)$ such that $UBV = \tilde{B}$ with B given by $A = \mathrm{diag}[I_{n-\tau}, B]$ and where $\tilde{B}$ is the lower right block of $\tilde{A}$ partitioned conformally. From now we focus on the case where $\nu_0 = 0$. (When $\nu_0 > 0$, it suffices to replace m_1 with ν_0 in the argument below.) When $\tau = 1$, one has $U = V = 1$; otherwise, writing $p = r + s$ with $r(\lambda, \epsilon) = \sum_{i<m_1} a_i(\epsilon)\lambda^i$ and $s(\lambda, \epsilon) = \sum_{i\geq m_1} a_i(\epsilon)\lambda^i$, we notice that all the $a_i(\epsilon)$'s in the "regular" part r have valuation in ϵ greater than $\tau - 1$ whereas the "singular" part s has at least one coefficient of valuation $\tau - 1$ or less. This allows to define

$$U_1^{(0)} = \left[\begin{array}{c|c|c} 1 & & \\ \hline & I_{\tau-2} & \\ \hline \lambda^{-m_1}\epsilon^{1-\tau}s(\lambda, \epsilon) & & 1 \end{array}\right]$$

and

$$V_1^{(0)} = \left[\begin{array}{c|c|c} \lambda^{m_1} & & \epsilon^{\tau-1} \\ \hline & I_{\tau-2} & \\ \hline -\epsilon^{1-\tau} & & 0 \end{array}\right]$$

and to transform

$$B = \left[\begin{array}{c|c|c} 1 & & \\ \hline & I_{\tau-2} & \\ \hline & & p(\lambda, \epsilon) \end{array}\right]$$

into

$$\begin{aligned} U_1^{(0)}BV_1^{(0)} &= \left[\begin{array}{c|c|c} \lambda^{m_1} & & \epsilon^{\tau-1} \\ \hline & I_{\tau-2} & \\ \hline -\epsilon^{1-\tau}r(\lambda, \epsilon) & & \lambda^{-m_1}s(\lambda, \epsilon) \end{array}\right] \\ &= \left[\begin{array}{c|c|c} \lambda^{m_1} & & \\ \hline & I_{\tau-2} & \\ \hline & & \lambda^{-m_1}s(\lambda, 0) \end{array}\right] + O(\epsilon). \end{aligned}$$

We have obtained the *first* diagonal entry of the target leading matrix $\tilde{B}(\lambda)$. More generally, let $\tau_j = \sigma_{m_j} + \cdots + \sigma_{m_q}$ for $1 \leq j \leq q$. One can then repeat the process and successively multiply on the left and on the right by, respectively, $\tau \times \tau$ unimodular matrices $\mathrm{diag}[I_{\tau-\tau_j+k}, U_j^{(k)}]$ and $\mathrm{diag}[I_{\tau-\tau_j+k}, V_j^{(k)}]$ where $U_j^{(k)}$, $V_j^{(k)}$ are defined for $0 \leq k \leq \sigma_{m_j}$ and $1 \leq j \leq q$ on the same model as $U_1^{(0)}$, $V_1^{(0)}$. (In particular, $U_{j+1}^{(0)} = U_j^{(\sigma_{m_j})}$ and $V_{j+1}^{(0)} = V_j^{(\sigma_{m_j})}$.) It follows that defining $U = U_q^{(\sigma_{m_q})} \cdots U_1^{(0)}$ and $V = V_1^{(0)} \cdots V_q^{(\sigma_{m_q})}$ allows to transform B into $\tilde{B} = UBV$. Note that the *last* diagonal entry of $\tilde{B}(\lambda)$ stems from the term λ^m of $p(\lambda, 0)$. $\square$

Theorem 5 thus defines a rational reduced form for equivalent square perturbed matrix polynomials over $\mathsf{F}[\lambda]$. The invariants of $A(\lambda, \epsilon)$ it gives access to are all the eigenvalue leading terms of $A(\lambda, \epsilon)$ of the form $\mu\epsilon^\beta$ with $\beta^{-1} \in \mathbb{N}^*$. Of course, this reduced form is not unique, for only its leading matrix is required to be normalized as $S(\lambda)_{max}$.

4.2 Case where $m > n$

In this case the maximal leading Smith form can still be misleading. Consider for example $A(\lambda, \epsilon) = \lambda^2 - \epsilon^2$. Here $(m, n) = (2, 1)$ and $S(\lambda)_{max} = A(\lambda)$ but Theorem 1 fails to recover the leading terms of the two ϵ-eigenvalues. There is thus no point in trying to systematically reduce $A(\lambda, \epsilon)$ itself. It is however always possible to reduce $B(\lambda, \epsilon) = \text{diag}[I_{m-n}, A(\lambda, \epsilon)]$. Returning to our example, it is not hard to see that $B(\lambda, \epsilon) = \text{diag}[1, \lambda^2 - \epsilon^2]$ would be transformed into the matrix $\tilde{A}(\lambda, \epsilon)$ shown in introduction. The possible need for embedding can be explained as follows: the perturbation theory of Section 2 is linear in the sense that only $A(\lambda, \epsilon)$ mod ϵ^2 is used. Embedding may thus be needed to "linearize" $A(\lambda, \epsilon)$ into an augmented reduced matrix to which Theorem 1 applies successfully.

5. CONCLUDING REMARKS

This study suggests two natural extensions. First, our results can easily be translated to hold in the general case where $A(\lambda)$ has some nonzero eigenvalues. In that case we can consider the local Smith form of $A(\lambda)$ at $\lambda = 0$ ([9, p. 330]) and use local equivalence of matrix functions rather than matrix polynomials. Second, the constructive proof of Theorem 5 yields a naive reduction algorithm that requires the computation of the Smith form $S(\lambda, \epsilon)$ of $A(\lambda, \epsilon)$ with $A(\lambda, \epsilon)$ viewed as a polynomial matrix over F. However, it would be interesting to know how to compute an equivalent perturbed matrix polynomial with maximal leading Smith form without resorting to the full knowledge of $S(\lambda, \epsilon)$. This is left for further research.

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