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Abstract. In this paper, we give bounds on the dichromatic number $\vec{\chi}(\Sigma)$ of a surface Σ, which is the maximum dichromatic number of an oriented graph embeddable on Σ. We determine the asymptotic behaviour of $\vec{\chi}(\Sigma)$ by showing that there exist constants a_1 and a_2 such that, $a_1\frac{\sqrt{-c}}{\log(-c)} \leq \vec{\chi}(\Sigma) \leq a_2\frac{\sqrt{-c}}{\log(-c)}$ for every surface Σ with Euler characteristic $c \leq -2$. We then give more explicit bounds for some surfaces with high Euler characteristic. In particular, we show that the dichromatic numbers of the projective plane $\mathbb{N}_1$, the Klein bottle $\mathbb{N}_2$, the torus $\mathbb{S}_1$, and Dyck's surface $\mathbb{N}_3$ are all equal to 3, and that the dichromatic numbers of the 5-torus $\mathbb{S}_5$ and the 10-cross surface $\mathbb{N}_{10}$ are equal to 4.

Keywords: dichromatic number, planar graphs, graphs on surfaces

1 Introduction

All surfaces considered in this paper are closed.

A graph is **embeddable** on a surface Σ if its vertices can be mapped onto distinct points of Σ and its edges onto simple curves of Σ joining the points onto which its endvertices are mapped, so that two edge curves do not intersect except in their common extremity. A **face** of an embedding $\tilde{G}$ of a graph G is a component of $\Sigma \setminus \tilde{G}$. Recall that an important theorem of the topology of surfaces, known as the Classification Theorem for Surfaces, states that every surface is homeomorphic to either the k-torus – a sphere with k-handles $\mathbb{S}_k$ or the k-cross surface – a sphere with k-cross-caps $\mathbb{N}_k$. The surface $\mathbb{S}_0 = \mathbb{N}_0$ is the sphere, and the surfaces $\mathbb{S}_1$, $\mathbb{S}_2$, $\mathbb{N}_1$, $\mathbb{N}_2$, $\mathbb{N}_3$ are also called the *torus*, the *double torus*, the *projective plane*, the *Klein bottle*, and *Dyck's surface*, respectively. The **Euler characteristic** of a surface homeomorphic to $\mathbb{S}_k$ is $2-2k$ and of a surface homeomorphic to $\mathbb{N}_k$ it is $2-k$. We denote the Euler characteristic of a surface Σ by $c(\Sigma)$.

Let G be a graph. We denote by $n(G)$ its number of vertices, and by $m(G)$ its number of edges. If G is embedded in a surface Σ, then we denote by $f(G)$ the number of faces of the embedding. Euler's Formula relates the numbers of vertices, edges and faces of a (connected) graph embedded in a surface.

Theorem 1. EULER'S FORMULA
Let G be a connected graph embedded on a surface Σ. Then

$$n(G) - m(G) + f(G) \geq c(\Sigma).$$

We denote by $\mathrm{Ad}(G) = 2m/n$ the average degree of a graph G. Euler's formula implies that graphs on surfaces have bounded average degree.

Theorem 2. *A connected graph G embeddable on a surface Σ satisfies:*

$$m(G) \leq 3n(G) - 3c(\Sigma) \quad \textit{and} \quad \mathrm{Ad}(G) \leq 6 - \frac{6c(\Sigma)}{n(G)}.$$

Moreover, there is equality if and only if G is a triangulation.

A k**-colouring** of a graph G is a partition of the vertex set of G into k disjoint **stable sets** (i.e. sets of pairwise non-adjacent vertices). A graph is k**-colourable** if it has a k-colouring. The **chromatic number** of a graph G, denoted by $\chi(G)$, is the least integer k such that G is k-colourable, and the **chromatic number** of a surface Σ, denoted by $\chi(\Sigma)$, is the least integer k such that every graph embeddable on Σ is k-colourable. Determining the chromatic number of surfaces attracted lots of attention, with its most important instance being the Four Colour Conjecture, which was eventually proved by Appel and Haken [2]. The chromatic numbers of the other surfaces were established earlier. Franklin [4] showed that the Klein bottle has chromatic number 6, and combined results of Heawood [6] and Ringel and Youngs [12] imply that if Σ is a surface different from the Klein bottle $\mathbb{N}_2$ with Euler characteristic c, then $\chi(\Sigma) \leq H(c) = \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor$.

In 1982, Neumann Lara [10] introduced the notion of directed colouring or dicolouring. A k**-dicolouring** of a digraph is a partition of its vertex set into k subsets inducing acyclic subdigraphs. A digraph is k**-dicolourable** if it has a k-dicolouring. The **dichromatic number** of a digraph D, denoted by $\vec{\chi}(D)$, is the least integer k such that D is k-dicolourable.

Let G be an undirected graph. The **bidirected graph** $\overleftrightarrow{G}$ is the digraph obtained from G by replacing each edge by a **digon**, that is a pair of oppositely directed arcs between the same end-vertices. Observe that $\chi(G) = \vec{\chi}(\overleftrightarrow{G})$ since any two adjacent vertices in $\overleftrightarrow{G}$ induce a directed cycle of length 2.

It is thus natural to consider **oriented graphs**, which are digraphs with no digons. Oriented graphs may be also seen as the digraphs which can be obtained from (simple) graphs by orienting every edge, that is replacing each edge by exactly one of the two possible arcs between its end-vertices. If $\vec{G}$ is obtained from G by orienting its edges, we say that G is the **underlying graph** of $\vec{G}$. It is easy to show that oriented planar graphs are 3-dicolourable and Neumann Lara [10] proposed the following conjecture.

Conjecture 1 (Neumann Lara [10]). Every oriented planar graph is 2-dicolourable.

This conjecture is part of an active field of research. It has been verified for planar oriented graphs on at most 26 vertices [7] and holds for planar digraphs of with no directed cycle of length 3 [9].

In this paper, we study the dichromatic number of surfaces. The **dichromatic number** of a surface Σ, denoted by $\vec{\chi}(\Sigma)$, is the least integer k such that every oriented graph embeddable on Σ is k-dicolourable. We first establish asymptotic bounds on the dichromatic number of surfaces.

Theorem 3. *There exist two positive constants a_1 and a_2 such that, for every surface Σ with Euler characteristic $c \leq -2$, we have*

$$a_1 \frac{\sqrt{-c}}{\log(-c)} \leq \vec{\chi}(\Sigma) \leq a_2 \frac{\sqrt{-c}}{\log(-c)}$$

Due to lack of space, we do not include the proof of this theorem. Like every other proofs missing in this paper, it can be found in the long version of the paper [1].

We then estimate the exact value of the dichromatic number of surfaces close to the sphere. Table 1 summarizes the main results.

Σ	$c(\Sigma)$	Bounds for $\vec{\chi}(\Sigma)$	Reference
Sphere $\mathbb{N}_0 = \mathbb{S}_0$	2	$2 \leq \vec{\chi} \leq 3$	Neumann Lara [10]
$\mathbb{N}_1, \mathbb{N}_2, \mathbb{S}_1, \mathbb{N}_3$	$\in \{1, 0, -1\}$	$\vec{\chi} = 3$	Theorem 5
$\mathbb{S}_2, \mathbb{N}_4, \mathbb{N}_5, \mathbb{S}_3, \mathbb{N}_6, \mathbb{N}_7, \mathbb{S}_4, \mathbb{N}_8, \mathbb{N}_9$	$\in \{-2, \dots, -7\}$	$3 \leq \vec{\chi} \leq 4$	Theorems 5 and 6
$\mathbb{S}_5, \mathbb{N}_{10}$	-8	$\vec{\chi} = 4$	Theorem 6

Table 1. Bounds on the dichromatic number of some surfaces.

In order to prove that the dichromatic number of a surface Σ is at most k, we shall prove that there is no $(k+1)$-dicritical digraph embeddable in Σ. A digraph D is $(k+1)$**-dicritical** if $\vec{\chi}(D) = k+1$ and $\vec{\chi}(H) \leq k$ for every proper subdigraph H of D. Kostochka and Stiebitz [8] prove the following.

Theorem 4 (Kostochka and Stiebitz [8]). *Let $\vec{G}$ be a 4-dicritical oriented graph then $3m(\vec{G}) \geq 10n(\vec{G}) - 4$. Moreover, if $\vec{G}$ is embeddable in a surface with Euler characteristic c, then $n(\vec{G}) \leq 4 - 9c$.*

2 The dichromatic number of $\mathbb{N}_1$, $\mathbb{N}_2$, $\mathbb{N}_3$, and $\mathbb{S}_1$

Theorem 5. $\vec{\chi}(\mathbb{N}_1) = \vec{\chi}(\mathbb{N}_2) = \vec{\chi}(\mathbb{N}_3) = \vec{\chi}(\mathbb{S}_1) = 3.$

Proof. $K_7 \setminus e$, the complete graph on seven vertices minus an edge, is embeddable in every surface other than the projective plane and the sphere. Neumann-Lara [11] proved that this graph has an orientation with dichromatic number 3. Hence $\vec{\chi}(\mathbb{N}_2)$, $\vec{\chi}(\mathbb{N}_3)$, $\vec{\chi}(\mathbb{S}_1) \geq 3$.

The complete graph on 6 vertices K_6 can be embedded as a **triangulation** of the projective plane, that is is an embedding of K_6 in the projective plane such that all faces are triangles. Let T be the orientation of K_6 displayed on the left of Figure 1. Let $\vec{G}$ be the oriented graph obtained from T by adding in each gray triangular face (which is a transitive tournament on three vertices with source s and sink t), the gadget graph depicted on the left of Figure 1. Observe that in any 2-dicolouring of the gadget graph, the vertices of the outer face do not have all the same colour.

Assume now for a contradiction that $\vec{G}$ admits a 2-dicolouring. Observe that either we have a monochromatic directed triangle in T or one of the gray triangles is

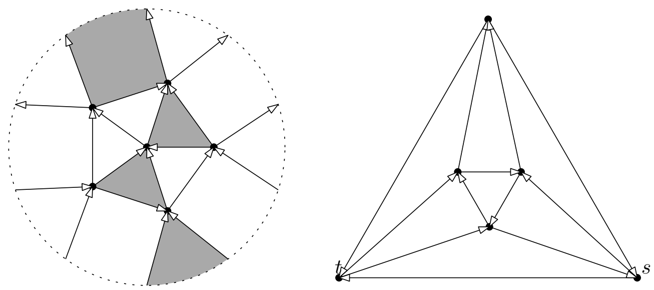

Fig. 1. Left: an orientation T of K_6 on the projective plane. Right: the gadget graph.

monochromatic. But then the 2-dicolouring cannot be extended to the gadget inside this transitive tournament by the above observation. Hence $\overrightarrow{G}$ is not 2-dicolourable. Hence $3 \leq \overrightarrow{\chi}(\mathbb{N}_1)$.

Suppose for a contradiction that there exists a 4-dicritical oriented graph $\overrightarrow{G}$ embeddable on $\mathbb{N}_3$. By Theorem 4, it has at most 13 vertices (because $c(\mathbb{N}_3) = -1$).

If G is not a triangulation of $\mathbb{N}_3$, then, by Theorem 2, $m(\overrightarrow{G}) \leq 3n(\overrightarrow{G}) + 2$, that is $3m(\overrightarrow{G}) \leq 9n(\overrightarrow{G}) + 6$. But $3m(\overrightarrow{G}) \geq 10n(\overrightarrow{G}) - 4$ by Theorem 4. Hence $n(\overrightarrow{G}) \leq 10$. But Neumann Lara [11] proved that every oriented graph of order at most 10 is 3-diciolorable. This is a contradiction.

So $\overrightarrow{G}$ is a triangulation of $\mathbb{N}_3$. By Theorem 4 and the abiove mentioned result of Neumann-Lara, $11 \leq n(\overrightarrow{G}) \leq 13$. Then, an exhaustive enumeration of the triangulations of order 11, 12 and 13 shows that there is no 4-dicritical oriented graph in $\mathbb{N}_3$.

Since every oriented graph embeddable in $\mathbb{N}_1$, $\mathbb{N}_2$, or $\mathbb{S}_1$ is also embeddable in $\mathbb{N}_3$, we get the result. □

2.1 The dichromatic number of $\mathbb{S}_5$ and $\mathbb{N}_{10}$

Theorem 6. $\overrightarrow{\chi}(\mathbb{S}_5) = \overrightarrow{\chi}(\mathbb{N}_{10}) = 4$.

Proof. The complete graph on 11 vertices is embeddable on $\mathbb{S}_5$ and $\mathbb{N}_{10}$ and Neumann-Lara [11] showed an orientation of this graph with dichromatic number 4. Therefore $\overrightarrow{\chi}(\mathbb{S}_5), \overrightarrow{\chi}(\mathbb{N}_{10}) \geq 4$.

It remains to prove that every oriented graph embeddable on $\mathbb{S}_5$ or $\mathbb{N}_{10}$ is 4-dicolourable. We now sketch this proof. The entire proof may be found in [1].

Assume for a contradiction that there is a 5-dicritical oriented graph $\overrightarrow{G}$ of order n which is embedded in $\mathbb{S}_5$ or $\mathbb{N}_{10}$.

Let T be the subdigraph induced by the vertices of degree 8 (i.e. in-degree 4 and out-degree 4). Set $H = \overrightarrow{G} - T$, $n_8 = n(T)$ and let $m(H,T)$ be the number of arcs with one end-vertex in H and the other in T. A result of Bang-Jensen et al. [3] implies that T is a **directed cactus**, that is an oriented graph in which each block is a single arc or a directed cycle. In particular, T is 2-dicolourable. Therefore H is not 2-dicolourable. In particular, one can prove that $m(H) \geq 20$.

Euler's Formula yields $8n_8+9(n-n_8)+\sum_{v\in V(H)}(d(v)-9) = 2m(\overrightarrow{G}) \leq 6n+48$ and so:

$$n_8 \geq 3(n-16) + \sum_{v\in V(H)} (d(v)-9) \geq 3(n-16) \tag{1}$$

On the other hand, we have $\sum_{v\in V(T)} d(v) = 8n_8 = 2m(T) + m(H,T)$ and $m(\overrightarrow{G}) = m(H) + m(H,T) + m(T)$. We deduce

$$m(H) = m(\overrightarrow{G}) + m(T) - 8n_8 \tag{2}$$

Since T is a directed cactus, we have $m(T) \leq \frac{3}{2}(n_8-1)$. Thus $20 \leq m(H) \leq m(\overrightarrow{G}) + \frac{3}{2}(n_8-1) - 8n_8$. Hence $13n_8 \leq 2m(\overrightarrow{G}) - 43$. With Eq. (1) and Euler's formula, it implies

$$3(n-16) \leq n_8 \leq \frac{2m(\overrightarrow{G})-43}{13} \leq \frac{6n+5}{13} \tag{3}$$

After simplifying, we get $n \leq 19$. Moreover, one can easily prove that every oriented graph of order at most 15 is 4-dicolourable, thus $n \geq 16$. We then distinguish few cases depending on the number n of vertices. We only sketch here the proof for $n = 19$. The details of this case and the proof of the other cases can be found in [1].

Case $n = 19$: By Eq. (3), we have $9 \leq n_8 \leq \frac{119}{13}$ and so $n_8 = 9$.

Assume first that $m(T) = \frac{3}{2}(n_8-1) = 12$. By as T is a directed cactus, T is connected and each block of T is a directed triangle. So T is Eulerian, i.e. $d_T^+(v) = d_T^-(v)$ for all $v \in V(T)$.

Since $n_8 = 9$, we have $n(H) = 10$. So, by a result of Neumann-Lara [11], H admits a 3-dicolouring ϕ with colour set $\{1,2,3\}$. Since all blocks of T are directed triangles, T contains a vertex v such that $d_T^+(v) = d_T^-(v) = 1$. So v has 3 out-neighbours in H. Let v_1, v_2 be two of these out-neighbours. Let us recolour v_1 and v_2 by setting $\phi(v_1) = \phi(v_2) = 4$ (since there is no digon, the resulting colouring is still proper). We then define for every vertex x of T:

$$L(x) = \{1,2,3,4\} \setminus \phi(N^+(x) \cap V(H))$$

Observe that an L-colouring of T extends the 4-colouring of H into a 4-colouring of G, so T is not L-colourable. Observe that $|L(x)| \geq 4-(4-d_T^+(x)) = \max\{d_T^+(x), d_T^-(x)\}$ because T is Eulerian. Moreover, since v_1 and v_2 are both coloured 4, $|L(v)| \geq 2 = \max\{d_T^+(x), d_T^-(x)\} + 1$. So T is L-dicolourable by a theorem of Harutyunyan and Mohar [5], a contradiction.

Therefore we have $m(T) \leq 11$. By Euler's Formula, $m(\overrightarrow{G}) \leq 3n+24$, and by Eq. (2) $m(H) = m(\overrightarrow{G}) - 8n_8 + m(T)$. Hence $m(H) \leq 20$. But H is not 2-dicolourable, so it contains a 3-dicritical oriented subgraph $\tilde{H}$, and $m(\tilde{H}) \leq 20$. One can show that there is a unique such 3-dicritical oriented graph with at most 20 arcs : it has 7 vertices and 20 arcs. Hence $n(\tilde{H}) = 7$, $m(\tilde{H}) = m(H) = 20$ and H is the disjoint union of $\tilde{H}$ and a stable set S' of size 3. Observe that each vertex of S' has degree at least 9, which implies that they are adjacent to every vertex of T and have degree exactly 9.

Now, $m(\tilde{H}) < m(K_7)$, so there are two non-adjacent vertices x, y in $\tilde{H}$. Thus $S = S' \cup \{x, y\}$ is a stable set of order 5 in H. Moreover, since it is a directed cactus, T has an acyclic subdigraph A of order 6. Pick $v \in V(T) \setminus V(A)$. The subdigraph B of $\overrightarrow{G}$ induced by $S \cup \{v\}$ is acyclic and has order 6. Let $G' = \overrightarrow{G} - (A \cup B)$. Observe that G' has order $19 - 6 - 6 = 7$. Recall that Neumann-Lara [11] showed that oriented graphs on at most 6 vertices are 2-dicolourable.

Let $w \in V(G') \cap V(T)$.

- If $|N(w) \cap V(A)| \leq 1$, then the subdigraph A' induced by $V(A) \cup \{w\}$ is acyclic. Hence G can be partitioned into two acyclic subdigraphs A' and B and $G - A' \cup B$ which has order 6 and so is 2-dicolourable. Thus $\overrightarrow{G}$ is 4-dicolourable, a contradiction.
- If $|N(w) \cap V(A)| \geq 2$, then as w is adjacent to all vertices of S', we have $d_{G'}(w) \leq 8 - 2 - 3 = 3$. Now, $G' - \{w\}$ is 2-dicolourable, and since $d_{G'}(w) = 3$, G' is also 2-dicolourable, and thus G is 4-dicolourable, a contradiction. □

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