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Abstract: We study continuous-time nonlinear systems, first in the case where there are continuous output measurements and next in the case where there are only discrete output measurements. When continuous measurements are available, we provide observers that converge in finite time. When only discrete measurements are available, we provide observers that do not converge in finite time, but which do converge asymptotically with a rate of convergence that is proportional to the negative of the logarithm of the size of the sampling interval. We illustrate our results in a pendulum example.

Keywords: Reduced order observer, finite time, discrete measurements

1. INTRODUCTION

Finite-time observers are efficient in practice because they provide the exact value of the state of a studied system in finite time. Many different types of finite-time observers are available in the literature; see in particular the contributions Engel and Kreisselmeier (2002), Lebastard et al. (2006), Lopez-Ramirez et al. (2018), Mazenc et al. (2015), Menard et al. (2010), Raff et al. (2005), Sanchez-Torres et al. (2012), and Sauvage et al. (2007). Some of them use sliding mode, or homogeneous functions, or delays, or dynamic extensions. The contribution by Raff and Allgower (2008) is significantly different from the others. It provides a continuous-discrete observer that possesses a key advantage, namely it does not incorporate delays and therefore may be easier to implement than observers that incorporate delays. Also, Raff and Allgower (2008) does not rely on homogeneity properties, and therefore may enjoy better robustness properties. However, it presents the two limitations that it only applies to linear systems and that although it has continuous-discrete type, it necessitates the knowledge of a continuous output, which is problematic when only discrete measurements are available.

In the present paper, we revisit the main result of Raff and Allgower (2008). We consider a family of nonlinear continuous-time observable systems and provide a twofold contribution. First, we propose a reduced order version of the observer in Raff and Allgower (2008) in the case where the measurements are continuous. The observer converges in finite time, after an instant which can be selected by the user. The limitation of this result is that it does not apply when the measurements are only available at discrete instants. Second, for a narrower family of nonlinear systems (that satisfies a Lipschitzness condition), we combine the reduced order observer of our first design with the key approach of Karafyllis and Kravaris (2009) (which is also used in Karafyllis and Jiang (2013)), to handle the case (which is very important in practice) where the measurements are only available at discrete instants. The price that is paid for considering discrete measurements is that this second observer does not converge to the solutions of the studied system in finite time. However, it is efficient in terms of speed of convergence when the size of the sampling intervals is small, insofar that its convergence speed is proportional to the negative of the logarithm of the size of the largest sampling interval.

We establish convergence for our second observer through a proof which relies on a recent stability analysis technique called the trajectory based approach that is developed in particular in the papers Ahmed et al. (2018) and Mazenc et al. (2017). We show the efficiency of our approach by applying it to a pendulum model that was discussed in Dinh et al. (2015) in the full order observer case.

The paper is organized as follows. The studied class of systems is presented in Section 2. A first observer is proposed in Section 3. A second observer is proposed in Section 4. An illustrative example is given in Section 5. Concluding remarks are drawn in Section 6.

Notation. We use standard notation, which is simplified when no confusion would arise, and where the dimensions of our Euclidean spaces are arbitrary unless otherwise

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noted. The standard Euclidean 2-norm, and the induced matrix norm, are denoted by $|\cdot|$, $|\cdot|_S$ is the essential supremum over any set S, and $|\cdot|_\infty$ is the usual $\mathcal{L}_\infty$ sup norm. For a piecewise continuous locally bounded function $\phi : [0, +\infty) \to \mathbb{R}^m$, we let $\phi(c^-)$ be the left limit $\phi(c^-) = \lim_{t \to c^-} \phi(t)$, and $\mathbb{N} = \{1, 2, \ldots\}$.

2. STUDIED SYSTEM

We consider the system

$$\dot{x} = Ax + f(r(t), u(t)) \tag{1}$$

with $x(t)$ valued in $\mathbb{R}^n$, $u(t)$ valued in $\mathbb{R}^q$ and $A \in \mathbb{R}^{n\times n}$, where f is a locally Lipschitz nonlinear function such that $f(0, 0) = 0$ and

$$r(t) = Cx(t) \tag{2}$$

with $r(t)$ valued in $\mathbb{R}^p$ and $C \in \mathbb{R}^{p\times n}$, and $p < n$.

We will first consider the case where the output is continuous, i.e. $y(t) = r(t)$, and next (in Section 4) the case where it is discrete. Throughout the paper, we assume:

Assumption A1: The rank of C is full. The pair (A, C) observable. □

For forty years, it has been well-known that, under Assumption A1, the system (1) can be transformed through a linear change of coordinates of the type

$$\begin{pmatrix} x_r(t) \\ r(t) \end{pmatrix} = \mho x(t) \tag{3}$$

with an invertible matrix $\mho$ into a system of the form

$$\begin{cases} \dot{r}(t) = F_{11}r(t) + F_{12}x_r(t) + f_1(r(t), u(t)) \\ \dot{x}_r(t) = F_{21}r(t) + F_{22}x_r(t) + f_2(r(t), u(t)) \end{cases} \tag{4}$$

with $F_{11} \in \mathbb{R}^{p\times p}$, $F_{12} \in \mathbb{R}^{p\times(n-p)}$, $F_{21} \in \mathbb{R}^{(n-p)\times p}$ and $F_{22} \in \mathbb{R}^{(n-p)\times(n-p)}$, where the pair (F_{22}, F_{12}) is observable; see pages 304-306 in Luenberger (1979). In Mazenc et al. (2015), it is proved that there are a matrix $L \in \mathbb{R}^{(n-p)\times p}$ and a constant ν (which can be taken to be arbitrarily large) such that with the choice

$$H = F_{22} + LF_{12} \in \mathbb{R}^{(n-p)\times(n-p)}, \tag{5}$$

the matrix

$$E = e^{-F_{22}\nu} - e^{-H\nu} \tag{6}$$

is invertible. We now introduce the sequence

$$t_i = i\nu \text{ for all } i \in \mathbb{N}, \tag{7}$$

the matrices

$$\begin{aligned} G &= F_{21} - F_{22}L + LF_{11} - LF_{12}L \\ &= F_{21} + LF_{11} - HL \in \mathbb{R}^{(n-p)\times p} \end{aligned} \tag{8}$$

and

$$\begin{aligned} R_1 &= E^{-1}e^{-\nu F_{22}} \in \mathbb{R}^{(n-p)\times(n-p)} \text{ and} \\ R_2 &= -E^{-1}e^{-\nu H} \in \mathbb{R}^{(n-p)\times(n-p)} \end{aligned} \tag{9}$$

and the $\mathbb{R}^{n-p}$-valued function

$$f_3 = f_2 + Lf_1. \tag{10}$$

3. OBSERVER WHEN THE OUTPUT IS CONTINOUS

In this section, we consider the system (1) with a continuous output $y(t) = Cx(t)$. Then (4) gives

$$\begin{cases} \dot{y}(t) = F_{11}y(t) + F_{12}x_r(t) + f_1(y(t), u(t)) \\ \dot{x}_r(t) = F_{21}y(t) + F_{22}x_r(t) + f_2(y(t), u(t)). \end{cases} \tag{11}$$

3.1 Observer

We consider the following dynamic extension:

$$\begin{cases} \dot{z}_1(t) = F_{21}y(t) + F_{22}z_1(t) + f_2(y(t), u(t)) \\ \qquad \text{if } t \in [t_k, t_{k+1}) \\ \dot{z}_2(t) = Hz_2(t) + Gy(t) + f_3(y(t), u(t)) \\ \qquad \text{if } t \in [t_k, t_{k+1}) \\ z_1(t_{k+1}) = R_1z_1(t_{k+1}^-) + R_2z_2(t_{k+1}^-) \\ \qquad -R_2Ly(t_{k+1}) - E^{-1}Ly(t_k) \\ z_2(t_{k+1}) = R_1z_1(t_{k+1}^-) + R_2z_2(t_{k+1}^-) \\ \qquad -R_2Ly(t_{k+1}) - E^{-1}Ly(t_k) \end{cases} \tag{12}$$

for all integers $k \geq 0$, with $z_1(0) = z_2(0) = 0$ (but our results remain true if we fix any other initial states for the z_i's at time 0). We state and prove the following result:

Theorem 1: Let the system (1) with the output $y(t) = Cx(t)$ satisfy Assumption A1 and let it be forward complete. Then the solutions of (11)-(12) are such that

$$z_1(t) = x_r(t) \tag{13}$$

for all $t \geq t_2$. □

Remark 1. The main difference between the observer (12) and the one proposed in Raff and Allgower (2008) is that the dimension of the z-subsystem in (12) is $2(n - p)$, whereas the dimension of the corresponding system in Raff and Allgower (2008) is $2n$. □

Remark 2. Since $y(t)$ and $z_1(t)$ are known for all $t \geq t_0$, Theorem 1 implies that $x(t)$ is known for all $t \geq t_2$ because, according to (3),

$$x(t) = \mho^{-1}\begin{pmatrix} x_r(t) \\ y(t) \end{pmatrix} \tag{14}$$

for all $t \geq t_2$. □

3.2 Proof of Theorem 1

Since we assume that the system (1) is forward complete, all the solutions of (11)-(12) are defined over $[0, +\infty)$. Next, let us introduce the variable

$$\xi(t) = x_r(t) + Ly(t). \tag{15}$$

Simple calculations give

$$\begin{aligned} \dot{\xi}(t) &= (F_{21} + LF_{11})y(t) + (F_{22} + LF_{12})x_r(t) \\ &\quad + f_3(y(t), u(t)) \\ &= (F_{21} + LF_{11})y(t) + (F_{22} \\ &\quad + LF_{12})(\xi(t) - Ly(t)) + f_3(y(t), u(t)). \end{aligned} \tag{16}$$

Thus we get

$$\begin{cases} \dot{x}_r(t) = F_{21}y(t) + F_{22}x_r(t) + f_2(y(t), u(t)) \\ \dot{\xi}(t) = H\xi(t) + Gy(t) + f_3(y(t), u(t)). \end{cases} \tag{17}$$

For any $k \in \mathbb{N}$, we integrate (12) and (17) over the interval $[t_k, t_{k+1})$ and obtain

$$\begin{aligned} & e^{-\nu F_{22}}x_r(t_{k+1}) = x_r(t_k) \\ & + \int_{t_k}^{t_{k+1}} e^{(t_{k+1}-\ell-\nu)F_{22}}\left[F_{21}y(\ell) + f_2(y(\ell), u(\ell))\right] d\ell \\ & \text{and } e^{-\nu H}\xi(t_{k+1}) = \xi(t_k) \\ & + \int_{t_k}^{t_{k+1}} e^{(t_{k+1}-\ell-\nu)H}\left[Gy(\ell) + f_3(y(\ell), u(\ell))\right] d\ell \\ & \text{and } e^{-\nu F_{22}}z_1(t_{k+1}^-) = z_1(t_k) \\ & + \int_{t_k}^{t_{k+1}} e^{(t_{k+1}-\ell-\nu)F_{22}}\left[F_{21}y(\ell) + f_2(y(\ell), u(\ell))\right] d\ell \\ & \text{and } e^{-\nu H}z_2(t_{k+1}^-) = z_2(t_k) \\ & + \int_{t_k}^{t_{k+1}} e^{(t_{k+1}-\ell-\nu)H}\left[Gy(\ell) + f_3(y(\ell), u(\ell))\right] d\ell. \end{aligned} \tag{18}$$

The first two equalities of (18) and our choice (6) of E give

$$\begin{array}{l} Ex_r(t_{k+1}) - e^{-\nu H}Ly(t_{k+1}) + Ly(t_k) \\ = \int_{t_k}^{t_{k+1}} \left[e^{(t_k-\ell)F_{22}}F_{21} - e^{(t_k-\ell)H}G\right] y(\ell)\mathrm{d}\ell \\ + \int_{t_k}^{t_{k+1}} \left[e^{(t_k-\ell)F_{22}}f_2(y(\ell),u(\ell)) \right. \\ \quad \left. -e^{(t_k-\ell)H}f_3(y(\ell),u(\ell))\right] \mathrm{d}\ell. \end{array} \tag{19}$$

Since $z_1(t_k) = z_2(t_k)$ for all $k \geq 1$, we deduce from the last two equations of (18) that

$$\begin{array}{l} e^{-\nu F_{22}}z_1(t_{k+1}^-) - e^{-\nu H}z_2(t_{k+1}^-) \\ = \int_{t_k}^{t_{k+1}} \left[e^{(t_k-\ell)F_{22}}F_{21} - e^{(t_k-\ell)H}G\right] y(\ell)\mathrm{d}\ell \\ + \int_{t_k}^{t_{k+1}} \left[e^{(t_k-\ell)F_{22}}f_2(y(\ell),u(\ell)) \right. \\ \left. -e^{(t_k-\ell)H}f_3(y(\ell),u(\ell))\right] \mathrm{d}\ell \end{array} \tag{20}$$

for all $k \geq 1$. Consequently,

$$\begin{array}{l} Ex_r(t_{k+1}) - e^{-\nu H}Ly(t_{k+1}) + Ly(t_k) \\ = e^{-\nu F_{22}}z_1(t_{k+1}^-) - e^{-\nu H}z_2(t_{k+1}^-). \end{array} \tag{21}$$

Since E is invertible, we have

$$\begin{array}{rl} x_r(t_{k+1}) = & E^{-1}e^{-\nu H}Ly(t_{k+1}) - E^{-1}Ly(t_k) \\ & +R_1z_1(t_{k+1}^-) + R_2z_2(t_{k+1}^-). \end{array} \tag{22}$$

From (12), it follows that

$$x_r(t_{k+1}) = z_1(t_{k+1}) \tag{23}$$

for all $k \geq 1$. From (11) and (12) and the existence and uniqueness of the solutions of ordinary differential equations, it follows that (13) holds for all $t \geq t_2$. This completes the proof.

4. OBSERVER WHEN THE OUTPUT IS DISCRETE

Throughout this section, we use the notation introduced in Sections 2 and 3. The main result of this section owes a great deal to the pioneering paper by Karafyllis and Kravaris (2009), because we use the dynamic extension introduced in Karafyllis and Kravaris (2009) to obtain an observer in the case where the measurements are discrete. However, our result can allow arbitrarily large convergence rates for the observer, which is a valuable feature that was beyond the scope of Karafyllis and Kravaris (2009).

We consider the case where the measurements are synchronous. We consider a constant $\mu > 0$, the sequence

$$s_i = i\mu \tag{24}$$

for all $i \in \mathbb{N}$, and the system

$$\begin{cases} \dot{x}(t) = Ax(t) + f(r(t),u(t)) \\ y(t) = Cx(s_j) \text{ if } t \in [s_j, s_{j+1}) \\ r(t) = Cx(t) \end{cases} \tag{25}$$

and let y be the output. We introduce assumptions:

Assumption A2. There is a constant $f_\dagger \geq 0$ such that

$$|f(m_1,u) - f(m_2,u)| \leq f_\dagger|m_1 - m_2| \tag{26}$$

for all $m_1 \in \mathbb{R}^p, m_2 \in \mathbb{R}^p$ and $u \in \mathbb{R}^q$. □

Assumption A3. There is $g \in \mathbb{N}$ such that $\nu = g\mu$, where ν satisfies the requirements from Section 2. □

Condition Assumption A3 is not restrictive at all because ν and g can be arbitrarily large.

According to Assumption A2 and the fact that the change of coordinates (3) is time-invariant, there are two constants $f_{\dagger,1} > 0$ and $f_{\dagger,2} > 0$ such that

$$|f_1(m_1,u) - f_1(m_2,u)| \leq f_{\dagger,1}|m_1 - m_2| \tag{27}$$

and

$$|f_2(m_1,u) - f_2(m_2,u)| \leq f_{\dagger,2}|m_1 - m_2| \tag{28}$$

hold for all $m_1 \in \mathbb{R}^p, m_2 \in \mathbb{R}^p$ and $u \in \mathbb{R}^q$.

4.1 Observer

We use this dynamic extension which is a candidate observer:

$$\left\{ \begin{array}{rcl} \dot{z}_1(t) &=& F_{21}w(t)+F_{22}z_1(t)+f_2(w(t),u(t)) \\ && \text{if } t \in [s_{gk}, s_{g(k+1)}) \\ \dot{z}_2(t) &=& Hz_2(t) + Gw(t) + f_3(w(t),u(t)) \\ && \text{if } t \in [s_{gk}, s_{g(k+1)}) \\ z_1(s_{g(k+1)}) &=& R_1z_1(s_{g(k+1)}^-) + R_2z_2(s_{g(k+1)}^-) \\ && -R_2Ly(s_{g(k+1)}) - E^{-1}Ly(s_{gk}) \\ z_2(s_{g(k+1)}) &=& R_1z_1(s_{g(k+1)}^-) + R_2z_2(s_{g(k+1)}^-) \\ && -R_2Ly(s_{g(k+1)}) - E^{-1}Ly(s_{gk}) \\ \dot{w}(t) &=& F_{11}w(t)+F_{12}z_1(t)+f_1(w(t),u(t)) \\ && \text{if } t \in [s_k, s_{k+1}) \\ w(s_k) &=& y(s_k) \end{array} \right. \tag{29}$$

for all integers $k \geq 0$, with $z_1(0) = z_2(0)$.

For a fixed constant $\nu > 0$ satisfying our requirements form Section 2, we will use the constants

$$\overline{E} = |E^{-1}|, \tag{30}$$

$$f_{\dagger,3} = f_{\dagger,2} + |L|f_{\dagger,1}, \tag{31}$$

and

$$\begin{array}{l} \beta(\nu) = e^{|F_{22}|\nu}(|F_{21}| + f_{\dagger,2}) + \\ \overline{E}\left(e^{\nu|F_{22}|+\nu|H|}(|G| + f_{\dagger,3}) + e^{2\nu|F_{22}|}(|F_{21}| + f_{\dagger,2})\right), \end{array} \tag{32}$$

and the function

$$\gamma(\mu) = e^{\mu|F_{11}|}\left(2\nu\beta(\nu) + 1\right)\max\left\{|F_{21}|, f_{\dagger,1}\right\}. \tag{33}$$

We fix a constant $\overline{\mu} > 0$ such that if $\mu \in (0, \overline{\mu})$, then

$$\mu\gamma(\mu) < 1. \tag{34}$$

Since we view $\nu > 0$ in (33) as being a fixed constant that satisfies the requirements from Section 2, satisfying the requirement $\mu \in (0, \bar{\mu})$ is equivalent to choosing the integer g in Assumption A3 such that $g = \nu/\mu > \nu/\bar{\mu}$. We are ready to state and prove the following result:

Theorem 2: Let the system (25) satisfy Assumptions A1 to A3 and $\mu \in (0, \overline{\mu})$. Then the solutions of (4) and (29) are defined over $[0, +\infty)$ and satisfy

$$\begin{array}{l} |x_r(t) - z_1(t)| \\ \leq e^{\frac{\ln(\mu\gamma(\mu))}{\mu+2\nu+m}(t-m)}\left(|x_r - z_1|_{[m-(\mu+2\nu),m]} \right. \\ \quad \left. +|w - r|_{[m-(\mu+2\nu),m]}\right) \end{array} \tag{35}$$

if $t \geq m \geq 4\nu$. □

Remark 3. The key feature of (35) is that it shows that the rate of convergence of the observer is proportional to $-\ln(\mu\gamma(\mu))$ or larger. We can always let $\mu = \nu$ by increasing μ is necessary. However, this choice will lead to less efficient observers in terms of speed of convergence, because when $\mu < \nu$ then $-\ln(\mu\gamma(\mu)) > -\ln(\nu\gamma(\nu))$. □

4.2 Proof of Theorem 2

Assumption A2 ensures that for the system (4) and (29), the finite escape time phenomenon does not occur. Thus the solutions of (4) and (29) are defined over $[0,+\infty)$. Now we decompose the proof in three parts.

First part of the proof: an expression for $x_r(t_{k+1})$.

To simplify the notation, let us define the sequence $t_k = s_{gk}$ for all $k \in \mathbb{N}$. According to Assumption A3, $t_k = k\nu$ for all $k \in \mathbb{N}$. Hence, this sequence is similar to the sequence t_k introduced in Section 2. Also, let us introduce the variable

$$\xi(t) = x_r(t) + Lr(t). \tag{36}$$

Then (4) and our choice $f_3 = f_2 + Lf_1$ give

$$\dot{\xi}(t) = (F_{21} + LF_{11})r(t) + Hx_r(t) + f_3(r(t),u(t)) \tag{37}$$

We deduce that

$$\left\{\begin{array}{rcl} \dot{x}_r(t) &=& F_{22}x_r(t) + F_{21}r(t) + f_2(r(t),u(t)) \\ \dot{\xi}(t) &=& H\xi(t) + Gr(t) + f_3(r(t),u(t)) \\ \dot{z}_1(t) &=& F_{21}w(t) + F_{22}z_1(t) + f_2(w(t),u(t)) \\ && \text{if } t \in [t_k,t_{k+1}) \\ \dot{z}_2(t) &=& Hz_2(t) + Gw(t) + f_3(w(t),u(t)) \\ && \text{if } t \in [t_k,t_{k+1}) \\ z_1(t_{k+1}) &=& R_1 z_1(t_{k+1}^-) + R_2 z_2(t_{k+1}^-) \\ && -R_2 Ly(t_{k+1}) - E^{-1}Ly(t_k) \\ z_2(t_{k+1}) &=& R_1 z_1(t_{k+1}^-) + R_2 z_2(t_{k+1}^-) \\ && -R_2 Ly(t_{k+1}) - E^{-1}Ly(t_k),\ k \geq 0. \end{array}\right. \tag{38}$$

Then, for any $k \in \mathbb{N}$, by integrating (38) over the interval $[t_k,t_{k+1})$, we obtain

$$\begin{array}{l} e^{-\nu F_{22}}x_r(t_{k+1}) = x_r(t_k) \\ + \int_{t_k}^{t_{k+1}} e^{(t_{k+1}-\ell-\nu)F_{22}}\left[F_{21}r(\ell) + f_2(r(\ell),u(\ell))\right]\mathrm{d}\ell \\ \text{and } e^{-\nu H}\xi(t_{k+1}) = \xi(t_k) \\ + \int_{t_k}^{t_{k+1}} e^{(t_{k+1}-\ell-\nu)H}\left[Gr(\ell) + f_3(r(\ell),u(\ell))\right]\mathrm{d}\ell \\ \text{and } e^{-\nu F_{22}}z_1(t_{k+1}^-) = z_1(t_k) \\ + \int_{t_k}^{t_{k+1}} e^{(t_{k+1}-\ell-\nu)F_{22}}\left[F_{21}w(\ell) + f_2(w(\ell),u(\ell))\right]\mathrm{d}\ell \\ \text{and } e^{-\nu H}z_2(t_{k+1}^-) = z_2(t_k) \\ + \int_{t_k}^{t_{k+1}} e^{(t_{k+1}-\ell-\nu)H}\left[Gw(\ell) + f_3(w(\ell),u(\ell))\right]\mathrm{d}\ell. \end{array} \tag{39}$$

Bearing in mind (38) and the fact that $z_1(t_j) - z_2(t_j) = 0$ for all $j \geq 1$, we deduce that

$$\begin{array}{l} e^{-\nu F_{22}}x_r(t_{k+1}) - e^{-\nu H}\xi(t_{k+1}) + Lr(t_k) \\ = \int_{t_k}^{t_{k+1}} e^{(t_k-\ell)F_{22}}F_{21}w(\ell)\mathrm{d}\ell \\ - \int_{t_k}^{t_{k+1}} e^{(t_k-\ell)H}Gw(\ell)\mathrm{d}\ell \\ + \int_{t_k}^{t_{k+1}} \left[e^{(t_k-\ell)H}G - e^{(t_k-\ell)F_{22}}F_{21}\right](w(\ell)-r(\ell))\mathrm{d}\ell \\ + \int_{t_k}^{t_{k+1}} e^{(t_k-\ell)F_{22}}f_2(r(\ell),u(\ell))\mathrm{d}\ell \\ - \int_{t_k}^{t_{k+1}} e^{(t_k-\ell)H}f_3(r(\ell),u(\ell))\mathrm{d}\ell \end{array} \tag{40}$$

and

$$\begin{array}{l} e^{-\nu F_{22}}z_1(t_{k+1}^-) - e^{-\nu H}z_2(t_{k+1}^-) \\ = \int_{t_k}^{t_{k+1}} e^{(t_k-\ell)F_{22}}F_{21}w(\ell)\mathrm{d}\ell \\ - \int_{t_k}^{t_{k+1}} e^{(t_k-\ell)H}Gw(\ell)\mathrm{d}\ell \\ + \int_{t_k}^{t_{k+1}} e^{(t_k-\ell)F_{22}}f_2(w(\ell),u(\ell))\mathrm{d}\ell \\ - \int_{t_k}^{t_{k+1}} e^{(t_k-\ell)H}f_3(w(\ell),u(\ell))\mathrm{d}\ell \end{array} \tag{41}$$

for all $k \geq 1$. As an immediate consequence, we have

$$\begin{array}{l} Ex_r(t_{k+1}) - e^{-\nu H}Lr(t_{k+1}) \\ = -Lr(t_k) + e^{-\nu F_{22}}z_1(t_{k+1}^-) - e^{-\nu H}z_2(t_{k+1}^-) \\ + \int_{t_k}^{t_{k+1}} \Lambda(t_k,\ell)(w(\ell) - r(\ell))\mathrm{d}\ell \\ + \int_{t_k}^{t_{k+1}} e^{(t_k-\ell)F_{22}}[f_2(r(\ell),u(\ell)) - f_2(w(\ell),u(\ell))]\mathrm{d}\ell \\ - \int_{t_k}^{t_{k+1}} e^{(t_k-\ell)H}[f_3(r(\ell),u(\ell)) - f_3(w(\ell),u(\ell))]\mathrm{d}\ell \end{array} \tag{42}$$

with E as defined in (6) and

$$\Lambda(m,\ell) = e^{(m-\ell)H}G - e^{(m-\ell)F_{22}}F_{21} \tag{43}$$

Thus, since (25) gives

$$r(t_{k+1}) = r(s_{g(k+1)}) = y(s_{g(k+1)}), \tag{44}$$

we have

$$\begin{array}{rcl} Ex_r(t_{k+1}) &=& e^{-\nu H}Lr(t_{k+1}) - Lr(t_k) \\ && + e^{-\nu F_{22}}z_1(t_{k+1}^-) - e^{-\nu H}z_2(t_{k+1}^-) \\ && + \int_{t_k}^{t_{k+1}} \Lambda(t_k,\ell)(w(\ell) - r(\ell)))\mathrm{d}\ell \\ && + \int_{t_k}^{t_{k+1}} e^{(t_k-\ell)F_{22}}\Delta_2(\ell)\mathrm{d}\ell \\ && - \int_{t_k}^{t_{k+1}} e^{(t_k-\ell)H}\Delta_3(\ell)\mathrm{d}\ell \\ &=& e^{-\nu H}Ly(t_{k+1}) - Ly(t_k) \\ && + e^{-\nu F_{22}}z_1(t_{k+1}^-) - e^{-\nu H}z_2(t_{k+1}^-) \\ && + \int_{t_k}^{t_{k+1}} \Lambda(t_k,\ell)(w(\ell) - r(\ell)))\mathrm{d}\ell \\ && + \int_{t_k}^{t_{k+1}} e^{(t_k-\ell)F_{22}}\Delta_2(\ell)\mathrm{d}\ell \\ && - \int_{t_k}^{t_{k+1}} e^{(t_k-\ell)H}\Delta_3(\ell)\mathrm{d}\ell \end{array} \tag{45}$$

where

$$\Delta_i(\ell) = f_i(r(\ell),u(\ell)) - f_i(w(\ell),u(\ell)). \tag{46}$$

Consequently, we have

$$\begin{array}{rcl} x_r(t_{k+1}) &=& R_1 z_1(t_{k+1}^-) + R_2 z_2(t_{k+1}^-) \\ && -R_2 Ly(t_{k+1}) - E^{-1}Ly(t_k) \\ && + E^{-1}\int_{t_k}^{t_{k+1}} \Lambda(t_k,\ell)(w(\ell) - r(\ell)))\mathrm{d}\ell \\ && + E^{-1}\int_{t_k}^{t_{k+1}} e^{(t_k-\ell)F_{22}}\Delta_2(\ell)\mathrm{d}\ell \\ && - E^{-1}\int_{t_k}^{t_{k+1}} e^{(t_k-\ell)H}\Delta_3(\ell)\mathrm{d}\ell \end{array} \tag{47}$$

Since (29) gives

$$\begin{array}{l} R_1 z_1(t_{k+1}^-) + R_2 z_2(t_{k+1}^-) \\ = z_1(t_{k+1}) + R_2 Ly(t_{k+1}) + E^{-1}Ly(t_k), \end{array} \tag{48}$$

we obtain

$$\begin{array}{rcl} x_r(t_{k+1}) &=& z_1(t_{k+1}) \\ && + E^{-1}\int_{t_k}^{t_{k+1}} \Lambda(t_k,\ell)(w(\ell) - r(\ell)))\mathrm{d}\ell \\ && + E^{-1}\int_{t_k}^{t_{k+1}} e^{(t_k-\ell)F_{22}}\Delta_2(\ell)\mathrm{d}\ell \\ && - E^{-1}\int_{t_k}^{t_{k+1}} e^{(t_k-\ell)H}\Delta_3(\ell)\mathrm{d}\ell \end{array} \tag{49}$$

for all $k \geq 1$.

Second part of the proof: an upper bound for $w(t) - r(t)$.

Let us introduce the variables

$$\tilde{w}(t) = w(t) - r(t) \tag{50}$$

and

$$\tilde{x}_r(t) = x_r(t) - z_1(t). \quad (51)$$

Then simple calculations based on (4) and (29) give

$$\begin{aligned} \dot{\tilde{w}}(t) &= F_{11}\tilde{w}(t) - F_{12}\tilde{x}_r(t) + f_1(w(t), u(t)) \\ &\quad -f_1(r(t), u(t)) \text{ if } t \in [s_k, s_{k+1}) \\ \tilde{w}(s_k) &= 0 \end{aligned} \quad (52)$$

for all $k \in \mathbb{N}$. By integrating the system (52) over $[s_k, t]$ with $t \in [s_k, s_{k+1})$, we obtain

$$\begin{aligned} &\tilde{w}(t) = -\int_{s_k}^{t} e^{F_{11}(t-m)}\left[\Delta_1(m) - F_{12}\tilde{x}_r(m)\right] \mathrm{d}m \\ &\text{if } t \in [s_k, s_{k+1}). \end{aligned} \quad (53)$$

From this equality and (27), it follows that

$$\begin{aligned} &|\tilde{w}(t)| \le e^{\mu|F_{11}|}\int_{s_k}^{t} f_{\dagger,1}|\tilde{w}(m)|\mathrm{d}m \\ &+e^{\mu|F_{11}|}|F_{12}|\int_{s_k}^{t}|\tilde{x}_r(m)|\mathrm{d}m \text{ if } t \in [s_k, s_{k+1}) \end{aligned} \quad (54)$$

From the definition of the sequence s_k, we deduce that

$$\begin{aligned} |\tilde{w}(t)| \le\ & e^{\mu|F_{11}|}f_{\dagger,1}\int_{t-\mu}^{t}|\tilde{w}(m)|\mathrm{d}m \\ &+e^{\mu|F_{11}|}|F_{12}|\int_{t-\mu}^{t}|\tilde{x}_r(m)|\mathrm{d}m \end{aligned} \quad (55)$$

for all $t \ge \mu$.

Third part of the proof: an upper bound for $\tilde{x}_r(t)$.

Bearing in mind (49), we can use (4) and (29) to get

$$\begin{aligned} \dot{\tilde{x}}_r(t) &= F_{22}\tilde{x}_r(t) - F_{21}\tilde{w}(t) + \Delta_2(t) \text{ if } t \in [t_k, t_{k+1}) \\ \tilde{x}_r(t_{k+1}) &= E^{-1}\int_{t_k}^{t_{k+1}}\Lambda(t_k, \ell)\tilde{w}(\ell)\mathrm{d}\ell \\ &\quad +E^{-1}\int_{t_k}^{t_{k+1}}e^{(t_k-\ell)F_{22}}\Delta_2(\ell)\mathrm{d}\ell \\ &\quad -E^{-1}\int_{t_k}^{t_{k+1}}e^{(t_k-\ell)H}\Delta_3(\ell)\mathrm{d}\ell \end{aligned} \quad (56)$$

for all integers $k \ge 1$. By integrating the system (56) over $[t_k, t]$ with $t \in [t_k, t_{k+1})$, we obtain:

$$\begin{aligned} \tilde{x}_r(t) =\ & e^{F_{22}(t-t_k)}\tilde{x}_r(t_k) \\ &+\int_{t_k}^{t}e^{F_{22}(t-\ell)}[-F_{21}\tilde{w}(\ell) + \Delta_2(\ell)]\mathrm{d}\ell \end{aligned} \quad (57)$$

for all $t \in [t_k, t_{k+1})$. As an immediate consequence of this equality and of (28), we have

$$\begin{aligned} |\tilde{x}_r(t)| \le\ & e^{\nu|F_{22}|}|\tilde{x}_r(t_k)| \\ &+\int_{t_k}^{t}e^{|F_{22}|(t-\ell)}[|F_{21}||\tilde{w}(\ell)| + f_{\dagger,2}|\tilde{w}(\ell)|]\mathrm{d}\ell \\ \le\ & e^{\nu|F_{22}|}|\tilde{x}_r(t_k)| \\ &+e^{|F_{22}|\nu}(|F_{21}| + f_{\dagger,2})\int_{t_k}^{t}|\tilde{w}(\ell)|\mathrm{d}\ell. \end{aligned} \quad (58)$$

On the other hand, by using the second equality in (56), to upper bound the $|\tilde{x}_r(t_k)|$ in (58), we have

$$\begin{aligned} &|\tilde{x}_r(t)| \le e^{\nu|F_{22}|}\overline{E}\int_{t_{k-1}}^{t_k}|\Lambda(t_{k-1}, \ell)||\tilde{w}(\ell)|\mathrm{d}\ell \\ &+\overline{E}\left(e^{2\nu|F_{22}|}f_{\dagger,2} + e^{\nu(|H|+|F_{22}|)}f_{\dagger,3}\right)\int_{t_{k-1}}^{t_k}|\tilde{w}(\ell)|\mathrm{d}\ell \\ &+e^{|F_{22}|\nu}(|F_{21}| + f_{\dagger,2})\int_{t_k}^{t}|\tilde{w}(\ell)|\mathrm{d}\ell \end{aligned} \quad (59)$$

for all $k \ge 2$ and $t \in [t_k, t_{k+1})$ with $f_{\dagger,3}$ defined in (31) and $\overline{E}$ defined in (30). It follows from our formula (43) for Λ that

$$\begin{aligned} &|\tilde{x}_r(t)| \le \\ &e^{\nu|F_{22}|}\overline{E}\left(e^{\nu|H|}|G| + e^{\nu|F_{22}|}|F_{21}|\right)\int_{t_{k-1}}^{t_k}|\tilde{w}(\ell)|\mathrm{d}\ell \\ &+\left[\overline{E}\left(e^{2\nu|F_{22}|}f_{\dagger,2} + e^{\nu(|H|+|F_{22}|)}f_{\dagger,3}\right)\right. \\ &\left.+e^{|F_{22}|\nu}(|F_{21}| + f_{\dagger,2})\right]\int_{t_{k-1}}^{t}|\tilde{w}(\ell)|\mathrm{d}\ell \end{aligned} \quad (60)$$

Consequently,

$$|\tilde{x}_r(t)| \le \beta(\nu)\int_{t-2\nu}^{t}|\tilde{w}(\ell)|\mathrm{d}\ell \quad (61)$$

with β defined in (32) for all $t \ge 2\nu$.

Fourth part of the proof: stability analysis.

Grouping (55) and (61), we have

$$\begin{aligned} |\tilde{x}_r(t)| \le\ & \beta(\nu)\int_{t-2\nu}^{t}|\tilde{w}(\ell)|\mathrm{d}\ell \\ |\tilde{w}(t)| \le\ & e^{\mu|F_{11}|}f_{\dagger,1}\int_{t-\mu}^{t}|\tilde{w}(m)|\mathrm{d}m \\ &+e^{\mu|F_{11}|}|F_{12}|\int_{t-\mu}^{t}|\tilde{x}_r(m)|\mathrm{d}m \end{aligned} \quad (62)$$

for all $t \ge 2\nu$. It follows that

$$\begin{aligned} &|\tilde{x}_r(t)| \le \\ &\beta(\nu)\int_{t-2\nu}^{t}\left(e^{\mu|F_{11}|}f_{\dagger,1}\int_{\ell-\mu}^{\ell}|\tilde{w}(m)|\mathrm{d}m\right. \\ &\left.+e^{\mu|F_{11}|}|F_{12}|\int_{\ell-\mu}^{\ell}|\tilde{x}_r(m)|\mathrm{d}m\right)\mathrm{d}\ell \end{aligned} \quad (63)$$

for all $t \ge 4\nu$. Consequently,

$$\begin{aligned} |\tilde{x}_r(t)| \le\ & \beta(\nu)e^{\mu|F_{11}|}f_{\dagger,1}\int_{t-2\nu}^{t}\mu|\tilde{w}|_{[\ell-\mu,\ell]}\mathrm{d}\ell \\ &+\beta(\nu)e^{\mu|F_{11}|}|F_{12}|\int_{t-2\nu}^{t}\mu|\tilde{x}_r|_{[\ell-\mu,\ell]}\mathrm{d}\ell \\ |\tilde{w}(t)| \le\ & e^{\mu|F_{11}|}f_{\dagger,1}\mu|\tilde{w}|_{[t-\mu,t]} \\ &+e^{\mu|F_{11}|}|F_{12}|\mu|\tilde{x}_r|_{[t-\mu,t]} \end{aligned} \quad (64)$$

for all $t \ge 4\nu$. It follows that

$$\begin{aligned} |\tilde{x}_r(t)| \le\ & 2\mu\nu\beta(\nu)e^{\mu|F_{11}|}\left(f_{\dagger,1}|\tilde{w}|_{[t-2\nu-\mu,t]}\right. \\ &\left.+|F_{12}||\tilde{x}_r|_{[t-2\nu-\mu,t]}\right) \\ |\tilde{w}(t)| \le\ & \mu e^{\mu|F_{11}|}\left(f_{\dagger,1}|\tilde{w}|_{[t-\mu,t]}\right. \\ &\left.+|F_{12}||\tilde{x}_r|_{[t-\mu,t]}\right) \end{aligned} \quad (65)$$

Let

$$\varsigma(t) = |\tilde{x}_r(t)| + |\tilde{w}(t)|. \quad (66)$$

Then the inequalites in (65) imply that

$$\begin{aligned} \varsigma(t) \le\ & \mu e^{\mu|F_{11}|}(2\nu\beta(\nu) + 1)\left[f_{\dagger,1}|\tilde{w}|_{[t-2\nu-\mu,t]}\right. \\ &\left.+|F_{12}||\tilde{x}_r|_{[t-2\nu-\mu,t]}\right] \end{aligned} \quad (67)$$

for all $t \ge 4\nu$. Consequently,

$$\begin{aligned} &\varsigma(t) \\ &\le \mu e^{\mu|F_{11}|}(2\nu\beta(\nu) + 1)\max\{|F_{21}|, f_{\dagger,1}\}|\varsigma|_{[t-2\nu-\mu,t]} \\ &= \mu\gamma(\mu)|\varsigma|_{[t-2\nu-\mu,t]} \text{ if } t \ge 4\nu \end{aligned} \quad (68)$$

with γ defined in (33). Then we can apply (Mazenc et al., 2017, Lemma 1) to the function $X(t) = \varsigma(t+m)$ to obtain

$$\varsigma(t) \le e^{\frac{\ln(\mu\gamma(\mu))}{\mu+2\nu+m}(t-m)}|\varsigma|_{[m-(\mu+2\nu),m]} \quad (69)$$

for all $t \ge m \ge 4\nu$. It follows that

$$\begin{aligned} &|\tilde{x}_r(t)| \le \\ &e^{\frac{\ln(\mu\gamma(\mu))}{\mu+2\nu+m}(t-m)}\left(|\tilde{x}_r|_{[m-(\mu+2\nu),m]} + |\tilde{w}|_{[m-(\mu+2\nu),m]}\right) \end{aligned} \quad (70)$$

for all $t \ge m \ge 4\nu$. This allows us to conclude.

5. ILLUSTRATION

In this section, we illustrate Theorem 2. As in Dinh et al. (2015), we study the pendulum model

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = \sin(x_1(t)), \\ y(t) = x_1(s_j) \text{ if } t \in [s_j, s_{j+1}) \end{cases} \tag{71}$$

with $x_1(t)$ valued in $\mathbb{R}$ and $x_2(t)$ valued in $\mathbb{R}$.

With the notation of the previous sections, we have

$$\begin{cases} \dot{r}(t) = x_r(t) \\ \dot{x}_r(t) = f_2(r(t)) \\ y(t) = r(s_j) \text{ if } t \in [s_j, s_{j+1}) \end{cases} \tag{72}$$

with $f_1(r) = 0$ and $f_2(r) = \sin(r)$. We can take $f_{\dagger,1} = 0$, $f_{\dagger,2} = 1$, $F_{11} = F_{22} = F_{21} = 0$, $F_{12} = 1$. We can choose $L = -1$ and any constant $\nu > 0$. Then $H = -1$, $G = -1$, $E = 1 - e^{\nu}$, $R_1 = \frac{1}{1-e^{\nu}}$, $R_2 = -\frac{e^{\nu}}{1-e^{\nu}}$ and $f_3 = f_2$.

Assumptions A1 to A3 are satisfied with any $g \in \mathbb{N}$,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } C = [1 \quad 0]. \tag{73}$$

Then Theorem 2 applies. It yields the observer

$$\begin{cases} \dot{z}_1(t) = \sin(w(t), \text{ if } t \in [s_{gk}, s_{g(k+1)}) \\ \dot{z}_2(t) = -z_2(t) - w(t) + \sin(w(t) \\ \qquad \text{if } t \in [s_{gk}, s_{g(k+1)}) \\ z_1(s_{g(k+1)}) = \frac{1}{1-e^{\nu}} z_1(s^-_{g(k+1)}) - \frac{e^{\nu}}{1-e^{\nu}} z_2(s^-_{g(k+1)}) \\ \qquad - \frac{e^{\nu}}{1-e^{\nu}} y(s_{g(k+1)}) + \frac{1}{1-e^{\nu}} y(s_{gk}) \\ z_2(s_{g(k+1)}) = \frac{1}{1-e^{\nu}} z_1(s^-_{g(k+1)}) - \frac{e^{\nu}}{1-e^{\nu}} z_2(s^-_{g(k+1)}) \\ \qquad - \frac{e^{\nu}}{1-e^{\nu}} y(s_{g(k+1)}) + \frac{1}{1-e^{\nu}} y(s_{gk}) \\ \dot{w}(t) = z_1(t) \text{ if } t \in [s_{gk}, s_{g(k+1)}) \\ w(s_{gk}) = y(s_{gk}) \end{cases} \tag{74}$$

with $z_1(0) = z_2(0) = 0$, where $s_i = i\mu$ for all integers $i \geq 0$, and where $\mu = \nu/g$. Then $f_{\dagger,3} = f_{\dagger,2} = 1$,

$$\beta(\nu) = \frac{3e^{\nu}}{e^{\nu} - 1}, \tag{75}$$

$$\text{and } \gamma(\mu) = 2\nu\beta(\nu) + 1 = 6\nu \frac{e^{\nu}}{e^{\nu} - 1} + 1, \tag{76}$$

so for our fixed choice of $\nu > 0$, $\gamma(\mu)$ is a constant. Hence, our requirement $\mu\gamma(\mu) < 1$ from (34) is

$$\mu < \frac{e^{\nu} - 1}{6\nu e^{\nu} + e^{\nu} - 1}. \tag{77}$$

For instance, if we choose $\nu = 1$ and $m = 4\nu = 4$, then Theorem 2 gives the convergence rate

$$-\frac{\ln(\mu\gamma(\mu))}{\mu + 6} = -\frac{\ln(\mu) + \ln(7e - 1) - \ln(e - 1)}{\mu + 6} \tag{78}$$

for the observer, and (78) converges to $+\infty$ as $\mu \to 0^+$, or equivalently, as $g = 1/\mu \to +\infty$ with $g \in \mathbb{N}$.

6. CONCLUSIONS

We have proposed two families of reduced order continuous-discrete observers for systems with continuous and respectively for systems discrete measurements. The simulations we performed illustrate their efficiency.

Many extensions of our results are expected. They include the case where the sampling of the output is asynchronous, systems with delay, time-varying systems, proofs of robustness of ISS type.

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