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Frederic Mazenc, Michael Malisoff, Miroslav Krstic

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Frédéric Mazenc * Michael Malisoff ** Miroslav Krstic ***

* *Inria EPI DISCO, L2S-CNRS-CentraleSupélec, 3 rue Joliot Curie, 91192, Gif-sur-Yvette, France (e-mail: frederic.mazenc@l2s.centralesupelec.fr)*
** *Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803-4918, USA (email: malisoff@lsu.edu)*
*** *Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093-0411, USA (email: krstic@ucsd.edu)*

Abstract: We present a generalized form of Halanay's inequality, having a time varying gain multiplying the delayed term and a constant decay rate. Unlike the usual Halanay's conditions where the decay rate is required to exceed an upper bound on the gain multiplying the delayed term, we provide less restrictive conditions that allow times when the decay rate can be strictly less than the gain. We include an application to continuous time systems with switched delay values. This illustrates the utility of our generalized Halanay's inequality conditions for proving asymptotic stability in significant cases that violate the contraction condition that was needed to prove asymptotic stability in previous trajectory based results, and which are also not amenable to previous Lyapunov function constructions.

Keywords: Stability, delay systems, switching

1. INTRODUCTION

We continue our development (which started in Ahmed et al. (2018); Mazenc and Malisoff (2015); Mazenc et al. (2017, 2018)) of contractivity and trajectory based approaches to proving asymptotic stability in cases that may not lend themselves to standard Lyapunov functional methods. See, e.g., Mazenc et al. (2017, 2018) for systems with discontinuous delays, and applications to switched systems where some subsystems satisfy asymptotic stability properties while other subsystems could be unstable. One setting where contractivity and trajectory-based methods are useful is for systems whose vector field could be discontinuous, which arise in numerous cases including dynamics that are asymptotically stabilized by piecewise constant feedbacks, and systems with switched delays. To prove asymptotic stability for these systems, some valuable tools include Halanay's approach (e.g., from Halanay (1966)) and extensions of Razumikhin's theorem (e.g., from Zhou and Egorov (2016)).

While many works have pursued Lyapunov function construction methods for proving asymptotic stability (including Malisoff and Mazenc (2009); Zhou (2019); Zhou et al. (2020)), it can sometimes be easier to compute constants $\rho \in (0,1)$ and $T_* > 0$ such that every solution ζ of a dynamics satisfies an inequality of the form $|\zeta(t)| \leq \rho \sup_{l\in[t-T_*,t]} |\zeta(l)|$ for all $t \geq T_*$, in which case ρ is called a contractivity constant and we say that the dynamics satisfy a contractivity condition. One can often verify contractivity conditions by first finding a nonnegative valued differentiable function V such that all solutions of a dynamics satisfy a Halanay inequality of the type $\dot{V}(\zeta(t)) \leq -cV(\zeta(t)) + d(t)\sup_{t-T\leq \ell\leq t} V(\zeta(\ell))$ for some positive constants c (called a decay rate) and T, for some nonnegative valued function $d(t)$ (called a gain), and for all $t \geq 0$. For a statement of the usual Halanay's inequality conditions, see (Fridman, 2014, Lemma 4.2), (Selivanov and Fridman, 2015, Lemma 1), or (Selivanov and Fridman, 2016, Lemma 1), which conclude that V converges exponentially to 0 provided $c > \sup_t d(t)$. If $c \leq \sup_t d(t)$, then the usual Halanay's inequality conditions do not allow us to prove exponential stability, and such case may not lend themselves to using previously reported contractivity conditions to prove exponential stability.

Therefore, this paper provides a relaxed version of Halanay's inequality. Our generalized Halanay's conditions allow the gain multiplying the delayed term to exceed the decay rate on arbitrarily long intervals, provided the gain is strictly less than the decay rate on other intervals whose lengths are large enough to compensate for the instants when the usual Halanay's condition does not hold. We illustrate our work in a class of dynamics with a piecewise constant switching delay that switches between a small and an arbitrarily large value. This illustrates the usefulness of our less restrictive versions of Halanay's conditions.

We use standard notation, where the dimensions of our Euclidean spaces are arbitrary unless otherwise noted, and which is simplified when no confusion would arise from the context. The standard Euclidean 2-norm, and the induced matrix norm, are both denoted by $|\cdot|$, $|\cdot|_S$ denotes the

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supremum over any set S, and $|\cdot|_\infty$ is the usual sup norm. We define Ξ_t by $\Xi_t(s) = \Xi(t+s)$ for all Ξ, $s \le 0$, and $t \ge 0$ such that $t+s$ is in the domain of Ξ. We set $\mathbb{Z}_{\ge 0} = \{0, 1, \ldots\}$ and $\mathbb{N} = \mathbb{Z}_{\ge 0} \setminus \{0\}$. Throughout this work, we consider sequences $t_i \in [0, +\infty)$ such that $t_0 = 0$ and there are two constants $\bar{T} > 0$ and $\underline{T} > 0$ such that

$$\underline{T} \le t_{i+1} - t_i \le \bar{T} \text{ for all } i \in \mathbb{Z}_{\ge 0}. \tag{1}$$

For square matrices M_1 and M_2 of the same size, we use $M_1 \le M_2$ to mean that $M_2 - M_1$ is a nonnegative definite matrix, and I is the identity matrix in the dimension under consideration. For delay systems, our initial conditions are assumed to be continuous initial functions.

2. GENERALIZED HALANAY'S CONDITIONS

We provide our extension of Halanay's inequality, whose significance follows because the study of switched systems with delays (and of observers for delay systems with sampled output values) often leads to generalized Halanay's inequality of the kind we study, as we illustrate below.

2.1 Assumptions

Let t_i be a sequence of instants that satisfies the conditions from Section 1 for some constant $\underline{T} > 0$. Let

$$E = \cup_{i\in\mathbb{N}}[t_i, t_i + T) \tag{2}$$

where $T > 0$ is a constant such that

$$\underline{T} > 2T. \tag{3}$$

Then (1) and (3) ensure that the intervals $[t_i, t_i + T)$ defining the set E are disjoint. We use constants

$$c > 0,\ \ \bar{\epsilon} \in [0, c), \text{ and } \overline{\varphi} > 0 \tag{4}$$

and the functions

$$\varphi(t) = \begin{cases} 0, \text{ if } t \notin E \\ \overline{\varphi}, \text{ if } t \in E \end{cases} \text{ and } \epsilon(t) = \begin{cases} \bar{\epsilon}, \text{ if } t \notin E \\ 0, \text{ if } t \in E\ . \end{cases} \tag{5}$$

Consider any continuous function $v : [-T, +\infty) \to [0, +\infty)$ such that

$$\dot{v}(t) \le -cv(t) + [\epsilon(t) + \varphi(t)]|v|_{[t-T,t]} \tag{6}$$

holds for all $t \ge 0$, under the following assumption:

Assumption 1. Either

$$\overline{\varphi} < c \tag{7}$$

or the inequality

$$\overline{\varphi}\left[e^{c(2T-\underline{T})} + \frac{\bar{\epsilon}}{c}\right]e^{2T\overline{\varphi}} + \frac{2T\overline{\varphi}}{\underline{T}} < c \tag{8}$$

is satisfied. □

2.2 Main result

Our main result is:

Theorem 1. Let $v(t)$ be a nonnegative valued solution of (6) under Assumption 1. Then we can construct positive constants $\bar{C}_1$ and $\bar{C}_2$ such that

$$v(t) \le \bar{C}_1 e^{-\bar{C}_2 t}|v|_{[-\underline{T},0]} \tag{9}$$

holds for all $t \ge 0$. □

Remark 1. Assumption 1 can be interpreted to mean that regardless of how large $\overline{\varphi}$ and T are, the function v exponentially converges to zero if $\underline{T}$ is sufficiently large and $\bar{\epsilon}$ is small enough. We can interpret the constant $\bar{\epsilon}$ as the amount by which (6) differs from being a Lyapunov decay condition of the type $\dot{v}(t) \le -cv(t)$ with decay rate $c > 0$ at times $t \notin E$. □

2.3 Sketch of proof of Theorem 1

We only sketch the proof here; see Mazenc et al. (2020) for complete proofs. We can assume that

$$\dot{v}(t) = -cv(t) + [\epsilon(t) + \varphi(t)]|v|_{[t-T,t]} \tag{10}$$

because if (10) is not satisfied, then we can apply a comparison lemma to prove exponential convergence of functions satisfying (6); see the appendix below. We can assume that $\overline{\varphi} \ge c$, because the case $\overline{\varphi} < c$ follows from the usual version of Halanay's inequality. Consider two cases.

First case: $t \notin E$. In this case, (10) gives

$$\dot{v}(t) = -cv(t) + \bar{\epsilon}|v|_{[t-T,t]}. \tag{11}$$

Second case: $t \in E$ and $t \ge t_1$. Then, there is $j \in \mathbb{N}$ such that $t \in [t_j, t_j + T)$ and

$$\dot{v}(t) = -cv(t) + \overline{\varphi}|v|_{[t-T,t]}. \tag{12}$$

Then

$$\dot{v}(t) \le -cv(t) + \overline{\varphi}|v|_{[t-T,t_j]} + \overline{\varphi}|v|_{[t_j,t]}. \tag{13}$$

Also, (12) gives $\dot{v}(t) \ge -cv(t) + \overline{\varphi}v(t) \ge 0$ for all $t \in [t_j, t_j + T)$ because $\overline{\varphi} \ge c$. It follows that $|v|_{[t_j,t]} = v(t)$ for all $t \in [t_j, t_j + T]$. Then (13) gives

$$\dot{v}(t) \le (\overline{\varphi} - c)v(t) + \overline{\varphi}|v|_{[t-T,t_j]}. \tag{14}$$

From (11), it follows that for all $\ell \in [t_{j-1} + T, t_j)$ and $s \in [t_{j-1} + T, \ell]$, we have

$$v(\ell) = e^{c(s-\ell)}v(s) + \bar{\epsilon}\int_s^\ell e^{c(m-\ell)}|v|_{[m-T,m]}\mathrm{d}m. \tag{15}$$

Let $\ell \in [t-T, t_j)$. Then, according to (3), we have $t \ge t_j \ge t_{j-1} + \underline{T} > t_{j-1} + 2T$, which gives $\ell \in [t_{j-1} + T, t_j)$. On the other hand, (3) implies that $t - \underline{T} + T < t - T \le \ell$. Also, $t - \underline{T} + T \ge t_j - \underline{T} + T \ge t_{j-1} + T$. Thus $t - \underline{T} + T \in [t_{j-1} + T, \ell)$. Also, (3) gives $t - T \ge t_j - T \ge t_{j-1} + \underline{T} - T \ge t_{j-1} + T$, and therefore $[t - T, t_j) \subseteq [t_{j-1} + T, t_j)$. Thus, we can set $s = t - \underline{T} + T$ in (15) to get

$$\begin{aligned} v(\ell) \le\ & e^{c(2T-\underline{T})}v(t - \underline{T} + T) \\ & + \bar{\epsilon}\int_{t-\underline{T}+T}^\ell e^{c(m-\ell)}|v|_{[m-T,m]}\mathrm{d}m. \end{aligned} \tag{16}$$

It follows that for all $\ell \in [t - T, t_j)$, we have

$$\begin{aligned} v(\ell) \le\ & e^{c(2T-\underline{T})}v(t - \underline{T} + T) \\ & + \bar{\epsilon}\int_{t-\underline{T}+T}^\ell e^{c(m-\ell)}\mathrm{d}m|v|_{[t-\underline{T},\ell]} \\ \le\ & \left[e^{c(2T-\underline{T})} + \frac{\bar{\epsilon}}{c}\right]|v|_{[t-\underline{T},\ell]}. \end{aligned} \tag{17}$$

Combining the last inequality in (17) with (14) gives

$$\dot{v}(t) \le (\overline{\varphi} - c)v(t) + \overline{\varphi}\left[e^{c(2T-\underline{T})} + \frac{\bar{\epsilon}}{c}\right]|v|_{[t-\underline{T},t]}. \tag{18}$$

General case. We deduce from (18) and (11) that

$$\dot{v}(t) \le \big(\varphi(t) - c\big)v(t) + \overline{\kappa}|v|_{[t-\underline{T},t]} \tag{19}$$

with

$$\overline{\kappa} = \overline{\varphi}\left[e^{c(2T-\underline{T})} + \tfrac{\bar{\epsilon}}{c}\right] \tag{20}$$

for all $t \ge t_1$ because our condition $\bar{\varphi} \ge c$ implies that $\overline{\varphi}\frac{\bar{\epsilon}}{c} \ge \bar{\epsilon}$. We can now use (19) to prove the exponential convergence assertion of the theorem.

To this end, first notice that

$$\frac{1}{\underline{T}}\int_{t-\underline{T}}^{t}\int_{\ell}^{t}\varphi(m)\mathrm{d}m\mathrm{d}\ell \leq \int_{t-\underline{T}}^{t}\varphi(m)\mathrm{d}m \leq 2T\overline{\varphi} \tag{21}$$

for all $t \geq \underline{T}$, where the second inequality follows by letting i be the largest index such that $t_i \leq t-\underline{T}$ and by separately considering the cases $t_{i+1} > t$ and $t_{i+1} \leq t$ and noting that $t_{i+2} - t_{i+1} \geq \underline{T}$. Hence, we can use (19) and then our dynamics (10) for v to conclude that the time derivative of

$$\mu(t) = e^{-\frac{1}{\underline{T}}\int_{t-\underline{T}}^{t}\int_{\ell}^{t}\varphi(m)\mathrm{d}m\mathrm{d}\ell}v(t) \tag{22}$$

satisfies

$$\begin{aligned}\dot{\mu}(t) &\leq e^{-\frac{1}{\underline{T}}\int_{t-\underline{T}}^{t}\int_{\ell}^{t}\varphi(m)\mathrm{d}m\mathrm{d}\ell}\Big[-cv(t) + \overline{\kappa}|v|_{[t-\underline{T},t]} \\ &\quad + \frac{1}{\underline{T}}\int_{t-\underline{T}}^{t}\varphi(m)\mathrm{d}m v(t)\Big] \\ &\leq \left(\frac{2T\overline{\varphi}}{\underline{T}} - c\right)\mu(t) + \overline{\kappa}|v|_{[t-\underline{T},t]}\end{aligned} \tag{23}$$

for all $t \geq \underline{T}$. We then conclude from (21) that

$$\dot{\mu}(t) \leq \left(\frac{2T\overline{\varphi}}{\underline{T}} - c\right)\mu(t) + \overline{\kappa}e^{2T\overline{\varphi}}|\mu|_{[t-\underline{T},t]} \tag{24}$$

for all $t \geq \underline{T}$. Since Assumption 1 ensures that

$$\overline{\kappa}e^{2T\overline{\varphi}} < c - \frac{2T\overline{\varphi}}{\underline{T}}, \tag{25}$$

it follows from the classical Halanay's result (e.g., (Fridman, 2014, Lemma 4.2)) that (24) and (25) imply that $\mu(t)$ converges exponentially to zero when t goes to $+\infty$, which readily implies the exponential convergence of v.

3. SYSTEMS WITH SWITCHING DELAYS

Let t_i be a sequence as defined in Section 1 and τ_l and τ_s be nonnegative constants such that $\tau_l > \tau_s$ and

$$\underline{T} > 2(\tau_l + \tau_s). \tag{26}$$

We study systems of the form

$$\dot{x}(t) = Mx(t) + Nx(t-\tau(t)) \tag{27}$$

where x valued in $\mathbb{R}^n$, $M \in \mathbb{R}^{n\times n}$ and $N \in \mathbb{R}^{n\times n}$ are constant matrices, and τ is a time varying piecewise continuous unknown delay such that

$$\begin{aligned}&0 \leq \tau(t) \leq \tau_s \ \text{ if } \ t \notin E \ \text{ and} \\ &0 \leq \tau(t) \leq \tau_l \ \text{ if } \ \ t \in E\end{aligned} \tag{28}$$

where E was defined by (2) for some constant $T \in (0, \underline{T}/2)$. We introduce two assumptions, the second of which is a largeness condition on $\underline{T}$ and a smallness condition on τ_s:

Assumption 2. There are a symmetric positive definite matrix $Q \in \mathbb{R}^{n\times n}$ and a constant $q > 0$ such that

$$Q(M+N) + (M+N)^\top Q \leq -qQ \tag{29}$$

and

$$I \leq Q \tag{30}$$

are satisfied. □

Assumption 3. Either

$$|N^\top QN| < \frac{q^2}{16} \tag{31}$$

or the inequality

$$\begin{aligned}&|N^\top QN|\left[e^{q(2T-\underline{T})/2} + \frac{2L\tau_s}{q}\right]e^{\frac{16T|N^\top QN|}{q}} \\ &+\frac{2T|N^\top QN|}{\underline{T}} < \frac{q^2}{16}\end{aligned} \tag{32}$$

with

$$L = \frac{2|N^\top QN|(|M|+|N|)^2}{q} \tag{33}$$

is satisfied. □

Our main result for (27) is:

Theorem 2. Let (27) satisfy Assumptions 2 and 3. Then its origin is a globally exponentially stable equilibrium point on $\mathbb{R}^n$. □

Proof: (Sketch.) For all $t \geq 0$, we have

$$\dot{x}(t) = (M+N)x(t) + N[x(t-\tau(t)) - x(t)]. \tag{34}$$

It follows that the time derivative of

$$U(x) = x^\top Qx \tag{35}$$

along all trajectories of (34) satisfies

$$\begin{aligned}\dot{U}(t) &\leq -qU(x(t)) + \left\{2x(t)^\top QN\Delta_\tau(x_t)\right\} \\ &\leq -\frac{q}{2}U(x(t)) + \frac{2}{q}(\Delta_\tau(x_t))^\top N^\top QN\Delta_\tau(x_t)\end{aligned} \tag{36}$$

for all $t \geq 0$, where $\Delta_\tau(x_t) = x(t-\tau(t)) - x(t)$ and where we used the triangle inequality to get

$$\begin{aligned}&2\left|\sqrt{q/2}\sqrt{Q}x(t)\right|\left|\frac{\sqrt{Q}}{\sqrt{q/2}}N(x(t-\tau(t)) - x(t))\right| \\ &\leq \frac{q}{2}U(x(t)) + \frac{2}{q}\left|\sqrt{Q}N(x(t-\tau(t)) - x(t))\right|^2\end{aligned} \tag{37}$$

to bound the term in curly braces in (36). It follows that

$$\dot{U}(t) \leq -\frac{q}{2}U(x(t)) + \frac{8|N^\top QN|}{q}\sup_{l\in[t-\tau_l,t]}U(x(l)) \tag{38}$$

where the last inequality is a consequence of (30). On the other hand, the last inequality from (36) yields

$$\dot{U}(t) \leq -\frac{q}{2}U(x(t)) + \frac{2}{q}|N^\top QN|\left|\int_{t-\tau(t)}^{t}\dot{x}(s)\mathrm{d}s\right|^2 \tag{39}$$

for all $t \geq 0$. Then we can use (30) and Jensen's inequality to obtain

$$\begin{aligned}\dot{U}(t) &\leq -\frac{q}{2}U(x(t)) \\ &\quad + \frac{2}{q}|N^\top QN|\left|\int_{t-\tau(t)}^{t}[|M|+|N|]\sup_{m\in[s-\tau_l,s]}|x(m)|\mathrm{d}s\right|^2 \\ &\leq -\frac{q}{2}U(x(t)) \\ &\quad + \frac{2\tau(t)|N^\top QN|(|M|+|N|)^2}{q}\sup_{m\in[t-\tau_l-\tau(t),t]}U(x(m)),\end{aligned}$$

which we can combine with (38) to get

$$\dot{U}(t) \leq -\frac{q}{2}U(x(t)) + L\tau_s\sup_{m\in[t-\tau_l-\tau_s,t]}U(x(m)) \tag{40}$$

with L defined in (33) when $t \notin E$ and

$$\dot{U}(t) \leq -\frac{q}{2}U(x(t)) + \frac{8|N^\top QN|}{q}\sup_{l\in[t-\tau_l,t]}U(x(l)) \tag{41}$$

when $t \in E$. Assumption 3 ensures that Theorem 1 applies to $U(x)$ with the choices

$$c = \frac{q}{2}, \;\; \bar{\epsilon} = L\tau_s, \text{ and } \overline{\varphi} = \tfrac{8|N^\top QN|}{q}. \tag{42}$$

Hence, $U(x(t))$ converges exponentially to zero, which readily yields the conclusion of the theorem. □

4. CONCLUSION

We proposed new stability results that complement both the Halanay's and the trajectory based approach. We have shown their usefulness in the context of switched systems with delays, without imposing any bound on the larger of the two delay values. A key feature of our work is that it allows cases where the gain multiplying the time varying term exceeds the decay rate in Halanay's inequality. In future work, we will apply our methods to observer design under scarce arbitrarily long sampling intervals in the scarcity sense of Mazenc (2019); see Mazenc et al. (2020). We also hope to find methods to maximize the rates of convergence in our theorems.

APPENDIX: COMPARISON LEMMA

We used the following lemma in our proof of Theorem 1:

Lemma A.1. Let $v : [-T, +\infty) \to [0, +\infty)$ be a nonnegative valued continuous solution of

$$\dot{v}(t) \leq -cv(t) + \Lambda(t)|v|_{[t-T,t]} \tag{A.1}$$

where $T > 0$ and $c > 0$ are constants, and where Λ is a piecewise constant function such that there is $\underline{\Lambda} > 0$ such that $\Lambda(t) > \underline{\Lambda}$ for all $t \geq 0$. Let w be a nonnegative valued solution of

$$\dot{w}(t) = -cw(t) + \Lambda(t)|w|_{[t-T,t]} \tag{A.2}$$

for all $t \geq 0$ such that there is a $t_0 \geq 0$ such that

$$v(m) < w(m) \text{ for all } m \in [t_0 - T, t_0]. \tag{A.3}$$

Then for all $t \geq t_0$, the inequality $v(t) < w(t)$ holds. □

Proof. For any continuous function $w : [t_0 - T, t_0] \to [0, +\infty)$, the continuous solution of (A.2) is uniquely defined on $[t_0 - T, +\infty)$; see (Hale and Verduyn Lunel, 1993, Chapt. 2). Consider v and w such that (A.3) is satisfied for all $t \in [t_0 - T, t_0]$. We proceed by contradiction. Suppose, for the sake of obtaining a contradiction, that the conclusion $v(t) < w(t)$ does not hold for all $t \geq t_0$. Then the continuity of v and w would imply that there is $t_c > t_0$ such that

$$v(m) < w(m) \text{ for all } m \in [t_0 - T, t_c) \tag{A.4}$$

and $v(t_c) = w(t_c)$. Moreover, (A.1) and (A.2) imply that for all $t \in [t_0, t_c)$, the function $\tilde{w}(t) = w(t) - v(t)$ satisfies

$$\dot{\tilde{w}}(t) \geq -c\tilde{w}(t) + \Lambda(t)[|w|_{[t-T,t]} - |v|_{[t-T,t]}]. \tag{A.5}$$

Setting $\varsigma(t) = e^{ct}\tilde{w}(t)$, we obtain the inequality $\dot{\varsigma}(t) \geq e^{ct}\Lambda(t)[|w|_{[t-T,t]} - |v|_{[t-T,t]}]$, which we can integrate over $[t, t_c]$ with $t \in [t_0, t_c)$ to obtain

$$\varsigma(t_c) - \varsigma(t) \geq \int_t^{t_c} e^{cm}\Lambda(m)[|w|_{[m-T,m]} - |v|_{[m-T,m]}]\mathrm{d}m \tag{A.6}$$

for all $t \in [t_0, t_c)$. Since $v(t_c) = w(t_c)$, we get $\varsigma(t_c) = 0$. Hence,

$$\varsigma(t) \leq - \int_t^{t_c} e^{cm}\Lambda(m)[|w|_{[m-T,m]} - |v|_{[m-T,m]}]\mathrm{d}m \tag{A.7}$$

holds for all $t \in [t_0, t_c)$. Since (A.4) and the continuity of v imply that $v(\ell) < |w|_{[m-T,m]}$ for all $\ell \in [m - T, m]$ and therefore also $|w|_{[m-T,m]} - |v|_{[m-T,m]} > 0$ for all $m \in [t_0, t_c)$, it now follows that

$$\varsigma(t) \leq -\underline{\Lambda}e^{ct} \textstyle\int_t^{t_c} [|w|_{[m-T,m]} - |v|_{[m-T,m]}]\mathrm{d}m \tag{A.8}$$

for all $t \in [t_0, t_c)$. This gives

$$\tilde{w}(t) \leq -\underline{\Lambda} \textstyle\int_t^{t_c} [|w|_{[m-T,m]} - |v|_{[m-T,m]}]\mathrm{d}m < 0 \tag{A.9}$$

for all $t \in [t_0, t_c)$. We conclude that $w(t) - v(t) < 0$ for all $t \in [t_0, t_c)$. This contradicts (A.4), so the lemma follows. □

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